Folded Hyperkähler Manifolds

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An in-depth formalism of Euclidean self-dual gravity is covered and then applied in the determination of several four-dimensional hyperkähler manifolds such as the Gibbons-Hawking ansatz, Plebański's "real heaven" background, and akin to Ashtekar's canonical reformulation of general relativity. A rigourous definition of a folded hyperkähler manifold is provided based on a canonical example, which is a particular form of the Gibbons-Hawking ansatz. Folded counterparts to Ashtekar's method for hyperkähler manifolds and also to the real heaven background are constructed and discussed.

Introduction

Recently, N. Hitchin has introduced the notion of a folded hyperkähler manifold, which is a determined by a 4-manifold in which the hyperkähler structure breaks down along a hypersurface known as the fold. In this report, we start by defining just what exactly a hyperkähler manifold is before determining a method of finding them via self-dual gravity in the Einstein-Cartan formulation of general relativity. From here several examples of hyperkähler manifolds are determined, with a foray into the Astekar-Jacobson-Smolin formulation for hyperkähler metrics. All of these examples are chosen specifically since later we study their folded counterparts.

The aim of this project is to study the notion of *folded hyperkähler manifolds*, i.e. a 4-dimensional manifold which is hyperkähler away from some folding hypersurface on which the hyperkähler structure degenerates and the metric is singular [1, 2]. In particular, it will be interesting to look into more examples of hyperkähler structures that admit a folding hypersurface, since the symplectic and Kähler versions of folding have already been studied in much more detail [3, 4].

From a physicist's point of view the topic of folded hyperkähler structures is still an interesting topic; the canonical example of a folded hyperkähler structure comes from a particular choice of the Gibbons-Hawking metric [1], and Biquard [2] has also constructed folded hyperkähler manifolds by modifying the work of Ashtekar, Jacobson and Smolin (ASJ) on half-flat solutions to Einstein's equations [5]. A specific feature of these two examples of folded hyperkähler manifolds is that the signature of the metric swaps from Euclidean $(++++)$ to anti-Euclidean $(----)$ as one travels across the fold; such a feature is a recurring theme in the physics literature on 5-dimensional supergravity, where hyperkähler manifolds act as the base space [6].

Background Theory

Hyperkähler Manifolds

A hyperkähler manifold is a Riemannian manifold of real dimension $4n$, which admits three covariantly orthogonal automorphisms, or almost complex structures I, J, and K of the tangent bundle that satisfy the quaternionic identies $I^2 = J^2 = K^2 = IJK = -1$ [7]. Since parallel transport preserves the almost complex structures on a hyperkähler manifold, its holonomy group is contained within the compact symplectic group $Sp(n) = GL(n, \mathbb{H})$ [8]. Due to the sequence of inclusions $Sp(n) \subset SU(2n) \subset U(2n) \subset SO(4n)$, each hyperkähler manifold is a Calabi-Yau manifold, which are also Kähler manifolds, and every Kähler manifold is orientable; therefore hyperkähler manifolds have become indispensable within the field of mathematical physics. Owing to this is the fact that any four-dimensional hyperkähler manifold is both Kähler and Ricci-flat, therefore solving the vacuum Einstein equations [7]; a sub-class of these hyperkähler manifolds are known in the physics literature as gravitational instantons [9, 10, 11], and we will come across an example later. In dimensions $n \geq 1$, hyperkähler manifolds also appear in nonlinear σ-models, since the action functionals are $N = 4$ supersymmetric if and only if the target manifold is hyperkähler ; the almost complex structures providing the additional three supersymmetries [12].

Self-Dual Gravity in Terms of the Self-Dual Spin Connection

In our approach to hyperkähler manifolds, we will consider complex general relativity first to simplify the process before taking certain reality conditions to arrive at our goal. This approach consists of a 4-manifold M and a metric $g_{\mu\nu}(x)$ on M in local coordinates x^μ. The metric can be decomposed into vierbeins/solder forms or tetrads $e^a{}_\mu(x)$ as

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu,$$
$$\eta^{ab} = g^{\mu\nu} e^a{}_\mu e^b{}_\nu.$$

Here Greek indices $\mu, \nu = 0, 1, 2, 3,$ transform as usual curved spacetime indices and are raised/lowered by $g^{\mu\nu}$, whereas the Latin indices $a, b = 1, 2, 3,$ are "internal" or flat indices which are raised/lowered by the Kronecker delta δ_{ab}. The $e^a{}_\mu$ can therefore be thought of as the "square root" of the metric g in a sense, with inverses defined by $E_a{}^\mu = g^{\mu\nu}\delta_{ab} e^b{}_\nu$. In terms of the flat indices, the torsion 1-form T^a and curvature 2-form $R^a{}_b$ are determined by the vierbeins as

$$T^a = de^a + \omega^a{}_b \wedge e^b, \qquad R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{1}$$

respectively, with $\omega^a{}_b$ as the spin-connection 1-form. Their respective tensors in terms of the spacetime indices are related to their flat index counterparts through multiplication by $e^a{}_\mu$, $E^\mu{}_a$

$$R^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu, \qquad T^a = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c = \frac{1}{2} T^a{}_{\mu\nu} dx^\mu \wedge dx^n, \tag{2}$$

with the Riemann tensor given by

$$R^\alpha{}_{\beta\mu\nu} = e^b{}_\beta E^\alpha{}_a R^a{}_{b\mu\nu}. \tag{3}$$

We will also assume the metricity condition $\omega^\mu{}_\nu = -\omega_\nu{}^\mu$ as well as the no torsion condition $T^a = 0$ in order for the Cartan formulation of geometry to be equivalent to the conventional Riemannian case [13]. The Riemann tensor is said to be *self-dual* is it satisfies

$$R^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2} \epsilon^{\mu\nu}{}_{\sigma\rho} R^{\sigma\rho}{}_{\alpha\beta}, \tag{4}$$

which then by virtue of the Bianchi identity $R_{\mu[\nu\alpha\beta]} = 0$ implies Einstein's vacuum equations

$$\epsilon_{\lambda\nu}{}^{\alpha\beta} R^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2}\epsilon_{\lambda\nu}{}^{\alpha\beta}\epsilon^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} = \mathcal{R}\delta_{\lambda}{}^{\mu} - 2\mathcal{R}_{\lambda}{}^{\mu} = 0, \tag{5}$$

where $\mathcal{R}_{ae}$ is the Ricci tensor [13]. For the hyperkähler structure that we are interested in we will have to impose reality conditions for the Euclidean signature on M, but for now we will consider complex general relativity to make our calculations easier [14]. In four-dimensions we can always decompose a 2-form into its self-dual and anti-self-dual parts

$$F = \frac{1}{2}(1+*)F + \frac{1}{2}(1-*)F = F^{+} + F^{-}, \tag{6}$$

since the Hodge involution operator acts as an involution on the vector space of 2-forms Λ^2 with eigenvalues $+1$ (for self-dual parts) and -1 (for anti-self-dual parts). For the field variables, we take a trio of two-forms Σ^a and an $\mathfrak{so}(3,\mathbb{C})$-valued connection 1-form A with corresponding curvature 2-form F defined by $F^a := dA^a + \frac{1}{2}\epsilon^{a}{}_{bc}A^b \wedge A^c$.

