EQUIVARIANT LOCALISATION AND FIXED-POINT THEOREMS

HODGE CLUB TALK NOTES

12th March 2021

ABSTRACT

Often in mathematics, we are tasked with the problem of evaluating an integral $\int_M \omega$ over some space M. Depending on the form ω, this integral can be related to calculating volume, finding topological or enumerative invariants, or integrating characteristic classes. Such computations are often difficult, but two notions can simplify them, which are *symmetry* and *localisation*.

By symmetry, we mean that we have a group G acting on M, and by identifying orbits we reduce the problem to that over a smaller space, M/G; this comes up in symplectic reduction, gauge theory, and integrable systems. By localisation, this means that we reduce global calculations to local ones; for example, as in the Poincaré-Hopf theorem. Symmetry and localisation synergise together through the Atiyah-Bott-Berline-Vergne fixed-point formula; when we have a smooth manifold M together with the action of a compact, connected Lie group G, then the integral on M localises on the xed-point set of the G-action.

In this talk, I want to introduce the equivariant cohomology of a G-manifold via the Cartan model - that is, equivariant de Rham theory - before going through some example calculations including the equivariant Riemann-Roch-Hirzebruch, the Duistermaat-Heckman, and the Lefschetz fixed-point theorems.

1 Introduction

Consider the following geometric sum:

$$\sum_{k=0}^{10000} q^k = 1 + q + q^2 + \ldots + q^{10001}$$

$$= \left(\frac{1-q}{1-q}\right) \cdot (1 + q + q^2 + \ldots + q^{10001})$$

$$= \frac{1 - q^{10001}}{1 - q}$$

$$= \frac{1}{1-q} + \frac{1 - q^{10000}}{1 - q^{-1}}.$$

To evaluate the left-hand side, we need to know the value of each term at the 10001 integral points inside of the closed interval $[0, 10000]$, whereas the right-hand side only needs the two terms to be evaluated. So we can say that this sum *localises* at the end points.

2 Equivariant Cohomology

Let G be a compact Lie group acting on a toplogical space M. If G acts freely on M, then the quotient space M/G is usually as nice as the space M is itself; for instance, if M is a manifold then so is M/G.

The idea behind an equivariant cohomology group, $H_G^*(M)$, is that the equivariant cohomology groups of M should just be the cohomology groups of M/G:

$$H_G^*(M) = H^*(M/G), \qquad \text{when the action is free.}$$

For example, if G acts on itself by left multiplication, then

$$H_G^*(G) = H^*(\text{pt}).$$

However, if the action is not free, then the space M/G might not behave very nicely from a cohomological point of view. Then the idea is that $H_G^*(M)$ should be the "correct" subsitute for $H^*(M/G)$. This concept is much more clear in the **Weil model** of equivariant cohomology, where $H_G^*(M)$ is defined using *"classifying bundles"*. However here, it will be much more convenient to use the **Cartan model** of equivariant cohomology; then the Lie group G is *compact*, the constructions are isomorphic.

2.1   The Cartan Model

Let M be an n-dimensional manifold acted on by a Lie group G with with Lie algebra $\mathfrak{g}$. A *G-equivariant differential form* on M is defined to be a polynomial map $\alpha : \mathfrak{g} \to \Omega(M)$ such that

$$\alpha(gX) = g \cdot \alpha(X), \qquad \text{for } g \in G.$$

Let $\mathbb{C}[\mathfrak{g}]$ denote the algebra of $\mathbb{C}$-valued polynomial functions on $\mathfrak{g}$. Then we can view the tensor product

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega(M),$$

as the algebra of polynomial maps from $\mathfrak{g}$ to Ω. The group G acts on an element $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ by the formula[1]

$$(g \cdot \alpha)(X) := g \cdot \left(\alpha(g^{-1} \cdot X) \right), \qquad \text{for all } g \in G, \text{ and } X \in \mathfrak{g}.$$

Let $A := \Omega_G^*(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega(M))^G$ be the subalgebra of G-invariant elements; an element $\alpha \in A$ thus satisfies $\alpha(g \cdot X) = g \cdot \alpha(M)$, hence is an equivariant differential form.

