# HYPERTORIC MANIFOLDS AND EQUIVARIANT LOCALISATION

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## 1. INDEX THEORY

### 1.1. Non-Equivariant Index Formula.

For a holomorphic vector bundle $\mathcal{L}$ over a complex $n$-dimensional variety $M$, the *index* $\mathrm{ind}(\bar{\partial}, \mathcal{L})$w is defined as

$$\mathrm{ind}(\bar{\partial}, \mathcal{L}) := \sum_{k=0}^{n} (-1)^k \dim H^k(M; \mathcal{L}).$$

Viewing the index $\mathrm{ind}(\bar{\partial}, \mathcal{L})$ as the Euler characteristic $\chi(M, \mathcal{L})$ of the vector bundle $\mathcal{L}$, we can apply the Atiyah-Singer index theorem, which we state below, to express the index as an integral over $M$ of the product of the Todd class $\mathrm{Td}(TM)$ of the tangent bundle $TM \to M$ over $M$, and the Chern character $\mathrm{Ch}(\mathcal{L}) := \exp(c_1(\mathcal{L}))$ of $\mathcal{L}$, where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$.

**Theorem 1.1** (Atiyah-Singer Index Theorem, [**?**])**.** *Let $M$ be a compact complex manifold, $\mathcal{L}$ a holomorphic vector bundle over $M$. Let*

$$\mathrm{Td}(TM) = \prod \frac{x_i}{1 - e^{-x_i}}$$

*be the Todd class of the complex vector bundle $TM \to M$, where the $x_i$ are the Chern roots of $TM$. Then the Euler characteristic $\chi(M, \mathcal{L})$ of the sheaf of germs of holomorphic sections of $\mathcal{L}$ is given by*

$$\chi(M, \mathcal{L}) = \int_M \mathrm{Td}(M) \cdot \mathrm{Ch}(\mathcal{L}).$$

**Example.** Let $M = \mathbb{CP}^1$ and let $\mathcal{L}$ be the line bundle $\mathcal{O}(k)$ for some positive integer $k$. If $\langle \xi \rangle = H^2(M; \mathbb{Z})$, *i.e.* $\xi$ is the generator of $H^2(\mathbb{CP}^1; \mathbb{Z})$, then $c_1(\mathcal{L}) = k\xi$, and thus the Chern character of $\mathcal{L}$ is

$$\mathrm{Ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = \sum_{j=0}^{\infty} (k\xi)^j = 1 + k\xi$$

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(the higher powers of $\xi$ vanish since $\dim_{\mathbb{C}} M = 1$).

For $n$-dimensional complex projective space $\mathbb{CP}^n$, both the total Chern class

$$c(\mathbb{CP}^n) := c(T\mathbb{CP}^n) := 1 + c_1 + c_2 + c_3 + \ldots,$$

and the Todd class $\mathrm{Td}(T\mathbb{CP}^n)$ for the tangent bundle $T\mathbb{CP}^n \to \mathbb{CP}^n$, can be calculated using the exact Euler sequence, along with the multiplicativity of

$$\{0\} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{CP}^n \longrightarrow \{0\},$$

the total Chern class and the Todd class,

$$c(\mathcal{F} \oplus \mathcal{G}) = c(\mathcal{F}) \cdot c(\mathcal{G}), \qquad \mathrm{Td}(\mathcal{F} \oplus \mathcal{G}) = \mathrm{Td}(\mathcal{F}) \cdot \mathrm{Td}(\mathcal{G}),$$

which yields

$$c(\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \mathcal{O}) = c(\mathcal{O}(1)^{\oplus(n+1)}) = (1 + \xi)^{n+1},$$

and

$$\mathrm{Td}(T\mathbb{CP}^n) = \mathrm{Td}(T\mathbb{CP}^n \oplus \mathcal{O}) = \mathrm{Td}(\mathcal{O}(1)^{\oplus(n+1)}) = \mathrm{Td}\left(\mathcal{O}(1)\right)^{n+1} = \left(\frac{\xi}{1 - e^{-\xi}}\right)^{n+1}.$$

This expression can be expanded as a formal power series which, for $n = 1$ in our example with the complex projective line $\mathbb{CP}^1$, gets us

$$c(\mathbb{CP}^1) = (1 + \xi)^2 = 1 + 2\xi, \qquad \mathrm{Td}(T\mathbb{CP}^1) = 1 + \tfrac{1}{2}c_1(T\mathbb{CP}^1) = 1 + \xi.$$

