# GEOMETRIC QUANTISATION OF HYPERTORIC MANIFOLDS BY SYMPLECTIC CUTTING

GENERAL NOTES

#### **ABSTRACT**

Lorem ipsum.

## 1 Introduction

Lorem ipsum.

## 2 Background

- 2.1 Hypertoric Geometry
- 2.2 Localisation and Symplectic Cutting

# 3 Hypertoric Varieties

#### 3.1 Compactification via Symplectic Cutting

We will use the  $S^1$ -action to perform a symplectic cut of the toric hyperkähler manifold  $\mathfrak M$  to compactify it, which has the effect of bounding the  $\|w\|^2$ -norm component of the real moment map  $\bar{\mu}_{\mathbb R}$  by above, and discarding the rest that lies above this bound. Consider the product  $\mathfrak M \times \mathbb C$ , and let  $S^1$  act on  $\mathfrak M \times \mathbb C$  via the diagonal product action, i.e.  $S^1$  acts on M by rotating the cotangent fibre coordinates, and on  $\mathbb C$  in the standard way:

$$e^{i\theta} \cdot ([z, w], \xi) = (e^{i\theta} \cdot [z, w], e^{i\theta} \xi) = ([z, e^{i\theta} w], e^{i\theta} \xi).$$

This action is Hamiltonian, and the corresponding moment map  $\Phi: \mathfrak{M} \times \mathbb{C} \to \mathbb{R}_{\geq 0}$  for the  $S^1$ -action is

$$\Phi \big( [z,w], \xi \big) = \phi[z,w] + |\xi|^2 = \|w\|^2 + |\xi|^2.$$

Then we have

$$\begin{split} \Phi^{-1}(\epsilon) &= \left\{ ([z,w],\xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = \epsilon \right\} \\ &= \left\{ [z,w] \in M : \|w\|^2 = \epsilon \right\} \bigsqcup \left\{ ([z,w],\xi) \in M \times \mathbb{C} : |\xi| = \pm \sqrt{\epsilon - \|w\|^2} \right\} \\ &= \left\{ [z,w] \in M : \|w\|^2 = \epsilon \right\} \bigsqcup \left\{ ([z,w],\xi) \in M \times \mathbb{C} : \xi = e^{i\arg(\xi)} \sqrt{\epsilon - \|w\|^2} \right\} \\ &= \phi^{-1}(\epsilon) \bigsqcup \left( \mathfrak{M} \times S^1 \right) \\ &=: \Sigma_1 \sqcup \Sigma_2, \end{split}$$

where we denote the level-set  $\phi^{-1}(\epsilon) \subseteq \mathfrak{M}$  by  $\Sigma_1$ , and  $\Sigma_2 \cong \mathfrak{M} \times S^1$  is the trivial  $S^1$ -bundle over  $\Sigma_2$  given by the globally defined section

$$\mathfrak{M} \to \mathfrak{M} \times S^1, \qquad [z,w] \longmapsto \left([z,w], e^{i\theta} \sqrt{\epsilon - \|w\|^2}\right), \qquad e^{i\theta} \in S^1.$$

Finally, taking the symplectic reduction of  $\Phi^{-1}(\epsilon)$  with respect to the  $S^1$ -action, we obtain the symplectic cut of  $\mathfrak{M}$  at level- $\epsilon$ ,

$$M_{\leq \epsilon} := \Phi^{-1}(\epsilon)/S^1 = \Sigma_1/S^1 \mid \Sigma_2/S^1,$$

where  $\Sigma_1/S^1 \cong \phi^{-1}(\epsilon)/S^1$  is just the usual symplectic reduction, and where  $\Sigma_2/S^1$  is diffeomorphic to  $\mathfrak{M}$  for  $||w||^2 < \epsilon$ , which we will denote by  $\mathfrak{M}_{<\epsilon}$ .

#### 3.2 The Combinatorics of the Cut Space, $\mathfrak{M}_{<\epsilon}$

Since the residual circle  $S^1$ -action acts as a subtorus  $S^1_A$  of the residual torus  $T^d$  on each component  $\mathcal{E}_A$  of the extended core, the hyperplane arrangement determined in  $(\mathfrak{t}^d)^*$  by the real moment map  $\bar{\mu}_{\mathbb{R}}$  is compactified by dropping in half-spaces with an inwards-pointing normal vector, given by  $v_A$  when taking the cut.

Recall from the previous section that  $j_A: S_1 \hookrightarrow T^n$  denoted the inclusion homomorphism of  $S^1$  into the original torus  $T^n$ . If we let  $j_{A,*}: \mathfrak{s}^1 \to \mathfrak{t}^n$  represent the differential of this inclusion, then

$$j_{A,*}(1) = \sum_{i \in A} e_i \in \mathfrak{t}^n,$$

and the generator  $\exp(v_A)$  of the one-parameter subgroup  $S_A^1$  in  $T^d$  is

$$\exp(v_A) = \exp\left(\pi_* \circ j_{A,*}(1)\right),\,$$

or to be more concise,

$$S_A^1 = \left\{ \exp\left(r \cdot \sum_{i \in A} u_i\right) \mid r \in \mathbb{R} \right\}.$$

Then the moment map for the restricted  $S^1$ -action to  $\mathcal{E}_A$  is

$$\phi_A[z,w] := \phi \Big|_{\mathcal{E}_A}[z,w] = (j_A^* \circ \mu_{\mathbb{R}})[z,w] = \left\langle \bar{\mu}_{\mathbb{R}}[z,w], \sum_{i \in A} u_i \right\rangle,$$

where  $j_A^*: (\mathfrak{t}^n)^* \to \mathbb{R}^*$  is the transposed differential of the inclusion,  $j_{A,*}$ .

