Folded Hyperkähler Manifolds

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An in-depth review of Euclidean self-dual gravity using Cartan geometry is covered and to used in the construction several 4-dimensional hyperkähler manifolds, in particular the Gibbons-Hawking ansatz, Plebański's "real heaven" background, and also Ashtekar's formulation of Einstein gravity. A rigourous definition of a folded hyperkähler manifold is then provided based on a particular form of the Gibbons-Hawking ansatz. The folded counterpart to Ashtekar's formulation provided by Biquard is described upon in detail. A novel result in this report is the folding of the real heaven background, where the fold hypersurface is an S^1-principal bundle over a base space of either $\mathbb{CP}^1$ or the Poincaré disk.

Introduction

Recently, Hitchin has introduced the notion of a *folded hyperkähler manifolds*, i.e. a 4-dimensional manifold which is hyperkähler away from some folding hypersurface, on which the hyperkähler structure degenerates and the metric is singular [1, 2]. In this report we rapidly cover what exactly a hyperkähler manifold is, before reviewing the method of constructing them via self-dual gravity in the Einstein-Cartan formulation of general relativity. From here several examples of hyperkähler manifolds are determined, with a foray into the Astekar-Jacobson-Smolin (ASJ) formulation for hyperkähler metrics. All of these examples are chosen specifically, as we later study their folded counterparts. The symplectic and Kähler versions of folding have already been studied in much more detail [3, 4], so it is interesting to see what similarities or differences folded hyperkähler admit.

From a physicist's point of view the topic of folded hyperkähler structures remains an interesting topic; the canonical example of a folded hyperkähler structure comes from a particular choice of the Gibbons-Hawking metric [1], and Biquard [2] has also constructed folded hyperkähler manifolds by modifying the ASJ constructing of half-flat solutions to Einstein's equations [5]. A specific feature of these two examples of folded hyperkähler manifolds is that the signature of the metric swaps from Euclidean $(++++)$ to anti-Euclidean $(----)$ as one travels across the fold; such a feature is a recurring theme in the physics literature on 5-dimensional supergravity, where hyperkähler manifolds act as the base space [6].

Background Theory

Hyperkähler Manifolds

A hyperkähler manifold is a Riemannian manifold of real dimension $4n$, that admits three covariantly orthogonal automorphisms, or almost complex structures[1] I, J, and K on the tangent bundle which satisfy the quaternionic identies $I^2 = J^2 = K^2 = IJK = -\mathbb{1}$, and are compatible with the Riemannian metric h. [8]. Since parallel transport preserves the almost complex structures on a hyperkähler manifold, its holonomy group is contained within the compact symplectic group $Sp(n) = GL(n, \mathbb{H})$ [9], from which there is the sequence of inclusions $Sp(n) \subset SU(2n) \subset U(2n) \subset SO(4n)$. From this is follows that each hyperkähler manifold is a Calabi-Yau manifold, which are also Kähler manifolds, and every Kähler manifold is orientable; therefore hyperkähler manifolds have become indispensable within the field of mathematical physics. Owing to this is the fact that any 4-dimensional hyperkähler manifold is both Kähler and Ricci-flat, thereby solving the vacuum Einstein equations [8]; one sub-class of these hyperkähler manifolds is known in the physics literature as gravitational instantons [10, 11, 12]. In dimensions with $n \geq 1$, hyperkähler manifolds also appear in nonlinear σ-models, since the action functionals involved in the theory are $N = 4$ supersymmetric if and only if the target manifold is hyperkähler; the three almost complex structures providing the additional three supersymmetries [7].

[1] Whilst here we use the term *almost complex structure* to refer to an endomorphism J on the tangent bundle that satisfies $J^2 = -\mathbb{1}$, it is a fact that if a Riemannian metric h has three closed 2-forms ω^i $(i = 1, 2, 3)$ compatible with three almost complex structures J^i, then the J^i are in fact integrable and h is hyperkähler [7].

Self-Dual Gravity in Terms of the Self-Dual Spin Connection

Our approach to constructing hyperkähler manifolds will follow that of complex general relativity, before enforcing certain reality conditions to simplify the process. This approach consists of a 4-manifold M and a metric $g_{\mu\nu}(x)$ on M in local coordinates x^μ. The metric can be decomposed into vierbeins or tetrads $e^a{}_\mu(x)$ as

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu,$$
$$\eta^{ab} = g^{\mu\nu} e^a{}_\mu e^b{}_\nu.$$

Here Greek indices $\mu, \nu = 0, 1, 2, 3$, transform as the usual curved, $SO(4, \mathbb{C})$ spacetime indices that are raised or lowered by $g^{\mu\nu}$, whereas the Latin indices $a, b = 1, 2, 3$, are "internal" or flat $SO(3, \mathbb{C})$ indices raised or lowered by the Kronecker delta δ_{ab} tensor. The $e^a{}_\mu$ can therefore be thought of as the "square root" of the metric g in a sense, with inverses defined by $E_a{}^\mu = g^{\mu\nu}\delta_{ab}e^b{}_\nu$. In terms of the flat indices, the torsion 1-form T^a and curvature 2-form $R^a{}_b$ are determined by the vierbeins as

$$T^a = de^a + \omega^a{}_b \wedge e^b, \qquad R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

respectively, with $\omega^a{}_b$ as the spin-connection 1-form. Their respective tensors in terms of the spacetime indices are related to their flat index counterparts through multiplication by $e^a{}_\mu$ or $E^\mu{}_a$

$$R^a{}_b = \frac{1}{2} R^a{}_{bcd} e^c \wedge e^d = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu, \qquad T^a = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c = \frac{1}{2} T^a{}_{\mu\nu} dx^\mu \wedge dx^n,$$

with the Riemann tensor given by

$$R^\alpha{}_{\beta\mu\nu} = e^b{}_\beta E^\alpha{}_a R^a{}_{b\mu\nu}.$$

We will also assume the metricity condition $\omega^\mu{}_\nu = -\omega_\nu{}^\mu$ as well as the zero torsion condition $T^a = 0$, so that the Cartan formulation of geometry is equivalent to the conventional Riemannian case [13]. The Riemann tensor is said to be *self-dual* if it satisfies

$$R^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2} \epsilon^{\mu\nu}{}_{\sigma\rho} R^{\sigma\rho}{}_{\alpha\beta},$$

which, by virtue of the Bianchi identity $R_{\mu[\nu\alpha\beta]} = 0$, implies that Einstein's vacuum equations

$$\epsilon_{\lambda\nu}{}^{\alpha\beta} R^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2}\epsilon_{\lambda\nu}{}^{\alpha\beta}\epsilon^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} = \mathcal{R}\delta_\lambda{}^\mu - 2\mathcal{R}_\lambda{}^\mu = 0,$$

are satisfied, where $\mathcal{R}_{ae}$ is the Ricci tensor [13]. For the hyperkähler structure that we are interested in we will have to impose reality conditions for the Euclidean signature on M, however for now we shall consider complex general relativity to simplify our calculations [14]. In 4 dimensions it is always possible to decompose a 2-form $F \in \Lambda^2$ into its self-dual $F^+ \in \Lambda^2_+$ and anti-self-dual components $F^- \in \Lambda^2_-$

$$F = \frac{1}{2}(1+*)F + \frac{1}{2}(1-*)F = F^+ + F^-,$$

where $*$ is the Hodge star operator that acts as an involution on Λ^2, with eigenvalues $+1$ for self-dual parts, and -1 for anti-self-dual parts. For the field variables, we take a triad of 2-forms Σ^a and an $\mathfrak{so}(3,\mathbb{C})$-valued connection 1-form A, with the corresponding curvature 2-form F defined by $F^a := dA^a + \frac{1}{2}\epsilon^a{}_{bc}A^b \wedge A^c$ [15]

