

HYPERTORIC MANIFOLDS AND EQUIVARIANT LOCALISATION

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1. INDEX THEORY

1.1. Non-Equivariant Index Formula. For a holomorphic vector bundle \mathcal{L} over a complex n -dimensional variety M , the *index* $\text{ind}(\bar{\partial}, \mathcal{L})$ is defined as

$$\text{ind}(\bar{\partial}, \mathcal{L}) := \sum_{k=0}^n (-1)^k \dim H^k(M; \mathcal{L}).$$

Viewing the index $\text{ind}(\bar{\partial}, \mathcal{L})$ as the Euler characteristic $\chi(M, \mathcal{L})$ of the vector bundle \mathcal{L} , we can apply the Atiyah-Singer index theorem, which we state below, to express the index as an integral over M of the product of the Todd class $\text{Td}(TM)$ of the tangent bundle $TM \rightarrow M$ over M , and the Chern character $\text{Ch}(\mathcal{L}) := \exp(c_1(\mathcal{L}))$ of \mathcal{L} , where $c_1(\mathcal{L})$ is the first Chern class of \mathcal{L} .

Theorem 1.1 (Atiyah-Singer Index Theorem, [?]). *Let M be a compact complex manifold, \mathcal{L} a holomorphic vector bundle over M . Let*

$$\text{Td}(TM) = \prod \frac{x_i}{1 - e^{-x_i}}$$

be the Todd class of the complex vector bundle $TM \rightarrow M$, where the x_i are the Chern roots of TM . Then the Euler characteristic $\chi(M, \mathcal{L})$ of the sheaf of germs of holomorphic sections of \mathcal{L} is given by

$$\chi(M, \mathcal{L}) = \int_M \text{Td}(M) \cdot \text{Ch}(\mathcal{L}).$$

Example. Let $M = \mathbb{CP}^1$ and let \mathcal{L} be the line bundle $\mathcal{O}(k)$ for some positive integer k . If $\langle \xi \rangle = H^2(M; \mathbb{Z})$, *i.e.* ξ is the generator of $H^2(\mathbb{CP}^1; \mathbb{Z})$, then $c_1(\mathcal{L}) = k\xi$, and thus the Chern character of \mathcal{L} is

$$\text{Ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = \sum_{j=0}^{\infty} (k\xi)^j = 1 + k\xi$$

(the higher powers of ξ vanish since $\dim_{\mathbb{C}} M = 1$).

For n -dimensional complex projective space \mathbb{CP}^n , both the total Chern class

$$c(\mathbb{CP}^n) := c(T\mathbb{CP}^n) := 1 + c_1 + c_2 + c_3 + \dots,$$

and the Todd class $\text{Td}(T\mathbb{CP}^n)$ for the tangent bundle $T\mathbb{CP}^n \rightarrow \mathbb{CP}^n$, can be calculated using the exact Euler sequence, along with the multiplicativity of

$$\{0\} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{CP}^n \longrightarrow \{0\},$$

the total Chern class and the Todd class,

$$c(\mathcal{F} \oplus \mathcal{G}) = c(\mathcal{F}) \cdot c(\mathcal{G}), \quad \text{Td}(\mathcal{F} \oplus \mathcal{G}) = \text{Td}(\mathcal{F}) \cdot \text{Td}(\mathcal{G}),$$

which yields

$$c(\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \mathcal{O}) = c(\mathcal{O}(1)^{\oplus(n+1)}) = (1 + \xi)^{n+1},$$

and

$$\text{Td}(T\mathbb{CP}^n) = \text{Td}(T\mathbb{CP}^n \oplus \mathcal{O}) = \text{Td}(\mathcal{O}(1)^{\oplus(n+1)}) = \text{Td}(\mathcal{O}(1))^{n+1} = \left(\frac{\xi}{1 - e^{-\xi}} \right)^{n+1}.$$

This expression can be expanded as a formal power series which, for $n = 1$ in our example with the complex projective line \mathbb{CP}^1 , gets us

$$c(\mathbb{CP}^1) = (1 + \xi)^2 = 1 + 2\xi, \quad \text{Td}(T\mathbb{CP}^1) = 1 + \frac{1}{2}c_1(T\mathbb{CP}^1) = 1 + \xi.$$

