EQUIVARIANT LOCALISATION AND FIXED-POINT THEOREMS

HODGE CLUB TALK NOTES

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ABSTRACT

1 Introduction

Often in mathematics, we are tasked with the problem of evaluating an integral over some space, that is trying to evaluate

$$\int_M \omega$$

for some space M. Depending on the form ω, this integral can be related to finding the volume, to finding topological or enumerative invariants, integrating characteristic classes, or computing partition functions of physical systems. Such computations can be difficult, and there are many ways that we can tackle the integral. Two methods are particularly fruitful - that of *localisation* and *symmetry*.

1.1 Symmetry and Localisation

By symmetry, we mean that we have a group G acting on M, and by identifying orbits we reduce the problem to that over a smaller space, M/G. Such an approach comes up in symplectic reduction, gauge theory, and integrable systems.

By localisation, informally this means that we relate global calculations to ones that are local. The Poincaré-Hopf theorem is an example of this, which relates the Euler characteristic of a compact manifold M to the sum of the indices of the zeros of a vector field on it:

$$\chi(M) = \int_M e(TM) = \sum_{\mathbf{V}(p)=0} \mathrm{Ind}(\mathbf{V}).$$

Unsurprisingly, it is often easier to consider a finite set of points rather than the global space M. The notion of localisation in algebra is a similar notion, in which we consider a single point (a prime) at a time.

Symmetry and localisation synergise through the Atiyah-Bott fixed-point theorem; in the situation that we have a smooth manifold M together with the action of a compact connected Lie group G, then the integral on M localises on the (isolated) fixed-point set $M^G := F \subseteq M$ of the G-action. If $i : F \hookrightarrow M$ is the inclusion, and $e(\nu_p)$ is the Euler class of the normal bundle ν_p to the fixed-point $p \in F$, then

$$\int_M \omega = \sum_{p\in F} \int_p \frac{i^*\omega}{e(\nu_p)}.$$

What I want to do in this talk is discuss some interesting cases when the fixed-point formula can be applied, and investigate how to use it in these examples. Consider the following geometric sum:

$$\begin{aligned}\sum_{k=0}^{10000} q^k &= 1 + q + q^2 + \ldots + q^{10001} \\ &= \left(\frac{1-q}{1-q}\right) \cdot (1 + q + q^2 + \ldots + q^{10001}) \\ &= \frac{1-q^{10001}}{1-q} \\ &= \frac{1}{1-q} + \frac{1-q^{10000}}{1-q^{-1}}.\end{aligned}$$

To evaluate the left-hand side, we need to know the value of each term at the 10001 integral points inside of the closed interval $[0, 10000]$, whereas the right-hand side only needs the two terms to be evaluated. So we can say that this sum *localises* at the end points.

2 Equivariant Cohomology

Let G be a compact Lie group acting on a toplogical space M. If G acts freely on M, then the quotient space M/G is usually as nice as the space M is itself; for instance, if M is a manifold then so is M/G.

The idea behind an equivariant cohomology group, $H^*_G(M)$, is that the equivariant cohomology groups of M should just be the cohomology groups of M/G:

$$H^*_G(M) = H^*(M/G), \qquad \text{when the action is free.}$$

For example, if G acts on itself by left multiplication, then

$$H^*_G(G) = H^*(\mathrm{pt}).$$

However, if the action is not free, then the space M/G might not behave very nicely from a cohomological point of view. Then the idea is that $H^*_G(M)$ should be the "correct" subsitute for $H^*(M/G)$.

2.1 Classifying Bundles

As cohomology is unchanged under homotopy equivalence, our guiding idea is that the equivariant cohomology of M should be the ordinary cohomology of M^*/G, where M^* is some topological space homotopy equivalent to M and on which G acts freely. The standard way of constructing such a space is to take it to be the product $M^* = M \times E$, where E is some contractible space on which G acts freely. Then the equivariant cohomology groups of M are defined by the recipe

$$H^*_G(M) := H^*\left((M \times E)/G\right).$$

Note that if G acts freely on M then the projection

$$(M \times E)/G \longrightarrow M/G$$

is a fibration with typical fibre E. Then as E is contractible, we get that

$$H^*_G(M) = H^*\left((M \times E)/G\right) = H^*(M/G),$$

so we arrive at the same situation if G acts freely on M.

