Polyptych Isotropy Weights

General Notes

Abstract

Calculations for the isotropy data of the compactified hypertoric manifolds.

1 Example: $M = T^*\mathbb{CP}^1$

1.1 Construction

Short exact sequence for the usual Delzant construction of $\mathbb{CP}^1$:

$$\{1\} \longrightarrow K \cong S^1 \xhookrightarrow{t \longmapsto (t,t)} T^2 \underset{(a,b)\longmapsto ab^{-1}}{\twoheadrightarrow} T^2/K \cong T^1 \longrightarrow \{1\}.$$

The induced action of K on $T^*\mathbb{C}^2$ is thus

$$t \cdot (z|w) \longmapsto (t,t) \cdot (z_1, z_2 \,|\, w_1, w_2) = (tz_1, tz_2 \,|\, t^{-1}w_1, t^{-1}w_2),$$

which is Hamiltonian with associated moment map

$$\mu_{\mathbb{R}} : T^*\mathbb{C} \longrightarrow \mathbb{R}; \qquad \mu_{\mathbb{R}}(z \,|\, w) = |z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2.$$

For some $a \in \mathbb{Z}_{>0}$, take the hyperkähler quotient of $T^*\mathbb{C}$ to get

$$M := T^*\mathbb{C} /\!/\!/\!/ K$$

2 Example: $M = T^*(\mathbb{CP}^2 \times \mathbb{CP}^2)$ (Non-Convex Core)

2.1 Construction

Short exact sequence for the usual Delzant construction of the non-convex $T^*(\mathbb{CP}^2 \times \mathbb{CP}^2)$:

$$\{1\} \longrightarrow K \cong T^2 \xhookrightarrow{(s,t)\longmapsto(s,st,s,t)} T^4 \underset{(a,b,c,d)\longmapsto(ac^{-1},bc^{-1}d^{-1})}{\twoheadrightarrow} T^4/K \cong T^2 \longrightarrow \{1\}.$$

The induced action of K on $T^*\mathbb{C}^4$ is thus

$$(s,t)\cdot(z|w) \longmapsto (s,st,s,t)\cdot(z_1,z_2,z_3,z_4 \,|\, w_1,w_2,w_3,w_4) = (sz_1, stz_2, sz_3, tz_4 \,|\, s^{-1}w_1, s^{-1}t^{-1}w_2, s^{-1}w_3, t^{-1}w_4),$$

which is Hamiltonian with associated moment map

$$\mu_{\mathbb{R}} : T^*\mathbb{C}^4 \longrightarrow \mathbb{R}^2; \qquad \mu_{\mathbb{R}}(z \,|\, w) = \begin{pmatrix} |z_1|^2 + |z_2|^2 + |z_3|^2 - |w_1|^2 - |w_2|^2 - |w_3|^2 \\ |z_2|^2 + |z_4|^2 - |w_2|^2 - |w_4|^2 \end{pmatrix}.$$

For some $(n,m) \in \mathbb{Z}^2_{>0}$, take the hyperkähler quotient of $T^*\mathbb{C}^4$ to get

$$M := T^*\mathbb{C}^4 /\!/\!/\!/_{(n,m)} K := \left(\mu_{\mathbb{R}}^{-1}(n,m) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/K.$$

Quotient relations arising from $K \cong T^2$:

$(s,1) \in S^1 \times \{e\} < T^2$:

$$[sz_1 : sz_2 : sz_3 : z_4 \,|\, s^{-1}w_1 : s^{-1}w_2 : s^{-1}w_3 : w_4] = [z_1 : z_2 : z_3 : z_4 \,|\, w_1 : w_2 : w_3 : w_4],$$

$(1,t) \in \{e\} \times S^1 < T^2$:

$$[z_1 : tz_2 : z_3 : tz_4 \,|\, w_1 : t^{-1}w_2 : w_3 : t^{-1}w_4] = [z_1 : z_2 : z_3 : z_4 \,|\, w_1 : w_2 : w_3 : w_4].$$

2.2 Figure

2.3 Isotropy Weights

2.3.1 Interior Points

P_{12}:

$$P_{12} = [0 : 0 : z_3 : z_4 \,|\, 0 : 0 : 0 : 0],$$

has isotropy weights

$$[sx_1 : tx_2 : z_3 : z_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : y_3 : y_4] \implies (sx_1, tx_2, s^{-1}y_1, t^{-1}y_2) \longleftrightarrow (s, t, s^{-1}, t^{-1}),$$

so tangent space weights (s, t) from (z_1, z_2) respectively, and cotangent space weights (s^{-1}, t^{-1}) coming from (w_1, w_2) respectively.

