Final Year Physics Project - Interim Report

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Introduction

Aims and Objectives

The aim of this project is to study the notion of *folded hyperkähler manifolds*, i.e. a 4-dimensional manifold which is hyperkähler away from some folding hypersurface on which the hyperkähler structure degenerates and the metric is singular [1, 2]. In particular, it will be interesting to look into more examples of hyperkähler structures that admit a folding hypersurface, since the symplectic and Kähler versions of folding have already been studied in much more detail [3, 4].

From a physicist's point of view the topic of folded hyperkähler structures still is an interesting topic; the canonical example of a folded hyperkähler structure comes from a particular choice of the Gibbons-Hawking metric [1], and Biquard [2] has also constructed folded hyperkähler manifolds by modifying the work of Ashtekar, Jacobson and Smolin (ASJ) on half-flat solutions to Einstein's equations [5]. A specific feature of these two examples of folded hyperkähler manifolds is that the signature of the metric swaps from Euclidean $(+ + ++)$ to anti-Euclidean $(- - --)$ as one travels across the fold. Such a feature is a recurring theme in the physics literature on 5-dimensional supergravity [6].

Background Theory

The work done in Term 1 was dedicated to understanding hyperkähler manifolds and examples of them. To this end, I followed the route of the Ashtekar formulation [7] of the self-dual Einstein equations in 4-dimensions since Biquard [2] uses a modified version of this to construct a folded hyperkähler manifold.

The basis of the construction is this: we take a real, 3-dimensional manifold Y with volume form α. Now consider the product 4-manifold $Y \times \mathbb{R}$, where $\tau \in \mathbb{R}$ can be considered as 'time'.

Claim 1 (Donaldson [8]). *Let V_i ($i = 1, 2, 3$) be time-dependent, volume-preserving, linearly independent vector fields on Y that satisfy Nahm's equations*

$$\frac{\partial V_i}{\partial \tau} = \frac{1}{2}\epsilon_{ijk}[V_j, V_k],$$

where ϵ_{ijk} is the fully-antisymmetric tensor. Then we can construct a complex symplectic structure on $Y \times \mathbb{R}$.

Proof. For brevity, write $\mathcal{M} = Y \times \mathbb{R}$. For each time τ, let $\epsilon_1, \epsilon_2, \epsilon_3$ be the basis of 1-forms dual to the V_i. Then, for some non-vanishing real function f on Y we have that $\alpha = f\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3$ for the volume form on Y. Define two 2-forms on $\mathcal{M}$ by

$$\theta_1 = f(d\tau \wedge \epsilon_1 + \epsilon_2 \wedge \epsilon_3), \tag{1a}$$

$$\theta_2 = f(d\tau \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_1). \tag{1b}$$

Then $\theta_1^2 = \theta_2^2 = f dt \wedge \alpha$, and $\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0$ and so if θ_1, θ_2 are closed on $\mathcal{M}$, then we have a complex symplectic structure on the manifold. To this end, we make use of the identity

$$d(\imath_X \imath_Y \phi) = \imath_{[X,Y]}\phi + \imath_X \mathcal{L}_Y \phi - \imath_Y \mathcal{L}_X \phi + \imath_X \imath_Y d\phi \tag{2}$$

for any two vector fields X and Y and form ϕ, and where $\mathcal{L}$ denotes the Lie derivative. We apply with identity to $d(\imath_{V_2}\imath_{V_3}\alpha)$, yielding

$$d(\imath_{V_2}\imath_{V_3}\alpha) = \imath_{[V_2,V_3]}\alpha + \imath_{V_2}\mathcal{L}_{V_3}\alpha - \imath_{V_3}\mathcal{L}_{V_2}\alpha + \imath_{V_2}\imath_{V_3}d\alpha$$

$$= \imath_{[V_2,V_3]}\alpha,$$

since the vector fields are volume-preserving. Furthermore, we have that

$$\imath_{V_3}\alpha = f\epsilon_1 \wedge e_2, \qquad \imath_{V_2}\imath_{V_3}\alpha = f\epsilon_1, \qquad \imath_{V_1}\alpha = f\epsilon_2 \wedge \epsilon_3,$$

