First Year Annual Report

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First-Year Report
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Chapter 1

Introduction

This is my report for the First Year Annual Review of my PhD at The University of Edinburgh. In this chapter, I will give a short overview of the other things I have done alongside my studies.

First of all, I was the First Year Postgraduate Representative for the School, a role that had more responsibilities tied to it than previous years with the introduction of the Bayes Centre, since now the whole of the first year cohort were based separately from the other students and academics in the department. My responsibilities included: being the medium through which feedback and suggestions would make their way to the Graduate School office, as well as the Head of School; organising the tea and coffee breaks at the Bayes Centre; assisting the other Postgraduate Representatives with the organisation of the Christmas party, as well as the Welcome Dinner for the new students.

There were also several Graduate Schools and Workshops that I attended throughout the year:

- March 16th - 22nd 2019: There was both a graduate school and workshop at the SCGP titled '*Challenges at the Interface of Hitchin Systems and String Theory*'.
- May 25th - 31st 2019: There was a graduate school at the SCGP titled '*Graduate Summer School on the Mathematics and Physics of Hitchin Systems*'.
- June 24th - 28th 2019: The VBAC 2019 conference at the Sandbjerg Estate with the theme '*GIT, Wall-Crossings and Moduli Spaces*'.
- July 29th - August 2nd 2019: There was a conference at the MPIM in Bonn titled

‘*Integrability, Geometry and Moduli*’.

Each of these was an inspiring experience for me, as I got to meet many academics whose papers I have or will be reading, even if some talks were too advanced for me to actively engage in at the time.

In terms of courses, I took two distant learning ones: a SMSTC course ‘*Algebraic Topology*’ and also one from the TCC, ‘*Combinatorial Algebraic Geometry*’. In addition to these courses, I also enrolled for two taught at Edinburgh, namely ‘*Topics in Noncommutative Algebra*’, and also ‘*Interactions in Algebra, Geometry and Topology*’. Finally, I gave a lecture for the ‘*Moment Maps*’ working seminar, in particular on ‘*Toric Symplectic Manifolds and the Delzant Construction*’. This latter talk is what has actually paved a way towards what my current postgraduate study is now on.

Finally, I am pursuing the teaching stream during my postgraduate study at Edinburgh, and over the second semester I tutored the ‘*Calculus and its Applications*’ workshops, as well as helped mark the exam scripts.

Chapter 2

Symplectic Toric Varieties and Delzant's Construction

The content of this chapter has now become quite standard, and as such can be found in several textbooks on symplectic and/or toric geometry, *e.g.* [1, 2, 3]. We shall mostly follow [2] for this chapter, as the reference is written with the end goal of geometric quantisation in mind for toric symplectic manifolds.

2.1 Basic Definitions and Theorems

Let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the vector space dual of $\mathfrak{g}$. Let (M, ω) be a symplectic manifold, and suppose that it is also a Hamiltonian G-space. If G does not act effectively on M, then by quotienting G by the kernel of the action, we end up with an effective action on M by the resulting quotient group.

Theorem 1. *If G acts effective on M, then* $\dim M \geq 2 \dim G$.

Later on in this section, we shall investigate the cases when equality holds. Before that however, we will give an overview of the Marsden-Weinstein reduction: let (M, ω_M) be a Hamiltonian G-space with moment map Φ, and let $M_0 = \Phi^{-1}(0)$. As Φ is G-equivariant, M_0 is G-invariant.

Theorem 2. *If G acts freely on M_0, then 0 is a regular value of Φ, and so M_0 is a closed submanifold of M of codimension equal to* $\dim G$.

Assuming still that G does act freely on M_0, then since G is compact the orbit space

$$B := M_0/G$$

is a Hausdorff manifold of codimension $2\dim G$, and the map from points to orbits

$$\pi : M_0 \longrightarrow B$$

is a principal G-fibration. Let $i : M_0 \to M$ be the inclusion map.

Theorem 3. *There exists a symplectic form ω_B on B such that*

$$i^*\omega_M = \pi^*\omega_B.$$

Definition 4. (B, ω_B) is the *reduction* of (M, ω_M).

Remark 5. When G is abelian (and so is isomorphic to a torus), its coadjoint action on $\mathfrak{g}^*$ is trivial. In this case, one can use any value $c \in \mathfrak{g}^*$ to reduce at, by using the moment map $\Phi - c$ in place of Φ.

2.2 Toric Symplectic Varieties and Delzant's Construction

Now let $G = T^d = \mathbb{R}^d/\mathbb{Z}^d$ and $\mathfrak{g} = \mathbb{R}^d$ and $\mathfrak{g}^* = (\mathbb{R}^d)^*$. We will construct a variety of examples of Hamiltonian G-spaces of dimension $2d$, on which G acts effectively. These are called:

Definition 6. A *symplectic toric manifold* is a compact connected symplectic manifold (M, ω) with an effective Hamiltonian action of a torus T such that

$$\dim T = \frac{1}{2}\dim M,$$

and with a choice of corresponding moment map $\mu : M \to \mathfrak{t}^*$.

Theorem 7 (Atiyah, Guillemin-Sternberg Convexity, [4, 5])**.** *Let (M, ω) be a symplectic toric manifold, and let T be a torus that acts on M in a Hamiltonian manner. Consider the moment map $\mu : M \to \mathfrak{t}^*$ for this T-action, then the following hold:*

- *the level sets $\mu^{-1}(c)$ are connected, for each $c \in \mathfrak{t}^*$;*

- *the image $\mu(M)$ is convex;*
- *the image $\mu(M)$ is the convex hull of the images of the fixed points of the action.*

Definition 8. A *Delzant polytope* [6] Δ in $(\mathbb{R}^d)^*$ is a convex polytope satisfying:

- simplicity: there are d edges meeting at each vertex;
- rationality: each edge that meets a vertex p is of the form $p + tu_i$, with $0 \leq t_i \leq \infty$ and $u_i \in (\mathbb{Z}^d)^*$ for each $i = 1, \ldots, d$;
- smoothness: for each vertex, the corresponding $u_1, \ldots, u_d$ can be chosen to be a $\mathbb{Z}$-basis of $(\mathbb{Z}^d)^*$.

It turns out that the moment polytope of a symplectic toric manifold is Delzant.

Proposition 9 ([6])**.** *For any symplectic toric manifold (M, ω), its moment polytope $\Delta_M := \mu(M)$ is a Delzant polytope.*

So this shows that any toric symplectic manifold has, as the image of its moment map, a Delzant polytope associated to it.

Theorem 10 ([6])**.** *Symplectic toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the moment map:*

$$\frac{\{\textit{symplectic toric manifolds}\}}{\{T^d\textit{-equivariant symplectomorphisms}\}} \longleftrightarrow \frac{\{\textit{Delzant polytopes}\}}{SL(d,\mathbb{Z}) \ltimes \mathbb{R}^d}$$
$$(M_\Delta^{2d}, \omega, T^d, \mu) \longleftrightarrow \mu(M_\Delta) = \Delta.$$

So Proposition 9 provided one direction of Theorem 10, and Delzant proved the opposite direction. Let us outline how one can associate a symplectic manifold X_Δ to every such Delzant polytope. To begin with, we start with the $(d-1)$-dimensional faces, or facets, of Δ, which can be defined by equations of the form

$$\langle u_i, v \rangle = \lambda_i, \qquad i = 1, \ldots, n$$

where $u_i \in \mathbb{Z}^d$. Without loss of generality, we can assume that the u_i's are *primitive*, *i.e.* , that they are *not* of the form $u_i = ku_i'$, $k \neq \pm 1$, and $u_i' \in \mathbb{Z}^d$. We can also assume that

