Polyptych Isotropy Weights

General Notes

Abstract

Calculations for the isotropy data of the compactified hypertoric manifolds.

1 Example: $M = T^*\mathbb{CP}^1$

Short exact sequence for the usual Delzant construction of $\mathbb{CP}^1$:

$$\{1\} \longrightarrow K \cong S^1 \xhookrightarrow{t \longmapsto (t,t)} T^2 \xrightarrow[(a,b) \longmapsto ab^{-1}]{} T^2/K \cong T^1 \longrightarrow \{1\}.$$

The induced action of K on $T^*\mathbb{C}^2$ is thus

$$t \cdot (z|w) \longmapsto (t,t) \cdot (z_1, z_2 \,|\, w_1, w_2) = (tz_1, tz_2 \,|\, t^{-1}w_1, t^{-1}w_2),$$

which is Hamiltonian with associated moment map

$$\mu_{\mathbb{R}} : T^*\mathbb{C} \longrightarrow \mathbb{R}; \qquad \mu_{\mathbb{R}}(z \,|\, w) = |z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2.$$

For some $a \in \mathbb{Z}_{>0}$, take the hyperkähler quotient of $T^*\mathbb{C}$ to get

$$M := T^*\mathbb{C} \;/\!\!/\!\!/\!\!/\; K$$

2 Example: $M = T^*(\mathbb{CP}^2 \times \mathbb{CP}^2)$ (Non-Convex Core)

2.1 Construction

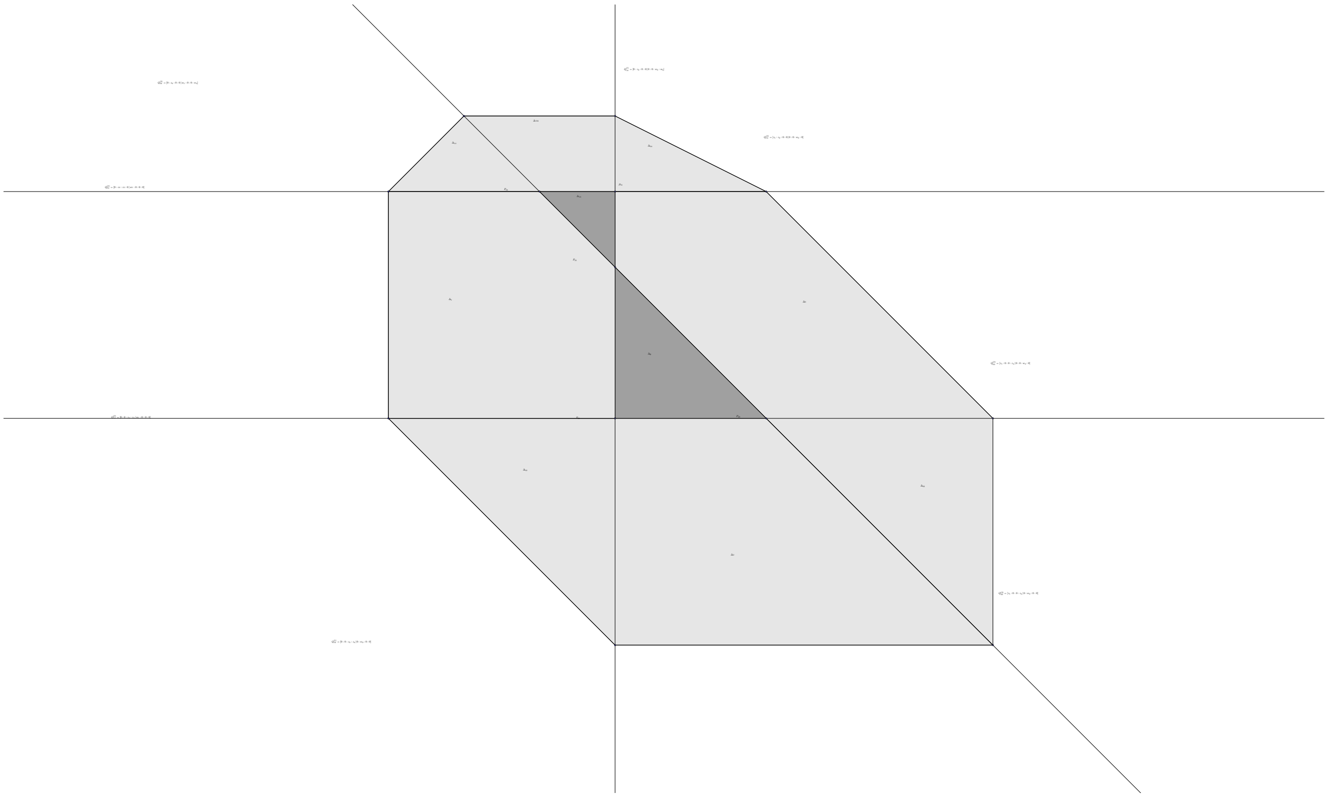
Quotient relations arising from $K \cong T^2$:

