
GEOMETRIC QUANTISATION OF HYPERTORIC MANIFOLDS BY SYMPLECTIC CUTTING

GENERAL NOTES

ABSTRACT

Lorem ipsum.

1 Introduction

Lorem ipsum.

2 Background

2.1 Hypertoric Geometry

2.2 Localisation and Symplectic Cutting

3 Hypertoric Varieties

3.1 Compactification via Symplectic Cutting

We will use the S^1 -action to perform a symplectic cut of the toric hyperkähler manifold \mathfrak{M} to compactify it, which has the effect of bounding the $\|w\|^2$ -norm component of the real moment map $\bar{\mu}_{\mathbb{R}}$ by above, and discarding the rest that lies above this bound. Consider the product $\mathfrak{M} \times \mathbb{C}$, and let S^1 act on $\mathfrak{M} \times \mathbb{C}$ via the diagonal product action, i.e. S^1 acts on M by rotating the cotangent fibre coordinates, and on \mathbb{C} in the standard way:

$$e^{i\theta} \cdot ([z, w], \xi) = (e^{i\theta} \cdot [z, w], e^{i\theta} \xi) = ([z, e^{i\theta} w], e^{i\theta} \xi).$$

This action is Hamiltonian, and the corresponding moment map $\Phi : \mathfrak{M} \times \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$ for the S^1 -action is

$$\Phi([z, w], \xi) = \phi[z, w] + |\xi|^2 = \|w\|^2 + |\xi|^2.$$

Then we have

$$\begin{aligned} \Phi^{-1}(\epsilon) &= \{([z, w], \xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = \epsilon\} \\ &= \{[z, w] \in M : \|w\|^2 = \epsilon\} \bigsqcup \{([z, w], \xi) \in M \times \mathbb{C} : |\xi| = \pm \sqrt{\epsilon - \|w\|^2}\} \\ &= \{[z, w] \in M : \|w\|^2 = \epsilon\} \bigsqcup \{([z, w], \xi) \in M \times \mathbb{C} : \xi = e^{i \arg(\xi)} \sqrt{\epsilon - \|w\|^2}\} \\ &= \phi^{-1}(\epsilon) \bigsqcup (\mathfrak{M} \times S^1) \\ &=: \Sigma_1 \sqcup \Sigma_2, \end{aligned}$$

where we denote the level-set $\phi^{-1}(\epsilon) \subseteq \mathfrak{M}$ by Σ_1 , and $\Sigma_2 \cong \mathfrak{M} \times S^1$ is the trivial S^1 -bundle over Σ_2 given by the globally defined section

$$\mathfrak{M} \rightarrow \mathfrak{M} \times S^1, \quad [z, w] \mapsto ([z, w], e^{i\theta} \sqrt{\epsilon - \|w\|^2}), \quad e^{i\theta} \in S^1.$$

Finally, taking the symplectic reduction of $\Phi^{-1}(\epsilon)$ with respect to the S^1 -action, we obtain the *symplectic cut of \mathfrak{M} at level- ϵ* ,

$$M_{\leq \epsilon} := \Phi^{-1}(\epsilon)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where $\Sigma_1/S^1 \cong \phi^{-1}(\epsilon)/S^1$ is just the usual symplectic reduction, and where Σ_2/S^1 is diffeomorphic to \mathfrak{M} for $\|w\|^2 < \epsilon$, which we will denote by $\mathfrak{M}_{< \epsilon}$.

3.2 The Combinatorics of the Cut Space, $\mathfrak{M}_{\leq \epsilon}$

Since the residual circle S^1 -action acts as a subtorus S_A^1 of the residual torus T^d on each component \mathcal{E}_A of the extended core, the hyperplane arrangement determined in $(\mathfrak{t}^d)^*$ by the real moment map $\bar{\mu}_{\mathbb{R}}$ is compactified by dropping in half-spaces with an inwards-pointing normal vector, given by v_A when taking the cut.

Recall from the previous section that $j_A : S^1 \hookrightarrow T^n$ denoted the inclusion homomorphism of S^1 into the original torus T^n . If we let $j_{A,*} : \mathfrak{s}^1 \rightarrow \mathfrak{t}^n$ represent the differential of this inclusion, then

$$j_{A,*}(1) = \sum_{i \in A} e_i \in \mathfrak{t}^n,$$

and the generator $\exp(v_A)$ of the one-parameter subgroup S_A^1 in T^d is

$$\exp(v_A) = \exp(\pi_* \circ j_{A,*}(1)),$$

or to be more concise,

$$S_A^1 = \left\{ \exp \left(r \cdot \sum_{i \in A} u_i \right) \mid r \in \mathbb{R} \right\}.$$

Then the moment map for the restricted S^1 -action to \mathcal{E}_A is

$$\phi_A[z, w] := \phi|_{\mathcal{E}_A}[z, w] = (j_A^* \circ \mu_{\mathbb{R}})[z, w] = \left\langle \bar{\mu}_{\mathbb{R}}[z, w], \sum_{i \in A} u_i \right\rangle,$$

where $j_A^* : (\mathfrak{t}^n)^* \rightarrow \mathbb{R}^*$ is the transposed differential of the inclusion, $j_{A,*}$.

