Symplectic Reduction, Geometric Invariant Theory, and the Kempf-Ness Theorem

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1 Moment Maps & Symplectic Reduction

1.1 Moment Maps

Suppose that G is a Lie group with Lie algebra $\mathfrak{g}$ that acts smooth on a symplectic manifold (M, ω). For this talk, we shall assume that G is also compact and connected (though this is not required for the definition of a moment map).

Definition 1. Given a symplectic manifold (M, ω) with a Lie group G acting smoothly on M, the *moment map* $\mu : M \to \mathfrak{g}^*$ is a smooth map satisfying:

1. For each $X \in \mathfrak{g}$, let
   - $\mu^X(p) := \langle \mu(p), X \rangle$, be the component on μ along X,
   - $X^\#$ be the vector field on M generated by the one-parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\} \subseteq G$.

   Then
   $$d\mu^X = \imath_{X^\#}\omega.$$

2. μ is equivariant with respect to the prescribed G-action on M, and the coadjoint action Ad_g^* on $\mathfrak{g}^*$, *i.e.*

$$\mu(g \cdot p) = \mathrm{Ad}_g^* \mu(p), \qquad \text{for all } g \in G, \text{ and } p \in M.$$

Example 2. Consider $(\mathbb{C}^n, \omega)$, where ω is the standard symplectic form, $\omega = \sum_{i=1}^n dz_i \wedge d\overline{z}_i$, and let the n-torus $T^n \cong (S^1)^n$ act as $t_i \cdot z_i \mapsto t_i z_i$. Then a moment map for this action is

$$\mu(z) = \frac{1}{2} \sum_{i=1}^n (|z_i|^2 + \lambda_i), \qquad \text{where } \lambda_i \in \mathbb{R}.$$

Here the coadjoint action on $(\mathfrak{t}^n)^*$ is trivial since T^n is abelian, and (if we change to polar coordinates) also

$$d\mu(r, \theta) = \sum_{i=1}^n r_i dr_i = \sum_{i=1}^n \imath_{\partial_{X_i}} r_i dr_i \wedge d\theta_i = \imath_X \omega,$$

where $X = (\partial_{\theta_1}, \ldots, \partial_{\theta_n})$ is the vector field generated by the action.

Remark 3. If we consider $N = \mathbb{R}^n$ and $M = T^*N$, where the cotangent coordinates p_i represent momentum and the base coordinates q_i represent position, and let $SO(n)$ act on M as rotations about the origin, then the moment map for this action is

$$\mu(p, q) = \sum_{i=1}^n p_i \times q_i,$$

which is the angular momentum of the system.

1.2 Symplectic Reduction

The properties of a moment map $\mu : M \to \mathfrak{g}^*$ imply that is the stabiliser G_α of any $\alpha \in \mathfrak{g}^*$ (with respect to the coadjoint action of G on $\mathfrak{g}^*$) preserves the level set $\mu^{-1}(\alpha)$, and that $\mu^{-1}(\alpha)$ is a submanifold of M and the symplectic form ω on M induces a symplectic structure on the quotient $\mu^{-1}(\alpha)/G_\alpha$.

Definition 4. The quotient $M_\alpha := \mu^{-1}(\alpha)/G_\alpha$ with the induced symplectic structure is called the *Marsden-Weinstein reduction*, or *symplectic reduction*.

One observes that if $\alpha = 0$, that is we reduce at the level zero so that the symplectic reduction is M_0, that the stabiliser of zero is the whole group $G_0 = G$. Intermediately, if the action of G_α on $\mu^{-1}(\alpha)$ is not free, then the quotient M_α still exists but is likely to have singularities.