$$d(\imath_{V_1}\alpha) = \mathcal{L}_{V_1}\alpha - \imath_{V_1}d\alpha = 0,$$

and so conclude that $\imath_{V_1}\alpha$ is a closed 2-form. Temporarily, let us write $\underline{d}$ for the exterior derivative of forms over $\mathcal{M}$ and d for the exterior derivative of forms over Y regarding time as a parameter. In this notation,

$$\underline{d}\psi = d\psi + dt \wedge \frac{\partial \psi}{\partial \tau},$$

and so

$$
\begin{aligned}
\underline{d}\theta_1 &= d\theta_1 + d\tau \wedge \frac{\partial \theta_1}{\partial \tau} \\
&= d(f\epsilon_2 \wedge \epsilon_3) + d\tau \wedge \left[\frac{\partial f}{\partial \tau}d\tau \wedge \epsilon_1 + \frac{\partial}{\partial \tau}(f\epsilon_2 \wedge e_3) \right] \\
&= d(\imath_{V_1}\alpha) + d\tau \wedge \left[d(f\epsilon_1) + \frac{\partial}{\partial \tau}(f\epsilon_2 \wedge e_3) \right] \\
&= 0 + d\tau \wedge \left[d(\imath_{V_2}\imath_{V_3}\alpha) + \frac{\partial}{\partial \tau}(\imath_{V_1}\alpha) \right],
\end{aligned}
$$

where we have used the fact that $\imath_{V_1}\alpha$ is closed on Y. Therefore θ_1 is closed on $\mathcal{M}$ if and only if

$$d(\imath_{V_2}\imath_{V_3}\alpha) + \frac{\partial}{\partial \tau}(\imath_{V_1}\alpha) = \imath_{[V_2,V_3]}\alpha + \imath_{\partial V_1/\partial \tau}\alpha = 0,$$

since α is time-independent. From the non-degeneracy of α, we conclude that θ_1 is closed on $\mathcal{M}$ if and only if $\frac{\partial V_1}{\partial \tau} + [V_2, V_3] = 0$, and the same argument for θ_2 proves that θ_2 is closed on $\mathcal{M}$ if and only if $\frac{\partial V_2}{\partial \tau} + [V_3, V_1] = 0$. Hence we have a complex symplectic structure on $\mathcal{M}$. $\qquad\square$

An immediate corollary of this claim is that we actually have three complex symplectic structures on $Y \times \mathbb{R}$.

Corollary 2. *With the same hypotheses as above, we have three complex symplectic structures on $Y \times \mathbb{R}$.*

Proof. Introduce a third 2-form on $\mathcal{M}$ given by

$$\theta_3 = f(d\tau \wedge \epsilon_3 + \epsilon_1 \wedge \epsilon_2).$$

By the same argument for both θ_1 and θ_2, θ_3 is also a closed 2-form on $\mathcal{M}$ and the three complex symplectic structures come from pairing together θ_1 with θ_2, θ_2 with θ_3, and finally θ_3 with θ_1. $\qquad \square$

With three complex symplectic forms, it looks promising that we can use them to define a hyperkähler structure of the product manifold $Y \times \mathbb{R}$. For the purpose of this, we state the following lemma without proof.

Lemma 3 (Hitchin [9]). *Let g be an almost hyperkähler metric, with 2-forms $\theta_1, \theta_2, \theta_3$ corresponding to almost complex structures I, J, K. Then g is hyperkähler if each θ_i is closed.*

In light of this lemma, we only need to define three almost complex structures that admit an action of the quaternions at each point in the tangent space, which are compatible with the metric. To proceed further, it will be convenient to introduce a fourth vector $V_0 = \frac{\partial}{\partial \tau}$, using Latin indices for space coordinates and Greek indices for spacetime coordinates. Now we can introduce the *'t Hooft symbols.* $\bar{\eta}^a_{\mu\nu}$

