Geometric Quantisation of Hypertoric Manifolds by Symplectic Cutting

General Notes

Abstract

Lorem ipsum.

1 Introduction

Lorem ipsum.

2 Background

2.1 Hypertoric Geometry

2.2 Localisation and Symplectic Cutting

3 Hypertoric Varieties

3.1 Compactification via Symplectic Cutting

We will use the S^1-action to perform a symplectic cut of the toric hyperkähler manifold $\mathfrak{M}$ to compactify it, which has the effect of bounding the $\|w\|^2$-norm component of the real moment map $\bar{\mu}_{\mathbb{R}}$ by above, and discarding the rest that lies above this bound. Consider the product $\mathfrak{M} \times \mathbb{C}$, and let S^1 act on $\mathfrak{M} \times \mathbb{C}$ via the diagonal product action, i.e. S^1 acts on M by rotating the cotangent fibre coordinates, and on $\mathbb{C}$ in the standard way:

$$e^{i\theta} \cdot \big([z,w],\xi\big) = \big(e^{i\theta} \cdot [z,w], e^{i\theta}\xi\big) = \big([z, e^{i\theta}w], e^{i\theta}\xi\big) .$$

This action is Hamiltonian, and the corresponding moment map $\Phi : \mathfrak{M} \times \mathbb{C} \to \mathbb{R}_{\geq 0}$ for the S^1-action is

$$\Phi\big([z,w],\xi\big) = \phi[z,w] + |\xi|^2 = \|w\|^2 + |\xi|^2.$$

Then we have

$$\begin{aligned}
\Phi^{-1}(\epsilon) &= \big\{([z,w],\xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = \epsilon\big\} \\
&= \big\{[z,w] \in M : \|w\|^2 = \epsilon\big\} \bigsqcup \big\{([z,w],\xi) \in M \times \mathbb{C} : |\xi| = \pm\sqrt{\epsilon - \|w\|^2}\big\} \\
&= \big\{[z,w] \in M : \|w\|^2 = \epsilon\big\} \bigsqcup \big\{([z,w],\xi) \in M \times \mathbb{C} : \xi = e^{i\arg(\xi)}\sqrt{\epsilon - \|w\|^2}\big\} \\
&= \phi^{-1}(\epsilon) \bigsqcup \big(\mathfrak{M} \times S^1\big) \\
&=: \Sigma_1 \sqcup \Sigma_2,
\end{aligned}$$

where we denote the level-set $\phi^{-1}(\epsilon) \subseteq \mathfrak{M}$ by Σ_1, and $\Sigma_2 \cong \mathfrak{M} \times S^1$ is the trivial S^1-bundle over Σ_2 given by the globally defined section

$$\mathfrak{M} \to \mathfrak{M} \times S^1, \qquad [z,w] \longmapsto \big([z,w], e^{i\theta}\sqrt{\epsilon - \|w\|^2}\big), \qquad e^{i\theta} \in S^1.$$

Finally, taking the symplectic reduction of $\Phi^{-1}(\epsilon)$ with respect to the S^1-action, we obtain the *symplectic cut of* $\mathfrak{M}$ *at level-*ϵ,

$$M_{\leq\epsilon} := \Phi^{-1}(\epsilon)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where $\Sigma_1/S^1 \cong \phi^{-1}(\epsilon)/S^1$ is just the usual symplectic reduction, and where Σ_2/S^1 is diffeomorphic to $\mathfrak{M}$ for $\|w\|^2 < \epsilon$, which we will denote by $\mathfrak{M}_{<\epsilon}$.

3.2 The Combinatorics of the Cut Space, $\mathfrak{M}_{\leq\epsilon}$

Since the residual circle S^1-action acts as a subtorus S^1_A of the residual torus T^d on each component $\mathcal{E}_A$ of the extended core, the hyperplane arrangement determined in $(\mathfrak{t}^d)^*$ by the real moment map $\bar{\mu}_{\mathbb{R}}$ is compactified by dropping in half-spaces with an inwards-pointing normal vector, given by v_A when taking the cut.

Recall from the previous section that $j_A : S_1 \hookrightarrow T^n$ denoted the inclusion homomorphism of S^1 into the original torus T^n. If we let $j_{A,*} : \mathfrak{s}^1 \to \mathfrak{t}^n$ represent the differential of this inclusion, then

$$j_{A,*}(1) = \sum_{i\in A} e_i \in \mathfrak{t}^n,$$

and the generator $\exp(v_A)$ of the one-parameter subgroup S^1_A in T^d is

$$\exp(v_A) = \exp\left(\pi_* \circ j_{A,*}(1)\right),$$

or to be more concise,

$$S^1_A = \left\{ \exp\left(r \cdot \sum_{i\in A} u_i \right) \,\middle|\, r \in \mathbb{R} \right\}.$$

Then the moment map for the restricted S^1-action to $\mathcal{E}_A$ is

$$\phi_A[z,w] := \phi\big|_{\mathcal{E}_A}[z,w] = (j_A^* \circ \mu_{\mathbb{R}})[z,w] = \left\langle \bar{\mu}_{\mathbb{R}}[z,w], \sum_{i\in A} u_i \right\rangle,$$

where $j_A^* : (\mathfrak{t}^n)^* \to \mathbb{R}^*$ is the transposed differential of the inclusion, $j_{A,*}$.