Example 5. Now consider $(\mathbb{P}^n, \omega_{FS})$, where ω_{FS} is the Fubini-Study Kähler form on $\mathbb{P}^n$,

$$\omega_{FS} = \frac{i}{2} \sum_{i=0}^n \frac{dz_i \wedge d\overline{z}_i}{(1 + \|z\|^2)^2}.$$

We let S^1 act on $\mathbb{P}^n$ linearly via the representation

$$\rho : S^1 \longrightarrow \mathrm{U}(n+1), \qquad t \longmapsto \mathrm{diag}(t^{-1}, t, \ldots, t),$$

and the moment map for the action is

$$\mu(z) = \frac{-|z_0|^2 + |z_1|^2 + \ldots + |z_n|^2}{\|z\|^2}, \qquad z = [z_0 : \ldots : z_n] \in \mathbb{P}^n.$$

Now a point $z \in \mu^{-1}(\xi)$ if and only if it satisfies

$$(1+\xi)|z_0|^2 = (1-\xi)(|z_1|^2 + \ldots + |z_n|^2),$$

so $\mu^{-1}(\xi) = \emptyset$ if $\xi > 1$, and also if $\xi < 1$, then for $z \in \mu^{-1}(\xi)$ we have $z_0 \neq 0$ and

$$|z_1/z_0|^2 + \ldots |z_n/z_0|^2 = (1+\xi)/(1-\xi).$$

Therefore $\mu^{-1}(\xi) \cong S^{2n-1}$ in $\mathbb{C}^n$, and as S^1 is abelian (and μ is equivariant), the stabiliser of $\xi \in (\mathfrak{s}^1)^* \cong \mathbb{R}$ is the whole group S^1. Thus the symplectic quotient is $\mu^{-1}(\xi)/\operatorname{Stab}_{S^1}(\xi) \cong S^{2n-1}/S^1 = \mathbb{P}^{n-1}$.

This example was chosen to highlight that the symplectic reduction in this case can be identified the with the topological quotient of the dense open subspace $\mathbb{C}^n \setminus \{0\} \subset \mathbb{P}^n$ by the action of the complexification $\mathbb{C}^*$ of S^1, *i.e.* $\mathbb{P}^{n-1} \cong (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$.

2 Geometric Invariant Theory

In this section, we shall show that the phenomena that occurred in the previous examples happens in much more generality. To start with, let us assume that a *complex* Lie group G is the complexification of a maximal compact subgroup K, so $G = K_{\mathbb{C}}$. This is one of the many equivalent definitions for a Lie group to be *reductive*, and is a necessary and sufficient condition for the coordinate subring of invariant functions under the action of G, denoted $R(X)^G$, for some algebraic variety, to be finitely generated (this is Nagata's theorem).

Mumford's geometric invariant theory associates to an action of G on X a projective 'quotient' variety, denoted $X /\!\!/ G$, which is the projective variety associated to the subring $R(X)^G$ of G-invariants in the coordinate ring $R(X)$ of X.

2.1 The Spec and Proj Constructions

Let R be a finitely generated integral domain over $\mathbb{C}$, which means that there exists a surjective $\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow R$ whose kernel $I \lhd \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal. For the purpose of this talk, we shall define $\operatorname{Spec}(R)$ to be the subset of $\mathbb{C}^n$ on which all the polynomials in I vanish. Of course, $\operatorname{Spec}(R)$ can be equipped with either the Zariski topology or the analytic topology, along with their respective sheaves of rings, though I will ignore these technicalities. If we do not want to list a generating set for R, then the definition for $\mathrm{Spec}(R)$ is that it is the set of maximal ideals in R.

Example 6. Let $R = \mathbb{C}[x_1, \ldots, x_n]$, then by the Nullstellensatz, any maximal ideal $I \lhd R$ is of the form $I = (x_1 - p_1, \ldots, x_n - p_n)$, with each $p_i \in \mathbb{C}$. Thus $\mathrm{Spec}(R) \cong \mathbb{C}^n$.

Suppose that a ring R is equipped with a grading

$$R = \bigoplus_{m=0}^{\infty} R_m,$$

which allows us to define an action of the group $\mathbb{C}^*$ on the ring R by setting $t \cdot f := t^m f$, for any $f \in R_m$ and $t \in \mathbb{C}^*$. This in turn induces an action of $\mathbb{C}^*$ on $\mathrm{Spec}(R)$ by setting $(t^{-1} \cdot f)(x) := f(t \cdot x)$ (the inverse exponent is required to make the action associative, or 'compatible'). A fixed point $x \in \mathrm{Spec}(R)$ is fixed if and only if $t \cdot x = x$ for all $t \in \mathbb{C}^*$, which is the same as

$$f(x) = f(t \cdot x) = (t^{-1} \cdot f)(x), \qquad \text{for every } f \in R.$$

If $f \in R_0$, then this is true trivially, whereas if $f \in R_m$ with $m \neq 0$, then this condition is equivalent to $f(x) = 0$. Hence the fixed-point set of $\mathbb{C}^*$ is equal to the vanishing set of the ideal generated by all the functions of non-zero weight.