Δ is the intersection of the half-spaces

$$\langle u_i, v\rangle \geq \lambda_i,$$

so that the u_i's are all *inward* pointing normal vectors to the facets. This condition along with primitivity determine the u_i's uniquely. Let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$ and consider the map

$$\pi : \mathbb{Z}^n \to \mathbb{Z}^d, \qquad e_i \mapsto u_i,$$

and its extension

$$\pi : \mathbb{R}^n \to \mathbb{R}^d.$$

We then get the induced quotient map by exponentiating

$$\pi : T^n \to T^d.$$

Denoting the kernel of π by N, one obtains the exact sequence:

$$1 \longrightarrow N \xhookrightarrow{\ i\ } T^n \xrightarrow{\ \pi\ } T^d \longrightarrow 1$$

where $i : N \hookrightarrow T^n$ is the inclusion homomorphism. The torus T^n acts on $\mathbb{C}^n$ by the multiplication mapping

$$e^{i\theta} \cdot z = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$$

and this action is Hamiltonian with moment map

$$\mu : \mathbb{C}^n \to (\mathbb{R}^n)^*, \qquad \mu(z) = \frac{1}{2}\big(|z_1|^2, \ldots, |z_n|^2\big) + \lambda,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^n)^*$ is an arbitrary constant, though soon we shall we that the λ_i's are those that delimit the Delzant polyhedron Δ. By restricting the action of T^n on $\mathbb{C}^n$ to N, one gets a Hamiltonian action of N on $\mathbb{C}^n$ whose moment map is the following. Recall the inclusion homomorphism $i : N \hookrightarrow T^n$, and let $\mathfrak{n}$ be the Lie algebra of N. Then by differentiating we get the inclusion $i : \mathfrak{n} \to \mathbb{R}^n$ on the Lie algebra level, and then by transposing we get $i^* : (\mathbb{R}^n)^* \to \mathfrak{n}^*$.

Lemma 11. *The moment map for the action of N on $\mathbb{C}^n$ is $i^* \circ \mu$.*

It can be further proven that

Theorem 12. $(i^* \circ \mu)^{-1}(0)$ *is a compact subset of* $\mathbb{C}^n$ *and* N *acts freely on this set.*

It is this theorem that lets us legitimately reduce $\mathbb{C}^n$ with respect to the action of N, which results in a compact symplectic manifold on which the quotient $T^d = T^n/N$ acts. Let $X_\Delta := (i^* \circ \mu)^{-1}(0)/N$ denote the resulting compact symplectic manifold, then its dimension is

$$\dim X_\Delta = \dim \mathbb{C}^n - 2\dim N = 2n - 2(n-d) = 2d = 2\dim T^d$$

Remark 13. By differentiating and dualising the exact sequence above, we get:

$$0 \longleftarrow \mathfrak{n}^* \xleftarrow{i^*} (\mathbb{R}^n)^* \xleftarrow{\pi^*} (\mathbb{R}^d)^* \longleftarrow 0,$$

so $\operatorname{Im}\pi^* = \ker i^*$ from the exactness. Let $\Delta' = \pi^*\Delta$, then $\Delta' \cong \Delta$ since π^* is injective.

Lemma 14. $(i^* \circ \mu)^{-1}(0) = \mu^{-1}(\Delta')$.

Proof. The image of μ is the set of points $x \in (\mathbb{R}^n)^*$ satisfying

$$\langle x, e_i \rangle \geq \lambda_i, \qquad i = 1, \ldots, n.$$

But $i^*(x) = 0 \iff v = \pi^*(y)$ for some $y \in (\mathbb{R}^d)^*$, and

$$\langle e_i, \pi^*(y) \rangle \geq \lambda_i, \qquad i = 1, \ldots n$$

implies

$$\langle \pi(e_i), y \rangle = \langle u_i, y \rangle \geq \lambda_i, \qquad i = 1, \ldots, n,$$

hence $y \in \Delta$. □

This lemma lets us interpret the resulting Delzant polytope as the intersection of the half-spaces determined by the λ_i's, further intersected with the $(n-d)$-dimensional affine subspace defined by $\ker i^*$.

Example 15. Let e_i, $i = 1, \ldots, 3$, be the standard basis of $\mathbb{R}^3$, and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$\pi(e_i) = \begin{cases} e_i, & \text{for } i = 1, 2, \\ -e_1 - e_2, & \text{for } i = 3, \end{cases}$$

and label $\pi(e_i) = u_i$ Observe that π is represented by the matrix

$$\pi = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

whose kernel is the span of the diagonal hyperplane, $\ker \pi = \mathrm{span}_{\mathbb{R}}(1,1,1) \subset \mathbb{R}^3$. Denoting $\mathfrak{n} := \ker \pi$, then the inclusion map $i : \mathfrak{n} \hookrightarrow \mathbb{R}^3$ is just the diagonal embedding, and its transpose $i^* : (\mathbb{R}^3)^* \to \mathfrak{n}^*$ is just summation. Exponentiating and letting T^{n+1} act on $\mathbb{C}^{n+1}$ diagonally (which is Hamiltonian), we get the moment map

$$\mu : \mathbb{C}^3 \longrightarrow (\mathbb{R}^3)^*, \qquad \mu(z) = \frac{1}{2}\Big(|z_1|^2, |z_2|^2, |z_3|^2\Big) - \lambda,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{R}^3)^*$ is a constant, which has to have integral components if we are to anticipate a Delzant polytope. The moment map for the Hamiltonian action of N on $\mathbb{C}^3$ via inclusion is subsequently

$$i^* \circ \mu : \mathbb{C}^3 \to \mathfrak{n}^*, \qquad (i^* \circ \mu)(z) = \frac{1}{2}\sum_{i=1}^{3}(|z_i|^2 - \lambda_i),$$

so that the zero level-set $(i^* \circ \mu)^{-1}(0) \subseteq \mathbb{C}^d$ is

$$(i^* \circ \mu)^{-1}(0) = \{z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 2(\lambda_1 + \lambda_2 + \lambda_2)\}.$$

For this example, we now set $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = k \in \mathbb{Z}_{\geq 0}$, and also letting $|z_i|^2 = 2x_i$, where we use $x_i \in (\mathbb{R}_{\geq 0})^*$ represent the image of μ in $(\mathbb{R}^3)^*$, image of the moment map $\phi : X \to (\mathbb{R}^2)^*$ for the symplectic quotient $X = (i^* \circ \mu)^{-1}(0)/N$ above is now

$$\{x \in (\mathbb{R}^3_{\geq 0})^* : x_1 + x_2 + x_3 = k\} \cong \{(x,y) \in \mathbb{R}^2_{\geq 0} : x + y \leq k\} =: \Delta \subseteq \mathbb{R}^2.$$

Here, Δ is an isosceles triangle in $\mathbb{R}^2$ with two of the sides with length k. Also, since $(i^* \circ \mu)^{-1}(0) \cong S^5$, and $N \cong S^1$ acts on this level-set diagonally, we see that

$$(i^* \circ \mu)^{-1}(0)/N \cong S^5/S^1 \cong \mathbb{CP}^2$$

that is the complex projective plane.

Mutatis mutandi, it is not hard to see that letting the same $N \cong S^1$ act on $\mathbb{C}^{n+1}$, we get

$$(i^* \circ \mu)^{-1}(0)/N \cong S^{2n+1}/S^1 \cong \mathbb{CP}^n.$$

Chapter 3

Toric Hyperkähler Manifolds

The goal of the first half of this chapter is to introduce the quaternionic analogue to toric symplectic manifolds, which are aptly called *toric hyperkähler manifolds*, which were first introduced in [7]. In fact, they introduced the more general hypertoric *varieties*, which also included the case of non-smooth spaces, or orbifolds. For brevity, we shall not concern ourselves with the non-smooth cases however. Our notation will follow more closely to that in [8], which specialises to types of hypertoric manifolds which are essentially analogues to the toric symplectic manifolds we considered in Chapter 2.

3.1 Hyperkähler Reduction and Hyperkähler Analogues

3.1.1 Introduction and Definitions

A *hyperkähler manifold* is a Riemannian manifold (M, g) equipped with three orthogonal, parallel complex structures J_1, J_2, J_3, satisfying the usual quaternion relations. These three complex structures give rise to three symplectic forms

$$\omega_1(v, w) = g(J_1 v, w), \quad \omega_2(v, w) = (J_2 v, w), \quad \omega_3(v, w) = g(J_3 v, w),$$

so that each (g, J_i, w_i) is in its own right a Kähler structure on M for $i = 1, 2, 3$. The complex-valued two-form $\omega_2 + \sqrt{-1}\omega_3$ is a closed, non-degenerate, and holomorphic two-form with respect to the complex structure J_1. Thus any hyperkähler manifold can be considered as a *holomorphic symplectic* manifold with complex structure J_1, real symplectic form $\omega_{\mathbb{R}} := \omega_1$, and holomorphic symplectic form $\omega_{\mathbb{C}} := \omega_2 + \sqrt{-1}\omega_3$.