The vacuum Einstein field equations may be derived from the first-order action functional

$$\mathcal{S}[\Sigma^a, A^a, \Psi^a{}_b, v] = \int_M \left[\Sigma^a \wedge F_a - \frac{1}{2}\Psi_{ab}\Sigma^a \wedge \Sigma^b + \Psi^a{}_a v\right], \tag{1}$$

where the symmetric $SO(3,\mathbb{C})$ tensor $\Psi_{ab} = \Psi_{(ab)}$ and 4-form v are Lagrange multipliers introduced in order to set constraints on the Σ^a [15]. The first term in the integrand is nothing more than the Einstein action when the connection form A satisfies its equation of motion. Minimising the action, we have the following equations

$$\frac{\delta\mathcal{S}}{\delta\Psi_{ab}} = \int_M \delta\Psi_{ab}(-(1/2)\Sigma^a \wedge \Sigma^b + \delta^{ab}v) = 0, \tag{2a}$$
$$\frac{\delta\mathcal{S}}{\delta\Sigma^a} = \int_M \delta\Sigma^a \wedge (F_a - \Psi_{ab}\Sigma^b) = 0, \tag{2b}$$
$$\frac{\delta\mathcal{S}}{\delta v} = \int_M \delta v \Psi^a{}_a = 0, \tag{2c}$$
$$\frac{\delta\mathcal{S}}{\delta A_a} = \int_M A_a \wedge D\Sigma^a + D(\Sigma^a \wedge \delta A_a) = \int_M \delta A_a \wedge D\Sigma^a = 0, \tag{2d}$$

where we have used the symmetry of Ψ_{ab} in (2c). In (2d), $\delta_A F^a = d\delta A^a + \epsilon^a{}_{bc}A^b \wedge \delta A^c = D\delta A^a$, where we have introduced the spin-covariant exterior derivative $D\Sigma^a := d\Sigma^a +$

$\omega^a{}_b \wedge \Sigma^b$ such that

$$D(\Sigma^a \wedge \delta A_a) = D\Sigma^a \wedge \delta A_a - \Sigma^a \wedge \delta F_a\,.$$

The term $\int_M D(\Sigma^a \wedge \delta A_a)$ is a total divergence, therefore only contributing to boundary terms which may be taken to be zero in (2d), since $\delta A_a = 0$ on ∂M. The equations of motion can now be read off easily as

$$\Sigma^a \wedge \Sigma^b = \frac{1}{3}\delta^{ab}v, \tag{3a}$$

$$D\Sigma^a \equiv d\Sigma^a + \epsilon^a{}_{bc}A^b \wedge \Sigma^c = 0, \tag{3b}$$

$$\Psi^a{}_a = 0, \tag{3c}$$

$$F^a = \Psi^a{}_b\Sigma^b, \tag{3d}$$

and express the content of the Einstein field equations [15, 14].

Indeed, the constraint (3a) on the two-forms Σ^a is a necessary and sufficient condition for the existence of a tetrad of 1-forms, such that Σ^a is equal to the self-dual part of the exterior product of two tetrad elements. To elucidate further on this, in introducing the complex linear combinations of 2-forms $Z^1 = \Sigma^1 + i\Sigma^2$ and $Z^2 = \Sigma^1 - i\Sigma^2$, and relabelling Σ^3 as Z^3, equation (3a) becomes the following algebraic constraints

$$Z^1 \wedge Z^1 = Z^2 \wedge Z^2 = 0, \tag{4a}$$

$$Z^1 \wedge Z^3 = Z^2 \wedge Z^3 = 0, \tag{4b}$$

$$Z^3 \wedge Z^3 = \frac{1}{2}Z^1 \wedge Z^2 = -\frac{1}{2}v. \tag{4c}$$

Condition (4a) states the the Z^1, Z^2 are decomposable, that is they can be written as an exterior product of two complex 1-forms θ^μ which we write in terms of the vierbeins e^μ as

$$\theta^0 = \frac{1}{\sqrt{2}}(e^3 + ie^0), \qquad \theta^1 = \frac{1}{\sqrt{2}}(e^2 + ie^1),$$
$$\theta^2 = \frac{1}{\sqrt{2}}(e^2 - ie^1), \qquad \theta^3 = \frac{1}{\sqrt{2}}(e^3 - ie^0),$$

which should be noted is not a unique choice; an $SL(2,\mathbb{C})$ transformation would keep the action (1) invariant whilst still changing the θ^μ. Condition (4a) states that Z^1 and Z^2 are decomposable, that is they can be written as an exterior product of two complex 1-forms

$$Z^1 = \theta^2 \wedge \theta^3 \quad \text{and} \quad Z^2 = \theta^0 \wedge \theta^1$$

therefore implying that from (4b), Z^3 must be of the form

$$Z^3 = \frac{1}{2}(\theta^0 \wedge (a\theta^2 + b\theta^3) + \theta^1 \wedge (c\theta^2 + d\theta^3))$$

with the constants satisfying $ad - bc = 1$ due to the condition (4c). Thus Z^3 is determined up to an $SL(2, \mathbb{C})$ gauge freedom [16], so in fixing $b = c = i$, $a = d = 0$ we may choose without losing any information that

$$Z^3 = \frac{i}{2}(\theta^0 \wedge \theta^3 + \theta^1 \wedge \theta^2).$$

The 2-forms $\{Z^1, Z^2, Z^3\}$ form a basis for the vector space of self-dual 2-forms $\Lambda^+ \cong \mathfrak{so}(3, \mathbb{C})$, hence the Σ^a are also self-dual since they are linear combinations of the Z^i [17]. We can therefore write the Σ^a as

$$\Sigma^a = \frac{1}{2}\eta^a{}_{\mu\nu}\Sigma^{\mu\nu}, \tag{5}$$

where we have introduced the *self-dual 't Hooft tensors* η, defined by

$$\eta^a{}_{\mu\nu} = \delta^0{}_\mu \delta^a{}_\nu - \delta^a{}_\mu \delta^0{}_\nu + \epsilon^{0a}{}_{\mu\nu}, \qquad \eta^a{}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}\eta^a{}_{\alpha\beta}. \tag{6}$$

They obey the following relations

$$\eta^a{}_{\mu\nu}\eta^b{}_{\mu\sigma} = \delta^{ab}\delta_{\nu\sigma} + \epsilon^{ab}{}_c\eta^c{}_{\nu\sigma}, \tag{7a}$$

$$\epsilon^a{}_{bc}\eta^b{}_{\mu\nu}\eta^c{}_{\alpha\beta} = \delta_{\mu\alpha}\eta^a{}_{\nu\beta} - \delta_{\mu\beta}\eta^a{}_{\nu\alpha} - \delta_{\nu\alpha}\eta^a{}_{\mu\beta} + \delta_{\nu\beta}\eta^a{}_{\mu\alpha}, \tag{7b}$$

$$\eta^a{}_{\alpha\beta}\eta_a{}^{\mu\nu} = \delta_\alpha{}^\mu\delta_\beta{}^\nu - \delta_\alpha{}^\nu\delta_\beta{}^\mu + \epsilon_{\alpha\beta}{}^{\mu\nu} \tag{7c}$$

amongst several others [18]. There is also an *anti-self-dual 't Hooft* tensor $\bar{\eta}$, defined by

$$\bar{\eta}^a{}_{\mu\nu} = -\delta^0{}_\mu \delta^a{}_\nu + \delta^a{}_\mu \delta^0{}_\nu + \epsilon^{0a}{}_{\mu\nu}, \qquad \bar{\eta}^a{}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}\bar{\eta}^a{}_{\alpha\beta}. \tag{8}$$

Their significance is this; the structure group which acts on the tangent space of $\mathcal{M}$ is

$SO(4,\mathbb{C})$, which is locally isomorphic to[2] $SO(3,\mathbb{C}) \times SO(3,\mathbb{C})$. This splitting is accompanied by the Hodge decomposition $\Lambda^2 \cong \Lambda^2_+ \oplus \Lambda^2_-$ of 2-forms into their self-dual and anti-self-dual components respectively. The self-dual 't Hooft tensors $\eta^a{}_{\mu\nu}$ just defined map self-dual $\mathfrak{so}(4,\mathbb{C})-$valued 2-forms $\Sigma^{\mu\nu}$ to their respective $\mathfrak{so}(3,\mathbb{C})$-valued 3-vectors Σ^a, which are easier to work with [18]. In particular, equation (7c) acts as a projection operator onto the self-dual component of an $\mathfrak{so}(4,\mathbb{C})-$valued 2-form, $\eta^a{}_{\mu\nu}\eta_a{}^{\alpha\beta} = \delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu + \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta} =: \frac{1}{2}(1+*)_{\mu\nu}{}^{\alpha\beta}$.

