HYPERTORIC MANIFOLDS

GENERAL NOTES

ABSTRACT

Notes on toric hyperkähler manifolds.

1 Cotangent Spaces to Extended Core Components

Let $M_\lambda = \left(\mu_{\mathbb{R}}^{-1}(\lambda) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/K$ be a toric hyperkähler manifold. Define

$$\mathbb{C}_A := \left\{ (z_i, w_i) \in \mathbb{C}^{2n} \;\middle|\; w_i = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \right\} \cong \mathbb{C}^n \subset \mathbb{H}^n.$$

Lemma 1.1 ([1]). *Let M_λ be a toric hyperkähler manifold. If $\mathcal{E}_A$ is non-empty, then its holomorphic cotangent bundle $T^*\mathcal{E}_A$ is contained in M_λ as an open subset.*

Fix a subset $A \subset \{ 1, \dots, n \}$, and define

$$(x_i^{(A)}, y_i^{(A)}) := \begin{cases} (z_i, w_i), & \text{if } i \in A, \\ (w_i, -z_i), & \text{if } i \notin A. \end{cases}$$

Then $x^{(A)} = (x_1^{(A)}, \dots, x_n^{(A)})$ is a point in the vector space $\mathbb{C}_A^n$, and $y^{(A)} = (y_1^{(A)}, \dots, y_n^{(A)})$ is a point in the dual space $(\mathbb{C}_A^n)^*$. That is, we identify the cotangent bundle $T^*\mathbb{C}_A^n$ with $\mathbb{H}^n$ as above.

2 Symplectic Cutting

2.1 Compactifying the Extended Core

Let S^1 act on M by rotating the cotangent fibres, that is, for $\tau \in S^1$,

$$\tau \cdot [z; w] = [z; \tau w].$$

This S^1-action is Hamiltonian, with moment map

$$\Phi : M \longrightarrow (\mathbb{R})^*; \qquad [z : w] \longmapsto \frac{1}{2}\|w\|^2.$$

Let S_A^1 denote the residual S^1-action on M restricted to the extended core component

$$\mathcal{E}_A = \{ [z_1 : \dots z_n; w_1, \dots, w_n] \mid w_0 = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \}.$$

Now the *global* S^1-action does not act on the cotangent fibres of M as a subtorus of T^n, but it does when *restricted* to each component of the extended core, $\mathcal{E}_A$. Indeed,

$$\tau \cdot [z; w] = [z; \tau w] = [z_1 : \dots : z_n; \tau w_1 : \dots : \tau w_n] = [\tau_1 z_1 : \dots : \tau_n z_n; \tau_1^{-1} w_1 : \dots : \tau_n^{-1} w_n],$$

where

$$\tau_i := \begin{cases} \tau^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A, \end{cases}$$

which shows that the S^1-action restricted to each individual $\mathcal{E}_A$ acts as a subtorus of the original torus T^n.

Denote by S_A^1 the image of S^1 in T^n when considered as a subtorus restricted to each individual $\mathcal{E}_A$, and let $\jmath_A : S^1 \hookrightarrow T^n$ be the respective inclusion homomorphism, so we have $S_A^1 := \jmath_A(S^1) \lhd T^n$.

On the Lie algebra level, we have that

$$(\jmath_A)_* : \mathrm{Lie}(S^1_A) \longrightarrow \mathfrak{t}^n; \qquad \xi \longmapsto (\xi_1, \ldots, \xi_n),$$

where analogously [UNSURE]

$$\xi_i := \begin{cases} -1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A. \end{cases}$$

[TODO: S^1_A and $S^1_{A'}$-action (say) on adjacent Δ_A's coincide along the common hyperplane H_i; formalise this!]

Since S^1_A acts as the subtorus $\jmath_A(S^1)$ of T^n on each $\mathcal{E}_A$, the moment map $\Phi_A := \Phi\big|_{\mathcal{E}_A}$ for this action is given by composing $\mu_{\mathbb{R}}$ with the dual of the inclusion $(\jmath_A)_*$, so

$$\Phi_A[z,w] = (\jmath_A^* \circ \mu_{\mathbb{R}})\,[z;w] = \jmath_A^* \left(\frac{1}{2} \sum_{i=1}^{n} \left(|z_i|^2 - |w_i|^2 \right) e^i \right).$$

3 Moment Polytychs

3.1 Example: $T^*\mathbb{CP}^2$

$\mathcal{E}_{123}$:

$$\mathcal{E}_{123} \cong \left\{ \, [z_1 : z_2 : z_3 : \xi] \in M_a \;\middle|\; |z_1|^2 + |z_2|^2 + |z_3|^2 + |\xi|^2 = k + a, \; |\xi|^2 = a \, \right\},$$

$\mathcal{E}_{12}$:

$$\mathcal{E}_{12} \cong \left\{ \, [z_1 : z_2 : w_3 : \xi] \in M_a \;\middle|\; |z_1|^2 + |z_2|^2 + |\xi|^2 = k + a, \; |w_3|^2 + |\xi|^2 = a \, \right\},$$

$\mathcal{E}_{1}$:

$$\mathcal{E}_{1} \cong \left\{ \, [z_1 : w_2 : w_3 : \xi] \in M_a \;\middle|\; |z_1|^2 + |\xi|^2 = k + a, \; |w_2|^2 + |w_3|^2 + |\xi|^2 = a \, \right\},$$

$\mathcal{E}_{13}$:

$$\mathcal{E}_{13} \cong \left\{ \, [z_1 : z_3 : w_2 : \xi] \in M_a \;\middle|\; |z_1|^2 + |z_3|^2 + |\xi|^2 = k + a, \; |w_2|^2 + |\xi|^2 = a \, \right\},$$

$\mathcal{E}_{3}$:

$$\mathcal{E}_{3} \cong \left\{ \, [z_3 : w_1 : w_2 : \xi] \in M_a \;\middle|\; |z_3|^2 + |\xi|^2 = k + a, \; |w_1|^2 + |w_2|^2 + |\xi|^2 = a \, \right\},$$

$\mathcal{E}_{23}$:

$$\mathcal{E}_{23} \cong \left\{ \, [z_2 : z_3 : w_1 : \xi] \in M_a \;\middle|\; |z_2|^2 + |z_3|^2 + |\xi|^2 = k + a, \; |w_1|^2 + |\xi|^2 = a \, \right\},$$

$\mathcal{E}_{2}$:

$$\mathcal{E}_{2} \cong \left\{ \, [z_2 : w_1 : w_3 : \xi] \in M_a \;\middle|\; |z_2|^2 + |\xi|^2 = k + a, \; |w_1|^2 + |w_3|^2 + |\xi|^2 = a \, \right\}.$$

References

[1] Hiroshi Konno. The topology of toric hyperKähler manifolds. In *Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999)*, volume 34 of *Adv. Stud. Pure Math.*, pages 173–184. Math. Soc. Japan, Tokyo, 2002.