Definition 4 ('t Hooft [10]). The 't Hooft symbols $\bar{\eta}^a_{\mu\nu}$ are given by the relations

$$\bar{\eta}^a_{\mu\nu} := \begin{cases} \epsilon_{a\mu\nu} & \text{if } \mu, \nu = 1, 2, 3 \\ \delta_{a\nu} & \text{if } \mu = 0 \\ -\delta_{a\mu} & \text{if } \nu = 0 \\ 0 & \text{otherwise.} \end{cases}$$

They obey the following identities which we will utilise:

$$\bar{\eta}^a_{\mu\nu} = \epsilon_{0a\mu\nu} + \delta_{0\mu}\delta_{a\nu} - \delta_{a\mu}\delta_{0\nu}, \tag{3a}$$

$$\bar{\eta}^a_{\mu\nu} = -\bar{\eta}^a_{\nu\mu}, \tag{3b}$$

$$\bar{\eta}^a_{\mu\nu}\bar{\eta}^b_{\mu\sigma} = \delta_{ab}\delta_{\nu\sigma} + \epsilon_{abc}\bar{\eta}^c_{\nu\sigma}. \tag{3c}$$

Following on from this, we are able to define three almost complex structures on $Y \times \mathbb{R}$ by

$$J^a(V_\mu) = \bar{\eta}^a_{\nu\mu}(V_\nu), \tag{4}$$

for $a = 1, 2, 3$. To see that this does indeed define three almost complex structures, observe that

$$
\begin{aligned}
J^a J^b(V_\mu) &= \bar{\eta}^a_{\nu\mu} \bar{\eta}^b_{\sigma\nu}(V_\sigma) \\
&= -\bar{\eta}^a_{\nu\mu} \bar{\eta}^b_{\nu\sigma}(V_\sigma) \\
&= -\delta_{ab}\delta_{\mu\sigma}(V_\sigma) + \epsilon_{abc}\bar{\eta}^c_{\mu\sigma}(V_\sigma) \\
&= -\delta_{ab}(V_\mu) + \epsilon_{abc}\bar{\eta}^c_{\sigma\mu}(V_\sigma) \\
&= -\delta_{ab}(V_\mu) + \epsilon_{abc}J^c(V_\mu),
\end{aligned}
$$

and so each point in the tangent space to $Y \times \mathbb{R}$ admits an action of the quaternions. It remains to define a metric which is compatible with respect to the three almost complex structures *i.e.* an almost hyperkähler structure. Such a metric can be given by

$$g(V_\mu, V_\nu) = f(d\tau \wedge \alpha)(V_0, V_1, V_2, V_3)\delta_{\mu\nu}, \tag{5}$$

so that $\theta_a(V_\mu, V_\nu) = g(J^a(V_\mu), V_\nu)$. Then, as each θ_a is closed by Lemma 3 we have a hyperkähler manifold.

We pause here to remark that the Kähler forms θ_a can be written in a more compact form; namely, that if we let $\omega = d\tau \wedge \alpha$ be the volume form on $Y \times \mathbb{R}$, then we write

$$\theta_a = \frac{1}{2}\bar{\eta}^a_{\mu\nu}\iota_{V_\mu}\iota_{V_\nu}\omega. \tag{6}$$

The benefit of this representation is that we recover the so-called "half-flat" condition [5, 11],

$$\frac{1}{2}\bar{\eta}^a_{\mu\nu}[V_\mu, V_\nu] = 0 \tag{7}$$

by taking the exterior derivative of the θ_a and using identity (2).

In order to generalise what has been covered, we make the following statements: in 4-dimensions the Riemann curvature tensor is half-flat, *i.e.* self-dual or anti-self dual, if and only if the metric is hyperkähler [12]. Furthermore the ASJ construction of a hy-

perkähler structure reduces to finding four linearly-independent vectors V_μ ($\mu = 0, 1, 2, 3$) and volume form ω on a 4-dimensional manifold $\mathcal{M}$ [7, 5]

$$\mathcal{L}_{V_a} \alpha = 0, \tag{8a}$$

$$\frac{1}{2} \bar{\eta}^a_{\mu\nu} [V_b, V_c] = 0. \tag{8b}$$

A hyperkähler metric on $\mathcal{M}$ is then given by $g(V_\mu, V_\mu) = f\omega(V_0, V_1, V_2, V_3)\delta_{\mu\nu}$, where f is a real, non-negative function on $\mathcal{M}$. The three complex structures J^a ($a = 1, 2, 3$) are defined by the action of the 't Hooft symbols on the tangent vectors at each point of $\mathcal{M}$ by