The vacuum Einstein field equations may be derived from the first-order action functional

$$\mathcal{S}[\Sigma^a, A^a, \Psi^a{}_b, v] = \int_M \left[\Sigma^a \wedge F_a - \frac{1}{2}\Psi_{ab}\Sigma^a \wedge \Sigma^b + \Psi^a{}_a v\right], \tag{7}$$

where the symmetric $SO(3,\mathbb{C})$ tensor $\Psi_{ab} = \Psi_{(ab)}$ and 4-form v are Lagrange multipliers [15]. The first term in the integrand is nothing more than the Einstein action when the connection form A satisfies its equation of motion, and the Lagrange multipliers are introduced in order to set constraints on the Σ^a. Minimising the action, we have the following:

$$\frac{\delta\mathcal{S}}{\delta\Psi_{ab}} = \int_M \delta\Psi_{ab}(-(1/2)\Sigma^a \wedge \Sigma^b + \delta^{ab}v) = 0, \tag{8a}$$

$$\frac{\delta\mathcal{S}}{\delta\Sigma^a} = \int_M \delta\Sigma^a \wedge (F_a - \Psi_{ab}\Sigma^b) = 0, \tag{8b}$$

$$\frac{\delta\mathcal{S}}{\delta v} = \int_M \delta v \Psi^a{}_a = 0, \tag{8c}$$

$$\frac{\delta\mathcal{S}}{\delta A_a} = \int_M A_a \wedge D\Sigma^a + D(\Sigma^a \wedge \delta A_a) = \int_M \delta A_a \wedge D\Sigma^a = 0, \tag{8d}$$

where we have used the symmetry of Ψ_{ab} in (8c). In (8d), $\delta F^a = d\delta A^a + \epsilon^a{}_{bc}A^b \wedge \delta A^c = D\delta A^a$, where we have defined the spin covariant exterior derivative $D\Sigma^a := d\Sigma^a + \omega^a{}_b \wedge \Sigma^b$

such that

$$D(\Sigma^a \wedge \delta A_a) = D\Sigma^a \wedge \delta A_a - \Sigma^a \wedge \delta F_a\,.$$

The term $\int_M D(\Sigma^a \wedge \delta A_a)$ is a total divergence, and therefore only contribute boundary terms which can be taken to be zero in (8d) since $\delta A_a = 0$ on ∂M. The equations of motion can now be read off easily as

$$\Sigma^a \wedge \Sigma^b = \frac{1}{3}\delta^{ab} v, \tag{9a}$$

$$D\Sigma^a \equiv d\Sigma^a + \epsilon^a{}_{bc} A^b \wedge \Sigma^c = 0, \tag{9b}$$

$$\Psi^a{}_a = 0, \tag{9c}$$

$$F^a = \Psi^a{}_b \Sigma^b, \tag{9d}$$

and express the content of the Einstein field equations [15, 14].

Indeed, the constraint (9a) on the two-forms Σ^a is the necessary and sufficient condition for the existence of a tetrad of 1-forms θ^μ such that Σ^a is equal to the self-dual part of the exterior product of two tetrad elements. It will be convenient to introduce four complex 1-forms θ^μ in terms of the e^μ as [16]

$$\theta^0 = \frac{1}{\sqrt{2}}(e^0 + e^3), \qquad \theta^1 = \frac{1}{\sqrt{2}}(e^1 - ie^2),$$
$$\theta^2 = \frac{1}{\sqrt{2}}(e^1 + ie^2), \qquad \theta^3 = \frac{1}{\sqrt{2}}(e^0 - e^3),$$

and fix the orientation of them by the requirement that the volume element is $v = -i\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$ (the reason for the minus sign is that we use x^0 as the local time coordinate, whereas [16] uses x^4). Introducing the complex linear combinations of 2-forms $Z^1 = \Sigma^1 + i\Sigma^2$ and $Z^2 = \Sigma^1 - i\Sigma^2$, and relabelling Σ^3 as Z^3, equation (9a) becomes the following algebraic constraints

$$Z^1 \wedge Z^1 = Z^2 \wedge Z^2 = 0, \tag{10a}$$

$$Z^1 \wedge Z^3 = Z^2 \wedge Z^3 = 0, \tag{10b}$$

$$Z^3 \wedge Z^3 = \frac{1}{2} Z^1 \wedge Z^2 = -v. \tag{10c}$$

Condition (10a) states the the Z^1, Z^2 are decomposable, that is they can be written as an exterior product of two complex 1-forms

$$Z^1 = \theta^2 \wedge \theta^3 \quad \text{and} \quad Z^2 = \theta^0 \wedge \theta^1$$

so from (10b), we can deduce that Z^3 must be of the form

$$Z^3 = \theta^0 \wedge (a\theta^2 + b\theta^3) + \theta^1 \wedge (c\theta^2 + d\theta^3) \tag{11}$$

with the constants satisfying $ad - bc = 1$ due to the condition (10c). Thus Z^3 is determined up to an $SO(3, \mathbb{C})$ gauge freedom [17], and therefore in fixing $b = -c = 1$, $a = d = 0$ we may choose without losing any information

$$Z^3 = \theta^0 \wedge \theta^3 - \theta^1 \wedge \theta^2. \tag{12}$$

From equations (11, 12) For the three Kähler 2-forms ω^a, we will consider the triple

$$\omega^1 = \Sigma^+ + \Sigma^-, \qquad \omega^2 = i(\Sigma^- - \Sigma^+),$$
$$\omega^3 = 2\Sigma^3,$$

however their integrability condition $d\omega^a = 0$ has yet to be established. Splitting the tetrad $e^\mu = \{e^0, e^a\}$ allows us to write the Σ^a explicitly in their self-dual forms:

$$\Sigma^a = e^0 \wedge e^a + \frac{1}{2}\epsilon^a{}_{bc} e^b \wedge e^c = \frac{1}{2}\eta^a{}_{\mu\nu} e^\mu \wedge e^\nu, \tag{13}$$

introducing the self-dual 't Hooft tensors $\eta^a_{\mu\nu}$ defined by

$$\eta^a{}_{\mu\nu} = \delta^0{}_\mu \delta^a{}_\nu - \delta^a{}_\mu \delta^0{}_\nu + \epsilon^{0a}{}_{\mu\nu}, \qquad\qquad \eta^a{}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}\eta^a{}_{\alpha\beta}, \tag{14}$$

and obey the following relations

$$\eta^a{}_{\mu\nu}\eta^b{}_{\mu\sigma} = \delta^{ab}\delta_{\nu\sigma} + \epsilon^{ab}{}_c \eta^c{}_{\nu\sigma}, \tag{15a}$$

$$\epsilon^a{}_{bc}\eta^b{}_{\mu\nu}\eta^c{}_{\alpha\beta} = \delta_{\mu\alpha}\eta^a{}_{\nu\beta} - \delta_{\mu\beta}\eta^a{}_{\nu\alpha} - \delta_{\nu\alpha}\eta^a{}_{\mu\beta} + \delta_{\nu\beta}\eta^a{}_{\mu\alpha}, \tag{15b}$$

$$\eta^a{}_{\alpha\beta}\eta_a{}^{\mu\nu} = \delta_\alpha{}^\mu \delta_\beta{}^\nu - \delta_\alpha{}^\nu \delta_\beta{}^\mu + \epsilon_{\alpha\beta}{}^{\mu\nu} \tag{15c}$$

amongst several others [18]. Their significance is this; the structure group that acts on the tangent space of $\mathcal{M}$ is $SO(4,\mathbb{C})$, which is locally isomorphic to[1] $SO(3,\mathbb{C})\times SO(3,\mathbb{C})$. This splitting is accompanied by the Hodge decomposition $\Lambda^2T^*\mathcal{M}\simeq\Lambda^2_+\oplus\Lambda^2_-$ of 2-forms on $\mathcal{M}$ into their self-dual and anti-self-dual components respectively. The 't Hooft tensors ${\eta^a}_{\mu\nu}$ just defined map the self-dual $\mathfrak{so}(4,\mathbb{C})$-valued 2-forms $\Sigma^{\mu\nu}$ to their respective $\mathfrak{so}(3,\mathbb{C})$-valued 3-vectors Σ^a, which are easier to work with [18].