Field equation (3b) identifies the curvature F^a with the self-dual part of the Riemann curvature tensor [14]; the tetrad e^α determines a metric compatible spin connection $\omega^\alpha{}_\beta$ by the torsion-free condition $de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0$, so equations (3a, 3b) imply

$$\begin{aligned}
D\Sigma^a\wedge =& d\Sigma^a \wedge + \epsilon^a{}_{bc}A^b\wedge\Sigma^c \\
=& \frac{1}{2}\eta^a{}_{\mu\nu}(de^\mu\wedge e^\nu - e^\mu\wedge de^\nu) + \frac{1}{2}\epsilon^a{}_{bc}\eta^c{}_{\mu\nu}A^b\wedge e^\mu\wedge e^\nu \\
=& \big(\eta^a{}_{\mu\nu}\omega^\mu{}_\alpha + \frac{1}{2}\epsilon^a{}_{bc}\eta^c{}_{\alpha\nu}A^b\big)\wedge e^\alpha\wedge e^\nu = 0 \\
\Longrightarrow&\ \eta^a{}_{\mu\nu}\omega^\mu{}_\alpha + \frac{1}{2}\epsilon^a{}_{bc}\eta^c{}_{\alpha\nu}A^b = 0 \qquad\qquad \text{(from 3a)} \\
\Longrightarrow&\ \eta_d{}^{\alpha\nu}\eta^a{}_{\mu\nu}\omega^\mu{}_\alpha = \epsilon^a{}_{db}\eta^b{}_{\alpha\mu}\omega^{\mu\alpha} = 2\epsilon^a{}_{db}A^b
\end{aligned}$$

Applying $\epsilon_a{}^{de}$ now to both sides of the last line and after relabelling the free indices, we arrive at

$$A^a = \frac{1}{2}\eta^a{}_{\mu\nu}\omega^{\mu\nu} = \omega^{0a} + \frac{1}{2}\epsilon^a{}_{bc}\omega^{bc} \implies A^{\alpha\beta} = \frac{1}{2}(1+*)^{\alpha\beta}{}_{\mu\nu}\omega^{\mu\nu} \in \Lambda^2_+,$$

with the implication following from (7c), and shows that A^a is determined entirely by the self-dual part of the spin connection $\omega^\alpha{}_\beta$ [15]. Moreover

$$\begin{aligned}
F^a &= dA^a + \frac{1}{2}\epsilon^a{}_{bc}A^b\wedge A^c \\
&= \frac{1}{2}\eta^a{}_{\mu\nu}d\omega^{\mu\nu} + \frac{1}{8}\epsilon^a{}_{bc}\eta^b{}_{\mu\nu}\eta^c{}_{\alpha\beta}\omega^{\mu\nu}\omega^{\alpha\beta} \\
&= \frac{1}{2}\eta^a{}_{\mu\nu}R^{\mu\nu} = R^{0a} + \frac{1}{2}\epsilon^{0a}{}_{bc}R^{bc} \implies F^{\alpha\beta} = \frac{1}{2}(1+*)^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}
\end{aligned}$$

[2] Actually $SO(4,\mathbb{C}) \cong SL(2,\mathbb{C})_L \times SL(2,\mathbb{C})_R/\mathbb{Z}_2$ but, since we are interesting only in local descriptions on the Lie algebra level, we can ignore the $\mathbb{Z}_2$ factor. The L and R subscripts correspond to the "left" and "right" chiral elements of $SO(4,\mathbb{C})$ respectively.

in having used (7b), which identifies F with the self-dual component of the Riemann curvature tensor. Equation (3c) forces Ψ to be traceless, and equation (3d) states that from the usual decomposition of the Riemann curvature tensor into its irreducible parts, one observes that Ψ must coincide with the self-dual Weyl tensor, as Ψ is symmetric [14].

In order to have a hyperkähler structure, we must fix the gauge $A^a = 0$ [19], implying that the self-dual Weyl tensor Ψ must vanish *identically* from field equation (3d), due to the non-degeneracy of the volume forms [14]. The Weyl tensor is conformally invariant and can be thought of being defined by an anti-self-dual structure[3], $[h]$ [20]. For the triple of hyperkähler 2-forms ω^i $(i = 1, 2, 3)$, we take the real-valued combinations

$$\omega^1 = i(Z^1 + Z^2) = e^0 \wedge e^1 + e^2 \wedge e^3, \qquad \omega^2 = Z^1 - Z^2 = e^0 \wedge e^2 + e^3 \wedge e^1,$$
$$\omega^3 = -2iZ^3 = e^0 \wedge e^3 + e^1 \wedge e^2,$$

which may be written out explicitly as

$$\omega^i = e^0 \wedge e^i + \frac{1}{2}\epsilon^i{}_{jk} e^j \wedge e^k = \frac{1}{2}\eta^i{}_{\mu\nu} e^\mu \wedge e^\nu \tag{9}$$

and satisfy the hyperkähler conditions from equations (3a, 3b)

$$\omega^1 \wedge \omega^1 = \omega^2 \wedge \omega^2 = \omega^3 \wedge \omega^3 \neq 0, \tag{10a}$$
$$d\omega^i = 0. \tag{10b}$$

Equation (10a) asserts the existence and non-degeneracy of the three hyperkähler forms, whereas equation (10b) is the integrability condition stating that each ω^i is closed [20].

Recall now that the E_μ represent the vectors dual to the e^μ, so the self-dual hyperkähler forms (9) can be rewritten via the anti-self-dual 't Hooft tensor (8) [21]

$$\omega^i = \frac{1}{2}\bar{\eta}^{i\mu\nu} \imath_{E_\mu} \imath_{E_\nu} v,$$

where $i_{E_\mu} v$ is the contraction of v with the vector E_μ. Furthermore, if we assert that the

[3] An *anti-self-dual structure* is a 4-dimensional conformal structure such that the self-dual Weyl tensor Ψ vanishes [20].

E_μ be volume preserving, that is if the Lie derivative of the volume-form v with respect to each vector field E_μ satisfies $\mathcal{L}_{E_\mu} v = 0$, then due to the closed property of each hyperkähler form ω^i it follows from the anti-self-dual 't Hooft tensor $\bar{\eta}$ (8) that

$$d\omega^i = \frac{1}{2}\bar{\eta}^{i\mu\nu} d(\imath_{E_\mu}\imath_{E_\nu} v) = \frac{1}{2}\bar{\eta}^{i\mu\nu}\imath_{[E_\mu,E_\nu]} v = 0,$$

in having employed the identity

$$d(\imath_{E_\mu}\imath_{E_\nu} v) = (\imath_{[E_\mu,E_\nu]} v + \imath_{E_\mu}\mathcal{L}_{E_\nu} v - \imath_{E_\nu}\mathcal{L}_{E_\mu} v + \imath_{E_\mu}\imath_{E_\nu} dv),$$

which comes from applying Cartan's homotopy formula twice, and the fact that $dv = 0$ [22]. From the non-degeneracy of v, it must hold that

$$\frac{1}{2}\bar{\eta}^{i\mu\nu}[E_\mu, E_\nu] = 0 \implies [E_0, E_i] = \frac{1}{2}\epsilon_i{}^{jk}[E_j, E_k], \tag{11}$$

thereby reducing the construction of a 4-dimensional hyperkähler manifolds to the search of four linearly-independent vector fields E_μ, that satisfy the following properties [21]

$$\mathcal{L}_{E_\mu} v = 0, \qquad \text{and} \qquad [E_0, E_i] = \frac{1}{2}\epsilon_i{}^{jk}[E_j, E_k]. \tag{12}$$

Examples of Hyperkähler Manifolds

This subsection will borrow largely from Refs. [23, 24], however our choice of vector fields, as we choose ones explicitly dual to the vierbeins e^μ. Note that since we have an anti-self-dual structure, it is possible to find a representative metric of the form $\hat{h} = \delta_{\mu\nu} e^\mu \otimes e^\nu$ within the conformal class of the hyperkähler metric, $[h]$. The vector fields must be volume-preserving with respect to some volume element $\hat{v}$, so in defining the function f by $\hat{v}(E_0, E_1, E_2, E_3) = f^2$, we will obtain the physical, Ricci-flat metric via the conformal transformation $h = f^2\hat{h}$ [25].

