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# GEOMETRIC QUANTISATION OF HYPERTORIC MANIFOLDS BY SYMPLECTIC CUTTING

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GENERAL NOTES

## ABSTRACT

Lorem ipsum.

## 1 Introduction

Lorem ipsum.

## 2 Hyperkähler Reduction and Hyperkähler Analogues

### 2.1 Introduction and Definitions

A *hyperkähler manifold* is a Riemannian manifold  $(M, g)$  equipped with three orthogonal, parallel complex structures  $J_1, J_2, J_3$ , satisfying the usual quaternion relations. These three complex structures give rise to three symplectic forms

$$\omega_1(v, w) = g(J_1 v, w), \quad \omega_2(v, w) = g(J_2 v, w), \quad \omega_3(v, w) = g(J_3 v, w),$$

so that each  $(g, J_i, \omega_i)$  is in its own right a Kähler structure on  $M$  for  $i = 1, 2, 3$ . The complex-valued two-form  $\omega_2 + \sqrt{-1}\omega_3$  is a closed, non-degenerate, and holomorphic two-form with respect to the complex structure  $J_1$ . Thus any hyperkähler manifold can be considered as a *holomorphic symplectic* manifold with complex structure  $J_1$ , real symplectic form  $\omega_{\mathbb{R}} := \omega_1$ , and holomorphic symplectic form  $\omega_{\mathbb{C}} := \omega_2 + \sqrt{-1}\omega_3$ .

An action of a Lie group  $G$  on a hyperkähler manifold  $M$  is called *hyperhamiltonian* if it is hamiltonian with respect to  $\omega_{\mathbb{R}}$ , and holomorphic hamiltonian with respect to  $\omega_{\mathbb{C}}$ , with a  $G$ -equivariant moment map

$$\mu_{HK} := \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} \longrightarrow \mathfrak{g}^* \oplus \mathfrak{g}_{\mathbb{C}}^*.$$

The following theorem describes the *hyperkähler quotient* construction, which is the quaternionic analogue of a Kähler quotient:

**Theorem 2.1** ([1]). *Let  $M$  be a hyperkähler manifold equipped with a hyperhamiltonian action of a compact Lie group  $G$ , with moment maps  $\mu_1, \mu_2, \mu_3$ . Suppose that  $\xi = \xi_{\mathbb{R}} \oplus \xi_{\mathbb{C}}$  is a central regular value for  $\mu_{HK}$ , and that  $G$  acts freely on  $\mu_{HK}^{-1}(\xi)/G$ . Then there is a unique hyperkähler structure on the hyperkähler quotient  $\mathfrak{M} = M \mathbin{/\!\!/}_{\xi} G := \mu_{HK}^{-1}(\xi)/G$ , with associated symplectic and holomorphic symplectic forms  $\omega_{\mathbb{R}}^{\xi}$  and  $\omega_{\mathbb{C}}^{\xi}$ , such that  $\omega_{\mathbb{R}}^{\xi}$  and  $\omega_{\mathbb{C}}^{\xi}$  pull-back to the restrictions of  $\omega_{\mathbb{R}}$  and  $\omega_{\mathbb{C}}$  on  $\mu_{HK}^{-1}(\xi)$ .*

In general, the action of  $G$  on  $\mu_{HK}^{-1}(\xi)$  will not be free, but only locally free. In this situation, we would end up with a *hyperkähler orbifold*. However in the sequel, we shall only concern ourselves when the action is free, and that  $\mathfrak{M}$  is smooth, i.e. a manifold.

Let us specialise to the case when  $M = T^*\mathbb{C}^n$ , and let  $G$  act on  $T^*\mathbb{C}^n$  with the induced action from a linear action of  $G$  on  $\mathbb{C}^n$ , with moment map  $\mu : \mathbb{C}^n \rightarrow \mathfrak{g}^*$ . We can identify  $\mathbb{H}^n$  with  $T^*\mathbb{C}^n$  such that the complex structure  $J_1$  on  $\mathbb{H}^n$  is given by right multiplication by  $i$ , and that  $J_1$  corresponds to the natural complex structure on  $T^*\mathbb{C}^n$ . With this identification in mind,  $T^*\mathbb{C}^n$  inherits a hyperkähler structure. The real symplectic form  $\omega_{\mathbb{R}}$  is obtained from the sum of the pull-backs of the standard Kähler forms on  $\mathbb{C}^n$  and  $(\mathbb{C}^n)^*$ , and the holomorphic symplectic form  $\omega_{\mathbb{C}}$  is  $\omega_{\mathbb{C}} = d\eta$ , where  $\eta$  is the canonical holomorphic one-form on  $T^*\mathbb{C}^n$ .