Let $A^{k,l}$ denote the space of elements of A which are homogeneous polynomial mappings of degree k, from $\mathfrak{g}$ to Ω^l. If $\omega \in A^{k,l}$, then

$$\left(\mathfrak{g} \ni X \longmapsto d(\omega(X)) \in \Omega^{l+1} \right) \in A^{k,l+1},$$

and

$$\left(\mathfrak{g} \ni X \longmapsto \imath_{X_M}\left(\omega(X) \right) \in \Omega^{l-1} \right) \in A^{k+1,l-1}.$$

We then equip $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ with the following $\mathbb{Z}$-grading,

$$\deg(P \otimes \alpha) := 2 \cdot \deg(P) + \deg(\alpha),$$

for the polynomial $P \in \mathbb{C}[\mathfrak{g}]$, and $\alpha \in \Omega(M)$.

Remark 1. *To justify this grading, consider (M, ω) a symplectic G-manifold with moment map $\Phi : M \to \mathfrak{g}^*$. The moment map condition means that Φ is G-equivariant and that*

$$d\langle \Phi, X \rangle =: d\Phi^X = \imath_{X_M}\omega.$$

Thus

$$\tilde{\omega} := \omega + \Phi^X$$

has degree-2 in the above grading.

Define the *equivariant exterior differential*, or *Cartan differential*, d_G by

$$(d_G\alpha)(X) := (d - \imath_{X_M})\alpha(X).$$

Proposition 2.1. *The Cartan differential d_G is closed on Ω_G^*, i.e. $d_G^2 = 0$.*

---

[1] G acts on $\Omega(M)$ by the induced G-action on M, and on $\mathfrak{g}$ by the adjoint action.

Corollary 2.2. *The space of equivariant differential forms* $\Omega_G^*(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G$, *equipped with the Cartan differential* d_G *forms a complex, called the* **Cartan complex**:

$$(\Omega_G^*(M), d_G) = \left((\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G, d_G \right).$$

Definition 2.3. *The* **equivariant cohomology** $H_G^*(M)$ *of M is the cohomology of the Cartan complex,* $(\Omega_G^*(M), d_G)$.

3 Characteristic Classes

Definition 3.1. *A G-equivariant vector bundle of a G-manifold M is a vector bundle $V \to M$ with an action of G on the total space V covering the action of G on M.*

Definition 3.2. *A* **prequantum line bundle with connection** $\mathcal{L} \to M$ *is a complex line bundle $\mathcal{L} \to M$ for which $c_1(\mathcal{L}) = [\omega]$, equipped with a connection θ satisfying $d\theta = \pi^*\omega$.*

Fact 1. *Imposing the condition $\mathcal{L}_{X_M}\theta = 0$ for the Lie derivative along X, we get*

$$\mathcal{L}_{X_M}\theta = (d \cdot \imath_{X_M} + \imath_{X_M} \cdot d)\theta = 0 \implies d\imath_{X_M}\theta = -\imath_{X_M} d\theta = -\imath_{X_M}\omega = -d\Phi^X.$$

So it's justified also impose the condition

$$\imath_{X_M}\theta = -\Phi^X.$$

Fact 2. *For $F \subseteq M^T$, then*

$$i^*\tilde{\omega}(X) = \omega\big|_F + \Phi^X(F).$$

3.1 Equivariant Euler Class

Definition 3.3. *The* **Euler class** *of a complex vector bundle of rank m is a characteristic class $e(E)$ associated to a real vector bundle $E \to M$ of rank $2m$.*

Definition 3.4. *If $E_{\mathbb{C}}$ is a complex vector bundle of complex rank m, then $e(E_{\mathbb{R}}) = c_m(E_{\mathbb{C}})$.*

Fact 3. *If $E = \mathcal{L}_1 \oplus \ldots \mathcal{L}_m$ is the direct sum of line bundles, then $e(E) = c_1(\mathcal{L}_1) \ldots c_1(\mathcal{L}_m)$.*

We can usually reduce to this situation by the **splitting principle**.