Example 1 (Gibbons-Hawking Ansatz). Let us write Euclidean space with standard coordinates $\mathbb{R}^4 = \{(\tau, x, y, z)\}$ as the underlying spacetime with volume form $\hat{v} = d\tau \wedge dx \wedge$

$dy \wedge dz$. Then let the tetrad of vector fields E_μ and their dual vierbeins e^μ be

$$\begin{aligned} E_0 &= V\frac{\partial}{\partial\tau}, & e^0 &= \frac{1}{V}(d\tau + \mathcal{A}) \\ E_i &= \frac{\partial}{\partial x^i} - \mathcal{A}_i\frac{\partial}{\partial\tau}, & e^i &= dx^i \end{aligned}$$

for the smooth functions $V \neq 0$, $\mathcal{A} = \mathcal{A}_i dx^i$, independent of τ, which ensure the E_μ are volume-preserving. Condition (11) translates to the equation

$$\underset{3}{*}dV = d\mathcal{A} \tag{13}$$

which is known as the *Bogomolny equation* or the *monopole equation* [20]. Here, $*_3$ is the Hodge star operator with respect to the Euclidean metric on $\mathbb{R}^3$, and states that V is a harmonic function on the space. From the vierbeins, a representative metric of $[h]$ is

$$\hat{h} = \delta_{\mu\nu} e^\mu \otimes e^\nu = V^{-2}(d\tau + \mathcal{A})^2 + (dx^2 + dy^2 + dz^2),$$

and $v(E_0, E_1, E_2, E_3) = V$, hence the conformal transformation

$$h = V\hat{h} = V^{-1}(d\tau + \mathcal{A})^2 + V(dx^2 + dy^2 + dz^2) \tag{14}$$

produces a Ricci-flat metric [20]; the hyperkähler 2-forms[4] are

$$\omega^i = (d\tau + \mathcal{A}) \wedge dx^i + \frac{1}{2}V\epsilon^i{}_{jk} dx^j \wedge dx^k. \tag{15}$$

It is a fact that any 4-dimensional hyperkähler metric which admits a triholomorphic Killing vector[5] can locally be put into the form (14) [20]. This example is the ansatz that Gibbons and Hawking considered when studying gravitational instantons that admit a triholomorphic Killing vector $\frac{\partial}{\partial\tau}$, whose action generates an S^1 symmetry [12].

[4] In our conformal rescaling the hyperkähler forms will also be scaled by the same factor as the representative metric. This can be seen from the relation between the metric h and the three almost complex structures J^a *i.e.* $\omega^a(\cdot,\cdot) = h(J^a(\cdot),\cdot)$. For brevity however we shall just write the Kähler forms as they appear in the literature, with the conformal scaling $\omega^a = f^2\hat{\omega}^a$ implied.

[5] A *triholomorphic Killing vector* K is a vector field generated by the group action on the manifold, that is an isometry and preserves the three complex structures, *i.e.* $\mathcal{L}_K h = 0$ and $\mathcal{L}_K \omega^i = 0$. In the Gibbons-Hawking ansatz, the triholomorphic Killing vector $K = \frac{\partial}{\partial\tau}$ generates an S^1 action; the Lie algebra of S^1 is $\mathfrak{u}(1) = \mathbb{R}$, and corresponds to a translation in the $\tau-$coordinate which does not change the metric (14).

Example 2 (Real Heaven Background). Let us take Euclidean space again with the same spacetimes coordinates and volume form as the previous example. This time however, let the vector fields E_μ and vierbeins e^μ be given by

$$
\begin{aligned}
E_0 &= e^{u/2}\left(u_z \cos(\tau/2)\frac{\partial}{\partial\tau} - \sin(\tau/2)\frac{\partial}{\partial z}\right), \qquad & E_2 &= \frac{\partial}{\partial x} - u_y\frac{\partial}{\partial \tau}\\
E_1 &= e^{u/2}\left(u_z \sin(\tau/2)\frac{\partial}{\partial\tau} + \cos(\tau/2)\frac{\partial}{\partial z}\right), \qquad & E_3 &= \frac{\partial}{\partial y} + u_x\frac{\partial}{\partial \tau},\\
e^0 &= \frac{1}{e^{u/2}u_z}\Big(\cos(\tau/2)(d\tau + u_y dx - u_x dy) - u_z\sin(\tau/2)dz\Big), \qquad & e^2 &= dx,\\
e^1 &= \frac{1}{e^{u/2}u_z}\Big(\sin(\tau/2)(d\tau + u_y dx - u_x dy) + u_z\cos(\tau/2)dz\Big), \qquad & e^3 &= dy,
\end{aligned}
$$

with u a smooth function that is independent of τ, so the volume-preserving condition is satisfied. Constraint (11) now becomes

$$
u_{xx} + u_{yy} + (e^u)_{zz} = 0, \tag{16}
$$

and is known as the *Boyer-Finley equation* or the $SU(\infty)$-*Toda equation* in the physics literature, due to its connection to solid state physics in the continuum limit [26, 27]. This time from the vierbeins, the representative metric of $[h]$ is

$$
\hat{h} = \delta_{\mu\nu}e^\mu \otimes e^\nu = dx^2 + dy^2 + e^{-u}(dz^2 + u_z^{-2}(d\tau + u_y dx - u_x dy)^2)
$$

and $\hat{v}(E_0, E_1, E_2, E_3) = e^u u_z$. After the conformal transormation again, the Ricci-flat metric h is

$$
h = e^u u_z \hat{h} = u_z(e^u(dx^2 + dy^2) + dz^2) + u_z^{-1}(d\tau + u_y dx - u_x dy)^2. \tag{17}
$$

This metric also admits the Killing vector $\frac{\partial}{\partial\tau}$ but in this case it is not triholomorphic, instead admitting only one rotational Killing symmetry rather than a translational symmetry [28]. This is why the current example differs from the Gibbons-Hawking ansatz previously considered, with the different symmetry reflected in the hyperkähler forms ω^i, that split into a $U(1)$ singlet

$$
\omega^1 = u_z e^u dx \wedge dy + dz \wedge (d\tau + u_y dx - u_x dy) \tag{18}
$$

and a $U(1)$ doublet [29]

$$\begin{pmatrix}\omega^2\\ \omega^3\end{pmatrix} = e^{u/2}\begin{pmatrix}\cos(\tau) & \sin(\tau)\\ -\sin(\tau) & \cos(\tau)\end{pmatrix}\begin{pmatrix}(d\tau - u_y dx + u_x dy)\wedge dx + u_z dy\wedge dz\\ (d\tau - u_y dx + u_x dy)\wedge dy + u_z dz\wedge dx\end{pmatrix}.$$

This can be verified by considering the Lie derivative with respect to ∂/∂_τ,

$$\mathcal{L}_{\frac{\partial}{\partial\tau}}\omega^1 = 0, \qquad \mathcal{L}_{\frac{\partial}{\partial\tau}}\omega^2 = \omega^3, \qquad \mathcal{L}_{\frac{\partial}{\partial\tau}}\omega^3 = -\omega^2,$$

thereby showing how the lack of invariance that ω^2 and ω^3 exhibit under the action of $\frac{\partial}{\partial\tau}$ explicitly.