P_{23}:

$$P_{23} = [z_1 : 0 : 0 : z_4 \,|\, 0 : 0 : 0 : 0],$$

has isotropy weights

$$\begin{aligned}[sz_1 : tx_2 : x_3 : z_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : y_3 : y_4] &\sim [z_1 : s^{-1}tx_2 : s^{-1}x_3 : z_4 \,|\, y_1 : st^{-1}y_2 : sy_3 : y_4] \\ &\implies (s^{-1}tx_2, s^{-1}x_3, st^{-1}y_2, sy_3) \longleftrightarrow (s^{-1}t, s^{-1}, st^{-1}, s),\end{aligned}$$

so tangent space weights $(s^{-1}t, s^{-1})$ from (z_2, z_3) respectively, and cotangent space weights (st^{-1}, s) coming from (w_2, w_3) respectively.

P_{13}:

$$P_{13} = [0 : z_2 : 0 : z_4 \,|\, 0 : 0 : 0 : 0],$$

has isotropy weights

$$\begin{aligned}[sx_1 : tz_2 : x_3 : z_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : y_3 : y_4] &\sim [st^{-1}x_1 : z_2 : t^{-1}x_3 : z_4 \,|\, s^{-1}ty_1 : y_2 : ty_3 : y_4] \\ &\implies (st^{-1}x_1, t^{-1}x_3, s^{-1}ty_1, ty_3) \longleftrightarrow (st^{-1}, t^{-1}, s^{-1}t, t),\end{aligned}$$

so tangent space weights (st^{-1}, t^{-1}) from (z_1, z_3) respectively, and cotangent space weights $(s^{-1}t, t)$ coming from (w_1, w_3) respectively.

P_{14}:

$$P_{14} = [0 : z_2 : 0 : 0 \,|\, 0 : 0 : w_3 : 0],$$

has isotropy weights

$$\begin{aligned}[sx_1 : tz_2 : x_3 : z_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : w_3 : y_4] &\sim [sx_1 : z_2 : x_3 : t^{-1}x_4 \,|\, s^{-1}y_1 : y_2 : w_3 : ty_4] \\ &\implies (sx_1, t^{-1}x_4, s^{-1}y_1, ty_4) \longleftrightarrow (s, t^{-1}, s^{-1}, t),\end{aligned}$$

so tangent space weights (s^{-1}, t^{-1}) from (w_1, z_4) respectively, and cotangent space weights (s, t) coming from (z_1, w_4) respectively.

P_{34}:

$$P_{34} = [0 : z_2 : 0 : 0 \,|\, w_1 : 0 : 0 : 0],$$

has isotropy weights

$$\begin{aligned}[sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}w_1 : t^{-1}y_2 : y_3 : y_4] &\sim [x_1 : s^{-1}tz_2 : s^{-1}x_3 : x_4 \,|\, w_1 : st^{-1}y_2 : sy_3 : y_4] \\ &\sim [x_1 : z_2 : s^{-1}x_3 : st^{-1}x_4 \,|\, w_1 : y_2 : sy_3 : s^{-1}ty_4] \\ &\implies (s^{-1}x_3, st^{-1}x_4, sy_3, s^{-1}ty_4) \longleftrightarrow (s^{-1}, st^{-1}, s, s^{-1}t),\end{aligned}$$

so tangent space weights (st^{-1}, s) from (z_4, w_3) respectively, and cotangent space weights $(s^{-1}, s^{-1}t)$ coming from (z_3, w_4) respectively.

2.3.2 Exterior Points

$Q_{12}^{(1)}$: Locally near $Q_{12}^{(1)}$, S_A^1 acts as $(\tau,\tau,1,1)$.

$$Q_{12}^{(1)} = ([0:0:z_3:z_4 \,|\, 0:w_2:0:0],\xi)\,,\ \text{with } |w_2|^2 = a,\ \xi = 0.$$

has isotropy weights

$$\begin{aligned}\big([sx_1:tx_2:z_3:z_4 \,|\, s^{-1}y_1:t^{-1}w_2],\xi\big) &\sim \big([sx_1:tx_2:z_3:z_4 \,|\, s^{-1}ty_1:w_2],t\xi\big)\\ &\implies (sz_1,tz_2,s^{-1}tw_1,t\xi) \longleftrightarrow (s,t,s^{-1}t,t),\end{aligned}$$

so normal weights $(s,s^{-1}t)$ from (z_1,w_1) respectively, and inwards-pointing weight t with multiplicity 2 coming from z_2 and ξ, since $|w_2|$ achieves its maximum at $Q_{12}^{(1)}$.

