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# POLYPTYCH ISOTROPY WEIGHTS

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## GENERAL NOTES

### ABSTRACT

Calculations for the isotropy data of the compactified hypertoric manifolds.

## 1 Example: $M = T^*\mathbb{CP}^1$

### 1.1 Construction

Short exact sequence for the usual Delzant construction of  $\mathbb{CP}^1$ :

$$\{1\} \longrightarrow K \cong S^1 \xrightarrow{t \mapsto (t,t)} T^2 \xrightarrow[(a,b) \mapsto ab^{-1}]{\quad} T^2/K \cong T^1 \longrightarrow \{1\}.$$

The induced action of  $K$  on  $T^*\mathbb{C}^2$  is thus

$$t \cdot (z|w) \mapsto (t, t) \cdot (z_1, z_2 | w_1, w_2) = (tz_1, tz_2 | t^{-1}w_1, t^{-1}w_2),$$

which is Hamiltonian with associated moment map

$$\mu_{\mathbb{R}} : T^*\mathbb{C} \longrightarrow \mathbb{R}; \quad \mu_{\mathbb{R}}(z|w) = |z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2.$$

For some  $a \in \mathbb{Z}_{>0}$ , take the hyperkähler quotient of  $T^*\mathbb{C}$  to get

$$M := T^*\mathbb{C} \mathbin{/\!\!/} K$$

## 2 Example: $M = T^*(\mathbb{CP}^2 \times \mathbb{CP}^2)$ (Non-Convex Core)

### 2.1 Construction

Short exact sequence for the usual Delzant construction of the non-convex  $T^*(\mathbb{CP}^2 \times \mathbb{CP}^2)$ :

$$\{1\} \longrightarrow K \cong T^2 \xrightarrow[(a,b,c,d) \mapsto (ac^{-1}, bc^{-1}, d^{-1})]{(s,t) \mapsto (s, st, s, t)} T^4 \xrightarrow{\quad} T^4/K \cong T^2 \longrightarrow \{1\}.$$

The induced action of  $K$  on  $T^*\mathbb{C}^4$  is thus

$$(s, t) \cdot (z|w) \mapsto (s, st, s, t) \cdot (z_1, z_2, z_3, z_4 | w_1, w_2, w_3, w_4) = (sz_1, stz_2, sz_3, tz_4 | s^{-1}w_1, s^{-1}t^{-1}w_2, s^{-1}w_3, t^{-1}w_4),$$

which is Hamiltonian with associated moment map

$$\mu_{\mathbb{R}} : T^*\mathbb{C}^4 \longrightarrow \mathbb{R}^2; \quad \mu_{\mathbb{R}}(z|w) = \begin{pmatrix} |z_1|^2 + |z_2|^2 + |z_3|^2 - |w_1|^2 - |w_2|^2 - |w_3|^2 \\ |z_2|^2 + |z_4|^2 - |w_2|^2 - |w_4|^2 \end{pmatrix}.$$

For some  $(n, m) \in \mathbb{Z}_{>0}^2$ , take the hyperkähler quotient of  $T^*\mathbb{C}^4$  to get

$$M := T^*\mathbb{C}^4 \mathbin{/\!\!/}_{(n,m)} K := (\mu_{\mathbb{R}}^{-1}(n, m) \cap \mu_{\mathbb{C}}^{-1}(0)) / K.$$

Quotient relations arising from  $K \cong T^2$ :

$$(s, 1) \in S^1 \times \{e\} < T^2:$$

$$[sz_1 : sz_2 : sz_3 : z_4 | s^{-1}w_1 : s^{-1}w_2 : s^{-1}w_3 : w_4] = [z_1 : z_2 : z_3 : z_4 | w_1 : w_2 : w_3 : w_4],$$

$$(1, t) \in \{e\} \times S^1 < T^2:$$

$$[z_1 : tz_2 : z_3 : tz_4 | w_1 : t^{-1}w_2 : w_3 : t^{-1}w_4] = [z_1 : z_2 : z_3 : z_4 | w_1 : w_2 : w_3 : w_4].$$

