Toric Hyperkähler Manifolds & the Equivariant Verlinde Formula
First Year Annual Review

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Structure of the Talk

1. Introduction & Motivation
2. My Progress
3. Outlook

Definition

(X^{2n}, ω) is a *toric symplectic manifold* if it is compact, connected, and equipped with an effective hamiltonian action of T^n, with moment map $\mu : X \to (\mathbb{R}^n)^*$.

Example

$(\mathbb{CP}^2, \omega_{FS})$, with T^2-action given by:

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2].$$

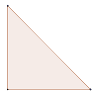
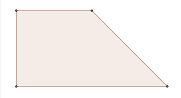
Moment map $\mu : \mathbb{CP}^2 \to (\mathbb{R}^2)^*$ is:

$$\mu([z_0 : z_1 : z_2]) = \frac{1}{2}\left(\frac{|z_1|^2}{\|z\|^2}, \frac{|z_2|^2}{\|z\|^2}\right) + c, \qquad c \in (\mathbb{R}^2)^*.$$

Definition

$\Delta \subset (\mathbb{R}^n)^*$ is a *Delzant polytope* if it is convex, simple, rational and smooth.

Examples

$\mathbb{CP}^2 \leftrightarrow$   $\leftrightarrow \widetilde{\mathbb{CP}^2}$

Remark

Given $(X^{2n}, \omega, T^n, \mu)$, the Atiyah, Guillemin-Sternberg convexity theorem asserts that $\mu(X^{2n}) = \Delta \subset (\mathbb{R}^n)^*$ is a Delzant polytope.

Theorem (Delzant)

$$\frac{(X^{2n}, \omega, T^n, \mu)}{T^n\text{-}\textit{equivariance}} \stackrel{1-1}{\longleftrightarrow} \frac{\textit{Delzant polytopes}}{SL(n, \mathbb{Z}) \ltimes \mathbb{R}^n}$$

Question

So for a given $\Delta \subset (\mathbb{R}^d)^*$, what is the respective X_Δ?

Let $u_i \in \mathbb{R}^d$ $(i = 1, \ldots, n)$ be the inward-pointing normals to to facets of Δ, and define $\pi(e_i) = u_i$.

$$0 \longrightarrow \mathfrak{n} := \ker \pi \xhookrightarrow{\;i\;} \mathbb{R}^n \xrightarrow{\;\pi\;} \mathbb{R}^d \longrightarrow 0$$

Can dualise:

$$0 \longleftarrow \mathfrak{n}^* \xleftarrow{\;i^*\;} (\mathbb{R}^n)^* \xleftarrow{\;\pi^*\;} (\mathbb{R}^d)^* \longleftarrow 0$$

Or exponentiate:

$$0 \longrightarrow N \xhookrightarrow{\;i\;} T^n \xrightarrow{\;\pi\;} T^d \longrightarrow 0$$

$$0 \longrightarrow N \xhookrightarrow{i} T^n \xrightarrow{\pi} T^d \longrightarrow 0$$

T^n acts diagonally on $\mathbb{C}^n$, with moment map:

$$J : \mathbb{C}^n \to (\mathbb{R}^n)^*, \qquad J(z) = \frac{1}{2}\sum_{i=1}^{n}(|z_i|^2 - \lambda_i)e_i, \quad \lambda \in (\mathbb{R}^n)^*.$$

N also acts on $\mathbb{C}^n$ via inclusion, with moment map $i^* \circ J : \mathbb{C}^n \to \mathfrak{n}^*$.

Fact

$X_\Delta := (i^* \circ J)^{-1}(0)/N$ is a toric symplectic manifold for the residual T^d-action, with Delzant polytope Δ, where

$$\Delta = \cap_{i=1}^{n}\{y \in \mathbb{R}^d : \langle y, u_i\rangle \geq \lambda_i \textit{ for all } i\}.$$

Question

For a (pre-quantum) line bundle $\mathcal{L}$ over (X, ω), can one find a canonical Hilbert space of (holomorphic) "wave-functions" $\mathcal{Q}(X)$ with

$$\mathcal{Q}(X) = H^0(X, \mathcal{L})?$$

Conjecture ("Quantisation Commutes with Reduction")

If $X_0 = \mu^{-1}(0)/G$*, then:*

$$\mathcal{Q}(X)^G = \mathcal{Q}(X_0)$$

Further, if $X_\alpha = \mu^{-1}(\alpha)/G$*:*

$$\dim \mathcal{Q}(X_\alpha) = \mathrm{mult}(\alpha)$$

Theorem

For toric symplectic manifolds: $\dim \mathcal{Q}(X) = \#(\Delta \cap \mathbb{Z}^n)$

