Compactifying Hypertoric Manifolds via Symplectic Cutting

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Group Actions

- G compact connect Lie group, $\mathrm{Lie}(G) = \mathfrak{g}$.
- $G \circlearrowright X$ gives rise to an infinitesimal action of $\mathfrak{g}$ which associates to each $\xi \in \mathfrak{g}$ a vector field $\xi^{\#}$.

Hamiltonian Functions

- (X, ω) symplectic manifold, and $G \circlearrowright X$ preserves ω.
- Say that G acts in a **Hamiltonian way** on X if every $\xi \in \mathfrak{g}$ has a function $\phi^{\xi} \in C^{\infty}(X)$ such that

$$\imath_{\xi^{\#}}\omega = d\phi^{\xi}.$$

Definition

Dual notion is the **moment mapping** $\mu : X \to \mathfrak{g}^*$, defined by

$$(\mu(p),\ \xi) = \phi^\xi(p),$$

for all $p \in X$ and $\xi \in \mathfrak{g}$.

Properties

- μ is G-equivariant when $G \circlearrowright \mathfrak{g}^*$ by the dual coadjoint action.
- If G is abelian (*i.e.* a torus), μ is unique up to a constant since dual coadjoint action is trivial.

Transitive Actions

- Recall: action is **transitive** if for any pair $x, y \in X$, there exists an element $g \in G$ such that $g \cdot x = y$.

Theorem (Kostant) [Gui94]: For compact G, all Hamiltonian G-spaces with transitive G-action are coadjoint orbits.

- For G abelian $\implies$ *no* positive dimensional Hamiltonian G-spaces with transitive action.

Effective Actions

Theorem [Gui94]: If G acts effectively, then $\dim X \geqslant 2 \dim G$.

- So for G abelian and acting effectively, simplest examples are when $\dim X = 2 \dim G$.

Definition

Definition: A **symplectic toric manifold** is a compact connected symplectic manifold (X^{2n}, ω) with an effective Hamiltonian action of a torus T^n, with moment map $\mu : X \to \mathbb{R}^n$.

Example

$T^n \circlearrowright \mathbb{C}^n$ diagonally, with moment map

$$\mu : \mathbb{C}^n \to \mathbb{R}^n, \quad \mu(z) = \frac{1}{2} \sum_{k=1}^{n} |z_k|^2 e_k.$$

Symplectic Reduction

Set-Up

- (X, ω) Hamiltonian G-space with moment map $\mu : X \to \mathbb{R}^n$.
- $X_0 := \mu^{-1}(0)$, for $c \in \mathbb{R}^n$; X_0 is G-invariant as μ is G-equivariant.

Theorem: If $G \circlearrowright X_0$ freely, 0 regular value of μ, then $X_0 \subseteq X$ closed submanifold and $\dim X_0 = \dim X - \dim G$.

Marsden-Weinstein Reduction

Theorem: If $G \circlearrowright X_0$ freely, $X /\!\!/ G := X_0/G$ is a symplectic manifold of dimension $\dim X - 2\dim G$.

Example

- Let $S^1 \circlearrowright \mathbb{C}^{n+1}$ as $t \cdot z_k = tz_k, \qquad k = 1, \dots n+1.$
- Choose the moment map $\mu : \mathbb{C}^{n+1} \to \mathbb{R}$ to be

$$\mu(z) = \sum_{k=1}^{n+1} |z_k|^2 - c^2, \qquad c \in \mathbb{R}.$$

- Then $X_0 = \mu^{-1}(0) = \{\|\mathbf{z}\| = c^2\} \cong S^{2n+1}$, so

$$\mathbb{C}^{n+1} /\!\!/ S^1 := \mu^{-1}(0)/S^1 \cong S^{2n+1}/S^1 \cong \mathbb{CP}^n,$$

  with residual $T^{n+1}/S^1 \cong T^n$-action.