Real, self-dual, Euclidean Einstein spaces with one rotational Killing symmetry arise as real Euclidean cross-sections of complex $\mathcal{H}$-spaces, which are solutions to the complex vacuum Einstein equations with a self-dual curvature called *heavens*, in the formalism of Plebański [16, 28]. These cross-sections are completely determined by the metric given by (17) and are called *real heavens*; hence this example will be referred to from now on as the real heaven background.

Remark 3. In linearising equation (16) via the perturbation $u \mapsto u + \epsilon V$ and keeping only the terms linear in ϵ, we would recover the *linearised Boyer-Finley* equation

$$V_{xx} + V_{yy} + (Ve^u)_{zz} = 0. \tag{19}$$

It can be shown that the metric arising from this is Ricci-flat if and only if $u_z = aV$ for some constant a. In particular, if the metric is Ricci-flat and $u = 0$ then the Gibbons-Hawking ansatz is recovered, with equation (19) becoming the monopole equation (13) [27].

There are several similarities between the two examples just presented; consider an open set $\mathcal{U} \subset \mathbb{R}^3$, and let $\mathcal{M} \xrightarrow{\pi} \mathcal{U}$ be a principal S^1-bundle over $\mathcal{U}$. Let $\mathcal{A}$ be the connection 1-form on $\mathcal{M}$ with curvature 2-form $\mathcal{F}$, then both of the examples are on the total space of an S^1-bundle. To see this, choose a local trivialisation of $\mathcal{M}$ so that τ is a fibre coordinate on the circle with period 2π, then $\mathcal{A}_{GH} = d\tau + xdy$ in the Gibbons-Hawking ansatz, whereas $\mathcal{A}_{RH} = d\tau + u_y dx - u_x dy$ for the real heaven background. This determines $\mathcal{M}$ and $\mathcal{F}$ up to gauge equivalence if $\mathcal{U}$ is simply connected [27].

The Ashtekar-Jacobson-Smolin Construction of Hyperkähler Manifolds

The section is dedicated to the ASJ construction of hyperkähler manifolds, of which the premise is as follows; one decomposes a 4-dimensional spacetime $\mathcal{M}$ into $\mathcal{M} = \mathbb{R} \times \mathcal{N}$, where $\mathcal{N}$ is a 3-dimensional manifold[6]. Let the leaves of the natural foliation of $\mathcal{M}$ be labelled by constant values of the coordinate τ, with $\frac{\partial}{\partial \tau}$ representing the normal vector field to each leaf. Then in labelling $V_0 = \frac{\partial}{\partial \tau}$, condition (11) becomes equivalent to *Nahm's equations* for the triad of orthogonal vector fields V_i

$$\frac{\partial V_i}{\partial \tau} = \frac{1}{2} \epsilon_i^{\ jk} [V_j, V_k], \tag{20}$$

and the volume-preserving condition holds, assuming that each V_i depend soley on τ [22]. It was from Ashtekar's Hamiltonian approach to general relativity, in which the Nahm's equations (20) for the Lie algebra of symplectomorphisms[7] on $\mathcal{N}$ represent a form of self-dual Einstein equations on $\mathcal{M}$ [30]. Further elucidation on the work of Ashtekar has been carried out by Mason and Newman, who consider Yang-Mills theory for the Lie algebra of symplectomorphisms on some 4-manifold [31], as well by Donaldson [22]. We will briefly cover the ASJ construction of hyperkähler manifolds, as Biquard adapts the procedure for an existence and uniqueness theorem for folded hyperkähler structures with real analytic data [2].

The ASJ construction is outlined as follows:

Proposition 4 (Donaldson [22]). *Let V_i be a triad of time-dependent, volume-preserving, linearly-independent vector fields on a smooth 3-manifold $\mathcal{N}$ that satisfy Nahm's equations*

$$\frac{\partial V_i}{\partial \tau} = \frac{1}{2} \epsilon_i^{\ jk} [V_j, V_k].$$

Then there exist three holomorphic symplectic structures[8] *on the product $\mathcal{M} = \mathbb{R} \times \mathcal{N}$.*

[6] If the 3-manifold $\mathcal{N}$ is orientable then there always exists three linearly-independent vector fields in the tangent space at each point. This is because $T\mathcal{N}$ is trivial, and so always admits a global section.

[7] A symplectomorphism on a manifold $\mathcal{N}$ is a volume-preserving diffeomorphism, and the space of them forms a Lie group SDiff($\mathcal{N}$) with an associated Lie algebra $\mathfrak{sdiff}(\mathcal{N})$ consisting of the volume-preserving vector fields on $\mathcal{N}$.

[8] A holomorphic symplectic form θ on a differentiable 4-manifold is a 2-form such that $\theta \wedge \bar{\theta} = 2\Omega$ is a volume form. Writing $\theta = \theta^1 + i\theta^2$, where θ^1 and θ^2 are real 2-forms, we have the algebraic conditions

$$\theta^1 \wedge \theta^1 = \theta^2 \wedge \theta^2 \neq 0, \qquad \theta^1 \wedge \theta^2 = 0.$$

The three holomorphic symplectic forms are closed, and fulfil the hyperkähler conditions (10b), implying the existence of a Riemannian metric h compatible with the complex structures [22]. There is a remarkable result that the converse holds as well:

Proposition 5 (Ashtekar [5] and Donaldson [22]). *Let $\mathcal{M}$ be a 4-dimensional hyperkähler manifold with volume form v, and let τ be a harmonic function that vanishes on some hypersurface $\mathcal{N} \subset \mathcal{M}$ of codimension 1. Then there exists a triad of time-dependent, volume-preserving, linearly-independent vector fields V_i that satisfy Nahm's equations (20).*

Results

A Canonical Example

We begin this section by discussing a canonical example of a folded hyperkähler manifold, in order to deduce the natural structure that arises and thereby formulate a concise definition. Recall the Gibbons-Hawking metric (14), albeit a particular choice for the function V [1]:

$$h = \frac{1}{z}(d\tau + \mathcal{A}_{GH})^2 + z(dx^2 + dy^2 + dz^2), \qquad d\mathcal{A}_{GH} = dx \wedge dy. \tag{21}$$

The hyperkähler forms are given by

$$\omega^1 = (d\tau + \mathcal{A}_{GH}) \wedge dz + z dx \wedge dy, \tag{22a}$$

$$\omega^2 = (d\tau + \mathcal{A}_{GH}) \wedge dx + z dy \wedge dz, \tag{22b}$$

$$\omega^3 = (d\tau + \mathcal{A}_{GH}) \wedge dy + z dz \wedge dx. \tag{22c}$$

The metric h is undefined at $z = 0$, and hence determines a hypersurface $\mathcal{Z}$ that divides the ambient manifold $\mathcal{M}$ into two disjoint ones; one with an Euclidean signature $(++++)$ when $z > 0$, and the other with an anti-Euclidean signature $(----)$ when $z < 0$. Under the involution $i : z \mapsto -z$ we observe that

$$i^*\omega^1 = -\omega^1, \tag{23a}$$

$$i^*\omega^2 = \omega^2, \qquad i^*\omega^3 = \omega^3, \tag{23b}$$

$$i^*h = -h. \tag{23c}$$

Furthermore whilst h is undefined along the fold at $z = 0$, the hyperkähler forms ω^i are smooth there. Pulling them back to $\mathcal{Z}$ we have that

$$\mathcal{Z}^*\omega^1 = 0, \qquad \mathcal{Z}^*\omega^2 = \varphi \wedge dx, \qquad \mathcal{Z}^*\omega^3 = \varphi \wedge dy, \qquad \text{where } \varphi \equiv d\tau + \mathcal{A}_{GH}. \tag{24}$$

Since $d\mathcal{A}_{GH} = dx \wedge dy$, it follows that

$$\varphi \wedge d\varphi = d\tau \wedge dx \wedge dy \neq 0, \tag{25}$$

and therefore φ determines a contact form on $\mathcal{Z}$.