Proposition 3.5. *If E is a complex vetor bundle with T-action, and $E = \sum_j \mathcal{L}_j$, where the $\mathcal{L}_j$ are complex line bundles with T-action given by weights $\beta_j : T \to S^1$, then the equivariant Euler class of E is*

$$e^T(E) = \prod_j c_1^T(\mathcal{L}_j),$$

which is represented in the Cartan model by

$$e^T(E)(X) = \prod_j (d\theta_j - \beta_j)(X).$$

Example 1. *If T acts on M and F is a component of M^T, then the normal bundle ν_F is a T-equivariant bundle over F. Assuming ν_F decomposes equivariantly as $\sum_j \nu_{F,j}$ with weights $\beta_{F,j} \in \mathfrak{t}^*$.*

Then the equivariant Euler class $e_F := e^T(\nu_F)$ is given by

$$e_F(X) = \prod_j (c_1(\nu_{F,j}) - \beta_{F,j}).$$

Example 2. *When the T-action on a symplectic manifold (M, ω) is Hamiltonian with moment map $\mu : M \to \mathfrak{t}^*$, the image of μ is a convex polytope, whose vertices are the images of the fixed-points of the T-action. As each fixed-point $p \in M^T$, isotropy weights on $TM\big|_p \cong \nu_p$ are then just given by the edge vectors $\alpha_{p,j} \in \mathfrak{t}^*$, $j = 1 \ldots, \dim_{\mathbb{C}} M$.*

4 Equivariant Localisation

4.1 The Berline-Vergne-Atiyah-Bott Fixed Point Theorem

When a manifold has a torus action, the equivariant localisation formula is a powerful tool for doing calculations in **ordinary** cohomology, despite being formulated in **equivariant** cohomology.

Theorem 4.1 (**Atiyah-Bott, Berline-Vergne Theorem**). *Suppose an n-dimensional torus T acts on a compact oriented manifold M with fixed-point set $F := M^T$. If ϕ is an equivariant closed form on M and $i_F : F \hookrightarrow M$ is the inclusion map, then*

$$\int_M \phi = \sum_{F \subseteq M^T} \int_F \frac{i_F^* \phi}{e^T(\nu_F)},$$

as elements of $H_T^*(\mathrm{pt}) = \mathbb{R}[u_1, \ldots, u_n]$. Here, ν_F is the normal bundle of F in M, and e^T is the T-equivariant Euler class.

In the case when the fixed-point set $M^T = \{p_i\}$ consists of isolated fixed-points, the localisation theorem simplifies greatly:

Theorem 4.2. *With the above hypotheses, if M^T consists of isolated fixed-points, then:*

$$\int_M \phi = \sum_{p \in M^T} \frac{\phi(p)}{\prod_i \lambda_{p,i}}.$$

5 Examples

5.1 The Index and Hirzebruch-Riemann Roch Theorems

Theorem 5.1 (**Hirzebruch-Riemann-Roch, (HRR)**). *Let $\mathcal{L} \to M$ a holomorphic line bundle over a **complex projective algebraic variety** M. Then the Euler characteristic, $\chi(M; \mathcal{L})$, is equal to the characteristic number*

$$\chi(M; \mathcal{L}) = \int_M e^{c_1(\mathcal{L})} \, \mathrm{Td}(TM).$$

Here, $c_1(\mathcal{L})$ is the 1st Chern class *of $\mathcal{L}$, and $\mathrm{Td}(M)$ is the* Todd class *of the complex vector bundle $TM \to M$.*

Definition 5.2. *The **Todd class** $\mathrm{Td}(TM)$ of a complex tangent bundle (TM, J) is*

$$\mathrm{Td}(TM) := \prod_{j=1}^n \frac{c_1}{1 + e^{-c_1}},$$

where $c_1 := c_1(TM)$ is the 1st Chern class of the tangent bundle.