Field equation (9b) allows us to identify the curvature F^a with the self-dual part of the Riemann curvature tensor [14]; the tetrad e^α determines a metric compatible spin connection ${\omega^\alpha}_\beta$ by the torsion-free condition $de^\alpha+{\omega^\alpha}_\beta\wedge e^\beta=0$, so equations (9a, 9b) imply

$$\begin{aligned}
D\Sigma^a\wedge =&d\Sigma^a\wedge+{\epsilon^a}_{bc}A^b\wedge\Sigma^c\\
=&\frac{1}{2}{\eta^a}_{\mu\nu}(de^\mu\wedge e^\nu-e^\mu\wedge de^\nu)+\frac{1}{2}{\epsilon^a}_{bc}{\eta^c}_{\mu\nu}A^b\wedge e^\mu\wedge e^\nu\\
=&\big({\eta^a}_{\mu\nu}{\omega^\mu}_\alpha+\frac{1}{2}{\epsilon^a}_{bc}{\eta^c}_{\alpha\nu}A^b\big)\wedge e^\alpha\wedge e^\nu=0\\
\implies& {\eta^a}_{\mu\nu}{\omega^\mu}_\alpha+\frac{1}{2}{\epsilon^a}_{bc}{\eta^c}_{\alpha\nu}A^b=0 \qquad\qquad \text{(from 9a)}\\
\implies& {\eta_d}^{\alpha\nu}{\eta^a}_{\mu\nu}{\omega^\mu}_\alpha={\epsilon^a}_{db}{\eta^b}_{\alpha\mu}\omega^{\mu\alpha}=2{\epsilon^a}_{db}A^b
\end{aligned}$$

Now applying ${\epsilon_a}^{de}$ to both sides of the last line and relabelling the free indices, we arrive at

$$A^a=\frac{1}{2}{\eta^a}_{\mu\nu}\omega^{\mu\nu}=\omega^{0a}+\frac{1}{2}{\epsilon^a}_{bc}\omega^{bc}\implies A^{\alpha\beta}=\frac{1}{2}{(1+*)^{\alpha\beta}}_{\mu\nu}\omega^{\mu\nu}\in\Lambda^2_+$$

where the implication follows from (15c), showing that A^a is determined purely by the self-dual part of the spin connection ${\omega^\alpha}_\beta$. Furthermore

$$\begin{aligned}
F^a&=dA^a+\frac{1}{2}{\epsilon^a}_{bc}A^b\wedge A^c\\
&=\frac{1}{2}{\eta^a}_{\mu\nu}d\omega^{\mu\nu}+\frac{1}{8}{\epsilon^a}_{bc}{\eta^b}_{\mu\nu}{\eta^c}_{\alpha\beta}\omega^{\mu\nu}\omega^{\alpha\beta}\\
&=\frac{1}{2}{\eta^a}_{\mu\nu}R^{\mu\nu}=R^{0a}+\frac{1}{2}{\epsilon^{0a}}_{bc}R^{bc}\implies F^{\alpha\beta}=\frac{1}{2}{(1+*)^{\alpha\beta}}_{\mu\nu}R^{\mu\nu}
\end{aligned}$$

having used (15b), identifying F with the self-dual component of the Riemann curvature

[1] Actually $SO(4,\mathbb{C})\simeq SL(2,\mathbb{C})_L\times SL(2,\mathbb{C})_R/\mathbb{Z}_2$ but, since we are interesting only in local descriptions on the Lie algebra level, we can ignore the $\mathbb{Z}_2$ factor. The L and R subscripts correspond to the "left" and "right" chiral elements of $SO(4,\mathbb{C})$ respectively. Alternatively one can observe $SO(3,\mathbb{C})\simeq SU(2,\mathbb{C})/\mathbb{Z}_2$.

tensor. The final equation of motion (9d) now states that the from the usual decomposition of the Riemann curvature tensor into its irreducible parts, one observes that Ψ must coincide with the self-dual Weyl tensor since Ψ is symmetric and traceless (from 9c) [14].

In order to have a hyperkähler structure, we must fix the gauge $A^a = 0$ [19], implying that the self-dual Weyl tensor Ψ must vanish *identically* from field equation (9d), because the volume form is non-degenerate [14]. The Weyl tensor is conformally invariant and can be thought of being defined by the anti-self-dual structure[2], $[h]$ [20]. Equations (9a, 9b) now become

$$\omega^1 \wedge \omega^1 = \omega^2 \wedge \omega^2 = \omega^3 \wedge \omega^3 = v \neq 0, \tag{16a}$$

$$d\omega^a = 0. \tag{16b}$$

Equation (16a) asserts the existence of the three Kähler forms and equation (16b) is the integrability condition that each ω^a is indeed Kähler .

Recall that E_μ represents the vector dual to e^μ, then (13) can be rewritten as [21]

$$\omega^a = -\frac{1}{2}\eta^{a\mu\nu}\imath_{E_\mu}\imath_{E_\nu}v. \tag{17}$$

Furthermore if we assert that the E_μ be volume preserving, that is if the Lie derivative of the volume-form v with respect to each vector field E_μ satisfies $\mathcal{L}_{E_\mu}v = 0$, then due to the closed property of each hyperkähler form ω^a it follows that

$$d\omega^a = -\frac{1}{2}\eta^{a\mu\nu}d(\imath_{E_\mu}\imath_{E_\nu}v) = -\frac{1}{2}\eta_a{}^{\mu\nu}\imath_{[E_\mu,E_\nu]}v = 0, \tag{18}$$

where we have made use of the identity [22]

$$d(\imath_{E_\mu}\imath_{E_\nu}v) = (\imath_{[E_\mu,E_\nu]}v + \imath_{E_\mu}\mathcal{L}_{E_\nu}v - \imath_{E_\nu}\mathcal{L}_{E_\mu}v + \imath_{E_\mu}\imath_{E_\nu}dv), \tag{19}$$

and the fact that $dv = 0$ since v is a volume form. From the non-degeneracy of v, we

[2]An anti-self-dual structure is a four-dimensional conformal structure such that the self-dual Weyl tensor Ψ vanishes [20].

conclude that

$$\frac{1}{2}\eta_a{}^{\mu\nu}[E_\mu, E_\nu] = 0 \implies [E_0, E_a] = \frac{1}{2}\epsilon_a{}^{bc}[E_b, E_c], \tag{20}$$

which reduces our search for hyperkähler manifolds in four-dimensions to that of finding four linearly independent vector fields E_μ that satisfy the following properties [21]

$$\mathcal{L}_{E_\mu} \upsilon = 0 \qquad \text{and} \qquad [E_0, E_a] = \frac{1}{2}\epsilon_a{}^{bc}[E_b, E_c]. \tag{21}$$

Examples of Hyperkähler Manifolds

This subsection borrows heavily from references [23, 24], but we use different vector fields E_μ in our approach; ones explicitly dual to the vierbeins e^μ. Since we have an anti-self-dual structure, it is possible to find a representative metric within the conformal class where a representative takes the form $\hat{h} = \delta_{\mu\nu} e^\mu \otimes e^\nu$. The vector fields are chosen to be volume-preserving with respect to some volume element $\hat{\upsilon}$, so in defining the function f by $\hat{\upsilon}(E_0, E_1, E_2, E_3) = f^2$ we obtain the physical, Ricci-flat metric by the conformal transformation $h = f^2 \hat{h}$ [25].

