Final Year Physics Project - Interim Report

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Introduction

Aims and Objectives

The aim of this project is to study the notion of *folded hyperkähler manifolds*, i.e. a 4-dimensional manifold which is hyperkähler away from some folding hypersurface, on which the hyperkähler structure degenerates and the metric is singular [1, 2]. The canonical example of a folded hyperkähler metric is a form of the Gibbons-Hawking ansatz on $\mathbb{R}^4 = \{(\tau, x, y, z)\}$ with coordinates [?]

$$h = \frac{1}{z}(d\tau + \psi)^2 + z(dx^2 + dy^2 + dz^2), \qquad d\psi = dx \wedge dy. \tag{1}$$

h is clearly undefined at $z = 0$ which defines the fold hypersurface $\mathcal{S}$, has signature $(++++)$ for $z > 0$, and signature $(----)$ for $z < 0$. The Kähler 2-forms are given by

$$\theta^1 = (d\tau + \psi) \wedge dx - z dy \wedge dz, \tag{2a}$$

$$\theta^2 = (d\tau + \psi) \wedge dy - z dz \wedge dx, \tag{2b}$$

$$\theta^3 = (d\tau + \psi) \wedge dz - z dx \wedge dy. \tag{2c}$$

Under the pullback of the involution $\imath : z \mapsto -z$, we have

$$\imath^* h = -h, \qquad \imath^* \theta^1 = \theta^1, \qquad \imath^* \theta^2 = \theta^2, \qquad \imath^* \theta^3 = -\theta^3. \tag{3}$$

We note that the 2-forms $\theta^1, \theta^2, \theta^3$ are smooth at $z = 0$, whilst h is undefined. Pulling back these 2-forms to $\mathcal{S}$, we have

$$\mathcal{S}^*\theta^1 = (d\tau + \psi) \wedge dx, \qquad \mathcal{S}^*\theta^2 = (d\tau + \psi) \wedge dy, \qquad \mathcal{S}^*\theta^3 = 0. \tag{4}$$

If we write $\eta = d\tau + \psi$, then we note that $d\eta = dx \wedge dy$, we have that

$$\eta \wedge d\eta = d\tau \wedge dx \wedge dy \neq 0, \tag{5}$$

i.e. it defines a volume form on $\mathcal{S}$ and hence η defines a contact form for $\mathcal{S}$.

Definition 1 ([2, ?])**.** A folded hyperkähler structure consists of a smooth 4-manifold $\mathcal{M}$, a smoothly imbedded hypersurface $\mathcal{S} \subset \mathcal{M}$, three smooth, closed 2-forms θ^i $(i = 1, 2, 3)$ on $\mathcal{M}$, and a smooth diffeomorphism $\imath : \mathcal{M} \to \mathcal{M}$ such that

1. $\mathcal{S}$ divides $\mathcal{M}$ into two disjoint connected components: $\mathcal{M} \backslash \mathcal{S} \cong \mathcal{M}^+ \cup \mathcal{M}^-$,
2. the 2-forms θ^i define a hyperkähler structure on $\mathcal{M}^\pm$ with hyperkähler metric $h^\pm$, where h^+ has signature $(++++)$ and h^- has signature $(----)$,
3. on the fold hypersurface $\mathcal{S} \subset \mathcal{M}$, one has $\mathcal{S}^*\theta^1 \neq 0$, $\mathcal{S}^*\theta^2 \neq 0$, $\mathcal{S}^*\theta^3 = 0$, and the distribution $\mathcal{D} \subset T\mathcal{S}$ given by $\mathcal{D} := \ker \mathcal{S}^*\theta^1 \oplus \ker \mathcal{S}^*\theta^2$ is a contact distribution,
4. $\imath$ is an involution that fixes $\mathcal{S}$ and maps $\mathcal{M}^\pm$ to $\mathcal{M}^\mp$ such that

$$\imath^* h^\pm = -h^\mp, \qquad \imath^*\theta^1 = \theta^1, \qquad \imath^*\theta^2 = \theta^2, \qquad \imath^*\theta^3 = -\theta^3. \tag{6}$$

Definition 2. Let $\mathcal{S}$ be a manifold of odd dimension $2n + 1$. A *contact structure* is a maximally non-integrable hyperplane field $\xi = \ker \theta \subset T\mathcal{S}$, *i.e.* the defining differential 1-form θ is required to satisfy

$$\theta \wedge (d\theta)^n \neq 0, \tag{7}$$

so it is nowhere vanishing. In other words, $\theta \wedge (d\theta)^n$ defines a volume form on $\mathcal{S}$.

Remark 3. An integrable hyperplane field means that for any point $p \in \mathcal{M}$ one can find a codimension 1 submanifold $\mathcal{S}$ whose tangent spaces coincide with the hyperplane field, *i.e.* such that $T_q\mathcal{S} = \xi_q$for all $q \in \mathcal{S}$.