As the S^1_A-action depends combinatorially on the component $\mathcal{E}_A$, the image of the real moment map in $(\mathfrak{t}^d)^*$ is compactified by inserting a half-space Z_A with inwards-pointing normal $v_A = \sum_{i\notin A} u_i$ determining the orientation, on each component Δ_A.

4 Hypertoric Subvarieties

4.1 Universal Modifications

Let $(M, \omega_{HK}, \Phi_{HK})$ be a tri-Hamiltonian K-manifold , where $\omega_{HK} = \omega_{\mathbb{R}} + \omega_{\mathbb{C}}$ and $\Phi_{HK} = \Phi_{\mathbb{R}} + \Phi_{\mathbb{C}}$. Define $j_{\mathbb{C}} : M \to M \times M \times T^* K_{\mathbb{C}}$ by $j(m) = (m, 1, \Phi_{\mathbb{C}}(m))$, and $j_{\mathbb{R}} : M \to M \times T^* K$ by $j_{\mathbb{R}}(m) = (m, 1, \Phi_{\mathbb{R}})$.

Lemma 4.1. *We have an isomorphism of holomorphic symplectic tri-Hamiltonian K-manifolds,*

$$j : M \xrightarrow{\sim} (M \times T^* K_{\mathbb{C}}) /\!/\!/\!/_0\, K. \tag{1}$$

Here, the right-hand side is the quotient with respect to the diagonal K-action , where K acts on the left on $T^* K_{\mathbb{C}}$. The K-action on $(M \times T^* K_{\mathbb{C}}) /\!/\!/\!/_0\, K$ is the one induced by the right action on $T^* K_{\mathbb{C}}$.

Proof. The map $m \mapsto (1, \Phi_{\mathbb{C}}(m))$ sends M to the Lagrangian $\mathfrak{k}^*_{\mathbb{C}} \subset T^* K_{\mathbb{C}}$, so $j_{\mathbb{C}}$ pulls back the complex-symplectic form $\omega_{\mathbb{C}} \times 0$ on $M \times \mathfrak{k}^* \subset M \times T^* K_{\mathbb{C}}$ to ω_{HK} on M. Thus $j_{\mathbb{C}}$ is a holomorphic-symplectic embedding.

The complex moment map for the diagonal left K-action on $M \times T^* K_{\mathbb{C}}$ is $\Psi^{\mathcal{L}}_{\mathbb{C}}(m, k, \lambda_{\mathbb{C}}) = \Phi_{\mathbb{C}} - k \cdot \lambda_{\mathbb{C}}$, so $j_{\mathbb{C}}$ maps M into

$$\Phi_{\mathbb{C}}^{-1}(0) = \{\, (m, k_{\mathbb{C}}, \lambda_{\mathbb{C}}) \in M \times T^* K_{\mathbb{C}} \mid \Phi_{\mathbb{C}}(m) = k_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} \,\}.$$

The induced map $\bar{j}_{\mathbb{C}}$ is a [NB: biholomorphism?], and moreover is a holomorphic symplectomorphism as $j_{\mathbb{C}}$ is a holomorphic symplectic embedding.

Moreover,

$$j_{\mathbb{C}}(k_{\mathbb{C}} \cdot m) = (k_{\mathbb{C}} \cdot m, e_{K_{\mathbb{C}}}, \Phi_{\mathbb{C}}(k_{\mathbb{C}} \cdot m)) = (k_{\mathbb{C}} \cdot m, e_{K_{\mathbb{C}}}, k_{\mathbb{C}} \cdot \Phi_{\mathbb{C}}(m)) = \mathcal{L}_{k_{\mathbb{C}}} \mathcal{R}_{k_{\mathbb{C}}} j_{\mathbb{C}}(m),$$

and

$$\Psi^{\mathcal{R}}_{\mathbb{C}}(j(m)) = \Psi^{\mathcal{R}}_{\mathbb{C}}(m, e_{K_{\mathbb{C}}}, \Phi_{\mathbb{C}}(m)) = \Phi_{\mathbb{C}}(m),$$

so $\bar{j}$ is K-equivariant and intertwines the complex moment maps on M and the complex-symplectic quotient, $\Psi_{\mathbb{C}}^{-1}(0)/K = (M \times T^* K_{\mathbb{C}}) /\!/ K$. $\square$

Proof.

$$\begin{tikzcd}
a \arrow[r, mapsto] & \phi(a) & \\
R \arrow[r, hook, "\phi"] \arrow[dr, hook, "\iota"'] & K & \phi(a) \\
a \arrow[dr, mapsto] & Q(R) \arrow[u, dotted, "\exists!\Phi"'] & \frac{a}{1} \arrow[u, mapsto] \\
& \frac{a}{1} &
\end{tikzcd}$$

$$\begin{tikzcd}
(M,\omega) \arrow[r, "j"] \arrow[dr, dotted, "\bar{\jmath}"] & (\Phi^{-1}(0),\omega) \arrow[d, "\pi"] \arrow[dr, hook, "i"] & \\
& (\Phi^{-1}(0)/G_{\mathbb{C}},\eta) & (M \times T^*G_{\mathbb{C}},\omega+\eta)
\end{tikzcd}$$

□

References

[1] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.