Example 1 (Gibbons-Hawking Ansatz). Let us write Euclidean space with standard coordinates $\mathbb{R}^4 = \{(\tau, x, y, z)\}$ as the underlying spacetime, with volume form $\upsilon = d\tau \wedge dx \wedge dy \wedge dz$. Then, considering the tetrad of vector fields E_μ with their corresponding vierbeins e^μ to be

$$\begin{aligned} E_0 &= V\frac{\partial}{\partial\tau}, & e^0 &= \frac{1}{V}(d\tau + A) \\ E_a &= \frac{\partial}{\partial x^a} - A_a\frac{\partial}{\partial\tau}, & e^a &= dx^a \end{aligned}$$

for the smooth functions $V \neq 0$, $A = A_a dx^a$ that are independent of τ, ensuring that the E_μ are volume preserving. Condition (20) translates to the equation

$$\underset{3}{*} dV = dA \tag{22}$$

which is known as the *Bogomolny equation* or the *monopole equation*. Here, $*_3$ is the Hodge star operator with respect to the Euclidean metric on $\mathbb{R}^3$ and states that V is a harmonic function on the space.

This is the ansatz that G. Gibbons and S. Hawking used when studying gravitational instantons that admit a tri-holomorphic Killing vector ∂/∂_τ, which in turn generates an S^1 symmetry [11]. A representative metric of $[h]$ is then

$$\hat{h} = \delta_{\mu\nu} e^\mu \otimes e^\nu = V^{-2}(d\tau + A)^2 + (dx^2 + dy^2 + dz^2),$$

with $v(E_0, E_1, E_2, E_3) = V$, hence we may perform the conformal transformation to obtain

$$h = V\hat{h} = V^{-1}(d\tau + A)^2 + V(dx^2 + dy^2 + dz^2) \tag{23}$$

which is now the Ricci-flat metricn [20]. The three Kähler 2-forms[3] are

$$\omega^a = (d\tau + A) \wedge dx^a + \frac{1}{2} V \epsilon^a{}_{bc} dx^b \wedge dx^c. \tag{24}$$

It is a fact that any four-dimensional hyperkähler metric which admits a tri-holomorphic Killing vector can locally be put into the form (23) [20]. This example is particularly important for when we begin to discuss folded hyperkähler manifolds, since we will use a specific example of the Gibbons-Hawking ansatz to formally define just how a hyperkähler structure should behave in order to admit a fold.

Example 2 (Real Heaven Background). Let us take Euclidean space again with the same coordinates and volume form as the previous example. This time however let us take the E_μ and e^μ to be given by

$$\begin{aligned}
E_0 &= e^{u/2}\left(u_z \cos(\tau/2)\frac{\partial}{\partial\tau} - \sin(\tau/2)\frac{\partial}{\partial z}\right), \qquad & E_2 &= \frac{\partial}{\partial x} - u_y \frac{\partial}{\partial \tau} \\
E_1 &= e^{u/2}\left(u_z \sin(\tau/2)\frac{\partial}{\partial\tau} + \cos(\tau/2)\frac{\partial}{\partial z}\right), \qquad & E_3 &= \frac{\partial}{\partial y} + u_x \frac{\partial}{\partial \tau}, \\
e^0 &= \frac{1}{e^{u/2}u_z}\left(\cos(\tau/2)(d\tau + u_y dx - u_x dy) - u_z \sin(\tau/2) dz\right), \qquad & e^2 &= dx, \\
e^1 &= \frac{1}{e^{u/2}u_z}\left(\sin(\tau/2)(d\tau + u_y dx - u_x dy) + u_z \cos(\tau/2) dz\right), \qquad & e^3 &= dy,
\end{aligned}$$

where u is a smooth function independent of τ in order for the volume-preserving condition

[3] In our conformal rescaling the hyperkähler forms will also be scaled by the same factor as the representative metric. This can be seen from the relation between the metric h and the three almost complex structures J^a *i.e.* $\omega^a(\cdot,\cdot) = h(J^a(\cdot),\cdot)$. For brevity however we shall just write the Kähler forms as they appear in the literature, with the conformal scaling $\omega^a = f^2\hat{\omega}^a$ implied.

to be satisfied. Constraint (20) now implies that u satisfies

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0, \tag{25}$$

which is known as the *Boyer-Finley equation* or the $SU(\infty)$-*Toda equation* in the physics literature, due to its connection to solid state physics in the continuum limit [26, 27]. A representative $[h]$ this time is

$$\hat{h} = \delta_{\mu\nu} e^\mu \otimes e^\nu = dx^2 + dy^2 + e^{-u}(dz^2 + u_z^{-2}(d\tau + u_y dx - u_x dy)^2)$$

with $v(E_0, E_1, E_2, E_3) = e^u u_z$, so conformally transforming again

$$h = e^u u_z \hat{h} = u_z(e^u(dx^2 + dy^2) + dz^2) + u_z^{-1}(d\tau + u_y dx - u_x dy)^2. \tag{26}$$

This metric also admits the Killing vector ∂/∂_τ but in this case it is not tri-holomorphic, instead admitting only one rotational Killing symmetry rather than translational symmetry [28]. This is why the current example differs from the Gibbons-Hawking ansatz previously considered and this different symmetry is reflected in the Kähler forms ω^a, which are split into an $SO(2)$ singlet

$$\omega^1 = u_z e^u dx \wedge dy + dz \wedge (d\tau + u_y dx - u_x dy) \tag{27}$$

and an $SO(2)$ doublet [29]

$$\begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix} = e^{u/2} \begin{pmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{pmatrix} \begin{pmatrix} (d\tau - u_y dx + u_x dy) \wedge dx + u_z dy \wedge dz \\ (d\tau - u_y dx + u_x dy) \wedge dy + u_z dz \wedge dx \end{pmatrix}. \tag{28}$$

This can be verified by considering the Lie derivative with respect to ∂/∂_τ,

$$\mathcal{L}_{\partial_\tau} \omega^1 = 0, \qquad \mathcal{L}_{\partial_\tau} \omega^2 = \omega^3, \qquad \mathcal{L}_{\partial_\tau} \omega^3 = -\omega^2,$$

explicitly showing how the lack of invariance that ω^2 and ω^3 exhibit under the action of ∂/∂_τ.