As  $G$  acts  $\mathbb{H}^n$ -linearly on  $T^*\mathbb{C}^n \cong \mathbb{H}^n$  from the left, the action is hyperhamiltonian with moment map  $\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$ , where

$$\mu_{\mathbb{R}}(z, w) = \mu(z) - \mu(w), \quad \text{and} \quad \mu_{\mathbb{C}}(z, w)(\hat{v}_z),$$

where  $w \in T_z^*\mathbb{C}^n$ ,  $v \in \mathfrak{g}_{\mathbb{C}}$ , and  $\hat{v}_z$  is the vector field in  $T_z\mathbb{C}^n$  induced by  $v$ . For a central element  $\alpha \in \mathfrak{g}^*$ , we call the specialised hyperkähler quotient

$$\mathfrak{M} = T^*\mathbb{C}^n \mathbin{/\!\!/}_{(\alpha, 0)} G := (\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))/G$$

the hyperkähler analogue of the corresponding Kähler quotient,

$$\mathfrak{X} = \mathbb{C}^n \mathbin{/\!\!/}_{\alpha} G = \mu^{-1}(\alpha)/G.$$

We quote the following propositions without proof:

**Proposition 2.2.** *Suppose that  $\alpha$  and  $(\alpha, 0)$  are regular values for  $\mu$  and  $\mu_{HK}$ , respectively. Then the cotangent bundle  $T^*\mathfrak{X}$  is isomorphic to an open subset of  $\mathfrak{M}$ , and is dense if it is non-empty.*

## 2.2 The $\mathbb{C}^*$ -Action and the Core of a Hyperkähler Analogue

Consider the action of  $\mathbb{C}^*$  on  $T^*\mathbb{C}^n$  given by

$$\hbar \cdot (z, w) = (z, \hbar w),$$

i.e. by scalar multiplication of the cotangent fibre. The holomorphic moment map  $\mu_{\mathbb{C}} : T^*\mathbb{C}^n \rightarrow \mathfrak{g}_{\mathbb{C}}^*$  is  $\mathbb{C}^*$ -equivariant with respect to the scalar action on  $\mathfrak{g}_{\mathbb{C}}^*$ , and hence the  $\mathbb{C}^*$ -action descends to  $\mu_{\mathbb{C}}^{-1}(0)$ . Further, this  $\mathbb{C}^*$ -action commutes with the linear action of  $G$  on  $\mathbb{C}^n$ , and consequently the action of  $\mathbb{C}^*$  is  $J_1$ -holomorphic on  $\mathfrak{M} = (\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))/G$ . However, the  $\mathbb{C}^*$ -action *does not* preserve the holomorphic symplectic form nor the hyperkähler structure on  $\mathfrak{M}$ ; rather it scales  $\mu_{\mathbb{C}}$  with “homogeneity one”, i.e.  $\hbar^* \omega_{\mathbb{C}} = \hbar \omega_{\mathbb{C}}$  for any  $\hbar \in \mathbb{C}^*$ .

