Folded Hyperkähler Manifolds

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Abstract

An example of a folded hyperkähler manifold is given based on a particular form of the Gibbons-Hawking ansatz, and its defining properties discussed. A folded analogue to Plebański's real heaven background is then constructed, with the structure of the fold hypersurface determined by solutions to the Boyer-Finley equation.

Introduction

A hyperkähler manifold is a Riemannian manifold of real dimension $4n$, that admits three covariantly orthogonal automorphisms, I, J, and K on the tangent bundle, which satisfy the quaternionic identities $I^2 = J^2 = K^2 = IJK = -\mathrm{id}$, and are compatible with the Riemannian metric h [1].

Recently, Nigel Hitchin has introduced the notion of a folded hyperkähler manifold, *i.e.* a 4-dimensional manifold which is hyperkähler away from some folding hypersurface, on which the hyperkähler structure degenerates and the metric is singular [2].

Method

To get an idea of how a hyperkähler manifold should admit a fold, we look at a particular example of the Gibbons-Hawking metric [3]. Consider a principal S^1-bundle $\mathcal{M}^4 \xrightarrow{\pi} \mathcal{U}$, where $\mathcal{U} \subset \mathbb{R}^3$ is an open set and consider a local trivialisation $\pi^{-1}(U) = \{(x, y, z, \tau) \in \mathcal{U} \times S^1\}$. The Gibbons-Hawking ansatz we consider is [2, 4]

$$h = z^{-1}(d\tau + \mathcal{A})^2 + z(dx^2 + dy^2 + dz^2), \qquad \mathcal{A} = (xdy - ydx)/2,$$

with the hyperkähler 2-forms given by

$$\omega^i = (d\tau + \mathcal{A}) \wedge dx^i + \frac{x^i}{2}\epsilon_{ijk}dx^j \wedge dx^k, \qquad i = 1, 2, 3.$$

The metric h is undefined at $z = 0$, and hence determines a hypersurface $\mathcal{Z} = \mathcal{M} \cap \{z = 0\}$ that divides the ambient manifold $\mathcal{M}$ into two disjoint ones; one with an Euclidean signature $(++++)$ when $z > 0$, and the other with an anti-Euclidean signature $(----)$ when $z < 0$. Under the involution $i : z \mapsto -z$ one observes that

$$i^*\omega^1 = \omega^1, \qquad i^*\omega^2 = \omega^2, \qquad i^*\omega^3 = -\omega^3, \qquad i^*h = -h.$$

Furthermore, whilst h is undefined along the fold $\mathcal{Z}$ the hyperkähler forms ω^i are smooth there. Pulling them back to $\mathcal{Z}$,

$$\mathcal{Z}^*\omega^1 = \varphi \wedge dx, \qquad \mathcal{Z}^*\omega^2 = \varphi \wedge dy, \qquad \mathcal{Z}^*\omega^3 = 0, \qquad \text{where } \varphi \equiv d\tau + \mathcal{A}.$$

Since $d\mathcal{A} = dx \wedge dy$, it follows that

$$\varphi \wedge d\varphi = d\tau \wedge dx \wedge dy \neq 0,$$

and so $(\mathcal{Z}, \varphi)$ determines a contact manifold. For a general 4-dimensional hyperkähler manifold $\mathcal{M}$, the quadruple $(\mathcal{M}, \mathcal{Z}, \omega^i, i)$ with the above properties defines a folded hyperkähler structure [4].