2.2 The Cartan Model

Let M be an n-dimensional manifold acted on by a Lie group G with with Lie algebra $\mathfrak{g}$. A *G-equivariant differential form* on M is defined to be a polynomial map $\alpha : \mathfrak{g} \to \Omega(M)$ such that

$$\alpha(gX) = g \cdot \alpha(X), \qquad \text{for } g \in G.$$

Let $\mathbb{C}[\mathfrak{g}]$ denote the algebra of $\mathbb{C}$-valued polynomial functions on $\mathfrak{g}$. Then we can view the tensor product

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega(M),$$

as the algebra of polynomial maps from $\mathfrak{g}$ to Ω. The group G acts on an element $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ by the formula[1]

$$(g \cdot \alpha)(X) := g \cdot \left(\alpha(g^{-1} \cdot X)\right), \qquad \text{for all } g \in G, \text{ and } X \in \mathfrak{g}.$$

Let $\Omega^G(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega(M))^G$ be the subalgebra of G-invariant elements; an element $\alpha \in \Omega^G(M)$ thus satisfies $\alpha(g \cdot X) = g \cdot \alpha(M)$, hence is an equivariant differential form. Equip $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M$ with the following $\mathbb{Z}$-grading,

$$\deg(P \otimes \alpha) := 2 \cdot \deg(P) + \deg(\alpha),$$

for the polynomial $P \in \mathbb{C}[\mathfrak{g}]$, and $\alpha \in \Omega(M)$. Define the *equivariant exterior differential*, or *Cartan differential*, d_G by

$$(d_G\alpha)(X) := (d - \imath_{X_M})\alpha(X),$$

where $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the usual de Rham differential, X_M is the fundamental vector field of $X \in \mathfrak{g}$ on M, and $\imath_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is the contraction of X on a differential form.

Proposition 2.1. *The Cartan differential d_G is closed on Ω^*_G,* i.e. $d_G^2 = 0$.

Proof. The derivations d and $\imath_v$ in $\Omega(M)$ are related to the **Lie derivative** $\mathcal{L}$, by means of the **homotopy formula**:

$$\mathcal{L}(v) := \left.\frac{d}{dt}\right|_{t=0} (e^{tv})^* = d \circ \imath_v + \imath_v \circ d.$$

Here e^{tv} is the flow in M after a time t of the velocity field equal to v.

Now for $X \in \mathfrak{g}$, if X_M represents the infinitesimal action of X in M, then

[TODO]

$\square$

Corollary 2.2. *The space of equivariant differential forms $\Omega^*_G(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G$, equipped with the Cartan differential d_G forms a complex, called the* ***Cartan complex****:*

$$(\Omega^*_G(M), d_G) = \left((\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G, d_G\right).$$

Definition 2.3. *The* ***equivariant cohomology*** *$H^*_G(M)$ of M is the cohomology of the Cartan complex, $(\Omega^*_G(M), d_G)$.*

[1] G acts on $\Omega(M)$ by the induced G-action on M, and on $\mathfrak{g}$ by the adjoint action.

2.3 Characteristic Classes

Let G and T be compact, connected Lie groups.

An ordinary characteristic class for a principal G-bundle on an n-dimensional manifold M is $[p(F_A)] \in H^{2n}(M)$, for a G -invariant degree n polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$. here F_A is the curvature of any connection A on the G-bundle.

To get a T-equivariant characteristic class for a principal G-bundle associated to a G-invariant, degree n polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$, we take $[p(F_{A,T})] \in H^{2n}_T(M)$, where now $F_{A,T}$ is the T-equivariant curvature of any T-equivariant connection A on the G-bundle.

Restricted to the T-fixed points M^T of M, the T-equivariant characteristic class associated to a polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$ is

$$p(F_A + \epsilon^a \rho(T_a)).$$

TODO: EXPLAIN WHAT ϵ^a, etc. ARE!

In particular, when V is a representation of G and p is the Chern character of the vector bundle V, then, if M is a point, the equivariant Chern characters are just the ordinary characters of the space V as a G-module.

2.4 The Euler Class

Here, let $G = \mathrm{SO}(2n)$ which preserves the Riemannian metric on an oriented real vector space V of dimension $\dim_{\mathbb{R}}(V) = 2n$.