Example

$X = \mathbb{C}^3$ and $\mathbb{CP}^2 = X_k = \mu^{-1}(k)/N$. For $s : \mathbb{C}^3 \to \mathbb{C}$ holomorphic, locally $s(z) = z_0^{j_0} z_1^{j_1} z_2^{j_2}$. For N-equivariance:

$$s(n \cdot z) = n^{j_0+j_1+j_2} s(z) \overset{?}{=} n \cdot s(z) = n^k s(z).$$

True for $H^0(\mathbb{CP}^2, \mathcal{O}(k))$, and solution set:

$$\{(j_0, j_1, j_2) : \sum j_i = k\} \overset{1-1}{\leftrightarrow} \{(j_0, j_2) : j_0 + j_1 \leq k\} \equiv (\Delta \cap \mathbb{Z}^2).$$

Remark

Case for $X = \mathbb{C}^4$ and $X_k = \mu^{-1}(k)/N \cong \mathbb{CP}^3$ very similar, with Delzant polytope $k\Delta \subset \mathbb{R}^3$ (Δ the standard 3-simplex).

$$\#(k\Delta \cap \mathbb{Z}^3) = \frac{(k+1)(k+2)(k+3)}{3!} = \frac{k^3}{6} + k^2 + \frac{11}{6}k + 1.$$

Fact

Let $M = \mathcal{M}_{flat}(\Sigma_2; SU(2))$, and $\mathcal{L}$ be the determinant line bundle. Then:

$$\dim H^0(M, \mathcal{L}^{\otimes k}) = \#(k\Delta \cap \mathbb{Z}^3).$$

In fact, $M \cong \mathbb{CP}^3$, and the above equation is known as the Verlinde formula for M.

Question

What about the moduli space for Higgs bundle $\mathcal{M}_{\text{Higgs}}(\Sigma_g; G)$, which is *non-compact*?

It has $T^*\mathcal{M}_{\text{flat}}(\Sigma_g; G) \subset \mathcal{M}_{\text{Higgs}}(\Sigma_g; G)$ as an open and dense subset. As $\mathcal{M}_{\text{Higgs}}$ is non-compact, $\dim \mathcal{Q}(\mathcal{M}_H) = \infty$. However $\mathcal{M}_H$ admits a $\mathbb{C}^*$-action by scaling the Higgs fields, and $S^1 \subset \mathbb{C}^*$ has compact fixed point loci $\implies$ $\mathbb{C}^*$-weight decomposition:

$$\mathcal{Q}(\mathcal{M}_H) = \bigoplus_{n \geq 0} \mathcal{Q}(\mathcal{M}_H)_n.$$

Proposition

The equivariant Verlinde formula is a recipe for calculating the n^{th} *graded part for* $\dim \mathcal{Q}(\mathcal{M}_{Higgs})$*, i.e.*

$$\dim_t H^0(\mathcal{M}_H, \mathcal{L}^k) = \sum_{n=0}^{\infty} \dim H^0_n(\mathcal{M}_H, \mathcal{L}^k) t^n.$$

In particular, the degree-0 (weight-0) part corresponds to the classical Verlinde formula.

Remark

There is also an equivariant Verlinde formula for *parabolic Higgs bundles.*

Complexification analogy for toric symplectic manifolds are *hypertoric* manifolds: consider $M = T^*\mathbb{C}^n$ with induced linear G-action from $\mathbb{C}^n$, and moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^*$.
Action is hyperhamiltonian with hyperkähler moment map $\mu_{HK} := \mu_\mathbb{R} \oplus \mu_\mathbb{C} : T^*\mathbb{C}^n \to \mathfrak{g}^* \oplus \mathfrak{g}^*_\mathbb{C}$, where:

$$\mu_\mathbb{R}(z,w) = \mu(z) - \mu(w), \qquad \mu_\mathbb{C}(z,w)(v) = w(\hat{v}_z),$$

with $w \in T^*_z\mathbb{C}^n$, $v \in \mathfrak{g}_\mathbb{C}$, and $\hat{v}_z \in T_z\mathbb{C}^n$ induced.

Definition

For $\alpha \in Z(\mathfrak{g}^*)$:

$$M := (\mu_\mathbb{R}^{-1}(\alpha) \cap \mu_\mathbb{C}^{-1}(0))/G$$

is the *hyperkähler analogue* to the Kähler $X = \mu^{-1}(\alpha)/G$.

Let $X = \mu^{-1}(\alpha)/N$ be a symplectic toric manifold with a residual $T^d = T^n/N$ action from the Delzant construction.