We call the following ideal

$$R_+ = \bigoplus_{m>0} R_m$$

the *irrelevant ideal* of R, which is exactly the vanishing ideal of the fixed-point set for the action of $\mathbb{C}^*$ on $\mathrm{Spec}(R)$. Thus the fixed-point set is equal to $\mathrm{Spec}(R/R_+) \cong \mathrm{Spec}(R_0)$.

Definition 7. For a graded ring $R = \oplus_{m=0}^{\infty} R_m$, define the *projective spectrum* of R to be

$$\mathrm{Proj}(R) := (\mathrm{Spec}(R) \setminus \mathrm{Spec}(R_0))/\mathbb{C}^*.$$

Example 8. For $R = \mathbb{C}[x_0, \ldots, x_n]$, where $\deg(x_i) = 1$ for each $0 \leq i \leq n$ (that is, $\mathbb{C}^*$ acts on each x_i with weight 1), we have that $R_0 = \mathbb{C}$ and the irrelevant ideal is $R_+ = (x_0, \ldots, x_n)$. Thus, we have a $\mathbb{C}^*$-equivariant embedding $\mathrm{Spec}(R) = \mathbb{C}^{n+1}$ with $\mathrm{Spec}(R/R_+) = \mathrm{Spec}(R_0) \cong \{0\}$, so

$$\mathrm{Proj}(R) = (\mathrm{Spec}(R) \setminus \mathrm{Spec}(R_0))/\mathbb{C}^* = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = \mathbb{P}^n.$$

2.2 Affine GIT Quotients

Suppose that R is a finitely generated integral domain over $\mathbb{C}$, so that $X = \mathrm{Spec}(R)$ defines a complex affine algebraic variety X. Further suppose that G is a reductive algebraic group acting on R, so that its $\mathbb{C}$-subalgebra of invariants R^G is finitely generated by Nagata's theorem.

Definition 9. The *affine GIT quotient* of X with respect to an action of G is

$$X /\!\!/ G := \operatorname{Spec}(R^G).$$

Proposition 10. *There is a surjective map from X to $X /\!\!/ G$, satisfying: any two points in X lie in the same fibre of this map if and only if the closures of their G-orbits intersect non-trivially.*

Let us investigate what happens if we naively take this usual quotient; let $X = \mathbb{C}^2 = \operatorname{Spec}(\mathbb{C}[x, y])$ and $G = \mathbb{C}^*$ act on X as

$$\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y).$$

Then $\mathbb{C}[x, y]^G = \mathbb{C}[xy] \cong \mathbb{C}[z]$ and the GIT quotient map should be

$$\varphi : X \longrightarrow X /\!\!/ G = \operatorname{Spec}(\mathbb{C}[z]) \cong \mathbb{C}, \qquad (x, y) \longmapsto xy = z.$$

However the closures of the non-closed three orbits

$$\mathbb{C}^* \cdot (x, y), \quad \mathbb{C}^* \cdot (0, y), \quad \mathbb{C}^* \cdot (0, 0),$$

are then identified, so $\varphi^{-1}(0)$ is equal to the union of the two coordinate axes. Thus away from the coordinate axes, the $\mathbb{C}^*$-orbits are closed and they can be separated, yet the $\mathbb{C}^*$-orbits of each coordinate axis are identified, and cannot be *separated* (the italics are to emphasise that separatedness in algebraic geometry is analogous to the Hausdorff property in topology).

This discussion motivates the following definitions:

Definition 11. Let $X = \operatorname{Spec}(R)$ and let G be a reductive algebraic group that acts on R, and thus on X. Then:

1. A point $x \in X$ is *stable* if its orbit is closed in X and $\dim G_x = 0$, *i.e.* its stabiliser G_x is finite. Denote by X^s the stable locus of points in X.