## 2.2 Figure

## 2.3 Isotropy Weights

### 2.3.1 Interior Points

$P_{12}$ :

$$P_{12} = [0 : 0 : z_3 : z_4 \mid 0 : 0 : 0 : 0],$$

has isotropy weights

$$\begin{aligned} [sx_1 : tx_2 : z_3 : z_4 \mid s^{-1}y_1 : t^{-1}y_2 : y_3 : y_4] \\ \implies (sx_1, tx_2, s^{-1}y_1, t^{-1}y_2) \longleftrightarrow (s, t, s^{-1}, t^{-1}), \end{aligned}$$

so tangent space weights  $(s, t)$  from  $(z_1, z_2)$  respectively, and cotangent space weights  $(s^{-1}, t^{-1})$  coming from  $(w_1, w_2)$  respectively.

$P_{23}$ :

$$P_{23} = [z_1 : 0 : 0 : z_4 \mid 0 : 0 : 0 : 0],$$

has isotropy weights

$$\begin{aligned} [sz_1 : tx_2 : x_3 : z_4 \mid s^{-1}y_1 : t^{-1}y_2 : y_3 : y_4] \sim [z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \mid y_1 : st^{-1}y_2 : sy_3 : y_4] \\ \implies (s^{-1}tx_2, s^{-1}x_3, st^{-1}y_2, sy_3) \longleftrightarrow (s^{-1}t, s^{-1}, st^{-1}, s), \end{aligned}$$

so tangent space weights  $(s^{-1}t, s^{-1})$  from  $(z_2, z_3)$  respectively, and cotangent space weights  $(st^{-1}, s)$  coming from  $(w_2, w_3)$  respectively.

$P_{13}$ :

$$P_{13} = [0 : z_2 : 0 : z_4 \mid 0 : 0 : 0 : 0],$$

has isotropy weights

$$\begin{aligned} [sx_1 : tz_2 : x_3 : z_4 \mid s^{-1}y_1 : t^{-1}y_2 : y_3 : y_4] \sim [st^{-1}x_1 : z_2 : t^{-1}x_3 : z_4 \mid s^{-1}ty_1 : y_2 : ty_3 : y_4] \\ \implies (st^{-1}x_1, t^{-1}x_3, s^{-1}ty_1, ty_3) \longleftrightarrow (st^{-1}, t^{-1}, s^{-1}t, t), \end{aligned}$$

so tangent space weights  $(st^{-1}, t^{-1})$  from  $(z_1, z_3)$  respectively, and cotangent space weights  $(s^{-1}t, t)$  coming from  $(w_1, w_3)$  respectively.

$P_{14}$ :

$$P_{14} = [0 : z_2 : 0 : 0 \mid 0 : 0 : w_3 : 0],$$

has isotropy weights

$$\begin{aligned} [sx_1 : tz_2 : x_3 : z_4 \mid s^{-1}y_1 : t^{-1}y_2 : w_3 : y_4] \sim [sx_1 : z_2 : x_3 : t^{-1}x_4 \mid s^{-1}y_1 : y_2 : w_3 : ty_4] \\ \implies (sx_1, t^{-1}x_4, s^{-1}y_1, ty_4) \longleftrightarrow (s, t^{-1}, s^{-1}, t), \end{aligned}$$

so tangent space weights  $(s^{-1}, t^{-1})$  from  $(w_1, z_4)$  respectively, and cotangent space weights  $(s, t)$  coming from  $(z_1, w_4)$  respectively.