$Q_{12}^{(2)}$: Locally near $Q_{12}^{(2)}$, S_A^1 acts as $(\tau,\tau,1,1)$.

$$Q_{12}^{(2)} = ([0:0:z_3:z_4 \,|\, w_1:0:0:0],\xi)$$

has isotropy weights

$$\begin{aligned}\big([sx_1:tx_2:z_3:z_4 \,|\, s^{-1}w_1:t^{-1}y_2],\xi\big) &\sim \big([sx_1:tx_2:z_3:z_4 \,|\, w_1:st^{-1}y_2],s\xi\big)\\ &\implies (sz_1,tz_2,st^{-1}w_2,s\xi) \longleftrightarrow (s,t,st^{-1},s).\end{aligned}$$

so normal weights (t,st^{-1}) from (z_2,w_2) respectively, and inwards-pointing weight s with multiplicity 2 coming from z_1 and ξ, since $|w_1|$ achieves its maximum at $Q_{12}^{(2)}$.

$Q_{23}^{(2)}$: Locally near $Q_{23}^{(2)}$, S_A^1 acts as $(1,\tau,\tau,1)$.

$$Q_{23}^{(2)} = ([z_1:0:0:z_4 \,|\, 0:0:w_3:0],\xi)$$

has isotropy weights

$$\begin{aligned}\big([sz_1:tx_2:x_3:z_4 \,|\, s^{-1}y_1:t^{-1}y_2:w_3:y_4],\xi\big) &\sim \big([z_1:s^{-1}tx_2:s^{-1}x_3:z_4 \,|\, y_1:st^{-1}y_2:sw_3:y_4],\xi\big)\\ &\sim \big([z_1:s^{-1}tx_2:s^{-1}x_3:z_4 \,|\, s^{-1}y_1:t^{-1}y_2:w_3:s^{-1}y_4],s^{-1}\xi\big)\\ &\implies (s^{-1}tz_2,s^{-1}z_3,t^{-1}w_2,s^{-1}\xi) \longleftrightarrow (s^{-1}t,s^{-1},t^{-1},s^{-1}).\end{aligned}$$

so normal weights $(s^{-1}t,t^{-1})$ from (z_2,w_2) respectively, and inwards-pointing weight s^{-1} with multiplicity 2 coming from z_3 and ξ, since $|z_1|$ achieves its maximum at $Q_{23}^{(2)}$.

$Q_{23}^{(3)}$:

$$Q_{23}^{(3)} = ([z_1:0:0:z_4 \,|\, 0:w_2:0:0],\xi)$$

has isotropy weights

$$\begin{aligned}\big([sz_1:tx_2:x_3:z_4 \,|\, s^{-1}y_1:t^{-1}w_2:y_3:y_4],\xi\big) &\sim \big([z_1:s^{-1}tx_2:s^{-1}x_3:z_4 \,|\, y_1:st^{-1}w_2:sy_3:y_4],\xi\big)\\ &\sim \big([z_1:s^{-1}tx_2:s^{-1}x_3:z_4 \,|\, s^{-1}ty_1:w_2:ty_3:s^{-1}ty_4],s^{-1}t\xi\big)\\ &\implies (s^{-1}z_3,s^{-1}tw_1,tw_3,s^{-1}t\xi) \longleftrightarrow (s^{-1},s^{-1}t,t,s^{-1}t).\end{aligned}$$

so normal weights (s^{-1},t) from (z_3,w_3) respectively, and inwards-pointing weight $s^{-1}t$ with multiplicity 2 coming from w_1 and ξ, since $|z_1|$ and $|z_4|$ achieve their maximum at $Q_{23}^{(3)}$. (???)