An action of a Lie group G on a hyperkähler manifold M is called *hyperhamiltonian* if it is hamiltonian with respect to $\omega_{\mathbb{R}}$, and holomorphic hamiltonian with respect to $\omega_{\mathbb{C}}$, with a G-equivariant moment map

$$\mu_{HK} := \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} \longrightarrow \mathfrak{g}^* \oplus \mathfrak{g}_{\mathbb{C}}^*.$$

The following theorem describes the *hyperkähler quotient* construction, which is the quaternionic analogue of a Kähler quotient:

Theorem 16 ([9]). *Let M be a hyperkähler manifold equipped with a hyperhamiltonian action of a compact Lie group G, with moment maps μ_1, μ_2, μ_3. Suppose that $\xi = \xi_{\mathbb{R}} \oplus \xi_{\mathbb{C}}$ is a central regular value for μ_{HK}, and that G acts freely on $\mu_{HK}^{-1}(\xi)/G$. Then there is a unique hyperkähler structure on the hyperkähler quotient $\mathfrak{M} = M /\!\!/\!\!/\!\!/_{\xi} G := \mu_{HK}^{-1}(\xi)/G$, with associated symplectic and holomorphic symplectic forms $\omega_{\mathbb{R}}^{\xi}$ and $\omega_{\mathbb{C}}^{\xi}$, such that $\omega_{\mathbb{R}}^{\xi}$ and $\omega_{\mathbb{C}}^{\xi}$ pull-back to the restrictions of $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$ on $\mu_{HK}^{-1}(\xi)$.*

In general, the action of G on $\mu_{HK}^{-1}(\xi)$ will not be free, but only locally free. In this situation, we would end up with a *hyperkähler orbifold.* However in the sequel, we shall only concern ourselves when the action is free, and that $\mathfrak{M}$ is smooth, *i.e.* a manifold.

Let us specialise to the case when $M = T^*\mathbb{C}^n$, and let G act on $T^*\mathbb{C}^n$ with the induced action from a linear action of G on $\mathbb{C}^n$, with moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^*$. We can identify $\mathbb{H}^n$ with $T^*\mathbb{C}^n$ such that the complex structure J_1 on $\mathbb{H}^n$ is given by right multiplication by i, and that J_1 corresponds to the natural complex structure on $T^*\mathbb{C}^n$. With this identification in mind, $T^*\mathbb{C}^n$ inherits a hyperkähler structure. The real symplectic form $\omega_{\mathbb{R}}$ is obtained from the sum of the pull-backs of the standard Kähler forms on $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$, and the holomorphic symplectic form $\omega_{\mathbb{C}}$ is $\omega_{\mathbb{C}} = d\eta$, where η is the canonical holomorphic one-form on $T^*\mathbb{C}^n$.

As G acts $\mathbb{H}^n$-linearly on $T^*\mathbb{C}^n \cong \mathbb{H}^n$ from the left, the action is hyperhamiltonian with moment map $\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$, where

$$\mu_{\mathbb{R}}(z, w) = \mu(z) - \mu(w), \qquad \text{and} \qquad \mu_{\mathbb{C}}(z, w)(\hat{v}_z),$$

where $w \in T_z^*\mathbb{C}^n$, $v \in \mathfrak{g}_{\mathbb{C}}$, and $\hat{v}_z$ is the vector field in $T_z\mathbb{C}^n$ induced by v. For a central element $\alpha \in \mathfrak{g}^*$, we call the specialised hyperkähler quotient

$$\mathfrak{M} = T^*\mathbb{C}^n \mathbin{/\!/\!/\!/}_{(\alpha,0)} G := \big(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\big)/G$$

the hyperkähler analogue of the corresponding Kähler quotient,

$$\mathfrak{X} = \mathbb{C}^n \mathbin{/\!/}_{\alpha} G = \mu^{-1}(\alpha)/G.$$

We quote the following propositions without proof:

Proposition 17. *Suppose that α and $(\alpha, 0)$ are regular values for μ and μ_{HK}, respectively. Then the cotangent bundle $T^*\mathfrak{X}$ is isomorphic to an open subset of $\mathfrak{M}$, and is dense if it is non-empty.*

3.1.2 The $\mathbb{C}^*$-Action and the Core of a Hyperkähler Analogue

Consider the action of $\mathbb{C}^*$ on $T^*\mathbb{C}^n$ given by

$$\hbar \cdot (z, w) = (z, \hbar w),$$

i.e. by scalar multiplication of the cotangent fibre. The holomorphic moment map $\mu_{\mathbb{C}} : T^*\mathbb{C}^n \to \mathfrak{g}_{\mathbb{C}}^*$ is $\mathbb{C}^*$-equivariant with respect to the scalar action on $\mathfrak{g}_{\mathbb{C}}^*$, and hence the $\mathbb{C}^*$-action descends to $\mu_{\mathbb{C}}^{-1}(0)$. Further, this $\mathbb{C}^*$-action commutes with the linear action of G on $\mathbb{C}^n$, and consequently the action of $\mathbb{C}^*$ is J_1-holomorphic on $\mathfrak{M} = \big(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\big)/G$. However, the $\mathbb{C}^*$-action *does not* preserve the holomorphic symplectic form nor the hyperkähler structure on $\mathfrak{M}$; rather it scales $\mu_{\mathbb{C}}$ with "homogeneity one", *i.e.* $\hbar^*\omega_{\mathbb{C}} = \hbar\omega_{\mathbb{C}}$ for any $\hbar \in \mathbb{C}^*$.

Given that $\mathfrak{M}$ is smooth, the action of the compact subgroup $S^1 \subset \mathbb{C}^*$ is hamiltonian with respect to the real symplectic two-form $\omega_{\mathbb{R}}$, with corresponding moment map $\Phi[z, w] = \frac{1}{2}\|w\|^2$. This map is a perfect Morse-Bott function, and its image is contained in $\mathbb{R}_{\geq 0}$. Further, we note that $\Phi^{-1}(0) = \mathfrak{X} \subset \mathfrak{M}$. The following proposition will be instrumental in the sequel, though again we quote it without proof:

Proposition 18. *If the original moment map for the G-action on $\mathbb{C}^n$, $\mu : \mathbb{C}^n \to \mathfrak{g}^*$, if proper, then so is the moment map for the S^1 action, $\Phi : \mathfrak{M} \to \mathbb{R}_{\geq 0}$.*

Next we shall define what is known as the *core* of a hyperkähler analogue, which will be essential in describing the fixed points of the $\mathbb{C}^*$-action of $\mathfrak{M}$.

Definition 19. Suppose that $\mathfrak{M}$ is smooth and Φ is proper. The *core* $\mathcal{L} \subset \mathfrak{M}$ of the hypertoric variety is defined to be the union of the $\mathbb{C}^*$ orbits whose closures are compact.

Let F be a connected component of $\mathfrak{M}^{S^1} = \mathfrak{M}^{\mathbb{C}^*}$, and let U_F be the closure of the set of points $p \in \mathfrak{M}$ such that $\lim\limits_{\hbar \to \infty} \hbar \cdot p \in F$.

Proposition 20 ([?]; Proposition 2.8)**.** *The core $\mathcal{L} \subset \mathfrak{M}$ has the following properties:*

1. *$\mathcal{L}$ is an S^1-equivariant deformation retract of M;*
2. *U_F is isotropic with respect to the holomorphic symplectic form $\omega_{\mathbb{C}}$;*
3. *Provided that $\mathfrak{M}$ is smooth at F, then* $\dim U_F = \frac{1}{2} \dim \mathfrak{M}$.

3.2 Hypertoric Manifolds

3.2.1 Definition

In this section, we shall specialise further now to when a hyperkähler analogue $\mathfrak{M}$ is the analogue to a toric symplectic manifold $\mathfrak{X} = \mu^{-1}(\alpha)/N$, *i.e.* we replace the compact Lie group G with the torus $N = \ker(\pi : T^n \to T^d)$, using the same notation as in the second chapter.

