The actual presentation should be displayed using the following URL, which is interactive:

bencwbrown.github.io/presentations/202009XX-glen-presentation/index.html

Compactifying Hypertoric Manifolds via Symplectic Cutting

Benjamin Brown

The University of Edinburgh

Structure Of This Presentation

- Delzant Polytopes and Toric Symplectic Manifolds
- Their Hypertoric Analogues
- Compactification via Symplectic Cutting
- Outlook

Delzant Polytopes

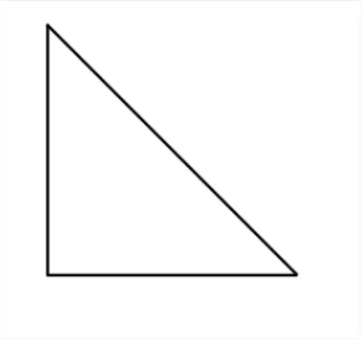
A **Delzant polytope** $\Delta \subseteq \mathbb{R}^n$ is a convex polytope satisfying:

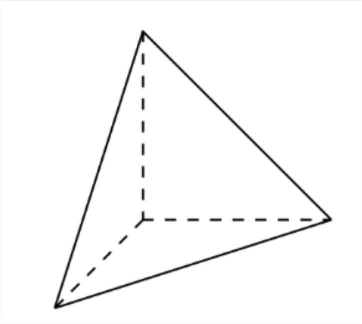
- **(simple)**; n edges meet at each vertex;
- **(rational)**; each edge meeting a vertex p is of the form $p + tu_i, t \geq 0, u_i \in \mathbb{Z}^n$;
- **(smooth)**; for each vertex, respective edge vectors $u_1, \ldots, u_n$, can be chosen to form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

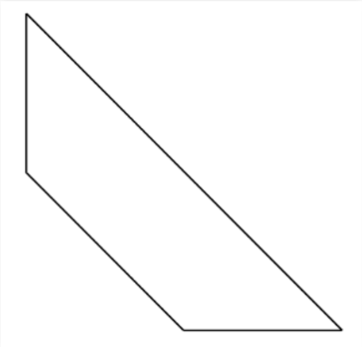
Each $\Delta \subseteq \mathbb{R}^n$ can be written as

$$\Delta = \bigcap_i \{x \in (\mathbb{R}^n)^* \ : \ \langle x,\ u_i \rangle + \lambda_i \geq 0\}, \qquad \lambda_i \in \mathbb{R},$$

where $u_i \in \mathbb{Z}^n$ are the inward-pointing normals to the facets of Δ.







Symplectic Toric Manifolds

Definition: A $2n$-dimensional symplectic toric manifold is a compact connected symplectic manifold (M^{2n}, ω) with an effective Hamiltonian action of an n-torus T^n, with corresponding moment map $\mu : M \to \mathrm{Lie}(T^n)^* \cong (\mathbb{R}^n)^*$.

This definition is easier to elaborate upon with an example.

Example

T^n acts on $\mathbb{C}^n$ diagonally:

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n),$$

Moment map for the action is: $\mu : \mathbb{C}^n \longrightarrow \mathbb{R}^n$,

$$\mu(z) = \frac{1}{2} \sum_{k=1}^{n} |z_k|^2 e_k \in \mathbb{R}^n.$$

Abuse of notation: Identify $(\mathbb{R}^n)^* = \mathbb{R}^n$ and omit constants for moment maps.

Toric Varieties

T^n acts diagonally on $\mathbb{C}^n$ preserving the Kähler structure.

Let $\{u_1, \ldots, u_n\}$ be inner-normals to some Delzant Δ. They generate a Lie algebra $\mathfrak{n}$ for some sub-torus $N \subseteq T^n$.

$\implies$ exact sequences, where $\pi : e_i \mapsto u_i$:

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^d \longrightarrow 0,$$

dualising

$$0 \longleftarrow \mathfrak{n}^* \xleftarrow{\iota^*} \mathbb{R}^n \xleftarrow{\pi^*} \mathbb{R}^d \longleftarrow 0,$$

or exponentiating

$$1 \longrightarrow N \xrightarrow{\iota} T^n \xrightarrow{\pi} T^d \longrightarrow 1.$$

N acts on $\mathbb{C}^n$ via ι, with moment map

$$\bar{\mu}(z) = (\iota^* \circ \mu)(z) = \frac{1}{2}\sum_{k=1}^{n} |z_k|^2 \alpha_k \in \mathfrak{n}^*,$$

with $\alpha_k = \iota^*(e_k)$.