Example 3. *Let $M = \mathbb{CP}^1$ and $\mathcal{L} = \mathcal{O}(k)$. Then $c_1(\mathcal{L}) = kH$, where H generates $H^2(\mathbb{CP}^1; \mathbb{Z})$. The Chern character of $\mathcal{L}$ is*

$$e^{c_1(\mathcal{L})} = 1 + kH.$$

The tangent bundle TM is a line bundle with 1st Chern class $c_1(TM) = 2H$, so its Todd class is

$$\mathrm{Td}(c_1(TM)) = \frac{c_1(TM)}{1 + e^{-c_1(TM)}} = 1 + \frac{1}{2} c_1(TM) = 1 + H.$$

Thus, the Riemann-Roch integral is

$$\int_M (1 + kH)(1 + H) = \int_M (k+1)H + 1 = k + 1.$$

In particular, if $k = 10000$, then we get $\chi(\mathbb{CP}^1; \mathcal{L}) = 10001$, which we saw at the start of this talk.

Remark 2. *The sections of $\mathcal{L}$ can be identified with degree k polynomials in the two coordinate variables of $\mathbb{CP}^1$, and hence the space of sections has dimension $k + 1$. The higher cohomology spaces vanish: $H^j(\mathbb{CP}^1; \mathcal{L}) = 0$ if $j \geq 1$ by Kodaira vanishing.*

This integral formula was originally proven for complex projective algebraic varieties by Hirzebruch, but the Atiyah-Singer index theorem generalises it to include complex analytic manifolds by the following argument:

The characteristic number $\chi(M; \mathcal{L})$ also equals

$$\chi(M; \mathcal{L}) = \sum (-1)^k \dim H^k(M; \mathcal{L}).$$

Provided that M has a Hermitian structure and hence metric, then this metric gives rise to an inner-product on the vector bundles $\Lambda^{0,k} = \Lambda^{0,k} TM$. Moreover, provided $\mathcal{L}$ also has a Hermitian structure, then there are Hermitian inner-products on the vector bundles $\mathcal{L} \otimes \Lambda^{0,k}$ also. These Hermitian products and the volume form give rise to differential operators,

$$\bar{\partial}^* : \Omega^{0,k}(M; \mathcal{L}) \longrightarrow \Omega^{0,k-1}(M; \mathcal{L}),$$

as the product gives rise to an adjoint to the Dolbeault operator, $\bar{\partial}$. By Hodge theory, one can show that

$$\operatorname{ind}(\bar{\partial}) := \ker(\bar{\partial} + \bar{\partial}^*) - \operatorname{coker}(\bar{\partial} + \bar{\partial}^*) = \sum (-1)^k \dim H^k(M; \mathcal{L}),$$

as virtual vector spaces.

5.2   The Equivariant Index Theorem

Suppose now that in addition to the hypotheses above, we now have a bundle automorphism γ of $\mathcal{L}$, which leaves all the given structures in variant. Then it induces an operator on $\Lambda^{0,*}$ which commutes with $\bar{\partial}$. Hence if γ comes from a representation of a G-action, we get a **virtual representation** on $\operatorname{Ind}(\bar{\partial})$, by lifting the G-action to the sections of $\mathcal{L}$. Letting χ denote the character of this representation and letting $G = T$ be a torus, then we have:

Theorem 5.3 (**The Equivariant Index Formula**)**.**

$$\operatorname{Ind}(\bar{\partial}; \mathcal{L}, T)(\xi) = \chi(e^{i\xi}) = \int_M e^{c_1^T(\mathcal{L})(\xi)} \operatorname{Td}^T(TM)(\xi), \qquad \xi \in \mathfrak{t}.$$