Given that  $\mathfrak{M}$  is smooth, the action of the compact subgroup  $S^1 \subset \mathbb{C}^*$  is hamiltonian with respect to the real symplectic two-form  $\omega_{\mathbb{R}}$ , with corresponding moment map  $\Phi[z, w] = \frac{1}{2}\|w\|^2$ . This map is a perfect Morse-Bott function, and its image is contained in  $\mathbb{R}_{\geq 0}$ . Further, we note that  $\Phi^{-1}(0) = \mathfrak{X} \subset \mathfrak{M}$ . The following proposition will be instrumental in the sequel, though again we quote it without proof:

**Proposition 2.3.** *If the original moment map for the  $G$ -action on  $\mathbb{C}^n$ ,  $\mu : \mathbb{C}^n \rightarrow \mathfrak{g}^*$ , is proper, then so is the moment map for the  $S^1$  action,  $\Phi : \mathfrak{M} \rightarrow \mathbb{R}_{\geq 0}$ .*

Next we shall define what is known as the *core* of a hyperkähler analogue, which will be essential in describing the fixed points of the  $\mathbb{C}^*$ -action of  $\mathfrak{M}$ .

**Definition 2.4.** *Suppose that  $\mathfrak{M}$  is smooth and  $\Phi$  is proper. The core  $\mathcal{L} \subset \mathfrak{M}$  of the hypertoric variety is defined to be the union of the  $\mathbb{C}^*$  orbits whose closures are compact.*

Let  $F$  be a connected component of  $\mathfrak{M}^{S^1} = \mathfrak{M}^{\mathbb{C}^*}$ , and let  $U_F$  be the closure of the set of points  $p \in \mathfrak{M}$  such that  $\lim_{\hbar \rightarrow \infty} \hbar \cdot p \in F$ .

**Proposition 2.5** ([?]; Proposition 2.8). *The core  $\mathcal{L} \subset \mathfrak{M}$  has the following properties:*

1.  $\mathcal{L}$  is an  $S^1$ -equivariant deformation retract of  $M$ ;
2.  $U_F$  is isotropic with respect to the holomorphic symplectic form  $\omega_{\mathbb{C}}$ ;
3. Provided that  $\mathfrak{M}$  is smooth at  $F$ , then  $\dim U_F = \frac{1}{2} \dim \mathfrak{M}$ .

## 3 Hypertoric Manifolds

### 3.1 Definition

In this section, we shall specialise further now to when a hyperkähler analogue  $\mathfrak{M}$  is the analogue to a toric symplectic manifold  $\mathfrak{X} = \mu^{-1}(\alpha)/N$ , i.e. we replace the compact Lie group  $G$  with the torus  $N = \ker(\pi : T^n \rightarrow T^d)$ , using the same notation as in the second chapter.

Recall the short exact sequence of tori:

$$1 \longrightarrow N \xhookrightarrow{i} T^n \xrightarrow{\pi} T^d \longrightarrow 1,$$

and extend the linear action of the torus  $N$  on  $\mathbb{C}^n$  to  $T^*\mathbb{C}^n$ . This action is trihamiltonian and we obtain the following hyperkähler moment map

$$\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : T^*\mathbb{C}^n \longrightarrow \mathfrak{n}^* \oplus \mathfrak{n}_{\mathbb{C}}^*,$$

where

$$\mu_{\mathbb{R}}(z, w) = i^* \left( \frac{1}{2} \sum_{i=1}^n (|z_i|^2 - |w_i|^2) \partial_i \right), \quad \text{and} \quad \mu_{\mathbb{C}}(z, w) = i_{\mathbb{C}}^* \left( \sum_{i=1}^n (z_i w_i) \partial_i \right).$$

Given an element  $\alpha \in \mathfrak{n}^*$  with a corresponding lift  $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{R}^n)^*$ , the Kähler quotient

$$\mathfrak{X} = \mathbb{C}^n //_{\alpha} N = \mu^{-1}(\alpha)/N$$

is our usual toric symplectic manifold with residual  $T^d$ -action from before, and moreover its hyperkähler analogue

$$\mathfrak{M} = T^*\mathbb{C}^n \mathrel{\mathcal{H}}_{(\alpha, 0)} N = (\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))/N$$

is what we shall call a *hypertoric manifold*<sup>1</sup>. The hypertoric manifold  $\mathfrak{M}$  also admits a residual action of the torus  $T^d$ , which is hyperhamiltonian with hyperkähler moment map

$$\phi_{HK} := \phi_{\mathbb{R}} \oplus \phi_{\mathbb{C}} : \mathfrak{M} \longrightarrow (\mathbb{R}^d)^* \oplus (\mathbb{C}^d)^*,$$

where

$$\begin{aligned} \phi_{\mathbb{R}}[z, w] &= \frac{1}{2} \sum_{i=1}^n (|z_i|^2 - |w_i|^2 - \lambda_i) \partial_i \in \ker(i^*) = (\mathbb{R}^d)^*, \\ \phi_{\mathbb{C}}[z, w] &= \sum_{i=1}^n (z_i w_i) \partial_i \in \ker(i_{\mathbb{C}}^*) = (\mathbb{C}^d)^*. \end{aligned}$$