Definition

A *toric hyperkähler manifold* M is the hyperkähler analogue to the Kähler quotient X.

Explicitly:

$$M = (\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))/N,$$

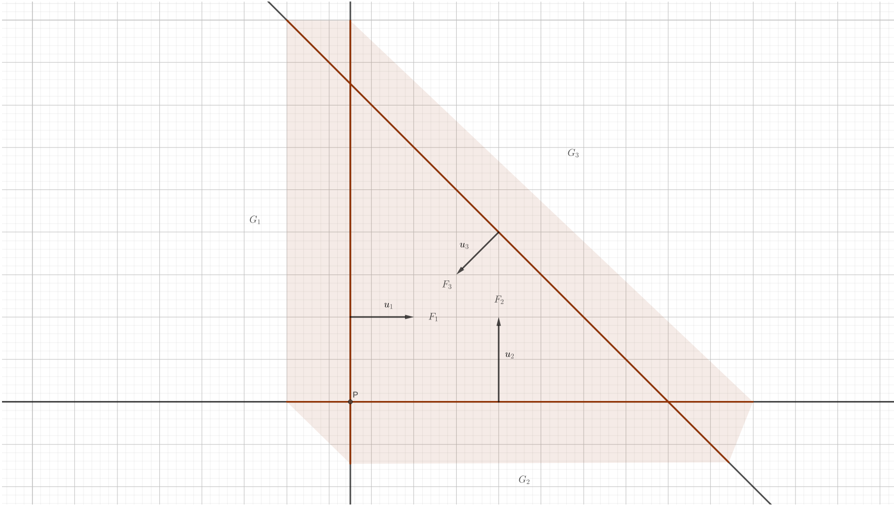
and, if $i^*(r_1, \ldots, r_n) = \alpha$ and $(\partial_i)_{i=1}^n$ is a basis for $(\mathbb{R}^d)^* = \ker i^*$, we get residual moment maps:

$$\phi_{\mathbb{R}}[z,w] = \frac{1}{2}\sum_{i=1}^{n}(|z_i|^2 - |w_i|^2 - r_i)\partial_i, \qquad \phi_{\mathbb{C}}[z,w] = \sum_{i=1}^{n}(z_i w_i)\partial_i.$$

Definition

The set
$$\mathcal{E} := \phi_{\mathbb{C}}^{-1}(0) = \{[z,w] \in M : z_i w_i = 0\}$$
is called the *extended core* of M.

For each $A \subseteq \{1, \ldots, n\}$, $\mathcal{E}$ breaks up into components:
$$\mathcal{E}_A = \{[z,w] \in M : w_i = 0 \text{ if } i \in A, \text{ and } z_i = 0 \notin A\},$$
with image
$$\Delta_A := \phi_{\mathbb{R}}(\mathcal{E}_A) = \bigcap_{i \in A} F_i \cap \bigcap_{i \notin A} G_i,$$
for
$$F_i = \{y \in (\mathbb{R}^d)^* : \langle y, u_i \rangle + r_i \geq 0\}, \quad G_i = \{y \in (\mathbb{R}^d)^* : \langle y, u_i \rangle + r_i \leq 0\}.$$



G_3
G_1
u_3
F_3
F_2
u_1
F_1
u_2
P
G_2

$$S^1\text{-action:} \qquad \hbar \cdot [z,w] = [z, \hbar w], \qquad \hbar \in S^1.$$

Descends to $\mathcal{E}$ and, on each $\mathcal{E}_A$, acts as a subgroup of T^n. On $\mathcal{E}_A$:

$$[z, \hbar w] = [\hbar^{-1} z_1, \ldots, \hbar^{-1} z_n; w] = [\hbar_1 z_1, \ldots, \hbar_n z_n; w],$$

with

$$\hbar_i := \begin{cases} \hbar^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A. \end{cases}$$

For $j_A : S^1\big|_{\mathcal{E}_A} \hookrightarrow T^n$, its moment map is

$$\Phi_A[z,w] = -\left\langle \mu_{\mathbb{R}}[z,w], \sum_{i \in A} u_i \right\rangle.$$

Globally $\Phi[z,w] = \frac{1}{2}\|w\|^2$ is proper; can "compactify" M: let S^1 act on $M \times \mathbb{C}$ diagonally. Moment map:

$$\mu_{\text{cut}} : M \times \mathbb{C} \to \mathbb{R}, \qquad ([z,w],\xi) \mapsto \Phi[z,w] + \tfrac{1}{2}|\xi|^2.$$