The Definition and an Existence and Uniqueness Theorem for Hyperkähler Manifolds

From the previous example a formal definition of a folded hyperkähler structure can be proposed:

Definition 6 ([1, 2])**.** A *folded hyperkähler structure* consists of a smooth 4-manifold $\mathcal{M}$, a smoothly embedded hypersurface $\mathcal{Z} \subset \mathcal{M}$, three smooth, closed, 2-forms ω^i $(i = 1, 2, 3)$ on $\mathcal{M}$, and an involution $\imath : \mathcal{M} \to \mathcal{M}$ which satisfy the following conditions

- $\mathcal{Z}$ divides $\mathcal{M}$ into two disjoint connected components: $\mathcal{M} \setminus \mathcal{Z} \simeq \mathcal{M}^+ \cup \mathcal{M}^-$.
- the 2-forms ω^i define a hyperkähler structure on $\mathcal{M}^\pm$ with hyperkähler metric $h^\pm$ where h^+ has Euclidean signature $(++++)$ and h^- has anti-Euclidean signature $(----)$.
- on the hypersurface $\mathcal{Z} \subset \mathcal{M}$ we have that $\mathcal{Z}^*\omega^1 = 0$, $\mathcal{Z}^*\omega^2 \neq 0$, and $\mathcal{Z}^*\omega^3 \neq 0$ with a contact distribution $\mathcal{H} \subset T\mathcal{Z}$ given by $\mathcal{H} = \ker \mathcal{Z}^*\omega^2 \oplus \ker \mathcal{Z}^*\omega^3$.

- the involution $\imath$ fixes $\mathcal{Z}$ and maps $\mathcal{M}^{\pm}$ to $\mathcal{M}^{\mp}$ such that

$$\imath^* h^{\pm} = -h^{\mp}, \qquad \imath^* \omega^1 = -\omega^1, \qquad \imath^* \omega^2 = \omega^2, \qquad \imath^* \omega^3 = \omega^3. \tag{26}$$

Several comments about this definition are in order; firstly it is of interest as to how it differs from the already well-established notion of folded structures in symplectic and Kähler geometry. A *folded symplectic form* on a $2n$-dimensional manifold $\mathcal{M}$ is a closed 2-form ω whose top form ω^n vanishes transversally on a submanifold $\mathcal{Z}$, and whose restriction as a form $\omega|_Z$ has maximal rank of $2n-2$. Then the pair $(\mathcal{M}, \omega)$ defines a *folded symplectic structure* [3]. The Kähler equivalent is similar with the additional fact that any compact smooth 4-manifold has a folded Kähler structure, such that the two components of $\mathcal{M} \setminus \mathcal{Z}$ determine Stein manifolds [7]. Similar to our hyperkähler definition is that the metric changes signature upon crossing the fold, however.

Evidently the folded symplectic and Kähler structures differ from how a hyperkähler structure admits a fold, since the latter case requires that the hyperkähler 2-form ω^1, say, must vanish as a form when restricted to the fold hypersurface. More precisely, in following the analysis of Hitchin [1]; suppose that $\mathcal{M}$ is a 4-dimensional hyperkähler manifold with hyperkähler 2-forms ω^i, and presume that we have $\omega^1 \wedge \omega^1 = 0$ at some point $p \in \mathcal{M}$. Then there are the algebraic constraints in $\Lambda^2 T_p^* M$ on the ω^i

$$\omega^1 \wedge \omega^1 = \omega^2 \wedge \omega^2 = \omega^3 \wedge \omega^3 = 0, \qquad \omega^1 \wedge \omega^2 = \omega^2 \wedge \omega^3 = \omega^3 \wedge \omega^1 = 0,$$

since the top-form $\omega^1 \wedge \omega^1$ has to vanish transversally. If the ω^i are linearly-independent at p then the condition $\omega^i \wedge \omega^i = 0$ means that the ω^i are decomposable. If the ω^i were folded in the symplectic sense then we could decompose them as $\omega^i = x dx \wedge \alpha^i + \beta^i \wedge \gamma^i$, with the $\beta^i \wedge \gamma^i$ non-vanishing on the fold at $z = 0$ so that the forms are of maximal rank 2. However this is not the case that we are interested in; our definition states that from (26), ω^1 vanishes as a form when restricted to the fold, whereas ω^2, ω^3 are both even. So if we take local coordinates about the fold, by an analogue of Darboux's theorem we must have to order z

$$\omega^1 = dz \wedge \alpha^1 + z\beta^1 \wedge \gamma^1, \tag{27a}$$
$$\omega^2 = zdz \wedge \alpha^2 + \beta^2 \wedge \gamma^2, \qquad \omega^3 = zdz \wedge \alpha^3 + \beta^3 \wedge \gamma^3,$$

where $i_{\partial/\partial z}(\alpha^i, \beta^i, \gamma^i) = 0$. From the condition that on $\mathcal{Z}$ we must have

$$\omega^1 \wedge \omega^2 = dz \wedge \alpha^1 \wedge \beta^2 \wedge \gamma^2 = 0, \tag{28a}$$
$$\omega^1 \wedge \omega^3 = dz \wedge \alpha^1 \wedge \beta^3 \wedge \gamma^3 = 0, \tag{28b}$$

so that $\alpha^1 \wedge \beta^2 \wedge \gamma^2 = 0 = \alpha^1 \wedge \beta^3 \wedge \gamma^3$, therefore take $\alpha^1 = \gamma^2 = \gamma^3 \equiv \varphi$. This will ensure the vanishing of equations (28a, 28b) due to the anti-symmetry of the wedge product. Moreover since ω^1 is closed we must have

$$d\omega^1 = dz \wedge (-d\varphi + \beta^1 \wedge \gamma^1) = 0,$$

and hence $\omega^1 = dz \wedge \varphi + zd\varphi$ to order z. As $\omega^1 \wedge \omega^1 = 2zdz \wedge \varphi \wedge d\varphi$ which vanishes transversally on the fold, we arrive back at the condition $\varphi \wedge d\varphi \neq 0$ along $z = 0$ and the contact structure along $\mathcal{Z}$ is recovered again. Relabel β^2, β^3 as $-\eta^1, -\eta^2$ respectively, then along the fold hypersurface $\mathcal{Z}$ we have that $\eta^1 \wedge \eta^2 \wedge \varphi \neq 0$. Letting $\imath : \mathcal{Z} \hookrightarrow \mathcal{M}$ be the inclusion of the fold hypersurface into the ambient manifold $\mathcal{M}$ we have that

$$\imath^*\omega^1 = 0,$$
$$\imath^*\omega^2 = \varphi \wedge \eta^1, \qquad \imath^*\omega^3 = \varphi \wedge \eta^2, \tag{29}$$

which sets the 2-forms such as those in (24).

We will now begin moving towards the existence and uniqueness theorem provided by Biquard which, whilst not an original result, is covered in full detail. However, first a lemma originally proven by Bryant is required [32].

Lemma 7 (Bryant [32])**.** *Let $\mathcal{N} \subset \mathcal{M}$ be a real hypersurface with a contact 2-plane $\mathcal{H} \subset T\mathcal{N}$, defined locally by the contact form θ^1 and holomorphic symplectic form $\beta = \beta^2 + i\beta^3$. Then there exists unique 1-forms $\theta^1, \theta^2, \theta^3$ on $\mathcal{N}$ such that*

- $\beta = \theta^1 \wedge (\theta^2 + i\theta^3)$ *and*

- $d\theta^1 = i(\theta^2 + i\theta^3) \wedge (\theta^2 - i\theta^3)$.

This lemma is required since it asserts that if the three 1-forms θ^1, θ^2 and θ^3 exist locally, then they are defined on the entirety of the hypersurface $\mathcal{N}$, therefore provide a canonical coframing for $T^*\mathcal{N}$ [32].