If 0 is a regular value for $\bar{\mu}$, then

$$X = \bar{\mu}^{-1}(0)/N = (\iota^* \circ \mu)^{-1}(0)/N$$

is a smooth Kähler quotient (assuming $\{u_1, \ldots, u_n\}$ come from Delzant Δ).

Convexity

Residual $T^d = T^n/N$ action on X, moment map $\phi : X \to (\mathbb{R}^d)^*$.

For X compact, *Atiyah-Guillemin-Sternberg theorem* $\implies \mathrm{Im}(\phi)$ is a convex polytope Δ, and fixed-points of T^d are its vertices.

Example

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathbb{R}^3 \xrightarrow{\pi} \mathbb{R}^2 \longrightarrow 0$$

$$u_1 = (1,0),\ u_2 = (0,1),\ u_3 = (-1,-1),$$
$$\ker(\pi) = \langle e_1 + e_2 + e_3 \rangle \subset \mathbb{R}^3$$
$$\implies \iota(t) = (t,t,t) \implies \iota^*(x,y,z) = x + y + z.$$

T^3 on $\mathbb{C}^3$ moment map: $\mu(z) = \frac{1}{2}\sum_{k=1}^{3} |z_k|^2 e_k$, so N moment map is:

$$\bar{\mu}(z) = (\iota^* \circ \mu)(z) = \frac{1}{2}\|z\|^2$$

$$X = \mu^{-1}(c)/N = \{\|z\|^2 = 2c\}/N \cong S^5/S^1 \cong \mathbb{CP}^2.$$

$$X = \mu^{-1}(c)/N \cong \mathbb{CP}^2$$

has residual $T^2 = T^3/N$ action:

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2].$$

Moment map

$$\phi(z) = \frac{1}{2}\left(\frac{|z_1|^2}{\|z\|^2}, \frac{|z_2|^2}{\|z\|^2}\right), \quad \text{with } \operatorname{Im}(\phi) = \Delta_2.$$

Fixed-points of T^2:

$$\begin{aligned} [1:0:0] &\mapsto (0,0) \\ [0:1:0] &\mapsto (1/2, 0) \\ [0:0:1] &\mapsto (0, 1/2) \end{aligned}$$

Hyperkähler Moment Maps

Analogous though now with $\mathbb{H}^n$.

Flat hyperkähler with three complex structures J_1, J_2, and J_3.

Fix J_1 so $\mathbb{H}^n \cong T^*\mathbb{C}^n$.

T^n-action on $\mathbb{C}^n$ induces T^n-action on $T^*\mathbb{C}^n$.

Hyperkähler moment maps

$$\mu_{\mathbb{R}}(z, w) = \frac{1}{2} \sum_{k=1}^{n} (|z_k|^2 - |w_k|^2) e_k \in \mathbb{R}^n,$$

$$\mu_{\mathbb{C}}(z, w) = \sum_{k=1}^{n} (z_k w_k) e_k \in \mathbb{C}^n.$$

Hypertoric Analogues

Choose $\{u_1, \dots, u_n\}$ to get $N \overset{\iota}{\hookrightarrow} T^n$.

Mutatis mutandi, same construction as before:

$$\bar{\mu}_{\mathbb{R}}(z, w) := (\iota^* \circ \mu_{\mathbb{R}})(z, w) = \frac{1}{2}\iota^* \left(\sum_{k=1}^{n} \left(|z_k|^2 - |w_k|^2\right) e_k \right),$$

$$\bar{\mu}_{\mathbb{C}}(z, w) := (\iota_{\mathbb{C}}^* \circ \mu_{\mathbb{C}})(z, w) = \iota_{\mathbb{C}}^* \left(\sum_{k=1}^{n} (z_k w_k) e_k \right).$$