Real, self-dual, Euclidean Einstein spaces with one rotational Killing symmetry arise as real Euclidean cross-sections of complex $\mathcal{H}$-spaces, which are solutions to the complex vacuum Einstein equations with a self-dual curvature often called *heavens*, in the formalism of Plebański [17, 28]. These cross-sections are completely determined by the metric given by (26), and are called *real heavens.*

Remark 3. If we were to linearise equation (25) via the perturbation $u \mapsto u + \epsilon V$ and keep only the terms linear in ϵ, we would recover the *linearised Boyer-Finley* equation

$$V_{xx} + V_{yy} + (Ve^u)_{zz} = 0. \tag{29}$$

It can be shown that the metric arising from this is Ricci-flat if and only if $u_z = aV$ for some constant a. In particular, if the metric is Ricci-flat and $u = 0$ then we recover the Gibbons-Hawking ansatz and from equation (29) the monopole equation (22) is recovered [27].

There are several similarities between the two examples just presented; consider an open set $\mathcal{U} \subset \mathbb{R}^3$, and let $\mathcal{M} \xrightarrow{\pi} \mathcal{U}$ be a principal S^1-bundle over $\mathcal{U}$. Let A be the connection 1-form on $\mathcal{M}$ with curvature 2-form F, then both of the examples are on the total space of the S^1-bundle. To see this, choose a local trivialisation of $\mathcal{M}$ so that τ is a fibre coordinate on the circle with period 2π, then $A = d\tau + \theta$ for some 1-form θ defined on $\mathcal{U}$. In the Gibbons-Hawking ansatz $\theta = xdy$, whereas for the real heaven background $\theta = u_y dx - u_x dy$, which determine $\mathcal{M}$ and F up to gauge equivalence if $\mathcal{U}$ is simply connected [27].

The Ashtekar-Jacobson-Smolin Construction of Hyperkähler Manifolds

The next example warrants a discussion and hence this section is dedicated to just that. The premise is as follows; one decomposes a 4-dimensional spacetime $\mathcal{M}$ into $\mathcal{M} = \mathbb{R} \times \mathcal{N}$, where $\mathcal{N}$ is a 3-dimensional manifold. Let the leaves of the natural foliation of $\mathcal{M}$ be labelled by constant values of the coordinate τ, with ∂/∂_τ representing the normal vector field to each leaf. Then in labelling $V_0 = \partial/\partial_\tau$, condition (20) is equivalent to *Nahm's equations* for the triad of orthogonal vector fields V_a

$$\frac{\partial V_a}{\partial \tau} = \frac{1}{2}\epsilon_a{}^{bc}[V_b, V_c], \tag{30}$$

and the volume preserving condition holds assuming that the V_a depend solely on τ. It was from Ashtekar's Hamiltonian approach to general relativity, in which the Nahm's equations (30) for the Lie algebra of symplectomorphisms on $\mathcal{N}$[4] represent a form of self-dual Einstein equations on $\mathcal{M}$ [30]. Further elucidation on the work of Ashtekar has been carried out by Mason and Newman, who consider Yang-Mills theory for the Lie algebra of symplectomorphisms on some 4-manifold [31], as well by Donaldson [22]. We will briefly cover the ASJ construction of hyperkähler manifolds, because Biquard adapts the formalism for a existence and uniqueness theorem of a folded hyperkähler structure for real analytic data [2].

The ASJ construction is outlined as follows:

Proposition 4 (Donaldson [22]). *Let V_a be a triad of time-dependent, volume-preserving, linearly-independent vector fields*[5] *on a smooth 3-manifold $\mathcal{N}$ that satisfy Nahm's equations*

$$\frac{\partial V_a}{\partial \tau} = \frac{1}{2}\epsilon_a{}^{bc}[V_b, V_c].$$

Then there exist three holomorphic symplectic structures[6] *on the product $\mathcal{M} = \mathbb{R} \times \mathcal{N}$.*

The three holomorphic symplectic forms are closed, and fulfil the hyperkähler conditions (16b) thus implying the existence of a compatible Riemannian metric h with the complex structures [22]. There is a remarkable result that the converse holds as well:

Proposition 5 (Ashtekar [5] and Donaldson [22]). *Let $\mathcal{M}$ be a hyperkähler manifold with volume form v, and let τ be a harmonic function that vanishes on some hypersurface $\mathcal{N}$. Then we can find a triad of time-dependent, volume-preserving, linearly-independent vector fields V_i that satisfy Nahm's equations (30).*

These results will be indispensable when it comes to discussing O. Biquard's theorem for folded hyperkähler manifolds, based on a modified version of the ASJ formalism.

[4] A symplectomorphism on a manifold $\mathcal{N}$ is a volume-preserving diffeomorphism, and the space of them forms a Lie group $\mathrm{SDiff}(\mathcal{N})$ with an associated Lie algebra $\mathfrak{sdiff}(\mathcal{N})$ consisting of the volume-preserving vector fields on $\mathcal{N}$.

[5] If the 3-manifold $\mathcal{N}$ is orientable then it is always possible to find three linearly-independent vector fields. This is because $T\mathcal{N}$ is trivial, and so always admits a global section.

[6] A holomorphic symplectic form θ on a differentiable 4-manifold is a 2-form such that $\theta \wedge \bar{\theta} = 2\Omega$ is a volume form. Writing $\theta = \theta^1 + i\theta^2$, where θ^1 and θ^2 are real 2-forms, we have the algebraic conditions

$$\theta^1 \wedge \theta^1 = \theta^2 \wedge \theta^2 \neq 0, \qquad \theta^1 \wedge \theta^2 = 0.$$

Results

A Canonical Example

We begin this section by discussing the motivating example of a folded hyperkähler manifold, in order to see the natural structure which arises and to formulate a concise definition. We begin by recalling the Gibbons-Hawking metric, albeit a particular choice for the function V [1]:

$$h = \frac{1}{z}(d\tau + A)^2 + z(dx^2 + dy^2 + dz^2), \qquad dA = dx \wedge dy. \tag{31}$$

The hyperkähler forms are given by

$$\omega^1 = (d\tau + A) \wedge dz + z dx \wedge dy, \tag{32a}$$
$$\omega^2 = (d\tau + A) \wedge dx + z dy \wedge dz, \tag{32b}$$
$$\omega^3 = (d\tau + A) \wedge dy + z dz \wedge dx. \tag{32c}$$

The metric h is undefined at $z = 0$, and hence determines a hypersurface $\mathcal{Z}$ that divides the ambient manifold $\mathcal{M}$ into two disjoint ones; one with an Euclidean signature $(++++)$ when $z > 0$ and the other with an anti-Euclidean signature $(----)$ when $z < 0$. We see that under the involution $i : z \mapsto -z$ that

$$i^*\omega^1 = -\omega^1, \tag{33a}$$
$$i^*\omega^2 = \omega^2, \qquad i^*\omega^3 = \omega^3, \tag{33b}$$
$$i^*h = -h. \tag{33c}$$

Furthermore whilst h is undefined along the fold at $z = 0$, the Kähler forms $\omega^1, \omega^2, \omega^3$ are smooth there. Pulling them back to $\mathcal{Z}$ we have that

$$\mathcal{Z}^*\omega^1 = 0, \qquad \mathcal{Z}^*\omega^2 = \varphi \wedge dx, \qquad \mathcal{Z}^*\omega^3 = \varphi \wedge dy, \qquad \text{where } \varphi \equiv d\tau + A. \tag{34}$$

Recalling that $dA = dx \wedge dy$, we observe that

$$\varphi \wedge d\varphi = d\tau \wedge dx \wedge dy \neq 0, \tag{35}$$

and therefore φ determines a contact form on $\mathcal{Z}$.