Recall the short exact sequence of tori:

$$1 \longrightarrow N \xhookrightarrow{i} T^n \xrightarrow{\pi} T^d \longrightarrow 1,$$

and extend the linear action of the torus N on $\mathbb{C}^n$ to $T^*\mathbb{C}^n$. This action is trihamiltonian and we obtain the following hyperkähler moment map

$$\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : T^*\mathbb{C}^n \longrightarrow \mathfrak{n}^* \oplus \mathfrak{n}^*_{\mathbb{C}},$$

where

$$\mu_{\mathbb{R}}(z, w) = i^* \left(\frac{1}{2} \sum_{i=1}^{n} (|z_i|^2 - |w_i|^2) \partial_i \right), \quad \text{and} \quad \mu_{\mathbb{C}}(z, w) = i^*_{\mathbb{C}} \left(\sum_{i=1}^{n} (z_i w_i) \partial_i \right).$$

Given an element $\alpha \in \mathfrak{n}^*$ with a corresponding lift $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{R}^n)^*$, the Kähler quotient

$$\mathfrak{X} = \mathbb{C}^n /\!\!/_{\alpha} N = \mu^{-1}(\alpha)/N$$

is our usual toric symplectic manifold with residual T^d-action from before, and moreover its hyperkähler analogue

$$\mathfrak{M} = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/_{(\alpha,0)} N = \big(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\big)/N$$

is what we shall call a *hypertoric manifold*[1]. The hypertoric manifold $\mathfrak{M}$ also admits a residual action of the torus T^d, which is hyperhamiltonian with hyperkähler moment map

$$\phi_{HK} := \phi_{\mathbb{R}} \oplus \phi_{\mathbb{C}} : \mathfrak{M} \longrightarrow (\mathbb{R}^d)^* \oplus (\mathbb{C}^d)^*,$$

where

$$\begin{aligned} \phi_{\mathbb{R}}[z,w] &= \frac{1}{2}\sum_{i=1}^{n}(|z_i|^2 - |w_i|^2 - \lambda_i)\partial_i \in \ker(i^*) = (\mathbb{R}^d)^*, \\ \phi_{\mathbb{C}}[z,w] &= \sum_{i=1}^{n}(z_i w_i)\partial_i \in \ker(i_{\mathbb{C}}^*) = (\mathbb{C}^d)^*. \end{aligned}$$

3.2.2 Hyperplane Arrangements

A fundamental difference between the toric manifold $\mathfrak{X}$ and the hypertoric manifold $\mathfrak{M}$ is that the hyperkähler moment map for $\mathfrak{M}$ is surjective, and that $\mathfrak{M}$ is non-compact. Despite this, we can still describe the image of the real moment map $\phi_{\mathbb{R}} : \mathfrak{M} \to (\mathbb{R}^d)^*$ combinatorially by means of a *hyperplane arrangement*. To describe this arrangement, recall that the map $\pi : \mathbb{R}^n \to \mathbb{R}^d$ was defined by $\pi(e_i) = u_i$, for $i = 1, \ldots, n$, where the u_i were the primitive, integral, inward-pointing normal vectors to the hyperplanes that determined our Delzant polytope. In the hypertoric case, they instead now describe a collection of *affine hyperplanes* $H_i \subset (\mathbb{R}^d)^*$ as follows: consider

$$H_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i = 0\},$$

[1]More generally, $\mathfrak{M}$ should be called a hypertoric variety, and only call $\mathfrak{M}$ a manifold when it is smooth. However, we shall restrict our attention to the smooth case for simplicity.

so that the $u_i \in \mathbb{Z}^d$ is the normal vector to the hyperplane H_i. The hyperplane H_i divides $(\mathbb{R}^d)^*$ into two half-spaces

$$\begin{aligned}F_i &= \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \geq 0\},\\ G_i &= \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \leq 0\}.\end{aligned}$$

Let

$$\Delta = \bigcap_{i=1}^{n} F_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \geq 0, \text{ for all } i = 1, \ldots, n\}$$

be the (possibly empty) polyhedron in $(\mathbb{R}^d)^*$ defined by the affine hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$. We note that choosing a different lift λ' of α corresponds combinatorially to translating the arrangement $\mathcal{A}$ inside of $(\mathbb{R}^d)^*$, and geometrically to shifting the Kähler and hyperkähler moment maps for the residual T^d-action by $\lambda' - \lambda \in \ker(i^*) = (\mathbb{R}^d)^*$.

We shall call that the arrangement $\mathcal{A}$ *simple* if every subset of m hyperplanes with non-empty intersection intersects with codimension m, and call $\mathcal{A}$ *smooth* if every collection of d linearly-independent vector $\{u_{i_1}, \ldots, u_{i_d}\}$ spans $(\mathbb{R}^d)^*$. The reason for this terminology is the following proposition.

Proposition 21. *The hypertoric variety $\mathfrak{M}$ is an orbifold if and only if $\mathcal{A}$ is simple, and $\mathfrak{M}$ is smooth if and only if $\mathcal{A}$ is smooth.*

As we wish to restrict our attention to the case where $\mathfrak{M}$ is a manifold, we shall assume in the sequel that $\mathcal{A}$ is a smooth arrangement of hyperplanes.

3.2.3 The Core of a Hypertoric Manifold

The holomorphic moment map $\phi_{\mathbb{C}} : \mathfrak{M} \to (\mathbb{C}^d)^*$ is $\mathbb{C}^*$-equivariant with respect to the scalar action of $\mathbb{C}^*$ on $(\mathbb{C}^d)^*$, hence both the core $\mathcal{L}$ and the fixed-point set $M^{\mathbb{C}^*}$ will be contained in

$$\mathcal{E} := \phi_{\mathbb{C}}^{-1}(0) = \Big\{[z, w] \in \mathfrak{M} : z_i w_i = 0,\ 1 \leq i \leq n\Big\}.$$

Definition 22. We shall call $\mathcal{E}$ the *extended core* of $\mathfrak{M}$.

The restriction of $\phi_{\mathbb{R}}|_{\mathcal{E}} : \mathcal{E} \to (\mathbb{R}^d)^*$ is surjective from the defining equations, and further the extended core naturally breaks up into components

$$\mathcal{E}_A := \Big\{ [z,w] \in \mathcal{E} : w_i = 0 \text{ for all } i \in A \text{ and } z_i = 0 \text{ for all } i \notin A \Big\},$$

where $A \subseteq \{1, \ldots, n\}$ is an indexing set. The hyperplanes $\{H_i\}_{i=1}^n$ divide $(\mathbb{R}^d)^*$ into a union of convex polyhedra

$$\Delta_A = \Big(\bigcap_{i \in A} F_i \Big) \cap \Big(\bigcap_{i \notin A} G_i. \Big),$$

some of which may be empty.

Lemma 23. *If $w_i = 0$ then $\mathrm{Im}(\phi_{\mathbb{R}}) \subseteq F_i$, and if $z_i = 0$ then $\mathrm{Im}(\phi_{\mathbb{R}}) \subseteq G_i$.*