Hyperkähler analogue M to the Kähler quotient X is

$$M := \left(\bar{\mu}_{\mathbb{R}}^{-1}(\lambda) \cap \bar{\mu}_{\mathbb{C}}^{-1}(0)\right)/N.$$

Hyperplane Arrangements

Residual $T^d = T^n/N$-action on M; has hyperkähler moment maps

$$\phi_{\mathbb{R}}[z,w] = \frac{1}{2}\sum_{k=1}^{n}(|z_k|^2 - |w_k|^2 - \lambda_k)\alpha_k,$$

$$\phi_{\mathbb{C}}[z,w] = \sum_{k=1}^{n}(z_k w_k)\alpha_k.$$

Image $\mathrm{Im}(\phi_{\mathbb{R}}) \subseteq \mathbb{R}^d$ decomposes into a hyperplane arrangement: for $y \in \mathbb{R}^d$,

$$F_i = \{y \cdot u_i + \lambda_i \geq 0\}, \qquad G_i = \{y \cdot u_i + \lambda_i \leq 0\},$$

$$H_i = F_i \cap G_i.$$

Example - Hypertoric Analogue for $\mathbb{CP}^2$

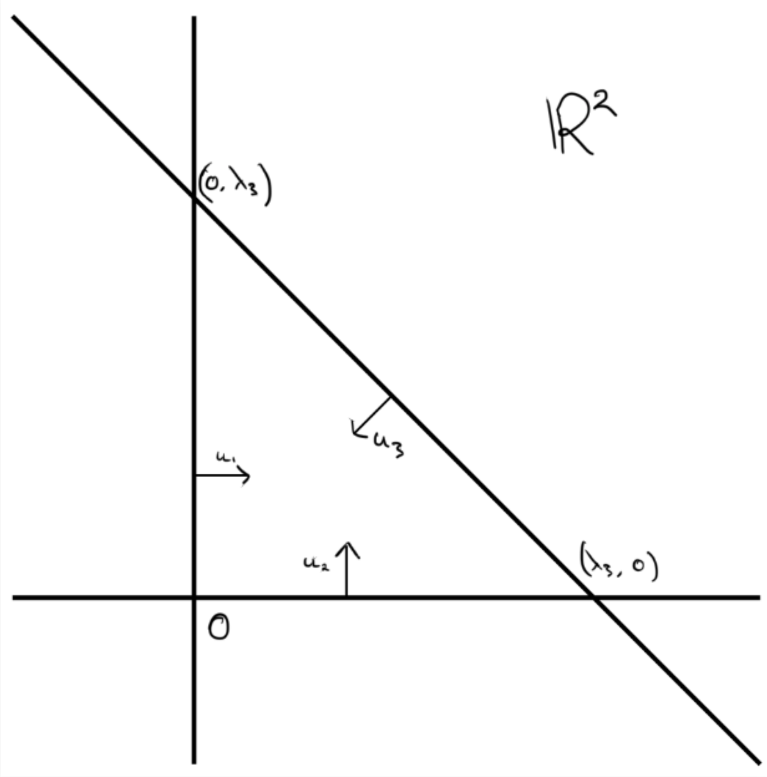
Extend T^3 diagonal action on $\mathbb{C}^3$ to $T^*\mathbb{C}^3$; now N acts as $t \cdot (z, w) = (tz, t^{-1}w)$.

Hyperkähler quotient

$$M = \left(\bar{\mu}_{\mathbb{R}}^{-1}(\lambda) \cap \bar{\mu}_{\mathbb{C}}^{-1}(0)\right)/N \cong T^*\mathbb{CP}^2$$

has residual T^2-action.

$$\phi_{\mathbb{R}}[z, w] = \frac{1}{2} \sum_{k=1}^{3} (|z_k|^2 - |w_k|^2) - \lambda_3.$$