Example Continued

- Residual torus $T^n \circlearrowright \mathbb{CP}^n$ diagonally, now with moment map
$$\bar{\mu}(\mathbf{z}) = \frac{1}{2}\left(\frac{|z_1|^2}{\|\mathbf{z}\|^2}, \ldots, \frac{|z_n|^2}{\|\mathbf{z}\|^2}\right) \in \mathbb{R}^n.$$
- Fixed-points for this action are the points with only one non-zero entry, *e.g.* $[1 : 0 : \ldots, 0]$.
- Image of $\bar{\mu}$ is the convex hull of the images of the fixed-points.
- This is a result of the *Atiyah-Huillemin-Sternberg convexity theorem.*

Convexity

Example

Definition

A **Delzant polytope** $\Delta \subset \mathbb{R}^n$ is a convex polytope such that [Del88]:

- (simple): n edges meet at each vertex;
- (rational): edges meeting a vertex p are of the form $p + tu_i$, $t \geqslant 0$ and $u_i \in \mathbb{Z}^n$;
- (smooth): for each p, the u_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Examples

Delzant's Theorem [Del88]

Toric manifolds are classified by Delzant polytopes, *i.e.* there is a one-to-one correspondence:

$$\frac{\text{toric manifolds}}{T^n - \text{equiv. symplectomorphisms}} \longleftrightarrow \frac{\text{Delzant polytopes}}{\mathrm{SL}(n;\mathbb{Z})}$$

Examples

Set-Up

- Start with a Delzant polytope, $\Delta = \bigcap_{k=1}^{n} H_k \subseteq \mathbb{R}^d$, where

$$H_k = \{x \in \mathbb{R}^d : \langle x, u_k \rangle \geqslant \lambda_k\}, \quad \lambda_k \in \mathbb{R}$$

  are inward-pointing half-spaces delimiting Δ.
- Define a surjective map $\pi : \mathbb{R}^n \to \mathbb{R}^d$, $\pi(e_k) = u_k$, where the e_k are basis vectors for $\mathbb{R}^n$.
- Define $\mathfrak{n} := \ker \pi$, and consider the inclusion $\imath : \mathfrak{n} \hookrightarrow \mathbb{R}^n$.

Short Exact Sequence

$$\{0\} \longrightarrow \mathfrak{n} \xhookrightarrow{\imath} \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^d \longrightarrow \{0\}$$

More Short Exact Sequences

$$\{0\} \longrightarrow \mathfrak{n} \xhookrightarrow{\imath} \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^d \longrightarrow \{0\}$$

- Can exponentiate to get our tori:

$$\{1\} \longrightarrow N \xhookrightarrow{\imath} T^n \xrightarrow{\pi} T^d \longrightarrow \{1\}.$$

- Or dualise:

$$\{0\} \longleftarrow \mathfrak{n}^* \xleftarrow{\imath^*} (\mathbb{R}^n)^* \xleftarrow{\pi^*} (\mathbb{R}^d)^* \longleftarrow \{0\}.$$

Subtorus Action

- If $T^n \circlearrowright \mathbb{C}^n$ diagonally, then $N \leqslant T^n$ also acts on $\mathbb{C}^n$ via the inclusion, $\imath$.
- Moment map for this action via inclusion is

$$\mu : \mathbb{C}^n \xrightarrow{J} (\mathbb{R}^n)^* \xrightarrow{\imath^*} \mathfrak{n}^*,$$

with $J : \mathbb{C}^n \to (\mathbb{R}^n)^*$ the usual moment map,

$$J(z) = \frac{1}{2} \sum_{k=1}^{n} |z_k|^2 e_k + (\lambda_1, \dots, \lambda_n).$$

Symplectic Reduction

Finally for the Delzant $\Delta \subset \mathbb{R}^n$,

$$X_\Delta := \mathbb{C}^n /\!\!/ N := \mu^{-1}(0)/N$$

is the corresponding toric manifold, with residual $T^n/N \cong T^d$-action.