The Definition and an Existence and Uniqueness Theorem for Hyperkähler Manifolds

The previous example from the physics literature allows us to define formally how a hyperkähler structure should behave if it is to allow folding behaviour:

Definition 6 ([1, 2])**.** A *folded hyperkähler structure* consists of a smooth 4-manifold $\mathcal{M}$, a smoothly embedded hypersurface $\mathcal{Z} \subset \mathcal{M}$, three smooth, closed, 2-forms ω^i $(i = 1, 2, 3)$ on $\mathcal{M}$, and an involution $\imath : \mathcal{M} \to \mathcal{M}$ which satisfy the following conditions

- $\mathcal{Z}$ divides $\mathcal{M}$ into two disjoint connected components: $\mathcal{M} \setminus \mathcal{Z} \simeq \mathcal{M}^+ \cup \mathcal{M}^-$.
- the 2-forms ω^i define a hyperkähler structure on $\mathcal{M}^\pm$ with hyperkähler metric $h^\pm$ where h^+ has Euclidean signature $(+ + ++)$ and h^- has anti-Euclidean signature $(- - --)$.
- on the hypersurface $\mathcal{Z} \subset \mathcal{M}$ we have that $\mathcal{Z}^*\omega^1 = 0$, $\mathcal{Z}^*\omega^2 \neq 0$, and $\mathcal{Z}^*\omega^3 \neq 0$ with a contact distribution $\mathcal{H} \subset T\mathcal{Z}$ given by $\mathcal{H} = \ker\mathcal{Z}^*\omega^2 \oplus \ker\mathcal{Z}^*\omega^3$.
- the involution $\imath$ fixes $\mathcal{Z}$ and maps $\mathcal{M}^\pm$ to $\mathcal{M}^\mp$ such that

$$\imath^* h^\pm = -h^\mp, \qquad \imath^*\omega^1 = -\omega^1, \qquad \imath^*\omega^2 = \omega^2, \qquad \imath^*\omega^3 = \omega^3. \tag{36}$$

Several comments about this definition are in order; firstly it is of interest as to how it differs from the already well-established notion of folded structures in symplectic and Kähler geometry. A *folded symplectic form* on a $2n$-dimensional manifold M is a closed 2-form ω whose top form ω^n vanishes transversally on a submanifold Z, and whose restriction as a form $\omega|_Z$ has maximal rank of $2n-2$. Then the pair (M, ω) defines a *folded symplectic structure*. The Kähler equivalent is similar with the additional fact that any compact smooth 4-manifold has a folded Kähler structure, such that the two components of $M \setminus Z$ determine Stein manifolds. Similarly to our hyperkähler definition is that the metric changes signature upon crossing the fold, however.

Evidently the folded symplectic and Kähler structures differ from how a hyperkähler structure admits a fold, since the latter case requires that the Kähler 2-form Σ^1, say, must vanish as a form when restricted to the fold hypersurface. More precisely, suppose that (M, h, ω^i) is a 4-dimensional hyperkähler manifold and presume that we have $\omega^1 \wedge \omega^1 = 0$ at some point $p \in M$. Then we have the algebraic constraints in $\Lambda^2 T_p^* M$ on the Kähler 2-forms

$$\omega^1 \wedge \omega^1 = \omega^2 \wedge \omega^2 = \omega^3 \wedge \omega^3 = 0, \qquad \omega^1 \wedge \omega^2 = \omega^2 \wedge \omega^3 = \omega^3 \wedge \omega^1 = 0.$$

If the ω^a are linearly-independent at p then the condition $\omega^a \wedge \omega^a = 0$ means that the ω^a are decomposable. If the ω^a were folded in the symplectic sense then we could decompose them as $\omega^a = x dx \wedge \alpha^a + \beta^a \wedge \gamma^a$, with the $\beta^a \wedge \gamma^a$ non-vanishing on the fold at $x = 0$ so that the forms are of maximal rank. However this is not the case that we are interested in; our definition states that from (36), ω^1 vanishes as a form when restricted to the fold, and that ω^2, ω^3 are both even. So if we take local coordinates about the fold we must have to order x

$$\omega^1 = dx \wedge \alpha^1 + x\beta^1 \wedge \gamma^1, \tag{37a}$$
$$\omega^2 = x dx \wedge \alpha^2 + \beta^2 \wedge \gamma^2, \qquad \omega^3 = x dx \wedge \alpha^3 + \beta^3 \wedge \gamma^3,$$

where $i_{\partial/\partial x}(\alpha^i, \beta^i, \gamma^i) = 0$. From the condition that on $\mathcal{Z}$ we must have

$$\omega^1 \wedge \omega^2 = dx \wedge \alpha^1 \wedge \beta^2 \wedge \gamma^2 = 0, \tag{38a}$$
$$\omega^1 \wedge \omega^3 = dx \wedge \alpha^1 \wedge \beta^3 \wedge \gamma^3 = 0, \tag{38b}$$

i.e. that $\alpha^1 \wedge \beta^2 \wedge \gamma^2 = 0 = \alpha^1 \wedge \beta^3 \wedge \gamma^3$, so take $\alpha^1 = \gamma^2 = \gamma^3 \equiv \varphi$. This will ensure the vanishing of equations (38a, 38b) due to the anti-symmetry of the exterior product. Furthermore, since ω^1 is closed we must have

$$d\omega^1 = d(dx \wedge \varphi + x\beta^1 \wedge \gamma^1) = dx \wedge (d\varphi + \beta^1 \wedge \gamma^1) = 0,$$

and hence $\omega^1 = dx \wedge \varphi - x d\varphi$ to order x. As $\omega^1 \wedge \omega^1 = 2x dx \wedge \varphi \wedge d\varphi$ and this vanishes transversally on the fold, we arrive back at the condition $\varphi \wedge d\varphi \neq 0$ along $x = 0$ and demonstrates the contact structure along $\mathcal{Z}$ once more. Let us relabel β^2, β^3 as $-\eta^1, -\eta^2$ respectively, then along the fold hypersurface Z we have that $\eta^1 \wedge \eta^2 \wedge \varphi \neq 0$. Letting $\imath : Z \hookrightarrow M$ be the inclusion of the fold hypersurface into the ambient manifold M we have that

$$\imath^*\omega^1 = 0,$$
$$\imath^*\omega^2 = \varphi \wedge \eta^1, \qquad \imath^*\omega^3 = \varphi \wedge \eta^2,$$

which sets the 2-forms such as those in (34).

We will now begin moving towards the existence and uniqueness theorem provided by Biquard which, whilst not an original result, is covered in full detail. We begin with a lemma originally proven by R. Bryant [32].

Lemma 7 (Bryant [32])**.** *Let $\mathcal{N} \subset \mathcal{M}$ be a real hypersurface with a contact 2-plane $\mathcal{H} \subset T\mathcal{N}$ defined locally by a contact form θ^1 and holomorphic symplectic 2-form $\beta = \beta^3 + i\beta^2$. Then there exists unique 1-forms $\theta^1, \theta^2, \theta^3$ on $\mathcal{N}$ such that*

- $\beta = \theta^1 \wedge (\theta^2 - i\theta^3)$ *and*
- $d\theta^1 = -i(\theta^2 - i\theta^3) \wedge (\theta^2 + i\theta^3)$.