Theorem 8 (Biquard [2]). *Given the real analytic data $(\mathcal{N}, \beta^2, \beta^3)$, where β^2 and β^3 are closed 2-forms on a 3-manifold $\mathcal{N}$ such that $\mathcal{H} = \ker\beta^2 \oplus \ker\beta^3$ is a contact distribution. Then there exists on a neighbourhood $(-\epsilon, \epsilon) \times \mathcal{N}$ a unique folded hyperkähler metric such that $i^*\omega^2 = \beta^3$ and $i^*\omega^3 = \beta^2$. This metric satisfies the parity condition (26).*

Proof. From the given data $(\mathcal{N}, \beta^2, \beta^3)$, consider the basis of 1-forms $(\theta^1, \theta^2, \theta^3)$ to $T^*\mathcal{N}$ with the associated dual frame of vector fields (X_1, X_2, X_3); these exist globally on $T\mathcal{N}$ due to Lemma (7). Let $\alpha = \theta^1 \wedge \theta^2 \wedge \theta^3$ be the volume form on $\mathcal{N}$, then X_2 and X_3 are volume-preserving, $\mathcal{L}_{X_2}\alpha = d\beta^2 = \mathcal{L}_{X_3}\alpha = d\beta^3 = 0$. To extend the frame to the neighbourhood $\mathcal{M} = (-\epsilon, \epsilon) \times \mathcal{N}$ for some $\epsilon > 0$ small enough, we want to solve Nahm's equations

$$\frac{\partial V_i}{\partial \tau} = \frac{1}{2}\epsilon_i^{\ jk}[V_j, V_k]$$

for the triad of time-dependent vector fields V_i, subject to the initial conditions

$$V_1(0) = 0, \qquad V_2(0) = X_2, \qquad V_3(0) = X_3.$$

As we are dealing with real analytic data, the Cauchy-Kovalevskaya Theorem [33] determines unique solutions to this system on $\mathcal{M}$. Moreover $(-V_1(-\tau), V_2(-\tau), V_3(-\tau))$ is also a solution with the same initial conditions, hence V_1 is odd and V_2, V_3 are even, implying the invariance of the involution (26) for the solution. As $d\theta^1 = \theta^2 \wedge \theta^3$, we must have

$$V_1(\tau) = \tau X_1 + \mathcal{O}(\tau^3),$$

for $i_{V_1} d\theta^1 = 0$. Define a fourth vector $V_0 = \frac{\partial}{\partial \tau}$, then the behaviour of the Riemannian metric on $(-\epsilon, \epsilon) \times \mathcal{N}$ can be deduced as

$$h = \tau(d\tau^2 + (\theta^2)^2 + (\theta^3)^2) + \tau^{-1}(\theta^1)^2 + \mathcal{O}(\tau^3)(d\tau, \tau^{-1}\theta^1, \theta^2, \theta^3)$$

where the final term is quadratic in $(d\tau, \tau^{-1}\theta^1, \theta^2, \theta^3)$ with coefficients of order τ^3. From the metric h we can determine the hyperkähler forms and metric as

$$\omega^i = d\tau \wedge h(V_i) + i_{V_i}\alpha, \qquad h(V_\mu, V_\nu) = \alpha(V_1, V_2, V_3)\delta_{\mu\nu}.$$

The existence of the involution (26) then follows from the parity of the V_i. $\square$

Since Theorem (8) considers real analytic data, it is possible to extend the folded hyperkähler manifold by analytic continuation onto the entirety of $\mathcal{M}$; even if we have only have C^∞ data, the result can nevertheless still be applied to the germs on $\mathcal{N}$ [2].

Contact Geometry of the Folded Gibbons-Hawking Manifold

Let us continue our investigation into the folded Gibbons-Hawking example by studying the contact structure of the fold hypersurface $\mathcal{Z}$, in particular the nature of $\mathcal{H} = \ker \mathcal{Z}^*\omega^2 \oplus \ker \mathcal{Z}^*\omega^3$. To find the connection 1-form $\mathcal{A}_{GH}$, consider

$$\underset{3}{*}dV = \underset{3}{*}dz = dx \wedge dy = d(xdy) = d\mathcal{A}_{GH} \implies \mathcal{A}_{GH} \equiv xdy, \tag{30}$$

determining $\mathcal{A}_{GH}$ modulo an exact form. Hence

$$\varphi = d\tau + xdy, \qquad d\varphi = dx \wedge dy,$$
$$\varphi \wedge d\varphi = d\tau \wedge dx \wedge dy \neq 0,$$

and the pullbacks of ω^2 and ω^3 to $\mathcal{Z}$ are

$$\mathcal{Z}^*\omega^2 = (d\tau + xdy) \wedge dx = d\tau \wedge dx + xdy \wedge dx,$$
$$\mathcal{Z}^*\omega^3 = (d\tau + xdy) \wedge dy = d\tau \wedge dy.$$

Their 1-dimensional kernels can be read off to determine the 2-dimensional hyperplane field $\mathcal{H}_{GH}$ as

$$\mathcal{H}_{GH} = \ker \mathcal{Z}^*\omega^2 \oplus \ker \mathcal{Z}^*\omega^3 = \mathrm{span}\left\{\frac{\partial}{\partial y} - x\frac{\partial}{\partial \tau},\ \frac{\partial}{\partial x}\right\}$$

Indeed, if we check the commutator of the basis vectors we observe that

$$\left[x\frac{\partial}{\partial \tau} - \frac{\partial}{\partial y},\ \frac{\partial}{\partial x}\right] = \frac{\partial}{\partial \tau} \notin \mathcal{H}_{GH},$$

so it is clear that $(\mathcal{Z}, \mathcal{H}_{GH})$ is a contact manifold. In fact, for the Reeb vector field R_φ defined by the equations

$$d\varphi(R_\varphi, \cdot) = 0, \qquad \varphi(R_\varphi) = 1, \tag{31}$$

one sets $R_\varphi = \frac{\partial}{\partial \tau}$, coinciding with the triholomorphic Killing vector field generated by the S^1 symmetry.

Folding the Real Heaven Background

Since our canonical example of a folded hyperkähler manifold comes from the Gibbons-Hawking ansatz, it is natural to ask whether real heaven background admits a fold hypersurface in the sense of our definition, since it also has an S^1 symmetry.

As the Boyer-Finley equation (16) determines the real heaven metric equation (17), it is natural to first find solutions to it. In separating the variables

$$u(x, y, z) = v(x, y) + w(z),$$

equation (16) becomes

$$(e^v)_{zz} = 2a, \tag{32a}$$

$$v_{xx} + v_{yy} + 2a(e^w)_{zz} = 0, \tag{32b}$$

with a a separation constant [26]. Equation (32a) can be solved immediately to get

$$e^v = az^2 + bz + c, \tag{33}$$

and equation (32b) is known as Liouville's equation [26]. Solutions to the Boyer-Finley equation (16) take the form

$$e^u = \frac{4(az^2 + bz + c)}{(1 + a(x^2 + y^2))^2},$$

$$u(x, y, z) = \log(az^2 + bz + c) - 2\log(1 + a(x^2 + y^2)) + \log(4)$$

with b, c constants. There are six cases: three on hyperbolic space ($b^2 - ac > 0$), two in flat space ($b^2 - 4ac = 0$), and one on the sphere ($b^2 - 4ac < 0$) [26]. Since the folded metric h and hyperkähler form ω^1 must satisfy the parity condition (26), only the two cases when $a \neq 0,\; b = 0$, and $c > 0$ shall be considered; the base spaces correspond to the 3-sphere S^3 when $a > 0$ and to hyperbolic 3-space $\mathcal{H}^3$ when $a < 0$. We calculate