$$[sz_1 : sz_2 : sz_3 : z_4 \,|\, s^{-1}w_1 : s^{-1}w_2 : s^{-1}w_3 : w_4] = [z_1 : z_2 : z_3 : z_4 \,|\, w_1 : w_2 : w_3 : w_4],$$

and

$$[z_1 : tz_2 : z_3 : tz_4 \,|\, w_1 : t^{-1}w_2 : w_3 : t^{-1}w_4] = [z_1 : z_2 : z_3 : z_4 \,|\, w_1 : w_2 : w_3 : w_4].$$

2.2 Figure



2.3 Isotropy Weights

$Q^{(1)}_{12}$:

$$Q^{(1)}_{12} = ([0:0:z_3:z_4\,|\,0:w_2:0:0],\xi)\,,\text{ with } |w_2|^2 = a,\, \xi = 0.$$

has isotropy weights

$$\begin{aligned}\big([sx_1:tx_2:z_3:z_4\,|\,s^{-1}y_1:t^{-1}w_2:y_3:y_4],\xi\big) &\sim \big([sx_1:tx_2:z_3:z_4\,|\,s^{-1}ty_1:w_2:ty_3:ty_4],t\xi\big)\\ &\implies (sz_1,tz_2,s^{-1}tw_1,t\xi) \longleftrightarrow (s,t,s^{-1}t,t),\end{aligned}$$

so normal weights $(s, s^{-1}t)$ from (z_1, w_1) respectively, and inwards-pointing weight t with multiplicity 2 coming from z_2 and ξ, since $|w_2|$ achieves its maximum at $Q^{(1)}_{12}$.

$Q^{(2)}_{12}$:

$$Q^{(2)}_{12} = ([0:0:z_3:z_4\,|\,w_1:0:0:0],\xi)$$

has isotropy weights

$$\begin{aligned}\big([sx_1:tx_2:z_3:z_4\,|\,s^{-1}w_1:t^{-1}y_2:y_3:y_4],\xi\big) &\sim \big([sx_1:tx_2:z_3:z_4\,|\,w_1:st^{-1}y_2:sy_3:sy_4],s\xi\big)\\ &\implies (sz_1,tz_2,s^{-1}tw_1,t\xi) \longleftrightarrow (s,t,st^{-1},s).\end{aligned}$$

so normal weights (t, st^{-1}) from (z_2, w_2) respectively, and inwards-pointing weight s with multiplicity 2 coming from z_1 and ξ, since $|w_1|$ achieves its maximum at $Q^{(2)}_{12}$.

To do (??)

$Q^{(2)}_{23}$:

$$Q^{(2)}_{23} = ([z_1:0:0:z_4\,|\,0:0:w_3:0],\xi)$$

has isotropy weights

$$\begin{aligned}\big([sz_1 : tx_2 : x_3 : z_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : w_3 : y_4], \xi\big) &\sim \big([z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \,|\, y_1 : st^{-1}y_2 : sw_3 : y_4], \xi\big)\\ &\sim \big([z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : w_3 : s^{-1}y_4], s^{-1}\xi\big)\\ &\implies (s^{-1}tz_2, s^{-1}w_1, t^{-1}w_2, s^{-1}\xi) \longleftrightarrow (s^{-1}t, s^{-1}, t^{-1}, s^{-1}).\end{aligned}$$

so normal weights $(s^{-1}t, t^{-1})$ from (z_2, w_2) respectively, and inwards-pointing weight s^{-1} with multiplicity 2 coming from w_1 and ξ, since $|z_1|$ achieves its maximum at $Q_{23}^{(2)}$.

$Q_{23}^{(3)}$:

$$Q_{23}^{(3)} = ([z_1 : 0 : 0 : z_4 \,|\, 0 : w_2 : 0 : 0], \xi)$$

has isotropy weights

$$\begin{aligned}\big([sz_1 : tx_2 : x_3 : z_4 \,|\, s^{-1}y_1 : t^{-1}w_2 : y_3 : y_4], \xi\big) &\sim \big([z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \,|\, y_1 : st^{-1}w_2 : sy_3 : y_4], \xi\big)\\ &\sim \big([z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \,|\, s^{-1}ty_1 : w_2 : ty_3 : s^{-1}ty_4], s^{-1}t\xi\big)\\ &\implies (s^{-1}z_3, s^{-1}tw_1, tw_3, s^{-1}t\xi) \longleftrightarrow (s^{-1}, s^{-1}t, t, s^{-1}t).\end{aligned}$$

so normal weights (s^{-1}, t) from (z_3, w_3) respectively, and inwards-pointing weight $s^{-1}t$ with multiplicity 2 coming from w_1 and ξ, since $|z_1|$ and $|z_4|$ achieve their maximum at $Q_{23}^{(3)}$. (???)