Proof. Let $y \in (\mathbb{R}^d)^*$ be the image of the moment map $\phi_{\mathbb{R}}$ for a point $[z,w] \in \mathcal{E}$, then

$$y \cdot u_i + r_i = \mu_{\mathbb{R}}(z,w) \cdot e_i = \frac{1}{2}\Big(|z_i|^2 - |w_i|^2 \Big),$$

and hence $y \geq 0$ if $i \in A$, and $y \leq 0$ if $i \notin A$. □

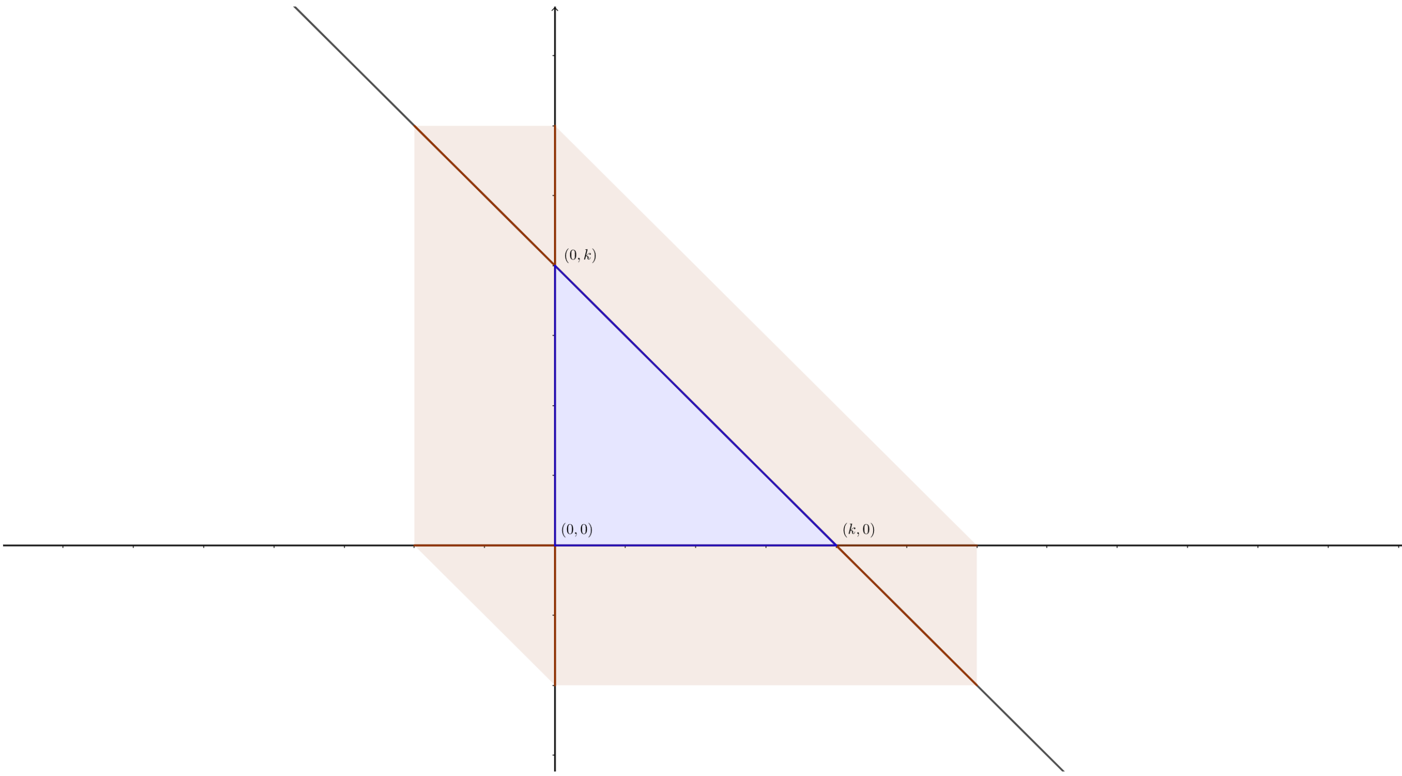

Figure 3.1: The image of $\phi_{\mathbb{R}}(M)$ for $\mathfrak{M} \cong T^*\mathbb{CP}^2$.

Lemma 24. *The component $\mathcal{E}_A$ of the extended core is isomorphic to the toric variety corresponding to the polytope Δ_A.*

The $\mathbb{C}^*$-action does not act as a sub-torus of $T^d_{\mathbb{C}}$ on M globally, but when restricted to each component $\mathcal{E}_A$ of the extended core it does in a combinatorial fashion that we shall now describe.

Consider a component $\mathcal{E}_A \subset \mathcal{E}$, then for some $[z,w] \in \mathcal{E}_A$ and $\hbar \in \mathbb{C}^*$:

$$\hbar \cdot [z,w] = [z, \hbar w] = [\hbar_1 z_1, \ldots, \hbar_n z_n, w_1, \ldots, w_n], \qquad \text{where } \hbar_i = \begin{cases} \hbar^{-1} & \text{if } i \in A, \\ 1 & \text{if } i \notin A. \end{cases}$$

This shows that the restriction of the $\mathbb{C}^*$-action to the component $\mathcal{E}_A$ is the one-dimensional sub-torus $(\mathbb{C}^*)_A := (\hbar_1, \ldots, \hbar_n)$ of the original torus $T^n_{\mathbb{C}}$, which we can then restrict to the circle subgroup $S_A \subset T^n$.

For $B \subseteq \{1, \ldots, n\}$, denote by $\mathcal{E}^B_A$ the subvariety of the extended core component $\mathcal{E}_A$ defined by

$$\mathcal{E}^B_A := \mathcal{E}_A \cap \{z_j = 0 = w_j\}$$

which may be empty. In $(\mathbb{R}^d)^*$, this corresponds to the intersection of the polyhedron Δ_A with the collection of hyperplanes $\{H_j\}_{j \in B}$, that is

$$\Delta^B_A = \left(\bigcap_{j \in B} H_j \right) \cap \Delta_A.$$

Proposition 25. *The fixed point set of the S^1-action on $\mathcal{E}_A$ is the union of those toric subvarieties $\mathcal{E}^B_A$ such that $\sum_{i \in A} u_i$ lies in $Span_{j \in B} u_j$.*

Corollary 26. *Every vertex $v \in (\mathbb{R}^d)^*$ of the polyhedral complex given by our arrangement is the image of an S^1-fixed point in $\mathfrak{M}$. Every component of $\mathfrak{M}^{S^1}$ maps to a face of the polyhedral complex.*

3.3 Compactifying the Hypertoric Variety via Symplectic Cutting

In the sequel, we shall want to try and associate to $\mathfrak{M}$ a "canonical Hilbert space" $\mathcal{Q}(\mathfrak{M})$, which will consist of a subspace of the space of global holomorphic sections to some "prequantum" line bundle, by means of the geometric quantisation construction which we shall discuss later. As $\mathfrak{M}$ is non-compact, the space $\mathcal{Q}(\mathfrak{M})$ will be infinite-dimensional. However, because of the $\mathbb{C}^*$-action, we can instead consider the weight spaces in $\mathcal{Q}(\mathfrak{M})$ of the $S^1 \subset \mathbb{C}^*$-action, which will have a compact fixed point loci. Hence the individual weight spaces will be finite dimensional, and a lot more tractable to work with. For this consideration, we can compactify $\mathfrak{M}$ by Lerman's symplectic cutting technique [?, ?] since the moment map for the S^1-action, $\Phi : \mathfrak{M} \to \mathbb{R}_{\geq 0}$ is proper. The contents of this section is original work, though a similar construction is done in [10] to compactify Higgs bundle moduli[2].

3.3.1 General Set-Up

We will use the S^1-action to symplectically cut the toric hyperkähler manifold $\mathfrak{M}$ in order to compactify it as follows: consider the product $\mathfrak{M} \times \mathbb{C}$, where now S^1 acts on $\mathfrak{M} \times \mathbb{C}$ as

$$e^{i\theta} \cdot \big([z, w], \xi\big) = \big([z, e^{i\theta}], e^{i\theta}\xi\big),$$

which is hamiltonian with moment map

$$\begin{aligned}\mu_{\text{cut}} &: \mathfrak{M} \times \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0},\\ \mu_{\text{cut}}\big([z, w], \xi\big) &= \Phi[z, w] + \tfrac{1}{2}|\xi|^2 - \epsilon,\end{aligned}$$

for some $\epsilon \in \mathbb{R}_{\geq 0}$. Then we have

$$\begin{aligned}\mu_{\text{cut}}^{-1}(0) &= \big\{([z, w], \xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = 2\epsilon\big\}\\ &= \big\{[z, w] \in M : \|w\|^2 = 2\epsilon\big\} \bigsqcup \big\{([z, w], \xi) \in M \times \mathbb{C} : |\xi| = \pm\sqrt{2\epsilon - \|w\|^2}\big\}\\ &= \big\{[z, w] \in M : \|w\|^2 = 2\epsilon\big\} \bigsqcup \big\{([z, w], \xi) \in M \times \mathbb{C} : \xi = e^{i\arg(\xi)}\sqrt{2\epsilon - \|w\|^2}\big\}\\ &= \Phi^{-1}(\epsilon) \bigsqcup (\mathfrak{M} \times S^1)\\ &=: \Sigma_1 \bigsqcup \Sigma_2,\end{aligned}$$

where Σ_1 is just the level-set of Φ at the level ϵ in $\mathfrak{M}$, and $\Sigma_2 = \mathfrak{M} \times S^1$ is exhibited as a trivial S^1-bundle over Σ_2, using the globally defined section

$$\mathfrak{M} \to \mathfrak{M} \times S^1, \qquad [z, w] \longmapsto \big([z, w], e^{i\theta}\sqrt{2\epsilon - \|w\|^2}\big), \qquad e^{i\theta} \in S^1.$$

[2]In fact, in section 4 of [10], it is stated that it would be an interesting task to apply the same idea to the toric hyperkähler varieties of [7].