$Q_{14}^{(4)}$:

$$Q_{14}^{(4)} = ([z_1:z_2:0:0 \,|\, 0:0:w_3:0],\xi)$$

has isotropy weights

$$\begin{aligned}\big([sz_1:tz_2:x_3:x_4 \,|\, s^{-1}y_1:t^{-1}y_2:w_3:y_4],\xi\big) &\sim \big([z_1:s^{-1}tz_2:s^{-1}x_3:x_4 \,|\, y_1:st^{-1}y_2:sw_3:y_4],\xi\big)\\ &\sim \big([z_1:z_2:s^{-1}x_3:st^{-1}x_4 \,|\, y_1:y_2:sw_3:s^{-1}ty_4],\xi\big)\\ &\sim \big([z_1:z_2:s^{-1}x_3:st^{-1}x_4 \,|\, s^{-1}y_1:s^{-1}y_2:w_3:s^{-2}ty_4],s^{-1}\xi\big)\\ &\implies (st^{-1}z_4,s^{-1}w_1,s^{-2}tw_4,s^{-1}\xi) \longleftrightarrow (st^{-1},s^{-1},s^{-2}t,s^{-1}).\end{aligned}$$

so normal weights $(st^{-1},s^{-2}t)$ from (z_4,w_4) respectively, and inwards-pointing weight s^{-1} with multiplicity 2 coming from w_1 and ξ, since $|z_1|$ achieves its maximum at $Q_{14}^{(4)}$.

$Q_{34}^{(4)}$:

$$Q_{34}^{(4)} = ([0 : z_2 : z_3 : 0 \,|\, w_1 : 0 : 0 : 0], \xi)$$

has isotropy weights

$$\begin{aligned}\big([sx_1 : tz_2 : z_3 : x_4 \,|\, s^{-1}w_1 : t^{-1}y_2 : y_3 : y_4], \xi\big) &\sim \big([sx_1 : z_2 : z_3 : t^{-1}x_4 \,|\, s^{-1}w_1 : y_2 : y_3 : ty_4], \xi\big)\\ &\sim \big([sx_1 : z_2 : z_3 : t^{-1}x_4 \,|\, w_1 : sy_2 : sy_3 : sty_4], s\xi\big)\\ &\implies (sz_1, t^{-1}z_4, stw_4, s\xi) \longleftrightarrow (s, t^{-1}, st, s).\end{aligned}$$

so normal weights (t^{-1}, st) from (z_4, w_4) respectively, and inwards-pointing weight s with multiplicity 2 coming from z_1 and ξ, since $|w_1|$ achieves its maximum at $Q_{34}^{(4)}$.

$Q_{34}^{(3)}$:

$$Q_{34}^{(3)} = ([0 : z_2 : 0 : 0 \,|\, w_1 : 0 : 0 : w_4], \xi)$$

has isotropy weights

$$\begin{aligned}&\big([sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}w_1 : t^{-1}y_2 : y_3 : w_4], \xi\big)\\ &\sim \big([x_1 : s^{-1}tz_2 : s^{-1}x_3 : x_4 \,|\, w_1 : st^{-1}y_2 : sy_3 : w_4], \xi\big)\\ &\sim \Big([s^{1/2}t^{-1/2}x_1 : z_2 : s^{-1/2}t^{1/2}x_3 : s^{1/2}t^{-1/2}x_4 \,|\, w_1 : s^{1/2}t^{-1/2}y_2 : sy_3 : w_4], \xi\Big)\\ &\implies (s^{-1/2}t^{-1/2}z_3, s^{1/2}t^{-1/2}z_4, sw_3, s^{1/2}t^{-1/2}\xi) \longleftrightarrow (s^{-1/2}t^{-1/2}, s^{1/2}t^{-1/2}, s, s^{1/2}t^{-1/2}).\end{aligned}$$

so normal weights $(s^{-1/2}t^{-1/2})$ from (z_3, w_3) respectively, and inwards-pointing weight $s^{1/2}t^{-1/2}$ with multiplicity 2 coming from z_4 and ξ, since $|w_4|$ achieves its maximum at $Q_{34}^{(3)}$.

$Q_{14}^{(1)}$:

$$Q_{14}^{(1)} = ([0 : z_2 : 0 : 0 \,|\, 0 : 0 : w_3 : w_4], \xi)$$

has isotropy weights

$$\begin{aligned}&\big([sx_1 : tz_2 : x_3 : x_4 \,|\, s^{-1}y_1 : t^{-1}y_2 : w_3 : w_4], \xi\big)\\ &\sim \Big([st^{-1/2}x_1 : z_2 : t^{-1/2}x_3 : t^{-1/2}x_4 \,|\, s^{-1}y_1 : t^{-1/2}y_2 : w_3 : w_4], t^{-1/2}\xi\Big)\\ &\implies (st^{-1/2}z_1, t^{-1/2}z_4, s^{-1}w_1, t^{-1/2}\xi) \longleftrightarrow (st^{-1/2}, t^{-1/2}, s^{-1}, t^{-1/2}).\end{aligned}$$

so normal weights $(st^{-1/2}, s^{-1})$ from (z_1, w_1) respectively, and inwards-pointing weight $t^{-1/2}$ with multiplicity 2 coming from z_4 and ξ, since $|w_4|$ achieves its maximum at $Q_{14}^{(1)}$.

