HYPERTORIC MANIFOLDS

GENERAL NOTES

ABSTRACT

Notes on toric hyperkähler manifolds.

1 Toric Hyperkähler Manifolds

1.1 Symplectic Quotients, [1]

Fix the standard Euclidean bilinear form on $\mathbb{C}^n$,

$$g(z, w) = \sum_{i=1}^{n} \left(\Re(z_i)\Re(w_i) + \Im(z_i)\Im(w_i) \right).$$

The corresponding Kähler form is

$$\omega(z, w) = g(iz, w) = \sum_{i=1}^{n} \left(\Re(z_i)\Im(w_i) - \Im(z_i)\Re(w_i) \right).$$

Let $A = [u_1, \ldots, u_n]$ be a $(d \times n)$-matrix whose $(d \times d)$-minors are relatively prime. Choose now an $n \times (n-d)$-matrix $B = [b_1, \ldots, b_n]^T$ that makes the following sequence exact:

$$\{0\} \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow \{0\}.$$

The choice of B is equivalent to choosing a basis in $\ker(A)$.

1.2 Hyperkähler Quotients

Let $\mathbb{H}$ be the quaternions, the 4-dimensional $\mathbb{R}$-vector space with basis $\{1, i, j, k\}$ equipped with an associative algebra structure defined by

$$i^2 = j^2 = k^2 = ijk = -1.$$

Left-multiplication by i (respectively j and k) define the following respective complex structures on $\mathbb{H}$,

$$I, J, K : \mathbb{H} \longrightarrow \mathbb{H}; \qquad I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{\mathbb{H}}.$$

Equipping $\mathbb{H}$ with the flat metric g arising from the standard Euclidean scalar-product on $\mathbb{H} \cong \mathbb{R}^4$, with $\{1, i, j, k\}$ providing an orthonormal basis. This is called a *hyperkähler metric* since it is a Kähler metric with respect to each individual complex structure, I, J, and K. This also means that the so-called *Kähler forms*, given by

$$\omega_I(X, Y) = g(IX, Y), \qquad \omega_J(X, Y) = g(JX, Y), \qquad \omega_K(KX, Y) = g(KX, Y), \qquad \text{for tangent vectors } X, Y,$$

are closed differential 2-forms.

A special orthogonal transformation with respect to this metric is said to *preserve the hyperkähler structure* if it commutes with all three complex structures, I, J, and K; or equivalently, it preserves the Kähler forms, ω_I, ω_J, and ω_K. The group of such transformations, the *unitary symplectic group* $\operatorname{Sp}(1)$, is generated by the right-multiplication action by the unit quaterions.

2 Cotangent Spaces to Extended Core Components

Let $M_\lambda = \left(\mu_{\mathbb{R}}^{-1}(\lambda) \cap \mu_{\mathbb{C}}^{-1}(0) \right) / K$ be a toric hyperkähler manifold. Define

$$\mathbb{C}_A := \left\{ (z_i, w_i) \in \mathbb{C}^{2n} \;\middle|\; w_i = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \right\} \cong \mathbb{C}^n \subset \mathbb{H}^n.$$

Lemma 2.1 ([2]). *Let M_λ be a toric hyperkähler manifold. If $\mathcal{E}_A$ is non-empty, then its holomorphic cotangent bundle $T^*\mathcal{E}_A$ is contained in M_λ as an open subset.*

Fix a subset $A \subset \{1, \ldots, n\}$, and define

$$(x_i^{(A)}, y_i^{(A)}) := \begin{cases} (z_i, w_i), & \text{if } i \in A, \\ (w_i, -z_i), & \text{if } i \notin A. \end{cases}$$

Then $x^{(A)} = (x_1^{(A)}, \ldots, x_n^{(A)})$ is a point in the vector space $\mathbb{C}_A^n$, and $y^{(A)} = (y_1^{(A)}, \ldots, y_n^{(A)})$ is a point in the dual space $(\mathbb{C}_A^n)^*$. That is, we identify the cotangent bundle $T^*\mathbb{C}_A^n$ with $\mathbb{H}^n$ as above.