2. A point $x \in X$ is *semi-stable* if zero does not belong in the closure of its orbit under the G-action, *i.e.* $0 \notin \overline{G \cdot x}$. Denote by X^{ss} the semi-stable locus of points in X.

3. A point $x \in X$ is *unstable* if it is not semi-stable.

So in the previous example, all the points disjoint from the x and y-coordinate axes were stable, yet the points in them were unstable. So after removing the unstable points, we would get a well-behaved quotient.

2.3 Projective GIT Quotients

To construct something similar for projective algebraic varieties, we need to remove some bad points from the orbit where the rational morphism induced by $R^G \hookrightarrow R$ is undefined. This is similar to the removal of the origin in $\mathbb{C}^{n+1}$ when we constructed $\mathbb{P}^n$.

Definition 12. For a linear action of a reductive group G on a projective variety $X \subseteq \mathbb{P}^n$, we denote by $X /\!\!/ G$ the projective variety $\mathrm{Proj}(R^G)$ associated to the finitely generated $\mathbb{C}$-algebra R^G of G-invariant functions, where $R = R(X)$ is the homogeneous coordinate ring of X. The inclusion $R^G \hookrightarrow R$ defines a rational map

$$\varphi : X \to X /\!\!/ G$$

which is undefined on the *null-cone*

$$N_{R^G}(X) := \{x \in X : f(x) = 0, \text{ for all } f \in R^G_+\}.$$

Define the *semi-stable locus* to be the complement in X of the null-cone, $X^{ss} := X \setminus N_{R^G}(X)$, then the *projective GIT quotient* for the action of G on $X \subseteq \mathbb{P}^n$ is the morphism $\varphi : X^{ss} \to X /\!\!/ G$.

Example 13. Let $G = \mathbb{C}^*$ act on $X = \mathbb{P}^n$ by

$$t \cdot [x_0 : x_1 : \ldots : x_n] = [t^{-1}x_0 : tx_1 : \ldots : tx_n].$$

Here, the homogeneous coordinate ring is $R = \mathbb{C}[x_0, \ldots, x_n]$ which is graded by degree. It is not hard to see that the G-invariant subring is

$$R^G = \mathbb{C}[x_0x_1 : \ldots, x_0x_n] \cong \mathbb{C}[y_1, \ldots, y_n],$$

which corresponds to the projective variety $X /\!\!/ G = \mathrm{Proj}(R^G) \cong \mathbb{P}^{n-1}$. Since we have chosen explicit generators for both R and R^G, we can write down the rational morphism

$$\varphi : \mathbb{P}^n = X \longrightarrow X /\!\!/ G = \mathbb{P}^{n-1}, \qquad [x_0 : x_1 : \ldots : x_n] \longmapsto [x_0x_1 : \ldots : x_0x_n],$$

from which it is clear that the null-cone is

$$N_{R^G}(X) = \{[x_0 : \ldots : x_n] \in \mathbb{P}^n : x_0 = 0, \text{ or } x_1, \ldots, x_n = 0\}$$

is the projective variety defined by the homogeneous ideal $I = (x_0x_1, \ldots, x_0x_n)$. Thus

$$X^{ss} = \{[x_0 : \ldots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ and } (x_1, \ldots, x_n) \neq (0, \ldots, 0)\} \cong \mathbb{C}^n \setminus \{0\}.$$

Therefore

$$\varphi : X^{ss} = \mathbb{C}^n \setminus \{0\} \longrightarrow X /\!\!/ G = \mathbb{P}^{n-1}.$$

3 The Kempf-Ness Theorem

At the start of this talk, we came across an example where the symplectic reduction with respect to an $K = S^1$ action on $\mathbb{P}^n$ via

$$S^1 \ni t \longmapsto \mathrm{diag}(t^{-1}, t, \ldots, t) \in U(n+1)$$

was equal to $\mathbb{P}^{n-1}$, *i.e.* $\mu^{-1}(\xi)/K_\xi \cong \mathbb{P}^{n-1}$. One observes that this is identical to the topological quotient of the dense open subset $\mathbb{C}^n \setminus \{0\}$ of $\mathbb{P}^n$, by the action of the complexification $\mathbb{C}^*$ of $K = S^1$.