This lemma is required since it asserts that if the three 1-forms θ^1, θ^2 and θ^3 exist locally, then they are defined on the entirety of the hypersurface $\mathcal{N}$, providing a canonical coframing for $T^*\mathcal{N}$.

Theorem 8 (Biquard [2])**.** *Given the real analytic data $(\mathcal{N}, \beta^2, \beta^3)$, where β^2 and β^3 are closed 2-forms on a 3-manifold $\mathcal{N}$ such that $\mathcal{H} = \ker\beta^2 \oplus \ker\beta^3$ is a contact distribution, there exists on a neighbourhood $(-\epsilon, \epsilon) \times \mathcal{N}$ a unique folded hyperkähler metric such that $\imath^*\omega^2 = \beta^2$ and $\imath^*\omega^3 = \beta^3$. This metric satisfies the parity condition (36).*

Proof. From the given data $(\mathcal{N}, \beta^2, \beta^3)$, consider the basis of 1-forms $(\theta^1, \theta^2, \theta^3)$ to $T^*\mathcal{N}$ with the associated dual frame of vector fields (X_1, X_2, X_3); these exist globally on $\mathcal{N}$ due to Lemma (7). Let $\alpha = \theta^1 \wedge \theta^2 \wedge \theta^3$ be the volume form on $\mathcal{N}$, then X_2 and X_3 are volume-preserving, $\mathcal{L}_{X_2}\alpha = d\beta^2 = \mathcal{L}_{X_3}\alpha = d\beta^3 = 0$. To extend the frame to the neighbourhood $\mathcal{M} = (-\epsilon, \epsilon) \times \mathcal{N}$ for some $\epsilon > 0$ small enough, we want to solve Nahm's equations

$$\frac{\partial V_i}{\partial \tau} = \frac{1}{2}\epsilon_i^{\ jk}[V_j, V_k]$$

for the triad of time-dependent vector fields V_i, subject to the initial conditions

$$V_1(0) = 0, \qquad V_2(0) = X_2, \qquad V_3(0) = X_3.$$

As we are dealing with real analytic data, the Cauchy-Kovalevskaya Theorem can be used to produce unique solutions on $\mathcal{M}$. Moreover $(-V_1(-\tau), V_2(-\tau), V_3(-\tau))$ is also a solution with the same initial conditions, hence V_1 is odd and V_2, V_3 are even, implying the invariance of the involution (36) for the solution. As $X_1 = [X_2, X_3]$, we must have

$$V_1(\tau) = \tau X_1 + \mathcal{O}(\tau^3),$$

and we define a fourth vector $V_0 = \frac{\partial}{\partial \tau}$. The behaviour of the metric can be deduced as

$$h = \tau(d\tau^2 + (\theta^2)^2 + (\theta^3)^2) + \tau^{-1}(\theta^1)^2 + \mathcal{O}(\tau^3)(d\tau, \tau^{-1}\theta^1, \theta^2, \theta^3) \tag{40}$$

where the final term is quadratic in $(d\tau, \tau^{-1}\theta^1, \theta^2, \theta^3)$ with coefficients of order τ^3. From the metric h we can determine the hyperkähler 2-forms and metric via

$$\omega^i = d\tau \wedge h(V_i) + i_{V_i}\alpha, \qquad h(V_\mu, V_\nu) = \alpha(V_1, V_2, V_3)\delta_{\mu\nu}. \tag{41}$$

□

Contact Geometry of the Folded Gibbons-Hawking Manifold

Let us continue our investigation into the folded Gibbons-Hawking example by studying the contact manifold determined by the fold, in particular the nature of $\mathcal{H} = \ker \mathcal{Z}^*\omega^2 \oplus \ker \mathcal{Z}^*\omega^3$. To find the connection 1-form A, consider

$$*_3 dV = *_3 dz = dx \wedge dy = d(xdy) = dA \implies A = xdy \tag{42}$$

which determines A modulo an exact form. Hence

$$\varphi = d\tau + xdy, \qquad d\varphi = dx \wedge dy,$$
$$\varphi \wedge d\varphi = d\tau \wedge dx \wedge dy \neq 0,$$

and the pulled-back 2-forms to $\mathcal{Z}$ are

$$\mathcal{Z}^*\omega^2 = (d\tau + xdy) \wedge dx = d\tau \wedge dx + xdy \wedge dx,$$
$$\mathcal{Z}^*\omega^3 = (d\tau + xdy) \wedge dy = d\tau \wedge dy.$$

Their 1-dimensional kernels can now easily be read off to determine the 2-dimensional hyperplane field $\mathcal{H}_{GH}$ as

$$\mathcal{H}_{GH} = \ker \mathcal{Z}^*\omega^2 \oplus \ker \mathcal{Z}^*\omega^3 = \mathrm{span}\left\{ \frac{\partial}{\partial y} - x\frac{\partial}{\partial \tau},\ \frac{\partial}{\partial x} \right\} \subset T\mathcal{Z}.$$

Indeed, if we check the commutator of the basis vectors we observe that

$$\left[x\frac{\partial}{\partial \tau} - \frac{\partial}{\partial y},\ \frac{\partial}{\partial x} \right] = \frac{\partial}{\partial \tau} \notin \mathcal{H}_{GH},$$

so it is clear that $(\mathcal{Z}, \mathcal{H})$ is a contact manifold. In fact, for the Reeb vector field R_φ defined by the equations

$$d\varphi(R_\varphi, \cdot) = 0, \qquad \varphi(R_\varphi) = 1, \tag{43}$$

one sets $R_\varphi = \partial_\tau$ which coincides with the Killing vector field generated by the S^1 isometry.

Folding of the Real Heaven Background

Since our canonical example of a folded hyperkähler manifold comes from the Gibbons-Hawking ansatz, it is natural to ask whether the generalised Gibbons-Hawking hyperkähler manifold can be folded in the sense of our definition. Recall that the Gibbons-Hawking metric is a special case of a hyperkähler metric which admits a triholomorphic Killing vector.

In order to see if the real heaven metric equation (26) can be folded, we must first find solutions to the $SU(\infty)$-Toda equation (25). One may write

$$u(x, y, z) = v(x, y) + w(z)$$

to separate the variables, so that (25) becomes

$$(e^v)_{zz} = 2a, \tag{44a}$$

$$v_{xx} + v_{yy} + 2a(e^w)_{zz} = 0, \tag{44b}$$

with a a separation constant [26]. Equation (44a) can be solved immediately to get

$$e^v = az^2 + bz + c, \tag{45}$$

and equation (44b) is known as Liouville's equation [26]. Solutions to the Toda field equation (25) now take the form

$$e^u = \frac{4(az^2 + bz + c)}{(1 + a(x^2 + y^2))^2}, \tag{46}$$

$$u(x, y, z) = \log(az^2 + bz + c) - 2\log(1 + a(x^2 + y^2)) + \log(4) \tag{47}$$