Comments

- $\dim X_\Delta = 2n - 2\dim N = 2(\dim T^n - \dim N) = 2\dim T^d$.
- The λ's determine the position of the half-spaces - translating λ by an element of $\ker \imath^* \subseteq (\mathbb{R}^n)^*$ gives the same result.
- The u's determine their (inwards-pointing) directions.

Old Example

- $T^3 \circlearrowright \mathbb{C}^3$ and $S^1 \hookrightarrow T^3$ diagonally:

$$\{1\} \longrightarrow S^1 \xhookrightarrow{\imath} T^3 \xrightarrow{\pi} T^2 \longrightarrow \{1\}.$$

- $u_1 = (1,0)$, $u_2 = (0,1)$, $u_3 = (-1,-1)$, so $\ker \pi = (t,t,t)$.
- Thus $\imath(t) = (t,t,t) \implies \imath^*(x,y,z) = x + y + z$.
- Moment map

$$\mathbf{z} \xmapsto{J} \frac{1}{2}(|z_1|^2, |z_2|^2, |z_3|^2) + \lambda \xmapsto{\imath^*} \frac{1}{2}\|\mathbf{z}\|^2 + \lambda_1 + \lambda_2 + \lambda_3.$$

- Then $X_\Delta \cong \mathbb{CP}^2$ with T^2-action, and moment map image $\mu(X_\Delta) = \{\langle x, \ u_i \rangle \geqslant \lambda_i\}$.

Example

$\mathbb{CP}^2$ *Example*

Definition and Properties

- A **hyperkähler manifold** is a Riemannian manifold (M, g) with three orthogonal, parallel complex structures J_1, J_2, J_3, that satisfy the quaternionic relations.
- Get three symplectic forms $\omega_1, \omega_2, \omega_3$, so each (g, J_i, ω_i) give a Kähler structure on M.
- Fixing J_1, we can write

$$\omega_{\mathbb{R}} := \omega_1, \qquad \omega_{\mathbb{C}} := \omega_2 + i \cdot \omega_3.$$

Examples

- Quaternionic space $\mathbb{H}^n$.
- Fixeds J_1, $T^*\mathbb{C}^n$ inherits a hyperkähler structure.

Induced Action

- For $M = T^*\mathbb{C}^n$, Hamiltonian G-action of $\mathbb{C}^n$ extends to hyperhamiltonian action on $T^*\mathbb{C}^n$.
- Original $\mu : \mathbb{C}^n \to \mathfrak{g}^*$, induced maps are

$$\mu_{\mathbb{R}}(z, w) = \mu(z) - \mu(w), \qquad \mu_{\mathbb{C}}(z, w)(v) = w(\hat{v}_z),$$

for $w \in T_z^*\mathbb{C}^n$, $v \in \mathfrak{g}_{\mathbb{C}}$, and $\hat{v}_z \in T_z\mathbb{C}^n$ induced by v.

Subtori Action

- If $N \overset{\imath}{\hookrightarrow} T^n$, get $N \circlearrowright T^*\mathbb{C}^n$ via inclusion as before.
- Maps $\mu_{\mathbb{R}}, \mu_{\mathbb{C}} \to \mathfrak{n}^*, \mathfrak{n}_{\mathbb{C}}^*$ then are:

$$\mu_{\mathbb{R}} = \imath^* \circ J_{\mathbb{R}}, \qquad \mu_{\mathbb{C}} = \imath_{\mathbb{C}}^* \circ J_{\mathbb{C}}.$$

Diagonal Torus Action

For $T^n \circlearrowright \mathbb{C}^n \rightsquigarrow T^n \circlearrowright T^*\mathbb{C}^n$ as $\tau \cdot (z, w) = (\tau z, \tau^{-1} w)$, thus

$$J_{\mathbb{R}}(z, w) = \sum_{k=1}^{n} (|z_k|^2 - |w_k|^2) e_k, \quad J_{\mathbb{C}}(z, w) = \sum_{k=1}^{n} (z_k w_k) e_k.$$