$$\begin{aligned}
u_x &= -\frac{4ax}{1+a(x^2+y^2)}, \qquad u_y = -\frac{4ay}{1+a(x^2+y^2)},\\
u_{xx} &= -\frac{4(1-x^2+y^2)}{(1+a(x^2+y^2))^2}, \qquad u_{yy} = -\frac{4(1+x^2-y^2)}{(1+a(x^2+y^2))^2},\\
(e^u)_{zz} &= \frac{8a}{(1+a(x^2+y^2))^2} = -(u_{xx}+u_{yy}),\\
u_z &= \frac{2az}{c+az^2},\\
u_z e^u &= \frac{8az}{(1+a(x^2+y^2))^2},\\
d\tau + u_y dx - u_x dy &= d\tau + 4a\frac{xdy - ydx}{1+a(x^2+y^2)} =: d\tau + \mathcal{A}_{RH},
\end{aligned}$$

where $\mathcal{A}_{RH}$ is the connection 1-form of the S^1-bundle above the base space. The metric and hyperkähler forms from equations (17, 18) and example 2 are

$$h = \frac{8az}{(1+a(x^2+y^2))^2}(dx^2+dy^2) + \frac{2az}{c+az^2}dz^2 + \frac{c+az^2}{2az}\psi^2, \tag{35a}$$

$$\omega^1 = \frac{8az}{(1+a(x^2+y^2))^2}dx\wedge dy + dz\wedge(d\tau + \mathcal{A}_{RH}), \tag{35b}$$

$$\begin{pmatrix}\omega^2\\ \omega^3\end{pmatrix} = e^{u/2}\begin{pmatrix}\cos(\tau) & -\sin(\tau)\\ \sin(\tau) & \cos(\tau)\end{pmatrix}\begin{pmatrix}(d\tau+\mathcal{A}_{RH})\wedge dx + \frac{2az}{c+az^2}dy\wedge dz\\ (d\tau+\mathcal{A}_{RH})\wedge dy + \frac{2az}{c+az^2}dz\wedge dx\end{pmatrix}. \tag{35c}$$

From this explicit representation, the parity relations (26) for h and the ω^i are satisfied. Let us restrict our attention to the fold at $z = 0$, which we will continue to call $\mathcal{Z}$. From equations (35a, 35b) we see that $\mathcal{Z}^*h$ is undefined and that $\mathcal{Z}^*\omega^1 = 0$, whereas from (35c)

$$\begin{aligned}
\mathcal{Z}^*\omega^2 &= \frac{2\sqrt{c}}{\big(1+a(x^2+y^2)\big)}(d\tau+\mathcal{A}_{RH})\wedge(\cos(\tau)dx - \sin(\tau)dy),\\
\mathcal{Z}^*\omega^3 &= \frac{2\sqrt{c}}{\big(1+a(x^2+y^2)\big)}(d\tau+\mathcal{A}_{RH})\wedge(\sin(\tau)dx + \cos(\tau)dy),
\end{aligned}$$

written out in a way to emphasise their similar form to that of (29). Indeed, this suggests that $\psi := d\tau + \mathcal{A}_{RH}$ is our contact form determining $\mathcal{Z}$. Computation yields

$$d\psi = \frac{8a}{(1+a(x^2+y^2))^2}dx\wedge dy, \qquad \psi\wedge d\psi = \frac{8a}{(1+a(x^2+y^2))^2}d\tau\wedge dx\wedge dy \neq 0$$

along $\mathcal{Z}$, so ψ is a contact form and $(\mathcal{Z}, \psi)$ is a contact manifold. The two vectors fields that annihilate ψ span

$$\mathcal{H}_{RH} = \ker \mathcal{Z}^*\omega^2 \oplus \ker \mathcal{Z}^*\omega^3 = \mathrm{span}\left\{ u_y \frac{\partial}{\partial \tau} - \frac{\partial}{\partial x},\ u_x \frac{\partial}{\partial \tau} + \frac{\partial}{\partial y} \right\},$$

determining the contact 2-plane $\mathcal{H}_{RH} \subset T\mathcal{Z}$. Their commutator bracket is

$$\left[u_y \frac{\partial}{\partial \tau} - \frac{\partial}{\partial x},\ u_x \frac{\partial}{\partial \tau} + \frac{\partial}{\partial y} \right] = \frac{8a}{(1 + a(x^2 + y^2))^2} \frac{\partial}{\partial \tau} \notin \mathcal{H}_{RH}$$

verifying the maximal non-integrability of the hyperplane field $\mathcal{H}_{RH}$. The Reeb field is the same as the folded Gibbons-Hawking one, $R_\psi = \frac{\partial}{\partial \tau}$.

Remark 9. Again, there is a similarity between the Gibbons-Hawking and the real heaven folds, which can be traced back to the fact that both space originate from a principal S^1-bundle $\mathcal{M} \xrightarrow{\pi} \mathcal{U} \subset \mathbb{R}^3$; in both examples the exterior derivative of the contact form φ coincided with the curvature 2-form $d\varphi = \mathcal{F} = d\mathcal{A}$ of the bundle. However the Gibbons-Hawking fold has a flat base space $\mathbb{R}^2$, yet the real heaven fold admits two, non-flat, possible base spaces, $S^2 \simeq \mathbb{C}P^1$ or $\mathcal{H}^2$, that depend on the sign of the constant a. To see this, identify $\mathbb{C} \simeq \mathbb{R}^2$ with $w = x + iy$ so that

$$d\psi = \pm \frac{i}{2} \frac{8a}{(1 + a|w|^2)^2} dw \wedge d\bar{w} = 4i\partial\bar{\partial} \log(1 \pm a|w|^2) = 8\pi\omega_\pm, \tag{36}$$

where $\omega_\pm$ is the *fundamental form* for either $\mathbb{C}P^1$ when $a > 0$ or $\mathcal{H}^2$ when $a < 0$. Without loss of generality suppose that $a = \pm 1$, then

$$(\mathbb{C}P^1): \qquad \omega_+ = \frac{i}{2\pi} \partial\bar{\partial} \log(1 + |w|^2), \tag{37a}$$

$$(\mathcal{H}^2): \qquad \omega_- = \frac{i}{2\pi} \partial\bar{\partial} \log(1 - |w|^2). \tag{37b}$$

Here ω_+ is known as the fundamental form for the Fubini-Study metric on $\mathbb{C}P^1$ in the literature, whereas ω_- is the fundamental form for the Poincaré disk model in hyperbolic space. It is perhaps not too surprising that this is the case; even before folding the real heaven background, the base spaces of the S^1-bundle was either S^3 or $\mathcal{H}^3$, so restricting to the fold is just equivalent to projecting $(\tau, x, y, z) \mapsto (\tau, x, y)$ onto the hypersurface $\mathcal{Z}$.

Conclusions

In studying a specific choice of the Gibbons-Hawking ansatz, we have defined how a hyperkähler manifold must behave if it is to admit a fold hypersurface. The definition must be different to the already well-studied symplectic and Kähler variants [3, 4], due to the additional structure enforced by the non-degeneracy of the hyperkähler forms away from the fold; this structure is implicit on the fold too through the contact structure it determines. A theorem provided by Biquard [2] was stated and proven which, whilst is not a new result, was elucidated upon and also provided the folded counterpart to the ASJ method, covered in the background material.

As far as the author is aware, the approach to folding the real heaven background in this report has not appeared in the literature, and provides a more general example of a folded hyperkähler manifold other than the Gibbons-Hawking one. Furthermore, although not being much of a surprising result, it is pleasing to see that the contact structure of the fold hypersurface $\mathcal{Z}$ of the real heaven background is determined by the connection 1-form $\mathcal{A}$ of an principal S^1-bundle, $\mathcal{Z} \xrightarrow{\pi} \mathbb{C}P^1$ or $\mathcal{Z} \xrightarrow{\pi} \mathcal{H}^2$, with the curvature 2-form $\mathcal{F}$ of each bundle adopting the familiar expression of either the Fubini-Study or Poincaré disk fundamental form respectively. It would be interesting to see if the folded Gibbons-Hawking ansatz appears as some asymptotic case of the real heaven background, in ode to the unfolded case and the linearised Boyer-Finley equation (16).

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