Remark 1. *For the exterior fixed-points $Q_{34}^{(3)}$ and $Q_{14}^{(1)}$, where there are two non-zero cotangent w coordinates, the relation*

$$\begin{aligned}\big([\tau z_1 : \tau^2 z_2 : \tau z_3 : \tau z_4 \,|\, w_1 : w_2 : w_3 : w_4], \xi\big) &\sim \big([z_1 : \tau z_2 : z_3 : \tau z_4 \,|\, \tau w_1 : \tau w_2 : \tau w_3 : w_4], \xi\big)\\ &\sim \big([z_1 : z_2 : z_3 : z_4 \,|\, \tau w_1 : \tau^2 w_2 : \tau w_3 : \tau w_4], \xi\big)\\ &\sim \big([z_1 : z_2 : z_3 : z_4 \,|\, w_1 : \tau w_2 : w_3 : w_4], \tau^{-1}\xi\big),\end{aligned}$$

is used.

3 Normal Bundles

For any subset $A \subseteq \{1, \ldots, n\}$, let

$$\mathbb{H}^n_A := \{\, (z, w) \in \mathbb{H}^n \mid z_i = 0 \text{ if } i \in A,\ w_i = 0 \text{ if } i \notin A \,\} \cong \mathbb{C}^n.$$

Then for $\mu_{\mathbb{C}} : T^*\mathbb{C}^n \to \mathfrak{k}^* \otimes \mathbb{C}$ given by

$$\mu_{\mathbb{C}}(z, w) = \sum_{j=1}^{n} z_j w_j \delta_j \in \mathfrak{k}^*_{\mathbb{C}}, \quad \text{with } \{\delta_j\}_{j=1}^n \text{ a basis for } \mathfrak{k}^*_{\mathbb{C}},$$

one has that

$$\mathbb{H}^n_A \subseteq \mu_{\mathbb{C}}^{-1}(0) = \left\{ (z,w) \in \mathbb{H}^n \;\middle|\; \sum_{j=1}^{n} z_j w_j \delta_j = 0 \right\}.$$

Now one also has the toric hyperkähler quotient manifold

$$M = \left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/K = T^*\mathbb{C}^n \,/\!\!/\!\!/\!\!/_{(\alpha,0)}\, K,$$

whose extended core is defined to be

$$\mathcal{E} := \{\, [z,w] \in M \mid z_i w_i = 0 \text{ for all } i \,\},$$

which decomposes into subvarieties, or components, of the extended core

$$\mathcal{E}_A := \{\, [z,w] \in M \mid z_i = 0 \text{ for all } i \in A,\ w_i = 0 \text{ for all } i \notin A \,\}.$$

Note that

$$\mathcal{E}_A = \left(\mathbb{H}^n_A \cap \mu_{\mathbb{R}}^{-1}(\alpha)\right)/K$$

and hence $\mathcal{E}_A$ is itself a Kähler subvariety of M with $\dim_{\mathbb{C}} \mathcal{E}_A = n - k = d$, and with a Hamiltonian residual T^d-torus action.

For $Z = \mu_{\mathbb{R}}^{-1}(\alpha) \subseteq \mathbb{H}^n$, denote its intersection with $\mathbb{H}^n_A$ by $Z_\alpha := \mu_{\mathbb{R}}^{-1}(\alpha) \cap \mathbb{H}^n_A$. Then the mapping

$$Z_A \longrightarrow Z_A/K = \left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mathbb{H}^n_A\right)/K \cong \mathcal{E}_A \subseteq M = \left(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/K$$

turns Z_A into a principal K-bundle. Moreover,

$$\dim_{\mathbb{R}}(Z_A/K) = \dim_{\mathbb{R}}(Z_A) - \dim_{\mathbb{R}}(K) = \dim_{\mathbb{R}}(\mathbb{H}^n_A) - \dim_{\mathbb{R}}(\mathfrak{k}^*) - \dim_{\mathbb{R}}(K) = 2n - 2k = 2d = \dim_{\mathbb{R}}(\mathcal{E}_A).$$

References