Finally, applying the Atiyah-Singer index theorem 1.1, we have

$$\chi(\mathbb{CP}^1, \mathcal{L}) = \int_{\mathbb{CP}^1} \mathrm{Td}(\mathbb{CP}^1) \cdot \mathrm{Ch}(\mathcal{L}) = \int_{\mathbb{CP}^1} (1+\xi) \cdot (1+k\xi) = \int_{\mathbb{CP}^1} 1 + (k+1)\xi = k+1.$$

**Example.** Now we let $M = \mathbb{CP}^2$, and let $\mathcal{L} = \mathcal{O}(k)$ and $\langle \xi \rangle = H^2(M, \mathbb{Z})$ again as above. Now we have

$$c(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + k\xi + k^2\xi^2,$$

and

$$c(T\mathbb{CP}^2) = 1 + c_1 + c_2 = (1 + \xi)^3 = 1 + 3\xi + 3\xi^2,$$

$$\mathrm{Td}(T\mathbb{CP}^2) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} = 1 + \frac{3}{2}\xi + \frac{9\xi^2 + 3\xi^2}{12} = 1 + \frac{3}{2}\xi + \xi^2.$$

Hence by the Atiyah-Bott index theorem 1.1,

$$\chi(M, \mathcal{L}) = \int_M \mathrm{Td}(TM) \cdot \mathrm{Ch}(\mathcal{L}) = \int_M \left(1 + \tfrac{3}{2}\xi + \xi^2\right) \cdot \left(1 + k\xi + k^2\xi^2\right)$$
$$= \int_M (k^2 + \tfrac{3}{2}k + 1)\xi^2 + O(\xi) = k^2 + \tfrac{3}{2}k + 1.$$

**Example.** Let $M = \mathbb{CP}^3$, and let $\mathcal{L}$, $\xi$, etc. be as above. Then

$$\mathrm{Ch}(\mathcal{L}) = 1 + k\xi + (k\xi)^2 + (k\xi)^3,$$
$$c(TM) = (1 + \xi)^4 = 1 + 4\xi + 6\xi^2 + 4\xi^3,$$
$$\mathrm{Td}(TM) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} = 1 + 2\xi + \frac{11}{6}\xi^2 + \xi^3.$$

Then by the Atiyah-Bott Index theorem 1.1,

$$\chi(M, \mathcal{L}) = \int_M \mathrm{Td}(TM) \cdot \mathrm{Ch}(\mathcal{L}) = \int_M \left(1 + 2\xi + \frac{11}{6}\xi^2 + \xi^3\right) \cdot \left(1 + k\xi + k^2\xi^2 + k^3\xi^3\right)$$
$$= \int_M \left(k^3 + 2k^2 + \frac{11}{6}k + 1\right)\xi^3 + O(\xi^2) =$$

1.2. **Equivariant Index Theorems.**

1.2.1. *Equivariant Characteristic Classes.*

## 2. Compactifying the Hypertoric Variety via Symplectic Cutting

2.1. **Set-Up.** We can use the residual $S^1$-action to perform a symplectic cut of the toric hyperkähler manifold $M$ in order to compactify it as follows: consider the product $M \times \mathbb{C}$, where now $S^1$ acts on $M \times \mathbb{C}$ as

$$e^{i\theta} \cdot \left([z, w], \xi\right) = \left([z, e^{i\theta}w], e^{i\theta}\xi\right),$$

which is a Hamiltonian action with associated moment map

$$\mu_{\mathrm{cut}} : M \times \mathbb{C} \longrightarrow \mathbb{R}_{\geqslant 0},$$
$$\mu_{\mathrm{cut}}\left([z, w], \xi\right) = \Phi[z, w] + \tfrac{1}{2}|\xi|^2 - \epsilon,,$$

where $\Phi : M \to \mathbb{R}_{\geqslant 0}$ is the moment map $\Phi[z, w] = \tfrac{1}{2}\|w\|^2$ for the residual $S^1$-action on $M$, and $\epsilon \in \mathbb{R}_{\geqslant 0}$.

Then we have

$$\begin{aligned}
\mu_{\text{cut}}^{-1}(0) &= \left\{([z,w],\xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = 2\epsilon\right\} \\
&= \left\{[z,w] \in M : \|w\|^2 = 2\epsilon\right\} \bigsqcup \left\{([z,w],\xi) \in M \times \mathbb{C} : |\xi| = \pm\sqrt{2\epsilon - \|w\|^2}\right\} \\
&= \left\{[z,w] \in M : \|w\|^2 = 2\epsilon\right\} \bigsqcup \left\{([z,w],\xi) \in M \times \mathbb{C} : \xi = e^{i\arg(\xi)}\sqrt{2\epsilon - \|w\|^2}\right\} \\
&= \Phi^{-1}(\epsilon) \bigsqcup (M \times S^1) \\
&=: \Sigma_1 \bigsqcup \Sigma_2,
\end{aligned}$$

where $\Sigma_1$ is just the level-set of $\Phi$ at the level $\epsilon$ in $M$, and $\Sigma_2 = M \times S^1$ is exhibited as a trivial $S^1$-bundle over $\Sigma_2$, using the globally defined section

$$M \to M \times S^1, \qquad [z,w] \longmapsto \left([z,w], e^{i\theta}\sqrt{2\epsilon - \|w\|^2}\right), \qquad e^{i\theta} \in S^1.$$