Finally, taking the quotient of $\mu_{\text{cut}}^{-1}(0)$ by the S^1-action, we obtain the symplectic cut

$$M_{\leq\epsilon} := \mu_{\text{cut}}^{-1}(0)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where $\Sigma_1/S^1 = \Phi^{-1}(\epsilon)/S^1$ is just the symplectic reduction, and where Σ_2/S^1 is diffeomorphic to M for $\|w\|^2 < 2\epsilon$, which we denote by $\mathfrak{M}_{<\epsilon}$.

3.3.2 Restriction to an Extended Core Component, $\mathcal{E}_A$

Since the residual circle S^1-action acts as a subgroup of the original torus T^n when restricted to each component $\mathcal{E}_A$ of the extended core $\mathcal{E}$, we can described combinatorially the resulting configuration of the hyperplane arrangement in $(\mathbb{R}^d)^*$ from taking the cut. For each component, let $j_A : \mathfrak{s}^1 \to \mathbb{R}^n$ be the derivative of the inclusion of S^1 into T^n on the Lie algebra level, that is

$$j_A(\xi) = (\xi_1, \dots, \xi_n), \qquad \text{where } \xi_i = \begin{cases} -1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A, \end{cases}$$

so that its image in $\mathbb{R}^n$ generates a circle subgroup S^1 in T^n that depends on each component $\mathcal{E}_A$. Then the moment map for this restriction for the S^1-action is

$$\Phi[z,w] = j_A^* \circ \mu_{\mathbb{R}}[z,w] = \left\langle \mu_{\mathbb{R}}(z,w), \sum_{i\in A} \xi_i u_i \right\rangle,$$

and so from our above discussion of how we constructed the symplectic cut, the image in $(\mathbb{R}^d)^*$ of the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ is

$$\phi_{\mathbb{R}}(\Phi^{-1}(\epsilon)) = \left\{ y \in \Delta_A : \left\langle y, \sum_{i\in A} \xi_i u_i \right\rangle + \epsilon = 0 \right\} =: H_A$$

which introduces an inward-pointing half-space

$$F_A := \left\{ y \in \Delta_A : \left\langle y, \sum_{i\in A} \hbar_i u_i \right\rangle + \epsilon \geq 0 \right\}$$

such that the image of the extended core component $\mathcal{E}_A$ after being compactified is the original convex polytope Δ_A intersected with H_A. One can also see clearly that the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ has the restricted S^1-action as its stabiliser subgroup since, by definition of H_A, the moment map $\Phi|_{\mathcal{E}_A}$ equals the hyperplane H_A, *i.e.* $\Phi|_{\mathcal{E}_A}$ is constant along $\Phi^{-1}(\epsilon)/S^1$.

Remark 27. If we had used instead the following action for S^1

$$e^{i\theta} \cdot \big([z, w], \xi\big) = \big([z, e^{i\theta} w], e^{-i\theta}\xi\big)$$

with respective moment map

$$\mu_{\text{cut}}([z, w], \xi) = \frac{1}{2}\|w\|^2 - \frac{1}{2}|\xi|^2 - \epsilon,$$

and taken the cut, then the resulting then we would obtain the other "discarded half" $\mathfrak{M}_{>\epsilon}$ of the hypertoric manifold $\mathfrak{M}$ along with the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ with the opposite orientation:

$$\mathfrak{M}_{\geq\epsilon} = \mathfrak{M}_{>\epsilon} \bigsqcup \Big(-(\Phi^{-1}(\epsilon)/S^1)\Big).$$

The component M_ϵ is non-compact however, so we focus on $M_{<\epsilon}$.

The following two figures show the resulting moment polytope after compactification, for the hypertoric varieties $T^*\mathbb{CP}^2$ and $T^*\mathbb{CP}^3$. For an interactive figure for the latter, go to: https://www.geogebra.org/classic/suqdctbh.

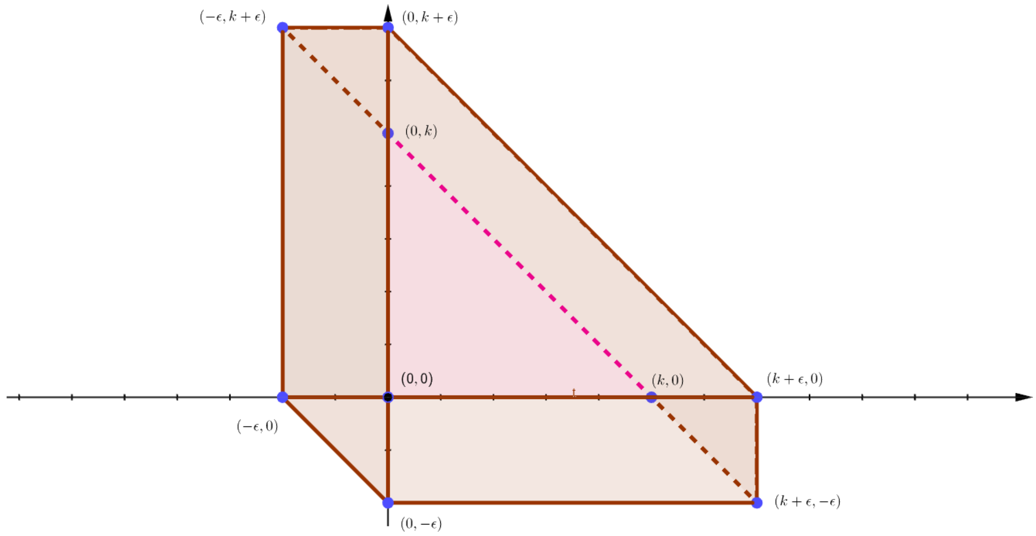

Figure 3.2: The image of $\phi_{\mathbb{R}}$ for $T^*\mathbb{CP}^2$ after cutting.

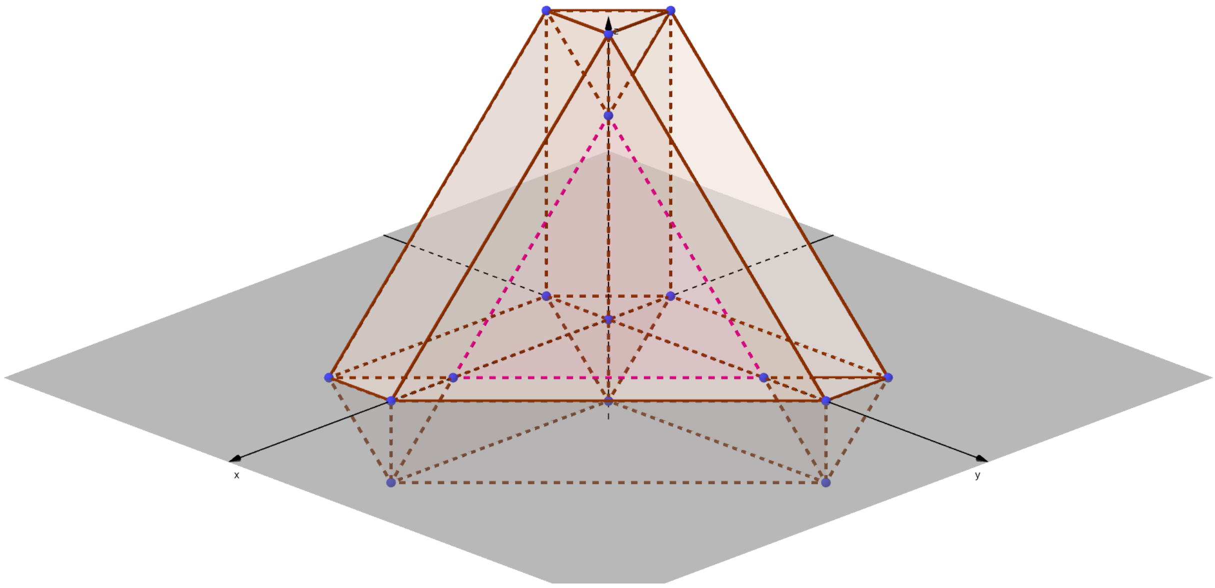

Figure 3.3: The image of $\phi_{\mathbb{R}}$ for $T^*\mathbb{CP}^3$ after cutting.