$\mathbb{R}^2$
$(0, \lambda_3)$
u_3
u_1
u_2
$(\lambda_3, 0)$
O

The $\{H_i\}$ divide $(\mathbb{R}^d)^*$ into a union of closed convex polyhedra

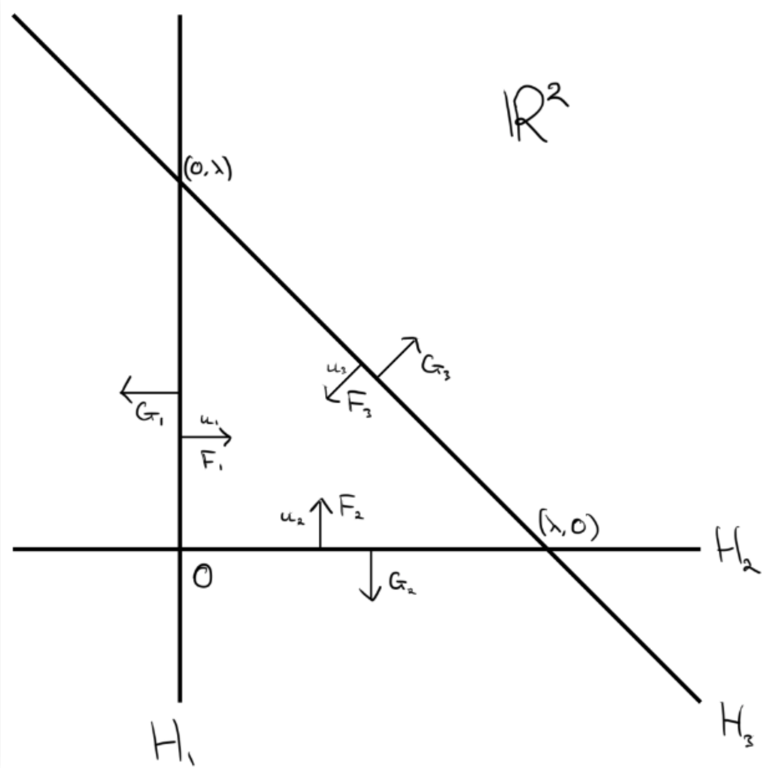
$$\Delta_A = \bigcap_{i \in A} F_i \cap \bigcap_{i \notin A} G_i .$$

Set $\mathcal{E} := \phi_{\mathbb{C}}^{-1}(0) = \{[z, w] \in M : z_i w_i = 0, \text{ for all } i\} \subseteq M$, which further decomposes

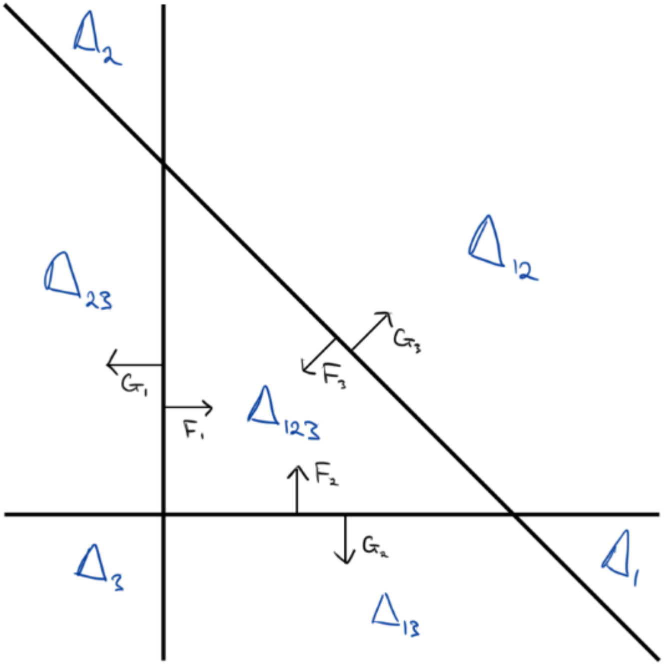
$$\mathcal{E}_A := \{w_i = 0 \text{ for all } i \in A, \text{ and } z_i = 0 \text{ for all } i \notin A\},$$

for subsets $A \subseteq \{1, \ldots, n\}$.

Lemma: If $w_i = 0$ then $\phi_{\mathbb{R}}[z, w] \in F_i$, and if $z_i = 0$, then $\phi_{\mathbb{R}}[z, w] \in G_i$.