Example 4. *In the case when the set M^T fixed points is isolated, we obtain the **equivariant Lefschetz fixed-point formula**:*

$$\operatorname{Ind}(\bar{\partial}; \mathcal{L}, T)(t) = \sum_{p \in M^T} \frac{\operatorname{Tr}_{\mathcal{L}_p}(t)}{\det_{T_p^{1,0}}(1 - t^{-1})}.$$

Example 5. *Let $M = \mathbb{CP}^1$ and $G = U^1$ act on M as*

$$t \cdot [z_0 : z_1] = [z_0 : t z_1],$$

and let $\mathcal{L} = \mathcal{O}(n)$ be a complex line bundle of degree n over $\mathbb{CP}^1$. Then the moment map for this action $\mu : \mathbb{CP}^1 \to \mathbb{R}^*$ is

$$\mu([z_0 : z_1]) = n \cdot \frac{|z_1|^2}{|z_0|^2 + |z_1|^2},$$

with image $\mu(\mathbb{CP}^1) = [0, n]$. The primitive edge vectors for the two fixed-points are

$$\alpha_{[1:0]} = +1, \qquad \alpha_{[0:1]} = -1,$$

and the images of μ at these two fixed-points are

$$\mu([1 : 0]) = 0, \qquad \mu([0 : 1]) = n,$$

giving us the weights on the fibres of $\mathcal{O}(n)$ at these two points. Thus

$$\operatorname{Ind}(\bar{\partial}; \mathcal{L}, \mathbb{CP}^1)(t) = \frac{1}{1 - t} + \frac{t^n}{1 - t^{-1}} = \sum_{k=0}^{n} t^n, \qquad n \geq 0.$$

Example 6. *Now for $M = \mathbb{CP}^2$ with T^2-action*

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2],$$

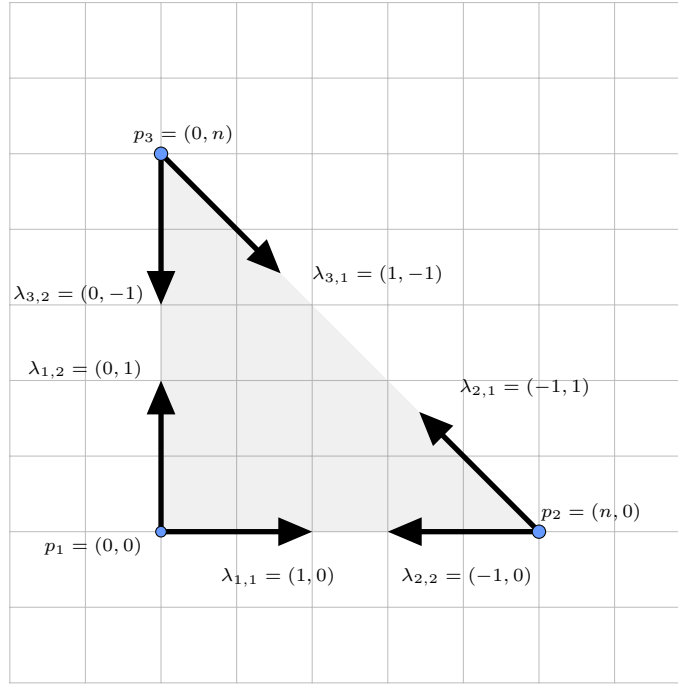*and moment map $\mu : M \to \mathbb{R}^2$ given by*

$$\mu([z_0 : z_1 : z_2]) = n \cdot \left(\frac{|z_1|^2}{\|z\|^2}, \frac{|z_2|^2}{\|z\|^2} \right),$$

for the bundle $\mathcal{L} = \mathcal{O}(n)$, so its moment map image is the dilated 2-simplex,

$$\mu(M) = n \cdot \Delta = \{(x,y) \in \mathbb{R}^2_{\geq 0} : x + y \leq n\}.$$