### 3.2 Hyperplane Arrangements

A fundamental difference between the toric manifold  $\mathfrak{X}$  and the hypertoric manifold  $\mathfrak{M}$  is that the hyperkähler moment map for  $\mathfrak{M}$  is surjective, and that  $\mathfrak{M}$  is non-compact. Despite this, we can still describe the image of the real moment map  $\phi_{\mathbb{R}} : \mathfrak{M} \rightarrow (\mathbb{R}^d)^*$  combinatorially by means of a *hyperplane arrangement*. To describe this arrangement, recall that the map  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^d$  was defined by  $\pi(e_i) = u_i$ , for  $i = 1, \dots, n$ , where the  $u_i$  were the primitive, integral, inward-pointing normal vectors to the hyperplanes that determined our Delzant polytope. In the hypertoric case, they instead now describe a collection of *affine hyperplanes*  $H_i \subset (\mathbb{R}^d)^*$  as follows: consider

$$H_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i = 0\},$$

so that the  $u_i \in \mathbb{Z}^d$  is the normal vector to the hyperplane  $H_i$ . The hyperplane  $H_i$  divides  $(\mathbb{R}^d)^*$  into two half-spaces

$$F_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \geq 0\},$$

$$G_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \leq 0\}.$$

Let

$$\Delta = \bigcap_{i=1}^n F_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \geq 0, \text{ for all } i = 1, \dots, n\}$$

be the (possibly empty) polyhedron in  $(\mathbb{R}^d)^*$  defined by the affine hyperplane arrangement  $\mathcal{A} = \{H_1, \dots, H_n\}$ . We note that choosing a different lift  $\lambda'$  of  $\alpha$  corresponds combinatorially to translating the arrangement  $\mathcal{A}$  inside of  $(\mathbb{R}^d)^*$ , and geometrically to shifting the Kähler and hyperkähler moment maps for the residual  $T^d$ -action by  $\lambda' - \lambda \in \ker(i^*) = (\mathbb{R}^d)^*$ .

We shall call that the arrangement  $\mathcal{A}$  *simple* if every subset of  $m$  hyperplanes with non-empty intersection intersects with codimension  $m$ , and call  $\mathcal{A}$  *smooth* if every collection of  $d$  linearly-independent vector  $\{u_{i_1}, \dots, u_{i_d}\}$  spans  $(\mathbb{R}^d)^*$ . The reason for this terminology is the following proposition.

**Proposition 3.1.** *The hypertoric variety  $\mathfrak{M}$  is an orbifold if and only if  $\mathcal{A}$  is simple, and  $\mathfrak{M}$  is smooth if and only if  $\mathcal{A}$  is smooth.*

As we wish to restrict our attention to the case where  $\mathfrak{M}$  is a manifold, we shall assume in the sequel that  $\mathcal{A}$  is a smooth arrangement of hyperplanes.

<sup>1</sup>More generally,  $\mathfrak{M}$  should be called a hypertoric variety, and only call  $\mathfrak{M}$  a manifold when it is smooth. However, we shall restrict our attention to the smooth case for simplicity.

