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# GEOMETRIC QUANTISATION OF HYPERTORIC MANIFOLDS BY SYMPLECTIC CUTTING

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GENERAL NOTES

## ABSTRACT

Lorem ipsum.

### 1 Introduction

Lorem ipsum.

### 2 Background

#### 2.1 Hypertoric Geometry

#### 2.2 Localisation and Symplectic Cutting

### 3 Hypertoric Varieties

#### 3.1 Compactification via Symplectic Cutting

We will use the  $S^1$ -action to perform a symplectic cut of the toric hyperkähler manifold  $\mathfrak{M}$  to compactify it, which has the effect of bounding the  $\|w\|^2$ -norm component of the real moment map  $\bar{\mu}_{\mathbb{R}}$  by above, and discarding the rest that lies above this bound. Consider the product  $\mathfrak{M} \times \mathbb{C}$ , and let  $S^1$  act on  $\mathfrak{M} \times \mathbb{C}$  via the diagonal product action, i.e.  $S^1$  acts on  $M$  by rotating the cotangent fibre coordinates, and on  $\mathbb{C}$  in the standard way:

$$e^{i\theta} \cdot ([z, w], \xi) = (e^{i\theta} \cdot [z, w], e^{i\theta} \xi) = ([z, e^{i\theta} w], e^{i\theta} \xi).$$

This action is Hamiltonian, and the corresponding moment map  $\Phi : \mathfrak{M} \times \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$  for the  $S^1$ -action is

$$\Phi([z, w], \xi) = \phi[z, w] + |\xi|^2 = \|w\|^2 + |\xi|^2.$$

Then we have

$$\begin{aligned} \Phi^{-1}(\epsilon) &= \{([z, w], \xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = \epsilon\} \\ &= \{[z, w] \in M : \|w\|^2 = \epsilon\} \bigsqcup \{([z, w], \xi) \in M \times \mathbb{C} : |\xi| = \pm \sqrt{\epsilon - \|w\|^2}\} \\ &= \{[z, w] \in M : \|w\|^2 = \epsilon\} \bigsqcup \{([z, w], \xi) \in M \times \mathbb{C} : \xi = e^{i \arg(\xi)} \sqrt{\epsilon - \|w\|^2}\} \\ &= \phi^{-1}(\epsilon) \bigsqcup (\mathfrak{M} \times S^1) \\ &=: \Sigma_1 \sqcup \Sigma_2, \end{aligned}$$

where we denote the level-set  $\phi^{-1}(\epsilon) \subseteq \mathfrak{M}$  by  $\Sigma_1$ , and  $\Sigma_2 \cong \mathfrak{M} \times S^1$  is the trivial  $S^1$ -bundle over  $\Sigma_2$  given by the globally defined section

$$\mathfrak{M} \rightarrow \mathfrak{M} \times S^1, \quad [z, w] \mapsto ([z, w], e^{i\theta} \sqrt{\epsilon - \|w\|^2}), \quad e^{i\theta} \in S^1.$$

Finally, taking the symplectic reduction of  $\Phi^{-1}(\epsilon)$  with respect to the  $S^1$ -action, we obtain the *symplectic cut of  $\mathfrak{M}$  at level- $\epsilon$* ,

$$M_{\leq \epsilon} := \Phi^{-1}(\epsilon)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where  $\Sigma_1/S^1 \cong \phi^{-1}(\epsilon)/S^1$  is just the usual symplectic reduction, and where  $\Sigma_2/S^1$  is diffeomorphic to  $\mathfrak{M}$  for  $\|w\|^2 < \epsilon$ , which we will denote by  $\mathfrak{M}_{< \epsilon}$ .

### 3.2 The Combinatorics of the Cut Space, $\mathfrak{M}_{\leq \epsilon}$

Since the residual circle  $S^1$ -action acts as a subtorus  $S_A^1$  of the residual torus  $T^d$  on each component  $\mathcal{E}_A$  of the extended core, the hyperplane arrangement determined in  $(\mathfrak{t}^d)^*$  by the real moment map  $\bar{\mu}_{\mathbb{R}}$  is compactified by dropping in half-spaces with an inwards-pointing normal vector, given by  $v_A$  when taking the cut.

Recall from the previous section that  $j_A : S^1 \hookrightarrow T^n$  denoted the inclusion homomorphism of  $S^1$  into the original torus  $T^n$ . If we let  $j_{A,*} : \mathfrak{s}^1 \rightarrow \mathfrak{t}^n$  represent the differential of this inclusion, then

$$j_{A,*}(1) = \sum_{i \in A} e_i \in \mathfrak{t}^n,$$

and the generator  $\exp(v_A)$  of the one-parameter subgroup  $S_A^1$  in  $T^d$  is

$$\exp(v_A) = \exp(\pi_* \circ j_{A,*}(1)),$$

or to be more concise,

$$S_A^1 = \left\{ \exp \left( r \cdot \sum_{i \in A} u_i \right) \mid r \in \mathbb{R} \right\}.$$

Then the moment map for the restricted  $S^1$ -action to  $\mathcal{E}_A$  is

$$\phi_A[z, w] := \phi|_{\mathcal{E}_A}[z, w] = (j_A^* \circ \mu_{\mathbb{R}})[z, w] = \left\langle \bar{\mu}_{\mathbb{R}}[z, w], \sum_{i \in A} u_i \right\rangle,$$

where  $j_A^* : (\mathfrak{t}^n)^* \rightarrow \mathbb{R}^*$  is the transposed differential of the inclusion,  $j_{A,*}$ .