$P_{34}$ :

$$P_{34} = [0 : z_2 : 0 : 0 \mid w_1 : 0 : 0 : 0],$$

has isotropy weights

$$\begin{aligned} [sx_1 : tz_2 : x_3 : x_4 \mid s^{-1}w_1 : t^{-1}y_2 : y_3 : y_4] \sim [x_1 : s^{-1}tz_2 : s^{-1}x_3 : x_4 \mid w_1 : st^{-1}y_2 : sy_3 : y_4] \\ \sim [x_1 : z_2 : s^{-1}x_3 : st^{-1}x_4 \mid w_1 : y_2 : sy_3 : s^{-1}ty_4] \\ \implies (s^{-1}x_3, st^{-1}x_4, sy_3, s^{-1}ty_4) \longleftrightarrow (s^{-1}, st^{-1}, s, s^{-1}t), \end{aligned}$$

so tangent space weights  $(st^{-1}, s)$  from  $(z_4, w_3)$  respectively, and cotangent space weights  $(s^{-1}, s^{-1}t)$  coming from  $(z_3, w_4)$  respectively.

### 2.3.2 Exterior Points

$Q_{12}^{(1)}:$

$$Q_{12}^{(1)} = ([0 : 0 : z_3 : z_4 \mid 0 : w_2 : 0 : 0], \xi), \text{ with } |w_2|^2 = a, \xi = 0.$$

has isotropy weights

$$\begin{aligned} ([sx_1 : tx_2 : z_3 : z_4 \mid s^{-1}y_1 : t^{-1}w_2 : y_3 : y_4], \xi) &\sim ([sx_1 : tx_2 : z_3 : z_4 \mid s^{-1}ty_1 : w_2 : ty_3 : ty_4], t\xi) \\ &\implies (sz_1, tz_2, s^{-1}tw_1, t\xi) \longleftrightarrow (s, t, s^{-1}t, t), \end{aligned}$$

so normal weights  $(s, s^{-1}t)$  from  $(z_1, w_1)$  respectively, and inwards-pointing weight  $t$  with multiplicity 2 coming from  $z_2$  and  $\xi$ , since  $|w_2|$  achieves its maximum at  $Q_{12}^{(1)}$ .

$Q_{12}^{(2)}:$

$$Q_{12}^{(2)} = ([0 : 0 : z_3 : z_4 \mid w_1 : 0 : 0 : 0], \xi)$$

has isotropy weights

$$\begin{aligned} ([sx_1 : tx_2 : z_3 : z_4 \mid s^{-1}w_1 : t^{-1}y_2 : y_3 : y_4], \xi) &\sim ([sx_1 : tx_2 : z_3 : z_4 \mid w_1 : st^{-1}y_2 : sy_3 : sy_4], s\xi) \\ &\implies (sz_1, tz_2, s^{-1}tw_1, t\xi) \longleftrightarrow (s, t, st^{-1}, s). \end{aligned}$$

so normal weights  $(t, st^{-1})$  from  $(z_2, w_2)$  respectively, and inwards-pointing weight  $s$  with multiplicity 2 coming from  $z_1$  and  $\xi$ , since  $|w_1|$  achieves its maximum at  $Q_{12}^{(2)}$ .

To do (??)

$Q_{23}^{(2)}:$

$$Q_{23}^{(2)} = ([z_1 : 0 : 0 : z_4 \mid 0 : 0 : w_3 : 0], \xi)$$

has isotropy weights

$$\begin{aligned} ([sz_1 : tx_2 : x_3 : z_4 \mid s^{-1}y_1 : t^{-1}y_2 : w_3 : y_4], \xi) &\sim ([z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \mid y_1 : st^{-1}y_2 : sw_3 : y_4], \xi) \\ &\sim ([z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \mid s^{-1}y_1 : t^{-1}y_2 : w_3 : s^{-1}y_4], s^{-1}\xi) \\ &\implies (s^{-1}tz_2, s^{-1}w_1, t^{-1}w_2, s^{-1}\xi) \longleftrightarrow (s^{-1}t, s^{-1}, t^{-1}, s^{-1}). \end{aligned}$$

so normal weights  $(s^{-1}t, t^{-1})$  from  $(z_2, w_2)$  respectively, and inwards-pointing weight  $s^{-1}$  with multiplicity 2 coming from  $w_1$  and  $\xi$ , since  $|z_1|$  achieves its maximum at  $Q_{23}^{(2)}$ .