### 3.3 The Core of a Hypertoric Manifold

The holomorphic moment map  $\phi_{\mathbb{C}} : \mathfrak{M} \rightarrow (\mathbb{C}^d)^*$  is  $\mathbb{C}^*$ -equivariant with respect to the scalar action of  $\mathbb{C}^*$  on  $(\mathbb{C}^d)^*$ , hence both the core  $\mathcal{L}$  and the fixed-point set  $M^{\mathbb{C}^*}$  will be contained in

$$\mathcal{E} := \phi_{\mathbb{C}}^{-1}(0) = \left\{ [z, w] \in \mathfrak{M} : z_i w_i = 0, 1 \leq i \leq n \right\}.$$

**Definition 3.2.** We shall call  $\mathcal{E}$  the extended core of  $\mathfrak{M}$ .

The restriction of  $\phi_{\mathbb{R}}|_{\mathcal{E}} : \mathcal{E} \rightarrow (\mathbb{R}^d)^*$  is surjective from the defining equations, and further the extended core naturally breaks up into components

$$\mathcal{E}_A := \left\{ [z, w] \in \mathcal{E} : w_i = 0 \text{ for all } i \in A \text{ and } z_i = 0 \text{ for all } i \notin A \right\},$$

where  $A \subseteq \{1, \dots, n\}$  is an indexing set. The hyperplanes  $\{H_i\}_{i=1}^n$  divide  $(\mathbb{R}^d)^*$  into a union of convex polyhedra

$$\Delta_A = \left( \bigcap_{i \in A} F_i \right) \cap \left( \bigcap_{i \notin A} G_i \right),$$

some of which may be empty. Note that  $\mathcal{E}_{\emptyset} = \mathfrak{X}$  and that in general, each variety  $\mathcal{E}_A$  is a  $d$ -dimensional Kähler subvariety of  $\mathfrak{M}$  with an effective Hamiltonian  $T^d$ -action, so is itself a toric variety.

**Lemma 3.3.** If  $w_i = 0$  then  $\text{Im}(\phi_{\mathbb{R}}) \subseteq F_i$ , and if  $z_i = 0$  then  $\text{Im}(\phi_{\mathbb{R}}) \subseteq G_i$ .

*Proof.* Let  $y \in (\mathbb{R}^d)^*$  be the image of the moment map  $\phi_{\mathbb{R}}$  for a point  $[z, w] \in \mathcal{E}$ , then

$$y \cdot u_i + r_i = \mu_{\mathbb{R}}(z, w) \cdot e_i = \frac{1}{2}(|z_i|^2 - |w_i|^2),$$

and hence  $y \geq 0$  if  $i \in A$ , and  $y \leq 0$  if  $i \notin A$ . □

**Lemma 3.4.** The component  $\mathcal{E}_A$  of the extended core is isomorphic to the toric variety corresponding to the polytope  $\Delta_A$ .

The  $S^1$ -action does not act as a subtorus of  $T^d$  on  $\mathfrak{M}$  globally, but does when restricted to each individual component  $\mathcal{E}_A$  of the extended core. Consider a component  $\mathcal{E}_A \subset \mathcal{E}$ , then for some  $[z, w] \in \mathcal{E}_A$  and  $\tau \in S^1$ ,

$$\tau \cdot [z, w] = [z, \tau w] = [\tau_1 z_1, \dots, \tau_n z_n | \tau_i^{-1} w_1, \dots, \tau_n^{-1} w_n], \quad \text{where } \tau_i = \begin{cases} \tau^{-1} & \text{if } i \in A, \\ 1 & \text{if } i \notin A, \end{cases}$$

since for each pair  $(z_i, w_i)$ , if  $i \in A$  then  $(\tau z_i, \tau^{-1} w_i) = (0, \tau w_i)$ , and if  $i \notin A$ , then  $(\tau z_i, \tau^{-1} w_i) = (z_i, 0)$ .

Thus when restricting our attention to each individual component  $\mathcal{E}_A$  of the extended core, the  $S^1$ -action acts on  $\mathcal{E}_A$  via an inclusion homomorphism onto a one-parameter subgroup of  $T^n$ , which consequently then descends to a  $T^d$ -action after taking the quotient of  $T^n$  by  $K$ .

Let us denote the restricted action of the image of  $S^1$  in  $T^d$  to  $\mathcal{E}_A$  by  $S_A^1$ ,

$$S^1 \xhookrightarrow{j_A} T^n \xrightarrow{\pi} S_A^1 < T^d$$

$$\tau \longmapsto (\tau_1^{-1}, \dots, \tau_n^{-1}) \longmapsto v_A := \sum_{i \in A} u_i$$

where we have denoted the generator of the one-parameter subgroup  $S_A^1$  in  $T^d$  by  $v_A = \sum_{i \in A} u_i$ .