$S^1 \hookrightarrow T^3 \circlearrowright T^*\mathbb{C}^n$

Example from before, with $S^1 \overset{\imath}{\hookrightarrow} T^3$ diagonally.

$$\mu_{\mathbb{R}}(z, w) = \sum_{k=1}^{3} (|z_k|^2 - |w_k|^2), \quad \mu_{\mathbb{C}}(z, w) = \sum_{k=1}^{3} (z_k w_k).$$

Hyperkähler Quotients [BD00]

- Nice hyperkähler quotient if G acts freely on $(\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}})^{-1}(\xi)$, for $\xi \in \mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}}$ regular.
- Can assume that the $\mathfrak{g}^*_{\mathbb{C}}$-component of ξ is zero, [BD00].

Recall: $\{1\} \longrightarrow N \xhookrightarrow{\imath} T^n \xrightarrow{\pi} T^d \longrightarrow \{1\}.$

Hyperkähler Analogues [Pro04]

For a toric $X = \mathbb{C}^n /\!\!/ N = \mu^{-1}(\lambda)/N$, its **hyperkähler analogue** is

$$M := T^*\mathbb{C}^n /\!\!/\!\!/\!\!/ N := (\mu_{\mathbb{R}}^{-1}(\lambda) \cap \mu_{\mathbb{C}}^{-1}(0))/N,$$

and is a **hypertoric variety**.

- Residual $T^d = T^n/N$ acts on M, with moment maps

$$\bar{\mu}_{\mathbb{R}}[z, w] = \frac{1}{2}\sum_{k=1}^{n}(|z_k|^2 - |w_k|^2 - \lambda_k)\partial_k \in \ker(\imath^*) \subseteq \mathbb{R}^d,$$

$$\bar{\mu}_{\mathbb{C}}[z, w] = \sum_{k=1}^{n}(z_k w_k)\partial_k \in \ker(\imath_{\mathbb{C}}^*) \subseteq \mathbb{C}^d.$$

- Image $\bar{\mu}_{\mathbb{R}}(M)$ given by a hyperplane arrangement:

$$F_k = \{y \in \mathbb{R}^d : \langle y,\ u_k\rangle + \lambda_k \geqslant 0\},$$
$$G_k = \{y \in \mathbb{R}^d : \langle y,\ u_k\rangle + \lambda_k \leqslant 0\},$$
$$H_k = \{y \in \mathbb{R}^d : \langle y,\ u_k\rangle + \lambda_k = 0\} = F_k \cap G_k.$$

$T^*\mathbb{CP}^2$

- From before, $S^1 \circlearrowright \mathbb{C}^3 \rightsquigarrow \mathbb{CP}^2 = \mu^{-1}(\lambda)/S^1$ with residual T^2-action and moment polytope $\bar{\mu}(\mathbb{CP}^2) = \Delta_2$.
- Same arrangement as toric case but now $\bar{\mu}_{\mathbb{R}}$ is surjective, for hypertoric $T^*\mathbb{CP}^2$.

Real Image

Residual S^1-Action

- $T^*\mathbb{C}^n$ has an S^1-action from rotating cotangent fibres: $\tau \cdot (z, w) = (z, \tau w)$.
- Descends to $\bar{\mu}_{\mathbb{C}}^{-1}(0)$ as $\bar{\mu}_{\mathbb{C}}$ is S^1-equivariant, so $M^{S^1} \subseteq \bar{\mu}_{\mathbb{C}}(M)$.