$\mathbb{R}^2$
$(0,\lambda)$
$(\lambda,0)$
O
H_1
H_2
H_3
G_1
u_1
F_1
u_2
F_2
G_2
u_3
F_3
G_3


Δ_2
Δ_{12}
Δ_{23}
G_3
F_3
G_1
F_1
Δ_{123}
F_2
G_2
Δ_3
Δ_1
Δ_{13}

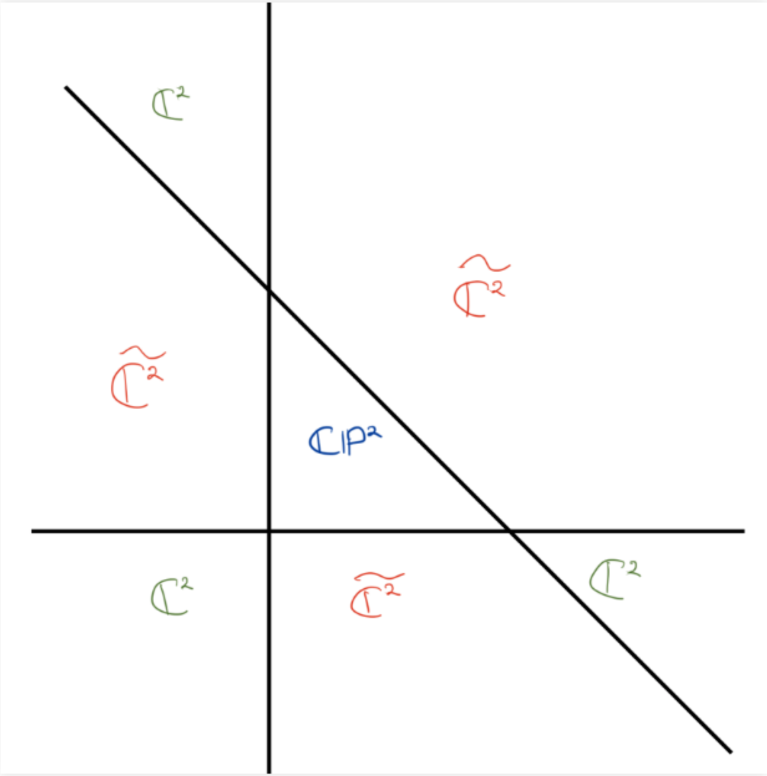
The Core and the Extended Core of M

Call $\mathcal{E}_A = \phi_{\mathbb{C}}^{-1}(0)$ the **extended core** of M:

Each $\mathcal{E}_A \subseteq M$ is a d-dimensional Kähler subvariety with effective Hamiltonian T^d-action.

Lemma: $\phi_{\mathbb{R}}(\mathcal{E}_A) \cong \bigcap_{i \in A} F_i \cap \bigcap_{i \notin A} G_i =: \Delta_A$, and Δ_A is corresponding Delzant polytope to $\mathcal{E}_A$.

We call $\mathcal{L} := \bigcup_{\Delta_A \text{ bounded}} \mathcal{E}_A$, the **core** of M.



$\mathbb{C}^2$
$\widetilde{\mathbb{C}^2}$
$\widetilde{\mathbb{C}^2}$
$\mathbb{CP}^2$
$\mathbb{C}^2$
$\widetilde{\mathbb{C}^2}$
$\mathbb{C}^2$

Residual S^1-Action

Additional S^1-action on $T^*\mathbb{C}^n$:

$$\tau \cdot (z, w) = (z, \tau w).$$

Descends to M, but **only** preserves J_1 structure, not J_2 nor J_3.

Does not act on M as a sub-torus of T^d, but does when restricted to each $\mathcal{E}_A$.

For $[z, w] \in \mathcal{E}_A$,

$$[z; \tau_1 w_1, \ldots, \tau_n w_n] = [\tau_1^{-1} z_1, \ldots, \tau_n^{-1} z_n; \tau_1 w_1, \ldots, \tau_n w_n],$$

$$\text{where } \tau_i = \begin{cases} \tau, & \text{if } i \in A, \\ 1, & \text{if } i \notin A. \end{cases}$$

Symplectic Cut

S^1-action on M has **proper** moment map $\Phi[z, w] = \frac{1}{2}\|w\|^2$.