**Example 1.** For  $\mathfrak{M} = T^*\mathbb{CP}^2$ ,

**Example 2.** For  $\mathfrak{M}$  whose core consists of two  $\mathbb{CP}^2$  intersecting at a point, so that the image of the real moment map is non-convex (todo)

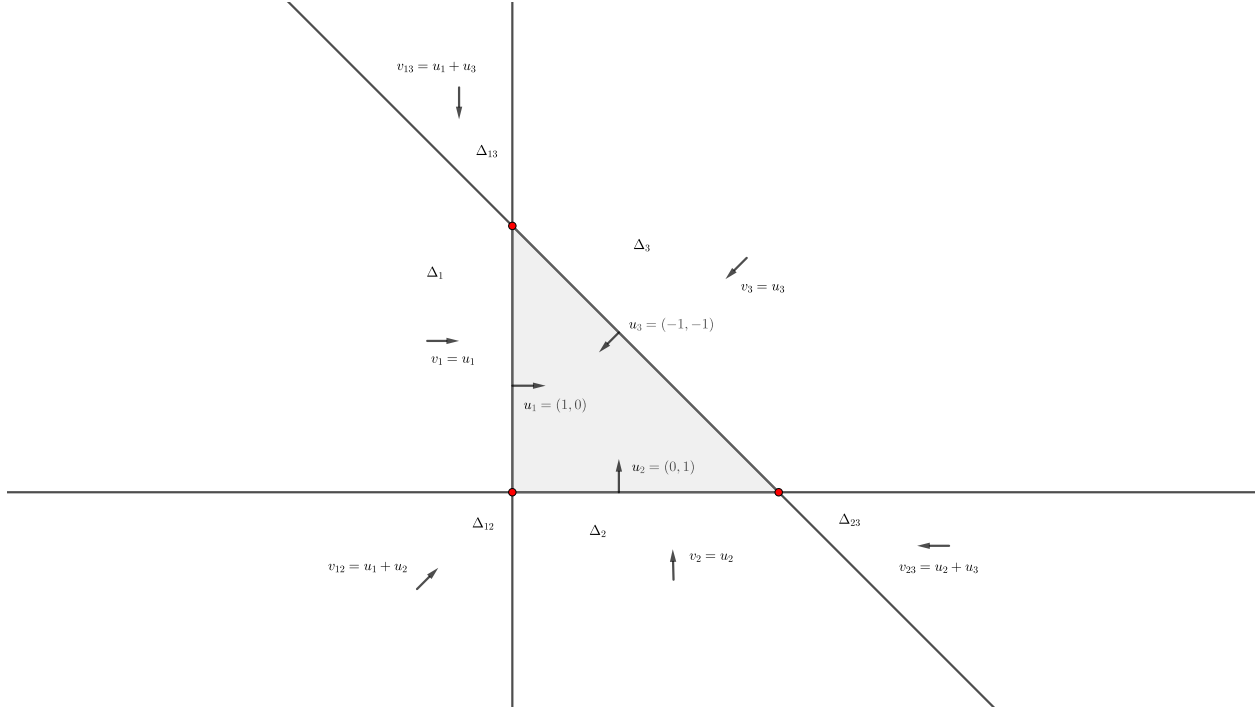


Figure 1: Combinatorics of the action of the residual  $S^1$ -action on the extended core  $\mathcal{E}_A$  of  $T^*\mathbb{CP}^2$ , represented by each generator  $v_A$  of  $S_A^1$  in  $T^2$ .