Extended Core of M

- $\mathcal{E} := \bar{\mu}_{\mathbb{C}}^{-1}(0) = \{[z, w] \in M : z_k w_k = 0, \forall k\}$.
- Breaks up further: for $A \subseteq \{1, \ldots, n\}$,

$$\mathcal{E}_A := \{[z, w] \in M : w_k = 0 \text{ for } k \in A,\ z_k = 0 \text{ for } k \notin A\}.$$

- $\bar{\mu}_{\mathbb{R}}(\mathcal{E}_A) =: \Delta_A$, polyhedra from arrangement.

Example

$\bar{\mu}_{\mathbb{R}}(T^*\mathbb{CP}^2)$

- S^1 does not acts as a subtorus of T^d globally, but does restricted to each $\mathcal{E}_A$:

$$\tau \cdot [z, w] = [z, \tau w] = [\tau_1^{-1} z_1, \ldots, \tau_n^{-1} z_n; \tau_1 w_1, \ldots, \tau_n w_n],$$

$$\text{where } \tau_i = \begin{cases} \tau, & \text{if } i \in A, \\ 1, & \text{otherwise.} \end{cases}$$

- Restricted to $\mathcal{E}_A$, circle S^1 acts a subtorus of *original* torus T^n.
- Moment map

$$\Phi\big|_A [z, w] = -\left\langle \mu_{\mathbb{R}}(z, w), \sum_{i \in A} u_i \right\rangle.$$

Set-Up

- Global moment map for this S^1-action is $\Phi[z, w] = \|w\|^2$.
- Product $M \times \mathbb{C}$ with S^1-action:

$$e^{i\theta} \cdot (m, \xi) = (e^{i\theta} \cdot m,\ e^{i\theta}\xi) \rightsquigarrow \rho(m, \xi) = \Phi[z, w] + |\xi|^2.$$

- Preimage at level ϵ for $(m, \xi) \in M \times \mathbb{C}$ is:

$$\begin{aligned} \rho^{-1}(\epsilon) &= \left\{(m, \xi) : \Phi(m) \leqslant \epsilon,\ |\xi| = e^{i\theta}\sqrt{\epsilon - \Phi(m)}\right\} \\ &= \left\{(m, \xi) : \Phi(m) < \epsilon,\ |\xi| = e^{i\theta}\sqrt{\epsilon - \Phi(m)}\right\} \\ &\quad \sqcup \left\{(m, 0) : \Phi(m) = \epsilon\right\} =: \Sigma_1 \sqcup \Sigma_2. \end{aligned}$$

Definition

Quotient $M_{\epsilon-\text{cut}} := \rho^{-1}(\epsilon)/S^1 \cong (\Sigma_1 \sqcup \Sigma_2)/S^1$ is called the **symplectic cut** of M at level-ϵ [Ler95].

What does it look like?

$$\Sigma_1 = \left\{(m, \xi) : \Phi(m) < \epsilon,\ |\xi| = e^{i\theta}\sqrt{\epsilon - \Phi(m)}\right\},$$
$$\Sigma_2 = \left\{(m, 0) : \Phi(m) = \epsilon\right\}.$$

- $\Sigma_1 \cong \{m \in M : \Phi(m) < \epsilon\} \times S^1$, so $\Sigma_1/S^1 \cong \{\Phi(m) < \epsilon\}$;
- $\Sigma_2 \cong \Phi^{-1}(\epsilon)/S^1$.
- $\Phi : M \to \mathbb{R}$ is proper, so the symplectic cut $M_{\epsilon-\text{cut}}$ is compact.

Moment Polyptychs

- Recall: S^1-action on $\mathcal{E}_A$ depends combinatorially on $A \subseteq \{1, \ldots, n\}$.
- $M_{\epsilon-\mathrm{cut}}$ is obtained by "throwing-away" the part of M with $\Phi(m) > \epsilon$ on each $\mathcal{E}_A$.
- Amounts to intersection each Δ_A with half-space [Pro04], with normal $-\sum_{i \in A} u_i$, from

$$\Phi\big|_{\mathcal{E}_A}[z, w] = -\left\langle \mu_{\mathbb{R}}(z, w), \sum_{i \in A} u_i \right\rangle.$$

- We call such an arrangement of polytopes a **moment polyptych**.