Chapter 4

Geometric Quantisation and Lattice Point Counting

Essentially all of the content from this chapter comes from [11], though [2] provides a more colloquial and digestible angle to the geometric quantisation construction with a larger focus on toric symplectic manifolds, and we shall also refer to it too regularly.

4.1 Pre-Quantisation, Polarisation, and Quantisation

Quantum mechanics associates to some symplectic manifold (M, ω) a quantum model $\mathcal{Q}(M, \omega)$, which is a Hilbert space. The space $\mathcal{Q}(M, \omega)$ is the quantum mechanical analogue of the classical phase space (M, ω), and is searched for as a vector space of smooth functions $C^\infty(M, \mathbb{R})$, which are to be thought of as the wave-functions for the corresponding quantum system to the classical phase space. The construction of geometric quantisation goes along of the lines of the following; given a symplectic manifold (M, ω) we associate to it a Hermitian line bundle $\mathcal{L} \to M$ with a connection ∇ on $\mathcal{L}$ whose curvature form is ω. If such a line bundle exists, then we say that (M, ω) is *pre-quantisable*. Pre-quantisability of (M, ω) is determined by the following theorem.

Theorem 28. *(M, ω) is pre-quantisable if and only if $[\omega]$ is in the image of the map*

$$\imath_* : H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R}).$$

In the case when (M, ω) is a toric symplectic manifold, *i.e.* it has an associated Delzant polytope, then we can state an equivalent necessary and sufficient condition for (M, ω) to be pre-quantisable.

Theorem 29 ([2]). *If Δ is a Delzant polytope, M_Δ is pre-quantisable if and only if the vertices of Δ are integer lattice points.*

Let us suppose from now that (M, ω) is pre-quantisable, then we can consider what sort of space of sections of the pre-quantum line bundle $\mathcal{L} \to M$ we should take as our Hilbert space $\mathcal{Q}(M, \omega)$. The first thing to note is that we cannot take the space of *all* sections of $\mathcal{L}$, since it is "too big", due to the works of Groenewold [12] and van Hove [13]. To remedy this we add a further structure called a *polarisation*, which informally means that we only consider sections of $\mathcal{L}$ that are "independent of half the variables".

Remark 30. In Hamiltonian mechanics, the state space is the cotangent bundle T^*X of the configuration space X. In quantum mechanics, the state space is the Hilbert space $L^2(X, \mathbb{C})$, *i.e.* the Hilbert space of complex-valued wave-functions. The correspondence

$$T^*X \longleftrightarrow L^2(X, \mathbb{C}),$$

which in physics would correspond to varying the size of Planck's constant $\hbar$, is studied in geometric quantisation within the context of symplectic geometry, and to define analogues of $L^2(X, \mathbb{C})$ for symplectic manifolds X other than cotangent bundles.

If we further assume now that (M, ω) is Kähler so that it has a complex structure that is compatible with the symplectic form, then a natural choice of polarisation is that of a *complex polarisation.* In local coordinates $z_1, \ldots, z_n$, a function f is polarised with respect to the complex polarisation if

$$\frac{\partial f}{\partial \bar{z}_1} = \ldots = \frac{\partial f}{\partial \bar{z}_n} = 0,$$

which is to say that f is a holomorphic function. It can then be shown that:

Theorem 31 ([?]). *Let (M, ω) be a complex manifold and ω be a closed $(1,1)$-form, and fix a pre-quantisation line bundle $(\mathcal{L}, \langle , \rangle, \nabla)$ for (M, ω). Equip (M, ω) with a complex polarisation. Then $\mathcal{L}$ is a holomorphic line bundle, and its polarised sections are holomorphic sections. Then the (virtual) vector space*

$$\mathcal{Q}(M, \omega) := \sum (-1)^i H^i(M, \mathcal{O}_{\mathcal{L}})$$

is a Hilbert space that we can use as our quantisation $\mathcal{Q}(M,\omega)$*, where* $\mathcal{O}_{\mathcal{L}}$ *is the* sheaf *of holomorphic sections.*

Remark 32. When ω is sufficiently positive, the higher cohomology groups $H^{>0}(M,\mathcal{O}_{\mathcal{L}})$ vanish by Kodaira's theorem, and we obtain $\mathcal{Q}(M) = H^0(M,\mathcal{O}_{\mathcal{L}})$, *i.e.* the space of global holomorphic sections of $\mathcal{L}$, which would seem a more natural choice at first for $\mathcal{Q}(M)$. The reason for the virtual representation definition of $\mathcal{Q}(M)$ is that if the curvature form ω is sufficiently negative and if M is compact, then there are no non-zero holomorphic sections by Kodaira vanishing [**?**].

The sheaf cohomology $H^k(M,\mathcal{O}_{\mathcal{L}})$ is equal to the cohomology $H^{0,k}(M,\mathcal{L})$ of the *twisted Dolbeault complex*

$$\ldots \longrightarrow \Omega^{0,k}(M,\mathcal{L}) \xrightarrow{\bar\partial} \Omega^{0,k+1}(M,\mathcal{L}) \longrightarrow \ldots,$$

so we have equivalently

$$\mathcal{Q}(M,\omega) = \sum (-1)^k H^{0,k}(M,\mathcal{L}) \tag{4.1}$$

Remark 33. It is shown in [11] that in the case where (M,ω) is Kähler, *i.e.* if we strengthen the hypothesis of the above theorem to include also an almost complex structure compatible with ω, then the higher cohomology groups $H^{0,>0}(M,\mathcal{L})$ vanish and we arrive at

$$\mathcal{Q}(M,\omega) = H^{0,0}(M,\mathcal{L}) = H^0(M,\mathcal{O}_{\mathcal{L}}).$$

The definition (4.1) is still not completely satisfactory when M is non-compact, since the quantisation should be a (virtual) Hilbert space and not just a (virtual) vector space. When M is non-compact we replace equation (4.1) with the alternating sum of L^2-cohomology groups:

$$\mathcal{Q}(M,\omega) = \sum (-1)^k H^{0,k}_{L^2}(M,\mathcal{L}). \tag{4.2}$$

We will not go into too much detail into how the L^2-cohomology groups $H^{0,*}_{L^2}(M,\mathcal{L})$ are defined, though we shall remark that they require the additional structure of a measure μ on M. When M is symplectic, we can just take μ to be the Liouville measure, *i.e.* $\mu = (\sqrt{-1})^n dz d\bar z$.

4.2 Quantisation Commutes with Reduction

Let (M, ω) be a symplectic manifold with a Hamiltonian action of some Lie group G, with moment map $\Phi : M \to \mathfrak{g}^*$. Then, under suitable conditions, the statement "quantisation commutes with reduction" roughly means that the processes of geometric quantisation and symplectic reduction can be interchanged without changing the final result. This result was proven first by Guillemin and Sternberg (insert ref) in the case that (M, ω) was a compact Kähler manifold, G was a compact Lie group, and M was equipped with the complex polarisation. To be precise:

Theorem 34 (G-S)**.** *Let (M, ω) be a compact Kähler manifold possessing a compact Lie group of symmetries, G. Let $\mathcal{Q}(M)_G$ be the set of fixed vectors of G in $\mathcal{Q}(M)$. Then*

$$\mathcal{Q}(M)_G = \mathcal{Q}(M_G),$$

M_G being the set of fixed points of G in M.