As the  $S_A^1$ -action depends combinatorially on the component  $\mathcal{E}_A$ , the image of the real moment map in  $(\mathfrak{t}^d)^*$  is compactified by inserting a half-space  $Z_A$  with inwards-pointing normal  $v_A = \sum_{i \notin A} u_i$  determining the orientation, on each component  $\Delta_A$ .

# 4 Hypertoric Subvarieties

#### 4.1 Universal Modifications

Let  $(M, \omega_{HK}, \Phi_{HK})$  be a tri-Hamiltonian K-manifold, where  $\omega_{HK} = \omega_{\mathbb{R}} + \omega_{\mathbb{C}}$  and  $\Phi_{HK} = \Phi_{\mathbb{R}} + \Phi_{\mathbb{C}}$ . Define  $j_{\mathbb{C}}: M \to M \times M \times T^*K_{\mathbb{C}}$  by  $j(m) = (m, 1, \Phi_{\mathbb{C}}(m))$ , and  $j_{\mathbb{R}}: M \to M \times T^*K$  by  $j_{\mathbb{R}}(m) = (m, 1, \Phi_{\mathbb{R}})$ .

Lemma 4.1. We have an isomorphism of holomorphic symplectic tri-Hamiltonian K-manifolds,

$$j: M \xrightarrow{\sim} (M \times T^* K_{\mathbb{C}}) /\!\!/\!/_{0} K. \tag{1}$$

Here, the right-hand side is the quotient with respect to the diagonal K-action G, where G acts on the left on G is the one induced by the right action on G is the right action of G is the one induced by the right action of G is the right action of G

*Proof.* The map  $m\mapsto (1,\Phi_{\mathbb C}(m))$  sends M to the Lagrangian  $\mathfrak k_{\mathbb C}^*\subset T^*K_{\mathbb C}$ , so  $j_{\mathbb C}$  pulls back the complex-symplectic form  $\omega_{\mathbb C}\times 0$  on  $M\times \mathfrak k^*\subset M\times T^*K_{\mathbb C}$  to  $\omega_{HK}$  on M. Thus  $j_{\mathbb C}$  is a holomorphic-symplectic embedding.

The complex moment map for the diagonal left K-action on  $M \times T^*K_{\mathbb{C}}$  is  $\Psi^{\mathcal{L}}_{\mathbb{C}}(m,k,\lambda_{\mathbb{C}}) = \Phi_{\mathbb{C}} - k \cdot \lambda_{\mathbb{C}}$ , so  $j_{\mathbb{C}}$  maps M into

$$\Phi_{\mathbb{C}}^{-1}(0) = \{ (m, k_{\mathbb{C}}, \lambda_{\mathbb{C}}) \in M \times T^*K_{\mathbb{C}} \mid \Phi_{\mathbb{C}}(m) = k_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} \}.$$

The induced map  $\bar{\jmath}_{\mathbb{C}}$  is a [NB: biholomorphism?], and moreover is a holomorphic symplectomorphism as  $j_{\mathbb{C}}$  is a holomorphic symplectic embedding.

Moreover,

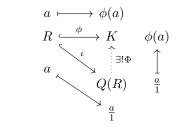
$$j_{\mathbb{C}}(k_{\mathbb{C}} \cdot m) = (k_{\mathbb{C}} \cdot m, e_{K_{\mathbb{C}}}, \Phi_{\mathbb{C}}(k_{\mathbb{C}} \cdot m)) = (k_{\mathbb{C}} \cdot m, e_{K_{\mathbb{C}}}, k_{\mathbb{C}} \cdot \Phi_{\mathbb{C}}(m)) = \mathcal{L}_{k_{\mathbb{C}}} \mathcal{R}_{k_{\mathbb{C}}} j_{\mathbb{C}}(m),$$

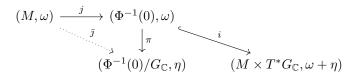
and

$$\Psi_{\mathbb{C}}^{\mathcal{R}}(j(m)) = \Psi_{\mathbb{C}}^{\mathcal{R}}(m, e_{K_{\mathbb{C}}}, \Phi_{\mathbb{C}}(m)) = \Phi_{\mathbb{C}}(m),$$

so  $\bar{\jmath}$  is K-equivariant and intertwines the complex moment maps on M and the complex-symplectic quotient,  $\Psi^{-1}_{\mathbb{C}}(0)/K = (M \times T^*K_{\mathbb{C}}) /\!\!/ K$ .

Proof.





References

[1] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.