Finally, applying the Atiyah-Singer index theorem 1.1, we have

$$\chi(\mathbb{CP}^1, \mathcal{L}) = \int_{\mathbb{CP}^1} \text{Td}(\mathbb{CP}^1) \cdot \text{Ch}(\mathcal{L}) = \int_{\mathbb{CP}^1} (1 + \xi) \cdot (1 + k\xi) = \int_{\mathbb{CP}^1} 1 + (k+1)\xi = k+1.$$

Example. Now we let $M = \mathbb{CP}^2$, and let $\mathcal{L} = \mathcal{O}(k)$ and $\langle \xi \rangle = H^2(M, \mathbb{Z})$ again as above. Now we have

$$c(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + k\xi + k^2\xi^2,$$

and

$$\begin{aligned} c(T\mathbb{CP}^2) &= 1 + c_1 + c_2 = (1 + \xi)^3 = 1 + 3\xi + 3\xi^2, \\ \text{Td}(T\mathbb{CP}^2) &= 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} = 1 + \frac{3}{2}\xi + \frac{9\xi^2 + 3\xi^2}{12} = 1 + \frac{3}{2}\xi + \xi^2. \end{aligned}$$

Hence by the Atiyah-Bott index theorem 1.1,

$$\begin{aligned}\chi(M, \mathcal{L}) &= \int_M \text{Td}(TM) \cdot \text{Ch}(\mathcal{L}) = \int_M (1 + \tfrac{3}{2}\xi + \xi^2) \cdot (1 + k\xi + k^2\xi^2) \\ &= \int_M (k^2 + \tfrac{3}{2}k + 1)\xi^2 + O(\xi) = k^2 + \tfrac{3}{2}k + 1.\end{aligned}$$

Example. Let $M = \mathbb{CP}^3$, and let \mathcal{L} , ξ , etc. be as above. Then

$$\begin{aligned}\text{Ch}(\mathcal{L}) &= 1 + k\xi + (k\xi)^2 + (k\xi)^3, \\ c(TM) &= (1 + \xi)^4 = 1 + 4\xi + 6\xi^2 + 4\xi^3, \\ \text{Td}(TM) &= 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1c_2}{24} = 1 + 2\xi + \frac{11}{6}\xi^2 + \xi^3.\end{aligned}$$

Then by the Atiyah-Bott Index theorem 1.1,

$$\begin{aligned}\chi(M, \mathcal{L}) &= \int_M \text{Td}(TM) \cdot \text{Ch}(\mathcal{L}) = \int_M \left(1 + 2\xi + \frac{11}{6}\xi^2 + \xi^3\right) \cdot (1 + k\xi + k^2\xi^2 + k^3\xi^3) \\ &= \int_M \left(k^3 + 2k^2 + \frac{11}{6}k + 1\right)\xi^3 + O(\xi^2) =\end{aligned}$$

1.2. Equivariant Index Theorems.

1.2.1. Equivariant Characteristic Classes.

2. COMPACTIFYING THE HYPERTORIC VARIETY VIA SYMPLECTIC CUTTING

2.1. Set-Up. We can use the residual S^1 -action to perform a symplectic cut of the toric hyperkähler manifold M in order to compactify it as follows: consider the product $M \times \mathbb{C}$, where now S^1 acts on $M \times \mathbb{C}$ as

$$e^{i\theta} \cdot ([z, w], \xi) = ([z, e^{i\theta}w], e^{i\theta}\xi),$$

which is a Hamiltonian action with associated moment map

$$\begin{aligned}\mu_{\text{cut}} : M \times \mathbb{C} &\longrightarrow \mathbb{R}_{\geq 0}, \\ \mu_{\text{cut}}([z, w], \xi) &= \Phi[z, w] + \tfrac{1}{2}|\xi|^2 - \epsilon, ,\end{aligned}$$

where $\Phi : M \rightarrow \mathbb{R}_{\geq 0}$ is the moment map $\Phi[z, w] = \frac{1}{2}\|w\|^2$ for the residual S^1 -action on M , and $\epsilon \in \mathbb{R}_{\geq 0}$.