On the other hand, we also saw that the projective GIT quotient for the ring $R = \mathbb{C}[x_0 : \dots x_n]$ with the following $G = \mathbb{C}^*$-action

$$t \cdot [x_0 : x_1 : \dots : x_n] = [t^{-1}x_0 : tx_1, \dots, tx_n],$$

also resulted in

$$\mathbb{P}^n /\!\!/ \mathbb{C}^* \cong \mathbb{P}^{n-1}.$$

This phenomena holds in much more generality and, quite surprisingly in fact, the symplectic reduction $\mu^{-1}(0)/K$ and the geometric invariant theoretic quotient $X /\!\!/ G$ are topologically the same.

Theorem 14 (Kempf-Ness Theorem)**.** *Let $X \subset \mathbb{P}^n$ be a projective variety with the action of a complex, reductive Lie group G with respective maximal compact subgroup K. Then any $x \in X$ is semi-stable if and only if the closure of its orbit meets $\mu^{-1}(0)$, and the inclusion of $\mu^{-1}(0)$ into X^{ss} induces a homeomorphism*

$$\mu^{-1}(0)/K \longrightarrow X /\!\!/ G.$$

3.1 Linearisations

For a proper projective algebraic variety X, we need to embed it into projective space $X \hookrightarrow \mathbb{P}^n$, and this embedding is determined by an ample line bundle $\mathcal{L}$ on X. Thus, to construct a GIT quotient of the projective variety X we also need to consider some extra data that comes in the form of a lift of the G-action to the line bundle $\mathcal{L}$ on X. Such a choice of lift is called a *linearisation* of the action.

Definition 15. Let $\pi : \mathcal{L} \to X$ be a line bundle on X. A *linearisation* of the action of G with respect to $\mathcal{L}$ is an action of G on $\mathcal{L}$ such that:

1. For all $g \in G$ and $l \in \mathcal{L}$, we have $\pi(g \cdot l) = g \cdot \pi(l)$, *i.e.* π is G-equivariant.
2. For all $x \in X$ and $g \in G$, the fibre map $\mathcal{L}_x \mapsto \mathcal{L}_{g \cdot x}$ is a linear map.

Example 16. Consider the trivial line bundle $\mathcal{L} = X \times \mathbb{A}^1$ on a variety X, then a linearisation of a G-action on X with respect to $\mathcal{L}$ corresponds to a character $\chi : G \to \mathbb{G}_m$. The character χ defines a lift of the action to $\mathcal{L}$ by

$$g \cdot (x, z) = (g \cdot x, \chi(g)z), \qquad \text{where } (x, z) \in X \times \mathbb{A}^1.$$

To finish, we remark that the freedom to choose a linearisation of a G-action on a line bundle $\mathcal{L}$ over a variety X is analogous to the choice of constant $\lambda \in Z(\mathfrak{g}^*)$ added onto the image of the moment map in the symplectic picture.

Example 17. Consider $\mathbb{C}^n$ with the diagonal action of S^1 on it, so that its moment map is

$$\mu(z) = \frac{1}{2}\sum_{i=1}^{n} |z_i|^2 - \lambda, \qquad \lambda \in \mathbb{Z}_{>0}.$$

In the GIT picture, this is equivalent to taking $R = \mathbb{C}[x_1, \ldots, x_n, y]$ with $\deg(x_i) = 0$ and $\deg(y) = 1$, so we still recover

$$\mathrm{Proj}(R) = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n]) \times \mathbb{P}^0 \cong \mathbb{C}^n.$$

However, in introducing a new, *distinct* action of $\mathbb{C}^*$ that lifts to R (which can be viewed as the space of sections over $\mathbb{C}^n$) as

$$t \cdot x_i = t^{-1} x_i, \qquad t \cdot y = t^{\lambda} y,$$

and then the maximal compact subgroup $S^1 \subset \mathbb{C}^*$ has the moment map as given above.