Example for $T^*\mathbb{CP}^2$

Example for $T^*\mathbb{CP}^3$

Verlinde Formula

- Let $\mathcal{N}$ be the moduli space of stable SL_2-bundles over Σ_2; it is isomorphic to $\mathbb{CP}^3$ [NR69].
- Geometric quantisation $\mathcal{Q}(\mathcal{N}) := H^0(\mathcal{N}; \mathcal{L}^{\otimes k})$; its dimension equals the **Verlinde formula** [Ver88]; [JW92],

$$\begin{aligned}\dim \mathcal{Q}(\mathcal{N}) = \mathrm{Ver}(k) &= \frac{k^3}{6} + k^2 + \frac{11k}{6} + 1 \\ &= \frac{(k+1)(k+2)(k+3)}{3!}.\end{aligned}$$

- Named after Dutch physicist Erik Verlinde, who was working on conformal field theories.

Riemann-Roch-Hirzebruch Theorem

For toric X, lattice point count of Δ_X equals to Euler characteristic,

$$\chi(X) = \int_X e^{c_1(\mathcal{O}(k))} \cdot \mathrm{Td}\,(TX).$$

Example

For $X = \mathbb{CP}^3$, have $e^{c_1(\mathcal{O}(k))} = 1 + kH + \frac{k^2}{2}H^2 + \frac{k^3}{6}H^3$, and $\mathrm{Td}\,(T\mathbb{CP}^3) = 1 + 2H + \frac{11}{6}H^2 + H^3$:

$$\chi(\mathbb{CP}^3) = \int_{\mathbb{CP}^3} \left(\frac{k^3}{6} + k^2 + \frac{11k}{6} + 1\right) \cdot H^3 + \ldots = \mathrm{Ver}(k).$$

Lattice Points

- So $\mathrm{Ver}(k) = \chi(\mathbb{CP}^3) = \#(k \cdot \Delta_3 \cap \mathbb{Z}^3)$.
- Anything similar for hypertoric manifolds?

Equivariant Verlinde Formula

- Recently, equivariant Verlinde formula for moduli spaces Higgs bundle, $\mathcal{M}$, popped up [GP17].
- $\dim \mathcal{Q}(\mathcal{M}) = \infty$, but $\mathcal{M}$ has a $\mathbb{C}^*$-action.
- Decompose into $\mathbb{C}^*$-weight spaces:

$$\dim \mathcal{Q}(\mathcal{M}) = \sum_n t^n \cdot \dim \mathcal{Q}_n(\mathcal{M}),$$

but now $\dim \mathcal{Q}_n(\mathcal{M}) < \infty$.

Fixed-Point Formula

$$\sum_{q\in\Delta} e^{\langle q,\phi\rangle} = \sum_{p\in M^T} \frac{e^{\langle p,\phi\rangle}}{\prod_{k=1}^{n}(1-e^{\langle \alpha_k^p,\phi\rangle})},$$

with edge vectors α_k^p, and $\langle \alpha_k^p, \phi\rangle \neq 0$, [Bar93].

- Letting $\phi \to 0$ gets the lattice point count (for Delzant Δ).

Example for $T^*\mathbb{CP}^3$

For $(T^*\mathbb{CP}^3)_{\epsilon-\mathrm{cut}}$, get

$$\frac{(\epsilon+1)(\epsilon+2)(\epsilon+3)}{3!}\cdot\frac{(k+\epsilon+1)(k+\epsilon+2)(k+\epsilon+3)}{3!}.$$

Observe that for $\epsilon = 0$, it becomes $\mathrm{Ver}(k)$.

Non-Convex Core

Want to see what happens for hypertoric manifolds with non-convex cores.

Example

Questions?

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