As before, we can read off the combinatorial data required from the polytope:

$$\mu([1:0:0]) = p_0 = (0,0), \qquad \alpha_{1,1} = (1,0), \qquad \alpha_{1,2} = (0,1),$$
$$\mu([0:1:0]) = p_1 = (n,0), \qquad \alpha_{2,1} = (-1,1), \qquad \alpha_{2,2} = (-1,0),$$
$$\mu([0:0:1]) = p_2 = (0,n), \qquad \alpha_{3,1} = (0,-1), \qquad \alpha_{3,2} = (1,-1).$$

Putting this all together into the index formula, we get

$$\text{Ind}(\bar{\partial}, M, \mathcal{O}(n))(t_1, t_2) = \frac{1}{1 - t_1 t_2} + \frac{t_1^n}{(1 - t_1^{-1})(1 - t_1^{-1} t_2)} + \frac{t_2^n}{(1 - t_2^{-1})(1 - t_1 t_2^{-1})}$$
$$= \sum_{\substack{0 \leq n_1, n_2 \\ n_1 + n_2 \leq n}} t_1^{n_1} t_2^{n_2}.$$

The last equality can be determined by expanding each rational polynomial expression into its Laurent series, then calculating the coefficients by the residue theorem.

[TODO]: Jupyter notebook.

5.3   Duistermaat-Heckman and the Stationary Phase Approximation

In physics, many integrals are of the form

$$\int_M e^{itf} d\mu,$$

*where M is a compact oriented n-manifold, f is a function, and μ is an n-form on M. Such an integral is called an **oscillatory integral**. Examples of situations in which they come up include Fourier transformations, path integrals, etc.*

*The method of stationary phase gives an approximation to such an oscillatory integral, based on the critical points of the function f; it is at these points that f does not change fast enough to cancel out its other values, contributing to the value to the integral. However in the case that (M, ω) is a symplectic manifold and S^1 acts symplectically on M with isolated fixed-point set $F = M^{S^1}$, the equivariant localisation formula gives an **exact** value:*

Theorem 5.4 (**Duistermaat-Heckman**). *Consider an n-dimensional manifold (M, ω, μ) with Hamiltonian S^1-action and moment map $\mu : M \to \mathbb{R}$. Then, for any $\hbar \in \mathbb{C}$:*

$$\int_M e^{\hbar\mu} \frac{\omega^n}{n!} \equiv \int_M e^{\omega + \hbar\mu} = \left(\frac{-1}{\hbar}\right)^n \sum_{p \in F} \frac{e^{\hbar\mu(p)}}{\prod_{j=1}^n \alpha_{p,j}(p)},$$

where the $\alpha_{p,j}$, $j = 1, \ldots, n$, are the isotropy weights for the S^1-action at the fixed point p.

Example 7. *Let S^1 act on (S^2, ω) by rotating the sphere around its z-axis in a counter-clockwise orientation (when viewed from above) with period 1 (or with speed 2π).*

[TODO]: Refresh remind yourself of spherical coordinates....

Remark 3. *Expanding out the Duistermaat-Heckman formula in terms of $\hbar$, we get:*

$$\int_M \cdot \frac{\omega^n}{n!} \left(1 + \hbar\mu + \ldots\right) = \left(\frac{-1}{\hbar}\right)^n \sum_{p \in F} \left(1 + \hbar\mu(p) + \ldots + \frac{\hbar^n \mu(p)^n}{n!} + \ldots\right) / \prod_{j=1}^n \alpha_{p,j}.$$

Of special interest is the degree-0 term which, after cancelling factors, gets us a formula for the volume of M in terms of its fixed-point data:

$$\mathrm{Vol}(M) = \int_M \omega^n = (-1)^n \sum_{p \in F} \frac{\mu(p)^n}{\prod_{j=1}^n \alpha_{p,j}}.$$