Definition 2.4. *Consider the following adjoint-invariant polynomial,*

$$\mathrm{Pf} : \mathfrak{so}(2n;\mathbb{R}) \longrightarrow \mathbb{R},$$

*of degree n on the Lie algebra $\mathfrak{so}(2n;\mathbb{R})$, called the **Pfaffian**.*

The case that we shall be interested in is when we have the $(2n \times 2n)$-antisymmetric matrix,

$$\mathrm{Pf}\begin{pmatrix} 0 & \lambda_1 & \dots & \dots & 0 & 0 \\ -\lambda_1 & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 & \lambda_n \\ 0 & 0 & \dots & \dots & -\lambda_n & 0 \end{pmatrix} = \lambda_1 \cdot \ldots \cdot \lambda_n.$$

Definition 2.5. *Let $P \to M$ be an $\mathrm{SO}(2n;\mathbb{R})$-principal bundle over M. The **Euler characteristic class** of P, $e(P)$, is given by*

$$e(P) := [\mathrm{Pf}(F)] \in H^{2n}(M;\mathbb{Z}).$$

Example 1. *If M is an oriented, $2n$-dimensional real manifold, then the **Euler characteristic** is given by*

$$e(M) = \int_M e(TM) = \int_M \mathrm{Pf}(R_\nabla),$$

where R_∇ is the curvature form of the tangent bundle TM, equipped with the Levi-Civita connection.

To upgrade the Euler characteristic class e to a T-equivariant one e_T, where T is a torus acting on a manifold M with isolated fixed-point set M^T, we need to investigate the polynomial

$$\mathrm{Pf}(F_A + \epsilon^a \rho(T_a)).$$

For simplicity, let $T = S^1$. Then, for a point $p \in M^{S^1}$, the S^1-action on T_pM gives rise to an S^1-representation,

$$\rho : S^1 \longrightarrow \mathrm{GL}(T_mM); \qquad g \longmapsto l_{g,*},$$

where $l_{g,*} : T_pM \to T_{g\cdot p}M = T_pM$ is the differential of the action of $g \in S^1$ on T_mM.

As p is isolated, ρ decomposes into a direct sum of 2-dimensional irreducible representations,

$$T_pM \cong L^{m_1} \oplus \ldots \oplus L^{m_n}.$$

Here, $L^m : S^1 \to \mathrm{GL}(2;\mathbb{R})$ is a representation of S^1 as m-fold rotations in $\mathbb{R}^2$,

$$L^m : g \longmapsto l_{g,*}; \qquad L^m(e^{it}) = \begin{bmatrix} -m\sin(mt) & -m\cos(mt) \\ m\cos(mt) & -m\sin(mt) \end{bmatrix}.$$

2.5 Chern Classes

Now let $G = \mathrm{U}(n)$, then $\mathfrak{g} = \mathfrak{u}(n)$ can be identified with the space of matrices of the form iA, where $A = A^T$. Define the polynomial c_k, of degree k in A to be the coefficient of $(-1)^k\lambda^{n-k}$ in the characteristic polynomial of A:

$$\det(\lambda - A) = \lambda^n - c_1(A)\lambda^{n-1} + \ldots + (-1)^n c_n(A).$$

In particular, $c_1(A) = \mathrm{Tr}(A)$ and $c_n(A) = \det(A)$. These polynomials are clearly adjoint invariant, thus the characteristic polynomial is.

The characteristic classes corresponding to the c_i for a complex vector bundle are called its **Chern classes**.

Remark 1. *If we consider a complex vector bundle $V_{\mathbb{C}}$ of $\dim_{\mathbb{C}}(V) = n$ then, by forgetting the complex structure on $V_{\mathbb{C}}$, we get an oriented real vector bundle $V_{\mathbb{R}}$ of real dimension $\dim_{\mathbb{R}}(V) = 2n$.*

By this correspondence, the Euler class $e(V)$ and the top Chern class $c_n(V)$ of V are related by

$$e(V_{\mathbb{R}}) = c_n(V_{\mathbb{C}}).$$

2.6 Equivariant Characteristic Classes

Definition 2.6. *A **G-equivariant vector bundle** of a G-manifold M is a vector bundle $V \to M$ with an action of G on the total space V covering the action of G on M.*

Definition 2.7. *Let (M, ω) be a symplectic manifold, and suppose that a torus T acts on M preserving ω. The action is **Hamiltonian** if there exists a **moment map** $\mu : M \to \mathfrak{t}^*$, whuch satisfies*

$$\imath_{\xi_M}\omega = d\langle\mu,\, \xi\rangle, \quad \textit{for all } \xi \in \mathfrak{t}.$$

Here, ξ_M is the induced vector field on M.