$$J^a(V_\mu) = \bar{\eta}^a_{\nu\mu}(V_\nu). \tag{9}$$

Example 5 (Gibbons-Hawking Metric). In order to demonstrate the construction of an explicit hyperkähler manifold using the method above, we consider Euclidean space with standard coordinates $\mathbb{R}^4 = \{(\tau, x^1, x^2, x^3)\}$ as the underlying spacetime with volume-form $\omega = d\tau \wedge dx^1 \wedge dx^2 \wedge dx^3$. Let ϕ and ψ_i ($i = 1, 2, 3$) be smooth functions and let the four vector fields V_μ be given by

$$V_0 = \phi \frac{\partial}{\partial \tau}, \tag{10a}$$

$$V_i = \frac{\partial}{\partial x^i} + \psi_i \frac{\partial}{\partial \tau}. \tag{10b}$$

Then the volume-preserving property (8a) implies that the functions ϕ and ψ_i are independent of τ, and half-flat condition (8b) implies the *monopole equations*

$$\underset{3}{*} d\phi = d\psi, \tag{11}$$

where $\psi = \sum_{i=1}^3 \psi_i dx^i$ and $*_3$ denotes the Hodge star operator on $\mathbb{R}^3 = \{(x^1, x^2, x^3)\}$ with its flat metric. These conditions are the same as the ansatz used by Gibbons and Hawking to construct hyperkähler metrics with a triholomorphic Killing vector, $\partial/\partial\tau$ [13].

Now is a good time to discuss why this particular route was taken, *i.e.* why we have largely followed Ashtekar's Hamiltonian approach to general relativity to understand hyperkähler manifolds. First and foremost, the canonical example of a folded hyperkähler manifold was based upon the Gibbons-Hawking metric by taking $\phi \equiv z$, so that there is a folded hypersurface defined by $z = 0$ [1].

Secondly, Biquard modifies the Ashtekar-Smolin-Jacobson construction of hyperkähler manifolds to define a folded hyperkähler metric on a product 4-dimensional manifold $X^3 \times (-\epsilon, \epsilon)$, for some ϵ small enough [2].

It will be interesting to review the construction of several other hyperkähler manifolds whose constructions are similar to those two mentioned, and to investigate whether or not they admit a folded structure.

Summary of work done in Term 1

The work done in Term 1 is as follows:

1. Understood and appreciated the definition of a hyperkähler manifold, *i.e.* its definition in terms of three complex structures which obey quaternionic multiplication and are also compatible with the Riemannian metric. Equivalently, one can say that the holonomy group of a 4-dimensional hyperkähler manifold is contained in $Sp(1)$.

2. Constructed a hyperkähler by considering a real 3-manifold Y which admits a one-parameter triad of linearly-independent vector fields which preserve the volume form and Nahm's equations. Then by taking the product 4-manifold $Y \times \mathbb{R}$ with three Kähler forms defined by the covectors dual to the triad as well as $d\tau \in T^*\mathbb{R}$, we showed that a hyperkähler structure for the product 4-manifold exists.

3. Investigated the more general construction of hyperkähler manifolds down the route of Ashtekar's approach to general relativity, that introduces a triad of volume-preserving vector fields that obey the half-flat condition (7). The manifold is then hyperkähler since the half-flat condition implies that the holonomy group of the manifold is $Sp(1)$.

4. As an example of the above, chose three vector fields that satisfied the conditions (8a) and (8b) to arrive at the ansatz used for the construction of the Gibbons-Hawking metric.

Plan of Work for Term 2

Over the course of Term 2, I aim to:

1. Elucidate the nature of the more "general construction" of hyperkähler structures mentioned above in point 3, as it was not sufficiently explained (in my opinion) at the end of Term 1. This method is discussed in [14, 15] and derives constraints of a tetrad of self-dual 2-forms, which satisfy specific constraints to minimise an action[1]. These constraints turn out to be that the Ricci curvature vanishes and that each 2-form is closed; hence they imply the existence of a hyperkähler structure. This should be covered by the end of Week 1.