$Q_{23}^{(3)}:$

$$Q_{23}^{(3)} = ([z_1 : 0 : 0 : z_4 \mid 0 : w_2 : 0 : 0], \xi)$$

has isotropy weights

$$\begin{aligned} ([sz_1 : tx_2 : x_3 : z_4 \mid s^{-1}y_1 : t^{-1}w_2 : y_3 : y_4], \xi) &\sim ([z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \mid y_1 : st^{-1}w_2 : sy_3 : y_4], \xi) \\ &\sim ([z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \mid s^{-1}ty_1 : w_2 : ty_3 : s^{-1}ty_4], s^{-1}t\xi) \\ &\implies (s^{-1}z_3, s^{-1}tw_1, tw_3, s^{-1}t\xi) \longleftrightarrow (s^{-1}, s^{-1}t, t, s^{-1}t). \end{aligned}$$

so normal weights  $(s^{-1}, t)$  from  $(z_3, w_3)$  respectively, and inwards-pointing weight  $s^{-1}t$  with multiplicity 2 coming from  $w_1$  and  $\xi$ , since  $|z_1|$  and  $|z_4|$  achieve their maximum at  $Q_{23}^{(3)}$ . (???)

$Q_{14}^{(4)}:$

$$Q_{14}^{(4)} = ([z_1 : z_2 : 0 : 0 \mid 0 : 0 : w_3 : 0], \xi)$$

has isotropy weights

$$\begin{aligned} ([sz_1 : tz_2 : x_3 : x_4 \mid s^{-1}y_1 : t^{-1}y_2 : w_3 : y_4], \xi) &\sim ([z_1 : s^{-1}tz_2 : s^{-1}x_3 : x_4 \mid y_1 : st^{-1}y_2 : sw_3 : y_4], \xi) \\ &\sim ([z_1 : z_2 : s^{-1}x_3 : st^{-1}x_4 \mid y_1 : y_2 : sw_3 : s^{-1}ty_4], \xi) \\ &\sim ([z_1 : z_2 : s^{-1}x_3 : st^{-1}x_4 \mid s^{-1}y_1 : s^{-1}y_2 : w_3 : s^{-2}ty_4], s^{-1}\xi) \\ &\implies (st^{-1}z_4, s^{-1}w_1, s^{-2}tw_4, s^{-1}\xi) \longleftrightarrow (st^{-1}, s^{-1}, s^{-2}t, s^{-1}). \end{aligned}$$

so normal weights  $(st^{-1}, s^{-2}t)$  from  $(z_4, w_4)$  respectively, and inwards-pointing weight  $s^{-1}$  with multiplicity 2 coming from  $w_1$  and  $\xi$ , since  $|z_1|$  achieves its maximum at  $Q_{14}^{(4)}$ .

$Q_{34}^{(4)}:$

$$Q_{34}^{(4)} = ([0 : z_2 : z_3 : 0 \mid w_1 : 0 : 0 : 0], \xi)$$

has isotropy weights

$$\begin{aligned} ([sx_1 : tz_2 : z_3 : x_4 \mid s^{-1}w_1 : t^{-1}y_2 : y_3 : y_4], \xi) &\sim ([sx_1 : z_2 : z_3 : t^{-1}x_4 \mid s^{-1}w_1 : y_2 : y_3 : ty_4], \xi) \\ &\sim ([sx_1 : z_2 : z_3 : t^{-1}x_4 \mid w_1 : sy_2 : sy_3 : sty_4], s\xi) \\ &\implies (sz_1, t^{-1}z_4, stw_4, s\xi) \longleftrightarrow (s, t^{-1}, st, s). \end{aligned}$$

so normal weights  $(t^{-1}, st)$  from  $(z_4, w_4)$  respectively, and inwards-pointing weight  $s$  with multiplicity 2 coming from  $z_1$  and  $\xi$ , since  $|w_1|$  achieves its maximum at  $Q_{34}^{(4)}$ .