Extend it to $M \times \mathbb{C}$ via

$$e^{i\theta} \cdot (m, \xi) = (e^{i\theta} \cdot m, e^{i\theta}\xi),$$

with moment map

$$\rho_{\mathrm{cut}} : M \times \mathbb{C} \to \mathbb{R}; \quad (m, \xi) \mapsto \Phi(m) + \frac{1}{2}|\xi|^2.$$

The **symplectic cut** is the quotient

$$M_{\epsilon-\mathrm{cut}} := \rho_{\mathrm{cut}}^{-1}(\epsilon)/S^1 \cong \{m \in M : \Phi(m) < \epsilon\} \sqcup (\Phi^{-1}(\epsilon)/S^1).$$

Compactification of M

S^1 acts on M, depending combinatorially on $A \subseteq \{1, \ldots, n\}$.
Recall:

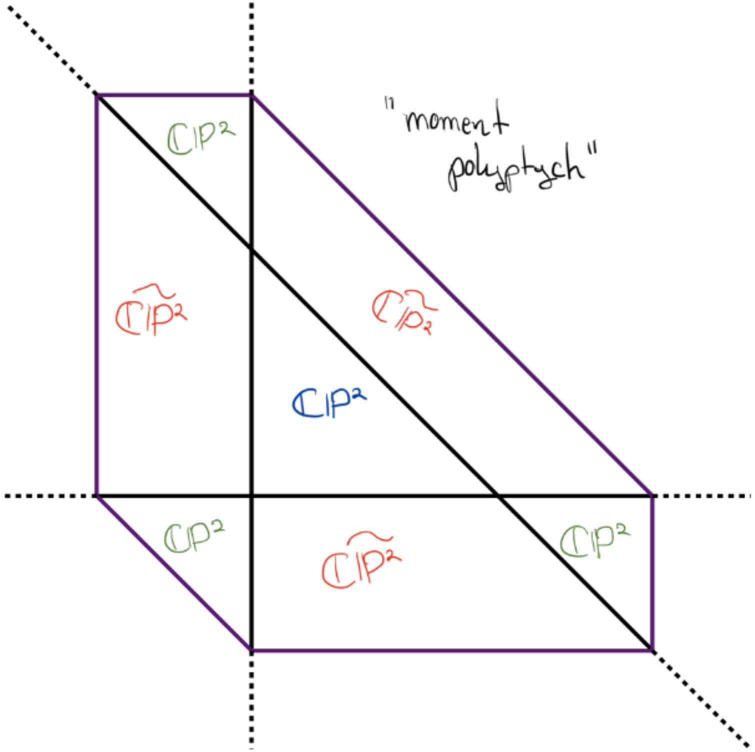
$$S^1_A = (\tau_1, \ldots, \tau_n), \text{ with } \tau_i = \tau \text{ if } i \in A; \tau_i = 1 \text{ otherwise.}$$

Action comes from inclusion $S^1_A \hookrightarrow T^n \to T^d$, with moment map

$$[z, w] \mapsto \left\langle \phi_{\mathbb{R}}[z, w], \sum_{i \in A} u_i \right\rangle.$$

Cutting introduces new half-spaces

$$\Delta_A^{(\epsilon)} := \Delta_A \cap \left\{ y \in \Delta_A : \left\langle y, \sum_{i \in A} u_i \right\rangle + \epsilon \geq 0 \right\}.$$



"moment polyptych"
$\mathbb{CP}^2$
$\widetilde{\mathbb{CP}^2}$
$\widetilde{\mathbb{CP}^2}$
$\mathbb{CP}^2$
$\mathbb{CP}^2$
$\widetilde{\mathbb{CP}^2}$
$\mathbb{CP}^2$

Outlook From Here

- Hypertoric manifolds with non-compact cores;
- Applying localisation formulae;
- Geometric quantisation and lattice point enumeration?

References

- Delzant, T. – Hamiltoniens périodiques et images convexes de l'application moment, *Bull. Soc. Math. France*, **116**, 1988.
- Bielawski, R. and Dancer, A. – The geometry and topology of toric hyperkähler manifolds, *Comm. Anal. Geom.*, **8**, 2000.
- Proudfoot, N. – Hyperkahler analogues of Kahler quotients, *Ph.D. Thesis*, Harvard University, 2004.
- Lerman, E. – Symplectic Cuts, *Math. Res. Lett.*, **2**, 1995.