2.1 Kähler Quotients

The Kähler quotient $X = \mu^{-1}(0)/N$ can be identified with the quotient of an open subset of $\mathbb{C}^n$ by the complexified torus $N^{\mathbb{C}}$ as follows: every orbit in $\mathbb{C}^n$ of $T_{\mathbb{C}}^n$ is of the form

$$\mathbb{C}_A^n = \left\{ (z_1, \ldots, z_n) \mid z_i = 0 \text{ if } i \in A \right\},$$

for some subset $A \subset \{1, \ldots, n\}$. If F is a face of Δ of codimension r, then F is defined by the intersection of r hyperplanes $\cap_j H_{i_j}$

3 Symplectic Cutting

3.1 Compactifying the Extended Core

Let S^1 act on M by rotating the cotangent fibres, that is, for $\tau \in S^1$,

$$\tau \cdot [z; w] = [z; \tau w].$$

This S^1-action is Hamiltonian, with moment map

$$\Phi : M \longrightarrow (\mathbb{R})^*; \qquad [z : w] \longmapsto \frac{1}{2}\|w\|^2.$$

Let S_A^1 denote the residual S^1-action on M restricted to the extended core component

$$\mathcal{E}_A = \left\{ [z_1 : \ldots z_n; w_1, \ldots, w_n] \mid w_0 = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \right\}.$$

Now the *global* S^1-action does not act on the cotangent fibres of M as a subtorus of T^n, but it does when *restricted* to each component of the extended core, $\mathcal{E}_A$. Indeed,

$$\tau \cdot [z; w] = [z; \tau w] = [z_1 : \ldots : z_n; \tau w_1 : \ldots : \tau w_n] = [\tau_1 z_1 : \ldots : \tau_n z_n; \tau_1^{-1} w_1 : \ldots : \tau_n^{-1} w_n],$$

where

$$\tau_i := \begin{cases} \tau^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A, \end{cases}$$

which shows that the S^1-action restricted to each individual $\mathcal{E}_A$ acts as a subtorus of the original torus T^n.

Denote by S_A^1 the image of S^1 in T^n when considered as a subtorus restricted to each individual $\mathcal{E}_A$, and let $\jmath_A : S^1 \hookrightarrow T^n$ be the respective inclusion homomorphism, so we have $S_A^1 := \jmath_A(S^1) \lhd T^n$.

On the Lie algebra level, we have that

$$(\jmath_A)_* : \mathrm{Lie}(S_A^1) \longrightarrow \mathfrak{t}^n; \qquad \xi \longmapsto (\xi_1, \ldots, \xi_n),$$

where we analogously define

$$\xi_i := \begin{cases} -1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A. \end{cases}$$

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Since S_A^1 acts as the subtorus $\jmath_A(S^1)$ of T^n on each $\mathcal{E}_A$, the moment map $\Phi_A := \Phi\big|_{\mathcal{E}_A}$ for this action is given by composing $\mu_{\mathbb{R}}$ with the dual of the inclusion $(\jmath_A)_*$, so

$$
\begin{aligned}
\Phi_A[z,w] = (\jmath_A^* \circ \mu_{\mathbb{R}})\,[z;w] &= \jmath_A^*\left(\frac{1}{2}\sum_{i=1}^{n}\left(|z_i|^2 - |w_i|^2\right)e^i\right) \\
&= -\frac{1}{2}\sum_{i\in A}|z_i|^2 \jmath_A^*(e^i) \\
&= \frac{1}{2}\sum_{i\notin A}|w_i|^2 \jmath_A^*(e^i) \\
&= \langle \mu_{\mathbb{R}}[z;w],\,\xi_A\rangle \\
&= \mu_{\mathbb{R}}^A[z;w],
\end{aligned}
$$

where $\xi_A = -\sum_{i\in A}\xi_i$, and $\mu_{\mathbb{R}}^A[z;w]$ is the component of $\mu_{\mathbb{R}}[z;w]$ in the ξ_A-direction.

References

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[3] Victor Guillemin. *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, volume 122 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1994.