Results

Plebański's real heaven background is a more general version on the Gibbons-Hawking ansatz [5]; with the same S^1-bundle as before, its hyperkähler metric and 2-forms are given by [6]

$$h = u_z(e^u(dx^2 + dy^2) + dz^2) + u_z^{-1}(d\tau + \mathcal{A}),$$
$$\begin{bmatrix}\omega^1\\ \omega^2\end{bmatrix} = e^{u/2}\begin{bmatrix}\cos(\tau) & -\sin(\tau)\\ \sin(\tau) & \cos(\tau)\end{bmatrix}\begin{bmatrix}(d\tau + \mathcal{A}) \wedge dx + u_z dy \wedge dz\\ (d\tau + \mathcal{A}) \wedge dy + u_z dz \wedge dx\end{bmatrix},$$
$$\omega^3 = u_z e^u dx \wedge dy + dz \wedge (d\tau + \mathcal{A}),$$

where $\mathcal{A} = -u_y dx + u_x dy$ such that $\psi \equiv d\tau + \mathcal{A}$ is the connection 1-form of the S^1-bundle, and where $u \in C^\infty(\mathcal{U})$ satisfies the Boyer-Finley equation

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0.$$

In separating the variables, the general solution is found to be

$$e^u = \frac{4(az^2 + bz + c)}{(1 + a(x^2 + y^2))^2} \qquad \Longrightarrow \qquad u_z = \frac{2az + b}{az^2 + bz + c}, \qquad e^u u_z = \frac{8az + 4b}{(1 + a(x^2 + y^2))^2},$$

with a, b and c constants [7]. In order to have a fold, we must have that u_z and $e^u u_z$ are odd in z, so choosing $a \neq 0, b = 0, c > 0$, and pulling back to the hypersurface $\mathcal{Z} = \mathcal{M} \cap \{z = 0\}$, the hyperkähler metric is singular, $\mathcal{Z}^*\omega^3 = 0$, and the 2-forms ω^1, ω^2 become

$$\mathcal{Z}^*\begin{bmatrix}\omega^1\\ \omega^2\end{bmatrix} = \frac{2\sqrt{c}}{1 + a(x^2 + y^2)}(d\tau + \mathcal{A}) \wedge \begin{bmatrix}\cos(\tau) & -\sin(\tau)\\ \sin(\tau) & \cos(\tau)\end{bmatrix}\begin{bmatrix}dx\\ dy\end{bmatrix}, \qquad \mathcal{A} = \frac{4a(xdy - ydx)}{1 + a(x^2 + y^2)}.$$

Writing the ω^i in this form suggests that we should take $\psi \equiv d\tau + \mathcal{A}$ to be our contact 1-form. Indeed,

$$d\psi = \frac{8a}{(1 + a(x^2 + y^2))^2}dx \wedge dy, \qquad \psi \wedge d\psi = \frac{8a}{(1 + a(x^2 + y^2))^2}d\tau \wedge dx \wedge dy \neq 0$$

along $\mathcal{Z}$, whence $(\mathcal{Z}, \psi)$ is a contact manifold. Let us identify $\mathbb{C} \cong \mathbb{R}^2$ via $w = x + iy$ and, as $d\psi$ is S^1-invariant and $i_{\partial_\tau} d\psi \equiv 0$, a closed 2-form α is induced on $\mathbb{C}$ by $d\psi = \pi^*\alpha$ [8], with the form

$$\alpha = \frac{4ai}{(1 + a|w|^2)^2}dw \wedge d\bar{w} = 4i\partial\bar{\partial}\log(1 + a|w|^2).$$

We see that if $a < 0$, then α is defined only on the open disk $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1/|a|\}$, which is conformally equivalent to hyperbolic space $\mathcal{H}^2$, whereas if $a > 0$ then the boundary extends to $\{\infty\}$, *i.e.* we may compactify to consider the Riemann sphere $\hat{\mathbb{C}} \cong S^2$, since $\alpha \to 0$ as $|w| \to \infty$.

Conclusion

Two non-trivial families of folded hyperkähler structures has been constructed, one where the embedded fold hypersurface descends to hyperbolic 2-space $\mathcal{H}^2$, and the other where it descends to the 2-sphere S^2.

To continue in the direction of this project, it would be interesting to:

- consider different solutions to the Boyer-Finley equation, and see if they admit a folded structure.
- identify whether the folded real heaven background can be linearised to recover the folded Gibbons-Hawking ansatz.
- generalise the definition of a folded hyperkähler manifold to higher dimensions.

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