Then we have

$$\begin{aligned}
\mu_{\text{cut}}^{-1}(0) &= \{([z, w], \xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = 2\epsilon\} \\
&= \{[z, w] \in M : \|w\|^2 = 2\epsilon\} \bigsqcup \{([z, w], \xi) \in M \times \mathbb{C} : |\xi| = \pm\sqrt{2\epsilon - \|w\|^2}\} \\
&= \{[z, w] \in M : \|w\|^2 = 2\epsilon\} \bigsqcup \{([z, w], \xi) \in M \times \mathbb{C} : \xi = e^{i\arg(\xi)}\sqrt{2\epsilon - \|w\|^2}\} \\
&= \Phi^{-1}(\epsilon) \bigsqcup (M \times S^1) \\
&=: \Sigma_1 \bigsqcup \Sigma_2,
\end{aligned}$$

where Σ_1 is just the level-set of Φ at the level ϵ in M , and $\Sigma_2 = M \times S^1$ is exhibited as a trivial S^1 -bundle over Σ_2 , using the globally defined section

$$M \rightarrow M \times S^1, \quad [z, w] \mapsto ([z, w], e^{i\theta}\sqrt{2\epsilon - \|w\|^2}), \quad e^{i\theta} \in S^1.$$

Finally, by taking the quotient of $\mu_{\text{cut}}^{-1}(0)$ by the S^1 -action, we obtain the *symplectic cut of M at level ϵ* ,

$$M_{\leq \epsilon} := \mu_{\text{cut}}^{-1}(0)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where $\Sigma_1/S^1 = \Phi^{-1}(\epsilon)/S^1$ is just the symplectic reduction, and where Σ_2/S^1 is diffeomorphic to M for $\|w\|^2 < 2\epsilon$, which we denote by $M_{< \epsilon}$.

2.2. Restriction to the Extended Core Component, \mathcal{E}_A . Since the residual circle S^1 -action acts as a subgroup of the original torus T^n when restricted to each component \mathcal{E}_A of the extended core \mathcal{E} , we can described combinatorially the resulting configuration of the hyperplane arrangement in $(\mathbb{R}^d)^*$ from taking the cut. For each component, let $j_A : \mathfrak{s}^1 \rightarrow \mathbb{R}^n$ be the derivative of the inclusion of S^1 into T^n on the Lie algebra level, that is

$$j_A(\xi) = (\xi_1, \dots, \xi_n), \quad \text{where } \xi_i = \begin{cases} -1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A, \end{cases}$$

so that its image in \mathbb{R}^n generates a circle subgroup S^1 in T^n that depends on each component \mathcal{E}_A . Then the moment map for this restriction for the S^1 -action is

$$\Phi[z, w] = j_A^* \circ \mu_{\mathbb{R}}[z, w] = \left\langle \mu_{\mathbb{R}}(z, w), \sum_{i \in A} \xi_i u_i \right\rangle,$$

and so from our above discussion of how we constructed the symplectic cut, the image in $(\mathbb{R}^d)^*$ of the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ is

$$\phi_{\mathbb{R}}(\Phi^{-1}(\epsilon)) = \left\{ y \in \Delta_A : \left\langle y, \sum_{i \in A} \xi_i u_i \right\rangle + \epsilon = 0 \right\} =: H_A$$

which introduces an inward-pointing half-space

$$F_A := \left\{ y \in \Delta_A : \left\langle y, \sum_{i \in A} \hbar_i u_i \right\rangle + \epsilon \geq 0 \right\}$$

such that the image of the extended core component \mathcal{E}_A after being compactified is the original convex polytope Δ_A intersected with H_A . One can also see clearly that the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ has the restricted S^1 -action as its stabiliser subgroup since, by definition of H_A , the moment map $\Phi|_{\mathcal{E}_A}$ equals the hyperplane H_A , *i.e.* $\Phi|_{\mathcal{E}_A}$ is constant along $\Phi^{-1}(\epsilon)/S^1$.

Remark 1. If we had used instead the following action for S^1

$$e^{i\theta} \cdot ([z, w], \xi) = ([z, e^{i\theta} w], e^{-i\theta} \xi)$$

with respective moment map

$$\mu_{cut}([z, w], \xi) = \frac{1}{2} \|w\|^2 - \frac{1}{2} |\xi|^2 - \epsilon,$$

and taken the cut, then the resulting then we would obtain the other “discarded half” $\mathfrak{M}_{>\epsilon}$ of the hypertoric manifold \mathfrak{M} along with the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ with the opposite orientation:

$$\mathfrak{M}_{\geq \epsilon} = \mathfrak{M}_{>\epsilon} \bigsqcup \left(-(\Phi^{-1}(\epsilon)/S^1) \right).$$

The component M_ϵ is non-compact however, so we focus on $M_{<\epsilon}$.

REFERENCES

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