As the S_A^1 -action depends combinatorially on the component \mathcal{E}_A , the image of the real moment map in $(\mathfrak{t}^d)^*$ is compactified by inserting a half-space Z_A with inwards-pointing normal $v_A = \sum_{i \notin A} u_i$ determining the orientation, on each component Δ_A .

4 Hypertoric Subvarieties

4.1 Universal Modifications

Let $(M, \omega_{HK}, \Phi_{HK})$ be a tri-Hamiltonian K -manifold, where $\omega_{HK} = \omega_{\mathbb{R}} + \omega_{\mathbb{C}}$ and $\Phi_{HK} = \Phi_{\mathbb{R}} + \Phi_{\mathbb{C}}$. Define $j_{\mathbb{C}} : M \rightarrow M \times M \times T^*K_{\mathbb{C}}$ by $j(m) = (m, 1, \Phi_{\mathbb{C}}(m))$, and $j_{\mathbb{R}} : M \rightarrow M \times T^*K$ by $j_{\mathbb{R}}(m) = (m, 1, \Phi_{\mathbb{R}})$.

Lemma 4.1. *We have an isomorphism of holomorphic symplectic tri-Hamiltonian K -manifolds,*

$$j : M \xrightarrow{\sim} (M \times T^*K_{\mathbb{C}}) \mathbin{\mathbb{H}}_0 K. \quad (1)$$

Here, the right-hand side is the quotient with respect to the diagonal K -action, where K acts on the left on $T^*K_{\mathbb{C}}$. The K -action on $(M \times T^*K_{\mathbb{C}}) \mathbin{\mathbb{H}}_0 K$ is the one induced by the right action on $T^*K_{\mathbb{C}}$.

Proof. The map $m \mapsto (1, \Phi_{\mathbb{C}}(m))$ sends M to the Lagrangian $\mathfrak{k}_{\mathbb{C}}^* \subset T^*K_{\mathbb{C}}$, so $j_{\mathbb{C}}$ pulls back the complex-symplectic form $\omega_{\mathbb{C}} \times 0$ on $M \times \mathfrak{k}_{\mathbb{C}}^* \subset M \times T^*K_{\mathbb{C}}$ to ω_{HK} on M . Thus $j_{\mathbb{C}}$ is a holomorphic-symplectic embedding.

The complex moment map for the diagonal left K -action on $M \times T^*K_{\mathbb{C}}$ is $\Psi_{\mathbb{C}}^{\mathcal{L}}(m, k, \lambda_{\mathbb{C}}) = \Phi_{\mathbb{C}} - k \cdot \lambda_{\mathbb{C}}$, so $j_{\mathbb{C}}$ maps M into

$$\Phi_{\mathbb{C}}^{-1}(0) = \{ (m, k_{\mathbb{C}}, \lambda_{\mathbb{C}}) \in M \times T^*K_{\mathbb{C}} \mid \Phi_{\mathbb{C}}(m) = k_{\mathbb{C}} \cdot \lambda_{\mathbb{C}} \}.$$

The induced map $\bar{j}_{\mathbb{C}}$ is a **[NB: biholomorphism?]**, and moreover is a holomorphic symplectomorphism as $j_{\mathbb{C}}$ is a holomorphic symplectic embedding.

Moreover,

$$j_{\mathbb{C}}(k_{\mathbb{C}} \cdot m) = (k_{\mathbb{C}} \cdot m, e_{K_{\mathbb{C}}}, \Phi_{\mathbb{C}}(k_{\mathbb{C}} \cdot m)) = (k_{\mathbb{C}} \cdot m, e_{K_{\mathbb{C}}}, k_{\mathbb{C}} \cdot \Phi_{\mathbb{C}}(m)) = \mathcal{L}_{k_{\mathbb{C}}} \mathcal{R}_{k_{\mathbb{C}}} j_{\mathbb{C}}(m),$$

and

$$\Psi_{\mathbb{C}}^{\mathcal{R}}(j(m)) = \Psi_{\mathbb{C}}^{\mathcal{R}}(m, e_{K_{\mathbb{C}}}, \Phi_{\mathbb{C}}(m)) = \Phi_{\mathbb{C}}(m),$$

so \bar{j} is K -equivariant and intertwines the complex moment maps on M and the complex-symplectic quotient, $\Psi_{\mathbb{C}}^{-1}(0)/K = (M \times T^*K_{\mathbb{C}}) \mathbin{\mathbb{H}} K$. \square

References

- [1] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.