$Q_{14}^{(4)}$:

$$Q_{14}^{(4)} = ([z_1 : z_2 : 0 : 0 \,|\, 0 : 0 : w_3 : 0], \xi)$$

has isotropy weights

$$\begin{aligned}\big([sz_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : w_3 : y_4], \xi\big) &\sim \big([z_1 : s^{-1}tz_2 : s^{-1}x_3 : x_4 \,|\, y_1 : st^{-1}y_2 : sw_3 : y_4], \xi\big)\\ &\sim \big([z_1 : z_2 : s^{-1}x_3 : st^{-1}x_4 \,|\, y_1 : y_2 : sw_3 : s^{-1}ty_4], \xi\big)\\ &\sim \big([z_1 : z_2 : s^{-1}x_3 : st^{-1}x_4 \,|\, s^{-1}y_1 : s^{-1}y_2 : w_3 : s^{-2}ty_4], s^{-1}\xi\big)\\ &\implies (st^{-1}z_4, s^{-1}w_1, s^{-2}tw_4, s^{-1}\xi) \longleftrightarrow (st^{-1}, s^{-1}, s^{-2}t, s^{-1}).\end{aligned}$$

so normal weights $(st^{-1}, s^{-2}t)$ from (z_4, w_4) respectively, and inwards-pointing weight s^{-1} with multiplicity 2 coming from w_1 and ξ, since $|z_1|$ achieves its maximum at $Q_{14}^{(4)}$.

$Q_{34}^{(4)}$:

$$Q_{34}^{(4)} = ([0 : z_2 : z_3 : 0 \,|\, w_1 : 0 : 0 : 0], \xi)$$

has isotropy weights

$$\begin{aligned}\big([sx_1 : tz_2 : z_3 : x_4 \,|\, s^{-1}w_1 : t^{-1}y_2 : y_3 : y_4], \xi\big) &\sim \big([sx_1 : z_2 : z_3 : t^{-1}x_4 \,|\, s^{-1}w_1 : y_2 : y_3 : ty_4], \xi\big)\\ &\sim \big([sx_1 : z_2 : z_3 : t^{-1}x_4 \,|\, w_1 : sy_2 : sy_3 : sty_4], s\xi\big)\\ &\implies (sz_1, t^{-1}z_4, stw_4, s\xi) \longleftrightarrow (s, t^{-1}, st, s).\end{aligned}$$

so normal weights (t^{-1}, st) from (z_4, w_4) respectively, and inwards-pointing weight s with multiplicity 2 coming from z_1 and ξ, since $|w_1|$ achieves its maximum at $Q_{34}^{(4)}$.

$Q_{34}^{(3)}$:

$$Q_{34}^{(3)} = ([0 : z_2 : 0 : 0 \,|\, w_1 : 0 : 0 : w_4], \xi)$$

has isotropy weights

$$\begin{aligned}\big([sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}w_1 : t^{-1}y_2 : y_3 : w_4], \xi\big) &\sim \big([sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}w_1 : t^{-1}y_2 : y_3 : w_4], \xi\big)\\ &\sim \big([sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}w_1 : t^{-1}y_2 : y_3 : w_4], \xi\big)\\ &\implies () \longleftrightarrow ().\end{aligned}$$

so normal weights () from () respectively, and inwards-pointing weight ? with multiplicity 2 coming from ? and ξ, since $|?|$ achieves its maximum at $Q_{34}^{(3)}$.

$Q^{(1)}_{14}$:

$$Q^{(1)}_{14} = ([0 : z_2 : 0 : 0 \,|\, 0 : 0 : w_3 : w_4], \xi)$$

has isotropy weights

$$\begin{aligned}\big([sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : w_3 : w_4], \xi\big) &\sim \big([sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : w_3 : w_4], \xi\big) \\ &\sim \big([sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : w_3 : w_4], \xi\big) \\ &\implies () \longleftrightarrow ().\end{aligned}$$

so normal weights () from () respectively, and inwards-pointing weight ? with multiplicity 2 coming from ? and ξ, since $|?|$ achieves its maximum at $Q^{(1)}_{14}$.

References