with b, c constants. In [26], there are six cases: three on hyperbolic space ($b^2 - ac > 0$), two in flat space ($b^2 - 4ac = 0$), and one on the sphere ($b^2 - 4ac < 0$). Since the folded metric h and Kähler form ω^1 must satisfy the parity condition (36) only the two cases when $a \neq 0$, $b = 0$, and $c > 0$, will be considered; they correspond to the 3-sphere S^3 when $a > 0$ and to hyperbolic 3-space $\mathcal{H}^3$ when $a < 0$. We calculate

$$\begin{aligned}
u_x &= -\frac{4ax}{1 + a(x^2 + y^2)}, \qquad u_y = -\frac{4ay}{1 + a(x^2 + y^2)}, \\
u_{xx} &= -\frac{4(1 - x^2 + y^2)}{(1 + a(x^2 + y^2))^2}, \qquad u_{yy} = -\frac{4(1 + x^2 - y^2)}{(1 + a(x^2 + y^2))^2}, \\
(e^u)_{zz} &= \frac{8a}{(1 + a(x^2 + y^2))^2} = -(u_{xx} + u_{yy}), \\
u_z &= \frac{2az}{c + az^2}, \\
u_z e^u &= \frac{8az}{(1 + a(x^2 + y^2))^2}, \\
d\tau + u_y dx - u_x dy &= d\tau + 4a\frac{xdy - ydx}{1 + a(x^2 + y^2)} =: \psi,
\end{aligned}$$

and so the metric and hyperkähler forms from equations (26–28) are

$$h = \frac{8az}{(1 + a(x^2 + y^2))^2}(dx^2 + dy^2) + \frac{2az}{c + az^2}dz^2 + \frac{c + az^2}{2az}\psi^2, \tag{49a}$$

$$\omega^1 = \frac{8az}{(1+a(x^2+y^2))^2} dx \wedge dy + dz \wedge \psi, \tag{49b}$$

$$\begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix} = e^{u/2} \begin{pmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{pmatrix} \begin{pmatrix} \psi \wedge dx + \frac{2az}{c+az^2} dy \wedge dz \\ \psi \wedge dy + \frac{2az}{c+az^2} dz \wedge dx \end{pmatrix}. \tag{50}$$

From this explicit representation, the parity relations (36) for h and the ω^i are satisfied.

Let us restrict our attention to the fold at $z = 0$, which we will continue to call $\mathcal{Z}$. From equations (49a, 49b) we see that $\mathcal{Z}^*h$ is undefined and that $\mathcal{Z}^*\omega^1 = 0$, whereas

$$\begin{aligned} \mathcal{Z}^*\omega^2 &= \frac{2\sqrt{c}}{\big(1+a(x^2+y^2)\big)} \psi \wedge \big(\cos(\tau)dx - \sin(\tau)dy\big), \\ \mathcal{Z}^*\omega^3 &= \frac{2\sqrt{c}}{\big(1+a(x^2+y^2)\big)} \psi \wedge \big(\sin(\tau)dx + \cos(\tau)dy\big), \end{aligned}$$

written out in a way to emphasise their similar form to that of (39). Indeed, this suggests that ψ is our contact form determining $\mathcal{Z}$. Computation yields

$$d\psi = (e^u)_{zz} dx \wedge dy, \qquad \psi \wedge d\psi = (e^u)_{zz} d\tau \wedge dx \wedge dy \neq 0$$

along $\mathcal{Z}$, so ψ is a contact form and $(\mathcal{Z}, \psi)$ is a contact manifold. The two vectors fields that annihilate ψ span

$$\mathcal{H}_{RH} = \ker \mathcal{Z}^*\omega^2 \oplus \ker \mathcal{Z}^*\omega^3 = \operatorname{span}\left\{ u_y \frac{\partial}{\partial \tau} - \frac{\partial}{\partial x},\ u_x \frac{\partial}{\partial \tau} + \frac{\partial}{\partial y} \right\}$$

determining the contact 2-plane $\mathcal{H}_{RH} \subset T\mathcal{Z}$, and their commutator bracket is

$$\left[u_y \frac{\partial}{\partial \tau} - \frac{\partial}{\partial x},\ u_x \frac{\partial}{\partial \tau} + \frac{\partial}{\partial y} \right] = (e^u)_{zz} \frac{\partial}{\partial \tau} \notin \mathcal{H}_{RH}$$

verifying the maximal non-integrability of the hyperplane field $\mathcal{H}_{RH}$. The Reeb field in the case is the same as the folded Gibbons-Hawking one, $R_\psi = \frac{\partial}{\partial \tau}$.

Remark 9. Again, there is a similarity between the Gibbons-Hawking fold and the real heaven fold, which can be traced back to the fact that both space originate from a principal S^1-bundle over $\mathbb{R}^3$; in both examples it was the connection 1-form that became the contact form ψ for the fold hypersurface, and thus $d\psi$ is its curvature 2-form. However the Gibbons-Hawking fold has a flat base space $\mathbb{R}^2$, whereas the real heaven fold admits two, non-flat, possible base spaces, $S^2 \simeq \mathbb{C}P^1$ or $\mathcal{H}^2$, depending on the sign of the constant a. To see this, identify $\mathbb{C} \simeq \mathbb{R}^2$ with $w = x + iy$ so that

$$d\psi = \pm \frac{i}{2} \frac{8a}{(1 + a|w|^2)^2} dw \wedge d\bar{w} = 4i\partial\bar{\partial} \log(1 \pm a|w|^2) = 8\pi\omega_\pm, \tag{51}$$

where $\omega_\pm$ is the *fundamental form* for either $\mathbb{C}P^1$ when $a > 0$ or $\mathcal{H}^2$ when $a < 0$. Without loss of generality suppose that $a = \pm 1$, then

$$(\mathbb{C}P^1): \qquad \omega_+ = \frac{i}{2\pi} \partial\bar{\partial} \log(1 + |w|^2), \tag{52a}$$

$$(\mathcal{H}^2): \qquad \omega_- = \frac{i}{2\pi} \partial\bar{\partial} \log(1 - |w|^2). \tag{52b}$$

ω_+ is known as the fundamental form for the Fubini-Study metric on $\mathbb{C}P^1$ in the literature, whereas ω_- is the fundamental form for the Poincaré disk model in hyperbolic space. It is perhaps not too surprising that this is the case; even before folding the real heaven background, the base spaces of the S^1-bundle was either S^2 or $\mathcal{H}^3$, so the restriction to the fold was just equivalent to just projecting the base spaces down a dimension.

Conclusions

From studying a particular example of the Gibbons-Hawking ansatz, we have defined how a hyperkähler manifold must behave if it is to admit of fold hypersurface. The definition must be different to the already studied symplectic and Kähler variants due to the additional structure enforced by the non-degeneracy of the hyperkähler forms away from the fold, and this structure is implicit on the fold too through the contact structure it determines. A theorem provided by Biquard [2] was stated and proven which, whilst is not a new result, was elucidated upon in this report and also provided the folded counterpart to the ASJ method, covered in the background material.

As far as the author is aware, the approach to folding the real heaven background in this report has not appeared in the literature, and provides a more general example of a folded hyperkähler manifold than that of the Gibbons-Hawking one. Furthermore, although not being much of a surprising result, it is pleasing to see that the fold hypersurfaces of the real heaven background take on the form of a circle bundle over either $\mathbb{C}P^1$ or $\mathcal{H}^2$, with the curvature forms of the bundle adopting the familiar expressions for the Fubini-Study and Poincaré disk fundamental forms respectively.

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