Finally, by taking the quotient of $\mu_{\text{cut}}^{-1}(0)$ by the $S^1$-action, we obtain the *symplectic cut of $M$ at level $\epsilon$*,

$$M_{\leqslant\epsilon} := \mu_{\text{cut}}^{-1}(0)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where $\Sigma_1/S^1 = \Phi^{-1}(\epsilon)/S^1$ is just the symplectic reduction, and where $\Sigma_2/S^1$ is diffeomorphic to $M$ for $\|w\|^2 < 2\epsilon$, which we denote by $M_{<\epsilon}$.

2.2. **Restriction to the Extended Core Component, $\mathcal{E}_A$.** Since the residual circle $S^1$-action acts as a subgroup of the original torus $T^n$ when restricted to each component $\mathcal{E}_A$ of the extended core $\mathcal{E}$, we can describe the resulting configuration of the hyperplane arrangement in $(\mathbb{R}^d)^*$ from taking the cut in a combinatorial way. For each component, let $j_A : \mathfrak{s}^1 \to \mathbb{R}^n$ be the derivative of the inclusion of $S^1$ into $T^n$ on the Lie algebra level, that is

$$j_A(\xi) = (\xi_1, \ldots, \xi_n), \qquad \text{where } \xi_i = \begin{cases} -1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A, \end{cases}$$

so that its image in $\mathbb{R}^n$ generates a circle subgroup $S^1$ in $T^n$ that depends on each component $\mathcal{E}_A$. Then the moment map for this restriction for the $S^1$-action is

$$\Phi[z,w] = j_A^* \circ \mu_{\mathbb{R}}[z,w] = \left\langle \mu_{\mathbb{R}}(z,w), \sum_{i \in A} \xi_i u_i \right\rangle,$$

and so from our above discussion of how we constructed the symplectic cut, the image in $(\mathbb{R}^d)^*$ of the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ is

$$\phi_{\mathbb{R}}(\Phi^{-1}(\epsilon)) = \left\{ y \in \Delta_A : \left\langle y, \sum_{i \in A} \xi_i u_i \right\rangle + \epsilon = 0 \right\} =: H_A$$

which introduces an inward-pointing half-space

$$F_A := \left\{ y \in \Delta_A : \left\langle y, \sum_{i \in A} \hbar_i u_i \right\rangle + \epsilon \geq 0 \right\}$$

such that the image of the extended core component $\mathcal{E}_A$ after being compactified is the original convex polytope $\Delta_A$ intersected with $H_A$. One can also see clearly that the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ has the restricted $S^1$-action as its stabiliser subgroup since, by definition of $H_A$, the moment map $\Phi|_{\mathcal{E}_A}$ equals the hyperplane $H_A$, i.e. $\Phi|_{\mathcal{E}_A}$ is constant along $\Phi^{-1}(\epsilon)/S^1$.

*Remark* 1. *If we had used instead the following action for $S^1$*

$$e^{i\theta} \cdot \big([z,w],\xi\big) = \big([z,e^{i\theta}w], e^{-i\theta}\xi\big)$$

*with respective moment map*

$$\mu_{cut}([z,w],\xi) = \frac{1}{2}\|w\|^2 - \frac{1}{2}|\xi|^2 - \epsilon,$$

*and taken the cut, then the resulting then we would obtain the other "discarded half" $\mathfrak{M}_{>\epsilon}$ of the hypertoric manifold $\mathfrak{M}$ along with the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ with the opposite orientation:*

$$\mathfrak{M}_{\geq \epsilon} = \mathfrak{M}_{>\epsilon} \bigsqcup \Big( -(\Phi^{-1}(\epsilon)/S^1)\Big).$$

*The component $M_\epsilon$ is non-compact however, so we focus on $M_{<\epsilon}$.*

## References

[1] Eugene Lerman and Susan Tolman. Hamiltonian torus actions on symplectic orbifolds and toric varieties. *Trans. Amer. Math. Soc.*, 349(10):4201–4230, 1997.

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