2. Look at the first example of a folded hyperkähler structure, which is a special case of the Gibbons-Hawking metric. Whilst a simple example, it allows us to concisely define in what way a hyperkähler structure can be said to be 'folded.' This should be covered by the end of Week 2.

3. Look at the second example of a folded hyperkähler structure, given by Biquard in [2]. The beauty about this example is that the hyperkähler structure folds in the same way as the folded Gibbons-Hawking metric does, which implies that there is a naturality to the definition. This should be covered by the end of Week 3.

4. The rest of the term will be dedicated to trying to find other examples of hyperkähler structures. The benefit of the claimed general construction mentioned in point 1. is that it provides a wealth of hyperkähler structures. Several examples of which are given in [11], such as the $SU(\infty)$-Toda equation and "heavenly" metric. This will cover Weeks 4 to 8 inclusively.

5. Weeks 9 and 10 will be dedicated to the writing of the final report, which is due in on the Thursday of Week 10 (16/03/2017).

References

[1] N. Hitchin, "Higgs bundles and diffeomorphism groups," *ArXiv e-prints*, Jan. 2015.

[2] O. Biquard, "Métriques hyperkähleriennes pliées," *ArXiv e-prints*, Mar. 2015.

[3] G. V. Cannas da Silva, A. and C. Woodward, "On the Unfolding of Folded Symplectic Structures," *Math. Res. Lett.*, vol. 7, p. 35, 2000.

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[1]This action is called the tetrad-Palatini action, and uses frame fields and spin connections as its independent variables.

[4] R. Baykur, "Kahler Decomposition of 4-Manifolds," *ArXiv e-prints*, Jan. 2006.

[5] A. Ashtekar, T. Jacobson, and L. Smolin, "A New Characterization of Half-Flat Solutions to Einstein's Equation," *Comm. Math. Phys.*, vol. 115, no. 4, p. 631, 1988.

[6] G. W. Gibbons and N. P. Warner, "Global Structure of Five-Dimensional Fuzzballs," *Class. Quant. Grav.*, vol. 31, Jan. 2014.

[7] A. Ashtekar, "New Hamiltonian Formulation of General Relativity," *Phys. Rev. D*, vol. 36, no. 6, 1987.

[8] S. K. Donaldson, "Complex Cobordism, Ashtekar's Equations and Diffeomorphisms," in *Symplectic Topology* (D. Salamon, ed.), London Math. Soc., 1992.

[9] N. J. Hitchin, "The Self-Duality Equations on a Riemann Surface," *Proc. London Math. Soc.*, vol. 55, no. 1, p. 59, 1987.

[10] G. 't Hooft, "Computation of the Quantum Effects due to a Four-Dimensional Pseudoparticle," *Phys. Rev. D*, vol. 14, no. 12, p. 3432, 1976.

[11] Y. Hashimoto, Y. Yasui, S. Miyagi, and T. Ootsuka, "Applications of the Ashtekar Gravity to Four Dimensional Hyperkähler Geometry and Yang-Mills Instantons," *Journ. Math. Phys.*, vol. 38, p. 5833, 1997.

[12] D. C. Robinson, "Half-Flat Solutions of Einstein's Equation and Hyperkähler Structures," in *New Perspectives on Canonical Gravity* (A. Ashtekar, ed.), Bibiopolis, 1988.

[13] G. Gibbons and S. Hawking, "Gravitational Multi-Instantons," *Phys. Lett. B*, vol. 78, no. 4, p. 430, 1978.

[14] R. Capovilla, J. Dell, T. Jacobson, and L. Mason, "Self-Dual 2-forms and Gravity," *Class. Quant. Grav.*, vol. 8, no. 1, p. 41, 1991.

[15] T. Jacobson and L. Smolin, "Covariant Action for Ashtekar's Form of Canonical Gravity," *Class. Quant. Grav.*, vol. 5, no. 4, p. 583, 1988.