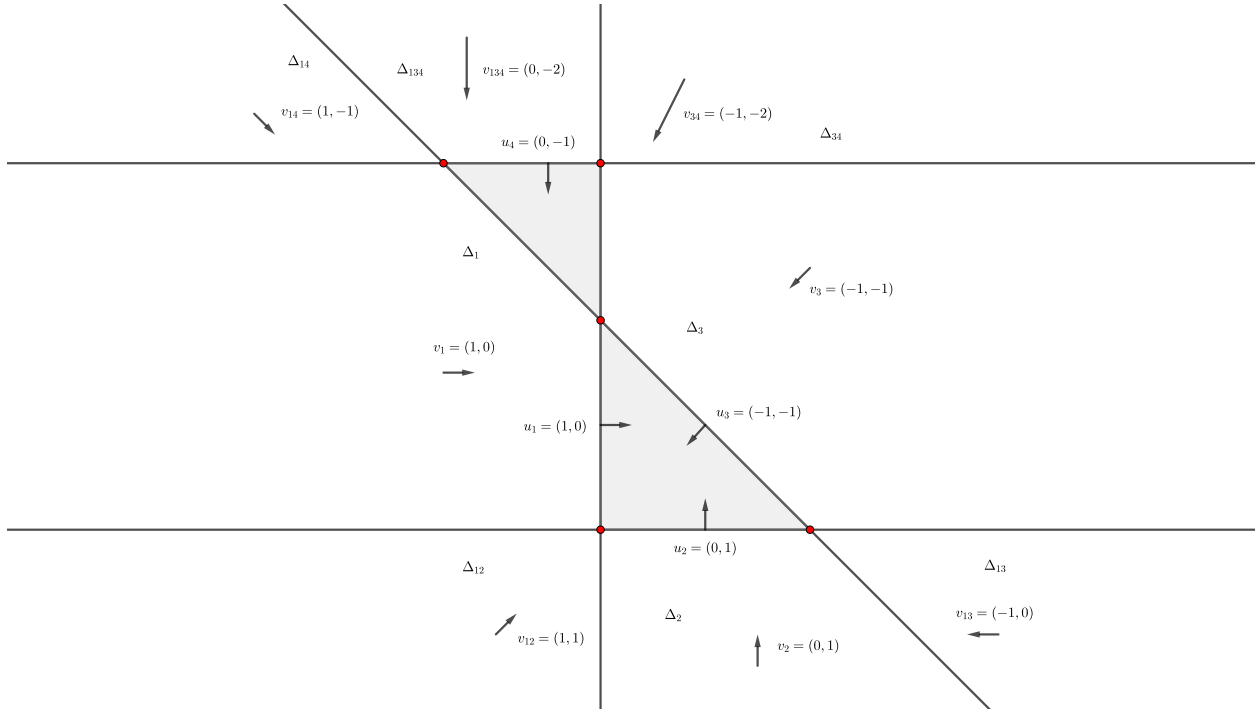


Figure 2: Combinatorics of the action of the residual  $S^1$ -action when the core consists of two non-convex components.

## 4 Compactifying the Hypertoric Variety via Symplectic Cutting

We will use the  $S^1$ -action to perform a symplectic cut of the toric hyperkähler manifold  $\mathfrak{M}$  to compactify it, which has the effect of bounding the  $\|w\|^2$ -norm component of the real moment map  $\bar{\mu}_{\mathbb{R}}$  by above, and discarding the rest that lies above this bound. Consider the product  $\mathfrak{M} \times \mathbb{C}$ , and let  $S^1$  act on  $\mathfrak{M} \times \mathbb{C}$  via the diagonal product action, i.e.  $S^1$  acts on  $M$  by rotating the cotangent fibre coordinates, and on  $\mathbb{C}$  in the standard way:

$$e^{i\theta} \cdot ([z, w], \xi) = (e^{i\theta} \cdot [z, w], e^{i\theta} \xi) = ([z, e^{i\theta} w], e^{i\theta} \xi).$$