$Q_{34}^{(3)}:$

$$Q_{34}^{(3)} = ([0 : z_2 : 0 : 0 \mid w_1 : 0 : 0 : w_4], \xi)$$

has isotropy weights

$$\begin{aligned} ([sx_1 : tz_2 : x_3 : x_4 \mid s^{-1}w_1 : t^{-1}y_2 : y_3 : w_4], \xi) \\ \sim ([x_1 : s^{-1}tz_2 : s^{-1}x_3 : x_4 \mid w_1 : st^{-1}y_2 : sy_3 : w_4], \xi) \\ \sim ([s^{1/2}t^{-1/2}x_1 : z_2 : s^{-1/2}t^{1/2}x_3 : s^{1/2}t^{-1/2}x_4 \mid w_1 : s^{1/2}t^{-1/2}y_2 : sy_3 : w_4], \xi) \\ \implies (s^{-1/2}t^{-1/2}z_3, s^{1/2}t^{-1/2}z_4, sw_3, s^{1/2}t^{-1/2}\xi) \longleftrightarrow (s^{-1/2}t^{-1/2}, s^{1/2}t^{-1/2}, s, s^{1/2}t^{-1/2}). \end{aligned}$$

so normal weights  $(s^{-1/2}t^{-1/2})$  from  $(z_3, w_3)$  respectively, and inwards-pointing weight  $s^{1/2}t^{-1/2}$  with multiplicity 2 coming from  $z_4$  and  $\xi$ , since  $|w_4|$  achieves its maximum at  $Q_{34}^{(3)}$ .

$Q_{14}^{(1)}:$

$$Q_{14}^{(1)} = ([0 : z_2 : 0 : 0 \mid 0 : 0 : w_3 : w_4], \xi)$$

has isotropy weights

$$\begin{aligned} ([sx_1 : tz_2 : x_3 : x_4 \mid s^{-1}y_1 : t^{-1}y_2 : w_3 : w_4], \xi) \\ \sim ([st^{-1/2}x_1 : z_2 : t^{-1/2}x_3 : t^{-1/2}x_4 \mid s^{-1}y_1 : t^{-1/2}y_2 : w_3 : w_4], t^{-1/2}\xi) \\ \implies (st^{-1/2}z_1, t^{-1/2}z_4, s^{-1}w_1, t^{-1/2}\xi) \longleftrightarrow (st^{-1/2}, t^{-1/2}, s^{-1}, t^{-1/2}). \end{aligned}$$

so normal weights  $(st^{-1/2}, s^{-1})$  from  $(z_1, w_1)$  respectively, and inwards-pointing weight  $t^{-1/2}$  with multiplicity 2 coming from  $z_4$  and  $\xi$ , since  $|w_4|$  achieves its maximum at  $Q_{14}^{(1)}$ .

*Remark 1. For the exterior fixed-points  $Q_{34}^{(3)}$  and  $Q_{14}^{(1)}$ , where there are two non-zero cotangent  $w$  coordinates, the relation*

$$\begin{aligned} ([\tau z_1 : \tau^2 z_2 : \tau z_3 : \tau z_4 \mid w_1 : w_2 : w_3 : w_4], \xi) &\sim ([z_1 : \tau z_2 : z_3 : \tau z_4 \mid \tau w_1 : \tau w_2 : \tau w_3 : w_4], \xi) \\ &\sim ([z_1 : z_2 : z_3 : z_4 \mid \tau w_1 : \tau^2 w_2 : \tau w_3 : \tau w_4], \xi) \\ &\sim ([z_1 : z_2 : z_3 : z_4 \mid w_1 : \tau w_2 : w_3 : w_4], \tau^{-1}\xi), \end{aligned}$$

*is used.*

## References