As the  $S_A^1$ -action depends combinatorially on the component  $\mathcal{E}_A$ , the image of the real moment map in  $(\mathfrak{t}^d)^*$  is compactified by inserting a half-space  $Z_A$  with inwards-pointing normal  $v_A = \sum_{i \notin A} u_i$  determining the orientation, on each component  $\Delta_A$ .

## 4 Hypertoric Subvarieties

### 4.1 Universal Modifications

Let  $(M, \omega_{HK}, \Phi_{HK})$  be a tri-Hamiltonian  $K$ -manifold, where  $\omega_{HK} = \omega_{\mathbb{R}} + \omega_{\mathbb{C}}$  and  $\Phi_{HK} = \Phi_{\mathbb{R}} + \Phi_{\mathbb{C}}$ . Define  $j_{\mathbb{C}} : M \rightarrow M \times M \times T^*K_{\mathbb{C}}$  by  $j(m) = (m, 1, \Phi_{\mathbb{C}}(m))$ , and  $j_{\mathbb{R}} : M \rightarrow M \times T^*K$  by  $j_{\mathbb{R}}(m) = (m, 1, \Phi_{\mathbb{R}})$ .

**Lemma 4.1.** *Let  $(M, I, \omega, K_{\mathbb{C}}, \mu)$  be a complex-symplectic  $K$ -Hamiltonian manifold. Then the complex-symplectic quotient  $(M \times T^*K_{\mathbb{C}}) //_0 K_{\mathbb{C}}$  is isomorphic to  $M$  as complex-symplectic manifolds. Here, the complex-symplectic quotient  $(M \times T^*K_{\mathbb{C}}) //_0 K_{\mathbb{C}}$  is taken with respect to the diagonal  $K_{\mathbb{C}}$ -action, where  $K_{\mathbb{C}}$  acts on  $T^*K_{\mathbb{C}}$  from the left.*

*Proof.* The  $K_{\mathbb{C}}$ -action on  $T^*K_{\mathbb{C}}$  is free and proper, and thus so is the diagonal action on the product  $M \times T^*K_{\mathbb{C}}$ , so we are in the position to apply the holomorphic Marsden-Weinstein Reduction Theorem. Identifying  $T^*K_{\mathbb{C}} \cong K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^*$  via polar decomposition, the moment map for the diagonal action is

$$\Phi : M \times K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^* \longrightarrow \mathfrak{k}_{\mathbb{C}}^*, \quad \Phi(m, k, \lambda) = \mu(m) - \text{Ad}_g^*(\lambda),$$

thus

$$(M \times T^*K_{\mathbb{C}}) //_0 K_{\mathbb{C}} = \Phi^{-1}(0)/K_{\mathbb{C}} = \{ (m, k, \lambda) \in M \times T^*K_{\mathbb{C}} \mid \mu(m) = \text{Ad}_k^*(\lambda) \} / K_{\mathbb{C}},$$

with  $K_{\mathbb{C}}$  acting as

$$g \cdot (m, k, \lambda) = (g \cdot m, gk, \text{Ad}_g^*(\lambda)).$$

Now define  $j : M \rightarrow \Phi^{-1}(0) \subset M \times T^*K_{\mathbb{C}}$  by  $j(m) = (m, e, \mu(m))$ , which descends to a biholomorphism  $\bar{j} : M \rightarrow \Phi^{-1}(0)/K_{\mathbb{C}}$ .

To prove that  $\bar{j}$  is also a symplectomorphism, i.e. if  $\eta$  represents the complex-symplectic form on  $(M \times T^*K_{\mathbb{C}}) //_0 K_{\mathbb{C}}$ , that  $\bar{j}^*\eta = \omega$ , let  $\nu$  denote the canonical form on  $T^*K_{\mathbb{C}}$  and let  $i : \Phi^{-1}(0) \hookrightarrow M \times T^*K_{\mathbb{C}}$  denote the inclusion, and  $q : \Phi^{-1}(0) \rightarrow \Phi^{-1}(0)/K_{\mathbb{C}}$  denote the quotient map.

$$\begin{array}{ccccc} (M, \omega) & \xrightarrow{j} & (\Phi^{-1}(0), \omega) & & \\ & \searrow \bar{j} & \downarrow q & \swarrow i & \\ & & (\Phi^{-1}(0)/K_{\mathbb{C}}, \eta) & & (M \times T^*K_{\mathbb{C}}, \omega + \nu) \end{array}$$

□

## References

- [1] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.