More generally, if M is not compact but is equipped with a *proper* moment map $\Phi : M \to \mathfrak{g}^*$ for a Hamiltonian G-action, then:

Theorem 35. *Let (M, ω, Φ) be a symplectic, Hamiltonian G-manifold with* proper *moment map $\Phi : M \to \mathfrak{g}^*$. Suppose that $\alpha \in \mathfrak{g}^*$ is a regular value for Φ and that G acts freely on $\Phi^{-1}(\alpha)$, so that the Marsden-Weinstein quotient $M_\alpha := \Phi^{-1}(\alpha)/G$ is well-defined and compact. Then*

$$\mathcal{Q}(M_\alpha) \cong \mathcal{Q}(M)^\alpha,$$

where $\mathcal{Q}(M)^\alpha$ denotes the subspace of vectors in $\mathcal{Q}(M)$ which transform with weight α.

In particular, the Guillemin-Sternberg conjecture can be interpreted as identifying $\mathcal{Q}(M)_G = \mathcal{Q}(M)^0$ being the subspace of vectors which transform under G with weight 0 (*i.e.* are fixed), and that $M_G = \Phi^{-1}(0)/G$ as the Marsden-Weinstein quotient.

When M_Δ is a toric symplectic manifold with corresponding Delzant polytope $\Delta \subset (\mathbb{R}^d)^*$, we have the particularly nice geometric viewpoint of the $[Q, R] = 0$ conjecture.

Proposition 36 ([14])**.** *Let M_Δ be a toric manifold with moment polytope $\Delta \subset \mathbb{R}^d$. Then the dimension of the quantisation space $\mathcal{Q}(M_\Delta)$ is equal to the number of integer lattice points in Δ,*

$$\dim \mathcal{Q}(M_\Delta) = \#(\Delta \cap \mathbb{Z}^d).$$

Example 37. This example also appears in [14]. Let $M_\Delta = \mathbb{CP}^n$ be obtained via the Delzant construction as before with $N \cong S^1$ acting on $\mathbb{C}^{n+1}$ diagonally, through the inclusion into T^{n+1}. For $\mathbb{CP}^n$ to be pre-quantisable, we require that the Marsden-Weinstein quotient $\mathbb{CP}^n = \mu^{-1}(k)/N$ to be reduced at an integral point $k \in (\mathbb{Z}^{n+1})^*$. The residual torus action $T^n \cong T^{n+1}/N$ has a moment map $\bar{\mu} : \mathbb{CP}^n \to (\mathbb{R}^n)^*$, whose image $k\Delta \subset (\mathbb{R}^n)^*$ is the standard n-dimensional simplex Δ, dilated by a factor of k.

Since $\mathbb{C}^{n+1}$ is simply-connected, the pre-quantum line bundle $\mathcal{L}$ is trivial, and in particular we can identify its global holomorphic sections with the space of homogeneous polynomials in $n+1$ variables. Suppose that $N_\mathbb{C}$ acts on $\mathcal{L}$ with weight k. Then if $s(z) = z_0^{\lambda_0} \dots z_n^{\lambda_n}$ is a trivialising section for $\mathcal{L} = \mathbb{C}^{n+1} \times \mathbb{C}_k$, then

$$s(t \cdot z) = t^{\lambda_0 + \ldots + \lambda_n} s(z),$$

whereas

$$t \cdot s(z) = t^k s(z).$$

Whence, for $s : \mathbb{C}^{n+1} \to \mathbb{C}_k$ to be an $N_\mathbb{C}$-equivariant and holomorphic section for $\mathcal{L}$, we must have

$$\lambda_0 + \ldots + \lambda_n = k,$$

whose solution set

$$\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} \lambda_i = k\}$$

is in a one-to-one correspondence with the set

$$\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 + \ldots + \lambda_n \leq k\} = (\mathbb{Z}^n \cap \Delta).$$

Whence, the subspace $\mathcal{Q}(\mathbb{C}^{n+1})_k$ of vectors in $\mathcal{Q}(\mathbb{C}^{n+1})$ that have multiplicity k can be identified with the space $H^0(\mathbb{CP}^n, \mathcal{O}(k))$, *i.e.* the space of homogenous polynomials of degree k in $n+1$ variables. It is well-known that

$$\dim H^0(\mathbb{CP}^n, \mathcal{O}(k)) = \binom{n+k}{n} = \frac{(k+1)(k+2), \ldots (k+n)}{n!},$$

and that this number coincides with $\#(\mathbb{Z}^n \cap \Delta)$ by theorem (36).

4.3 Counting Integer Points in Delzant Polytopes

Let us work out by hand some ways that we can determine $\#(\mathbb{Z}^n \cap \Delta)$, where Δ is the moment polytope for $\mathbb{CP}^n$ from the previous section, for small values of n.

Example 38. Let $n = 2$, then the Delzant polytope $k\Delta$ is the 2-dimensional simplex Δ dilated by a factor of k, *i.e.* a right-angled triangle with base and height lengths equal to k. It can be shown that

$$\#(k\Delta \cap \mathbb{Z}^2) = \sum_{m=0}^{k} m = \frac{(k+1)(k+2)}{2} = \frac{1}{2}k^2 + \frac{3}{2}k + 1.$$

Example 39. Let $n = 3$, so that now the Delzant polytope is the 3-dimensional simplex $k\Delta$ that again is a dilation of the standard simplex by a factor of k. In this case, we have

$$\#(k\Delta \cap \mathbb{Z}^3) = \frac{(k+1)(k+2)(k+3)}{3!} = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1. \tag{4.3}$$

We finish this section by remarking that the cubic polynomial in k in equation (4.3) coincides with what is known as the *Verlinde formula* [15], which is the dimension of the space of global holomorphic sections of the k^{th} tensor power of the determinant line bundle $\mathcal{L}$, over the moduli space of flat $SU(2)$ connections over a Riemann surface of genus 2, $\mathcal{M}_{\text{flat}}(\Sigma_2; SU(2))$ [16], *i.e.*

$$\dim H^0(\mathcal{M}_{\text{flat}}(\Sigma_2; SU(2)), \mathcal{L}^k) = \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1.$$

Chapter 5

Outlook

In this report we have discussed the construction of a toric symplectic manifold if given a Delzant polytope, as well as the opposite direction if one is provided with a toric symplectic manifold, by means of the AGS convexity theorem. This allowed us to then gently introduce the quaternionic analogue to a toric manifold, which are aptly named hyperkähler analogues and which are special cases of the more general toric hyperkähler varieties.

We then discussed the author's original work, which was a discussion into how one can compactify a hyperkähler analogue by the means of the residual $\mathbb{C}^*$-action on the cotangent fibres. The cases for $T^*\mathbb{CP}^2$ and $T^*\mathbb{CP}^3$ were mostly discussed: the first because its associated hyperplane arrangement is in $(\mathbb{R}^2)^*$, and thus easy to visualise; and the latter was considered with foresight to geometric quantisation and the (equivariant) Verlinde formula.

It should not be too much of a surprise however that the number of integral lattice points in the moment polytope $k\Delta \subset (\mathbb{R}^3)^*$ to $\mathbb{CP}^3 = \mu^{-1}(k)/S^1$ coincided with the Verlinde formula for $\mathcal{M}_{\text{flat}}(\Sigma_2; SU(2))$, since the two spaces can be identified [17], *i.e.*

$$\mathbb{CP}^3 \cong \mathcal{M}_{\text{flat}}(\Sigma_2; SU(2)).$$

Finally, we now ask what $\mathcal{Q}(\mathfrak{M})$ is, or rather what the dimension is for the weight spaces of the $\mathbb{C}^*$-action, for any general hypertoric variety/analogue to a Kähler $\mathfrak{X}$. The case for

$T^*\mathbb{CP}^3$ will serve as our toy model, since then

$$T^*\mathbb{CP}^3 \cong \mathcal{M}_{\text{Higgs}}(\Sigma_2; SU(2)),$$

where the right-hand side is now the moduli space of $SU(2)$-Higgs bundles over a genus 2 Riemann surface Σ_2. The reason that this will serve as our toy model is that recently the "*equivariant Verlinde formula*" has been defined for the Higgs bundle moduli [16], and the author hopes that the dimension of the $\mathbb{C}^*$-weight spaces should coincide with that of the equivariant Verlinde formula.

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