This action is Hamiltonian, and the corresponding moment map  $\Phi : \mathfrak{M} \times \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$  for the  $S^1$ -action is

$$\Phi([z, w], \xi) = \phi[z, w] + |\xi|^2 = \|w\|^2 + |\xi|^2.$$

Then we have

$$\begin{aligned} \Phi^{-1}(\epsilon) &= \{([z, w], \xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = \epsilon\} \\ &= \{[z, w] \in M : \|w\|^2 = \epsilon\} \bigsqcup \{([z, w], \xi) \in M \times \mathbb{C} : |\xi| = \pm \sqrt{\epsilon - \|w\|^2}\} \\ &= \{[z, w] \in M : \|w\|^2 = \epsilon\} \bigsqcup \{([z, w], \xi) \in M \times \mathbb{C} : \xi = e^{i \arg(\xi)} \sqrt{\epsilon - \|w\|^2}\} \\ &= \phi^{-1}(\epsilon) \bigsqcup (\mathfrak{M} \times S^1) \\ &=: \Sigma_1 \sqcup \Sigma_2, \end{aligned}$$

where we denote the level-set  $\phi^{-1}(\epsilon) \subseteq \mathfrak{M}$  by  $\Sigma_1$ , and  $\Sigma_2 \cong \mathfrak{M} \times S^1$  is the trivial  $S^1$ -bundle over  $\Sigma_2$  given by the globally defined section

$$\mathfrak{M} \rightarrow \mathfrak{M} \times S^1, \quad [z, w] \mapsto ([z, w], e^{i\theta} \sqrt{\epsilon - \|w\|^2}), \quad e^{i\theta} \in S^1.$$

Finally, taking the symplectic reduction of  $\Phi^{-1}(\epsilon)$  with respect to the  $S^1$ -action, we obtain the *symplectic cut of  $\mathfrak{M}$  at level- $\epsilon$* ,

$$M_{\leq \epsilon} := \Phi^{-1}(\epsilon)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where  $\Sigma_1/S^1 \cong \phi^{-1}(\epsilon)/S^1$  is just the usual symplectic reduction, and where  $\Sigma_2/S^1$  is diffeomorphic to  $\mathfrak{M}$  for  $\|w\|^2 < \epsilon$ , which we will denote by  $\mathfrak{M}_{< \epsilon}$ .

### 4.1 The Combinatorics of the Cut Space, $\mathfrak{M}_{\leq \epsilon}$

Since the residual circle  $S^1$ -action acts as a subtorus  $S_A^1$  of the residual torus  $T^d$  on each component  $\mathcal{E}_A$  of the extended core, the hyperplane arrangement determined in  $(\mathfrak{t}^d)^*$  by the real moment map  $\bar{\mu}_{\mathbb{R}}$  is compactified by dropping in half-spaces with an inwards-pointing normal vector, given by  $v_A$  when taking the cut.

Recall from the previous section that  $j_A : S^1 \hookrightarrow T^n$  denoted the inclusion homomorphism of  $S^1$  into the original torus  $T^n$ . If we let  $j_{A,*} : \mathfrak{s}^1 \rightarrow \mathfrak{t}^n$  represent the differential of this inclusion, then

$$j_{A,*}(1) = \sum_{i \in A} e_i \in \mathfrak{t}^n,$$

and the generator  $\exp(v_A)$  of the one-parameter subgroup  $S_A^1$  in  $T^d$  is

$$\exp(v_A) = \exp(\pi_* \circ j_{A,*}(1)),$$

or to be more concise,

$$S_A^1 = \left\{ \exp \left( r \cdot \sum_{i \in A} u_i \right) \mid r \in \mathbb{R} \right\}.$$

Then the moment map for the restricted  $S^1$ -action to  $\mathcal{E}_A$  is

$$\phi_A[z, w] := \phi|_{\mathcal{E}_A}[z, w] = (j_A^* \circ \mu_{\mathbb{R}})[z, w] = \left\langle \bar{\mu}_{\mathbb{R}}[z, w], \sum_{i \in A} u_i \right\rangle,$$

where  $j_A^* : (\mathfrak{t}^n)^* \rightarrow \mathbb{R}^*$  is the transposed differential of the inclusion,  $j_{A,*}$ .

As the  $S_A^1$ -action depends combinatorially on the component  $\mathcal{E}_A$ , the image of the real moment map in  $(\mathfrak{t}^d)^*$  is compactified by inserting a half-space  $Z_A$  with inwards-pointing normal  $v_A = \sum_{i \notin A} u_i$  determining the orientation, on each component  $\Delta_A$ .

## 5 Hypertoric Subvarieties

### References

- [1] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.