Proposition 2.8. *Let (M, ω, μ) be a symplectic manifold with a Hamiltonian action of a torus T and associated moment map $\mu : M \to \mathfrak{t}^*$. Set*

$$\tilde{\omega} := \omega + \mu.$$

Then $\tilde{\omega}$ is a T-equivariantly closed two-form.

Proof. For $\tilde{\omega}$ to be equivariantly closed under d_T, we have

$$d_T\tilde{\omega} = 0 \iff (d - \imath_\xi)(\omega + \mu^\xi) = d\omega - \imath_\xi\omega + d\mu^\xi - \imath_\xi\mu^\xi = -\imath_\xi\omega + d\mu^\xi = 0 \iff \imath_\xi\omega = d\mu^\xi.$$

□

So given an ordinary characteristic class $[\omega] \in H^2(M)$ and a moment map $\mu : M \to \mathfrak{g}^*$, we can elevate it to an equivariant characteristic class by the substitution

$$H^2(M) \ni [\omega] \longmapsto [\omega_G] = [\omega + \mu] \in H^2_G(M).$$

Proposition 2.9. *If E is a complex vector bundle with a T-action and $E \cong \bigoplus_j \mathcal{L}_j$, where $\mathcal{L}_j$ are complex line bundles with T-action given by weights $\lambda_j : T \to U(1)$, then the equivariant Euler class of E is*

$$e^T(E) = \prod_j c_1^T(\mathcal{L}_j).$$

In the Cartan model, this is represented by

$$e^T(E)(\xi) = \prod_j (F_j - \lambda_j)(\xi).$$

Example 2. *If T acts on M and F is a component of M^T, then the normal bundle ν_F is a T-equivariant bundle over V. Assume that ν_F decomposes equivariantly as $\nu_F \cong \bigoplus_j \nu_{F,j}$ with weights $\lambda_{F,j} \in \mathfrak{t}^*$. Then the equivariant Euler class $e^T(\nu_F)$ is*

$$e^T(\nu_F) = \prod_j \left(c_1(\nu_{F,j}) + \beta_{F,j}\right).$$

When the fixed-point set M^T consists of isolated fixed-points then, for $p \in M^T$,

$$e^T(\nu_p)$$

3 Equivariant Localisation

3.1 The Berline-Vergne-Atiyah-Bott Fixed Point Theorem

When a manifold has a torus action, the equivariant localisation formula is a powerful tool for doing calculations in **ordinary** cohomology, despite being formulated in **equivariant** cohomology.

Theorem 3.1 (Atiyah-Bott, Berline-Vergne Theorem). *Suppose an n-dimensional torus T acts on a compact oriented manifold M with fixed-point set $F := M^T$. If ϕ is an equivariant closed form on M and $i_F : F \hookrightarrow M$ is the inclusion map, then*

$$\int_M \phi = \sum_{F \subseteq M^T} \int_F \frac{i_F^* \phi}{e^T(\nu_F)},$$

as elements of $H_T^*(\mathrm{pt}) = \mathbb{R}[u_1, \dots, u_n]$. Here, ν_F is the normal bundle of F in M, and e^T is the T-equivariant Euler class.

In the case when the fixed-point set $M^T = \{p_i\}$ consists of isolated fixed-points, the localisation theorem simplifies greatly:

Theorem 3.2. *With the above hypotheses, if M^T consists of isolated fixed-points, then:*

$$\int_M \phi = \sum_{p \in M^T} \frac{\phi(p)}{\prod_i \lambda_{p,i}}.$$

4 Examples

4.1 Stationary Phase and Duistermaat-Heckman

Let M be a compact, oriented $2n$-manifold, $f : M \to \mathbb{R}$ a function, and $\tau \in \Omega^{2n}(M)$.

TODO: SEE LORING TU'S BOOK.

4.2 Riemann-Roch-Hirzebruch Theorem

For $\mathcal{L} \to M$ a holomorphic line bundle over a complex manifold M. The Hirzebruch-Riemann-Roch Theorem then states that the Euler characteristic, $\chi(M; \mathcal{L})$, is equal to the characteristic number

$$\chi(M; \mathcal{L}) = \int_M e^{c_1(\mathcal{L})} \ \mathrm{Td}(TM).$$

Here, $c_1(\mathcal{L})$ is the *1st Chern class* of $\mathcal{L}$, and $\mathrm{Td}(M)$ is the *Todd class* of the complex vector bundle $TM \to M$.