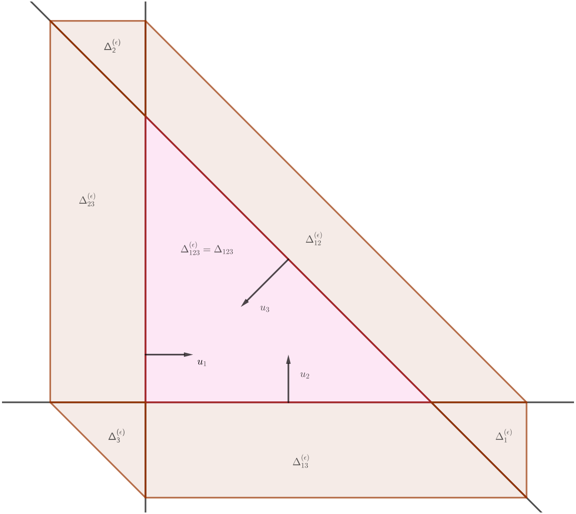
Resulting in:

$$M_\epsilon := \mu_{\text{cut}}^{-1}(\epsilon)/S^1 \cong \{m \in M : \Phi(m) < \epsilon\} \sqcup \Phi^{-1}(\epsilon)/S^1$$

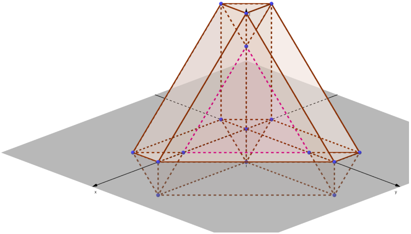
On each component $\mathcal{E}_A^{(\epsilon)} := \mathcal{E}_A \cap M_\epsilon$, we find that:

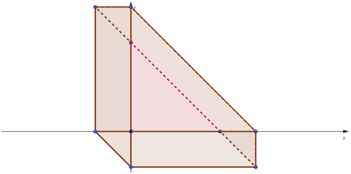
$$\phi_{\mathbb{R}}(\mathcal{E}_A^{(\epsilon)}) = \Delta_A \cap \{v \in (\mathbb{R}^d)^* : \langle v, \textstyle\sum_{i\in A} u_i\rangle \leq \epsilon\}$$

Example for $T^*\mathbb{CP}^2$



Example for $T^*\mathbb{CP}^3$





Recall: $G = SU(2)$, $G_{\mathbb{C}} = SL(2,\mathbb{C})$, $g = 2$,

and $\mathcal{M}_{\text{Higgs}}(\Sigma_2; SU(2)) \cong T^*\mathbb{CP}^3$.

Example

For "large enough k":

$$\begin{aligned}\dim \mathcal{Q}(\mathcal{M}_{\text{Higgs}}(\Sigma_2; G, k)) &= \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1 \\ &+ \left(\frac{1}{2}k^3 + 3k^2 - \frac{1}{2}k - 3\right)t \\ &+ (k^3 + 8k^2 - 3k + 6)t^2 + \dots\end{aligned}$$

The degree-0 part is our classical Verlinde formula for $\mathbb{CP}^3$.

There are many striking analogies between hypertoric varieties and Higgs moduli spaces:

Hypertoric Varieties, M	Higgs Bundle Moduli, $\mathcal{M}_{\text{Higgs}}$
$\mathbb{C}^*$-action	Hitchin $\mathbb{C}^*$-action
$\Phi[z,w] = \frac{1}{2}\|w\|^2$, perfect Morse	$\mu(\mathcal{E},\phi) = \|\phi\|^2_{L^2}$, perfect Morse
$T^*X \subset M$ open, dense	$T^*\mathcal{M}_{\text{flat}} \subset \mathcal{M}_H$ open, dense
$X = \Phi^{-1}(0)$ core, deformation retract of M	$\mu^{-1}(0) = \mathcal{M}_{\text{flat}}$ "nilpotent cone" deformation retract of $\mathcal{M}_H$

Goal (for now)

Find a combinatorial method to calculate

$$\dim \mathcal{Q}(M)_n, \qquad \textit{for} \qquad \mathcal{Q}(M) = \bigoplus_{n\geq 0} \mathcal{Q}(M)_n,$$

with M a hypertoric manifold, which should coincide with the equivariant Verlinde formula for our toy model

$$M = \mathcal{M}_{Higgs}(\Sigma_2; SU(2)) \cong T^*\mathbb{CP}^3.$$

Question

- What about for more complicated hypertoric varieties, *e.g.* non-convex core, orbifolds, etc.
- Consider other Higgs bundle moduli spaces, *e.g.* $g \geq 2$, $G \neq SU(2)$, etc.

The End.