Week 9 Lecture

Given a 3-dimensional manifold Y, we have three symplectic forms θ^a defined on the product manifold $\mathbb{R}\times Y$ by

$$\theta^1 = f(dt\wedge\epsilon_1 + \epsilon_2\wedge\epsilon_2), \tag{8a}$$

$$\theta^2 = f(dt\wedge\epsilon_2 + \epsilon_3\wedge\epsilon_1), \tag{8b}$$

$$\theta^3 = f(dt\wedge\epsilon_3 + \epsilon_1\wedge\epsilon_2), \tag{8c}$$

where f is a real, non-zero valued function of $\mathbb{R}\times Y$. Let $\omega = f^2 dt\wedge dx\wedge dy\wedge dz$ be the volume form on $\mathbb{R}\times Y$, so that

$$\theta^1\wedge\theta^1 = \theta^2\wedge\theta^2 = \theta^3\wedge\theta^3 = 2\omega. \tag{9}$$

Recall the 't Hooft eta tensors $\bar{\eta}^a_{\mu\nu}$ $(a = 1,2,3)$ defined in Ref. [3] by

$$\bar{\eta}^a_{\mu\nu} = \begin{cases} \epsilon_{a\mu\nu}, & \text{if } \mu,\nu = 1,2,3 \\ \delta_{a\nu}, & \text{if } \mu = 0 \\ -\delta_{a\mu}, & \text{if } \nu = 0 \\ 0, & \text{otherwise,} \end{cases}$$

and which obey the following identities,

$$\bar{\eta}^a_{\mu\nu} = \epsilon_{0a\mu\nu} + \delta_{0\mu}\delta_{a\nu} - \delta_{a\mu}\delta_{0\nu}, \tag{10a}$$

$$\bar{\eta}^a_{\mu\nu} = -\bar{\eta}^a_{\nu\mu}, \tag{10b}$$

$$\bar{\eta}^a_{\mu\nu}\bar{\eta}^b_{\mu\sigma} = \delta_{ab}\delta_{\nu\sigma} + \epsilon_{abc}\bar{\eta}^c_{\nu\sigma} \tag{10c}$$

so that three almost complex structures J^a on $\mathbb{R}\times Y$ can be given by

$$J^a(V_\mu) = \bar{\eta}^a_{\nu\mu}(V_\nu). \tag{11}$$

Indeed, we observe through an explicit calculation that

$$\begin{aligned}
J^a J^b(V_\mu) &= \bar{\eta}^a_{\nu\mu}\bar{\eta}^b_{\sigma\nu}(V_\sigma)\\
&= -\bar{\eta}^a_{\nu\mu}\bar{\eta}^b_{\nu\sigma}(V_\sigma)\\
&= -(\delta_{ab}\delta_{\mu\sigma} + \epsilon_{abc}\bar{\eta}^c_{\mu\sigma})(V_\sigma)\\
&= -\delta_{ab}(V_\mu) + \epsilon_{abc}\bar{\eta}^c_{\sigma\mu}(V_\sigma)\\
&= (-\delta_{ab} + \epsilon_{abc}J^c)(V_\mu),
\end{aligned}$$

so the endomorphisms defined in (11) obey the quaternionic multiplication relations, therefore providing three almost complex structures on $\mathbb{R} \times Y$. For $\mathbb{R} \times Y$ to be a hyperkähler manifold, we still require a metric that is compatible with each of the J^a. To this end, we can define the metric g to be given by $g_{\mu\nu} = \delta_{\mu\nu}\omega(V_0, V_1, V_2, V_3)$. Then the three symplectic forms given in 8a are compatible with the metric g, since

Claim 4. *For each symplectic form θ^a, $(a = 1, 2, 3)$, induced by the volume-preserving, linearly-independent vector fields V_μ, $(\mu = 0, 1, 2, 3)$ on the product manifold $\mathbb{R} \times Y$, we may write*

$$\theta^a = \frac{1}{2}\bar{\eta}^a_{\mu\nu}\imath_{V_\mu}\imath_{V_\nu}\omega, \tag{12}$$

where $\omega = f dt \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3$ is the volume form on $\mathbb{R} \times Y$, and $\imath_{V_\mu}$ is interior multiplication (equivalently contraction) by the vector V_μ.

Proof. By using the first identity in 10a and the anticommutativity of the interior multiplication $\imath_{V_\mu}\imath_{V_\nu} = -\imath_{V_\nu}\imath_{V_\mu}$, it follows immediately that

$$\begin{aligned}
\frac{1}{2}\bar{\eta}^a_{\mu\nu}\imath_{V_\mu}\imath_{V_\nu}\omega &= \frac{1}{2}(\epsilon_{0a\mu\nu} + \delta_{0\mu}\delta_{a\nu} - \delta_{a\mu}\delta_{0\nu})\imath_{V_\mu}\imath_{V_\nu}\omega\\
&= f\left(\frac{1}{2}\epsilon_{0a\mu\nu}\imath_{V_\mu}\imath_{V_\nu} + \imath_{V_0}\imath_{V_a}\right)dt \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3\\
&= \begin{cases} f(dt \wedge \epsilon_1 + \epsilon_2 \wedge \epsilon_3), & \text{if } a = 1\\ f(dt \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_1), & \text{if } a = 2\\ f(dt \wedge \epsilon_3 + \epsilon_1 \wedge \epsilon_2), & \text{if } a = 3 \end{cases}\\
&= \theta^a.
\end{aligned}$$

□

Corollary 5 (Half-flat condition)**.** *The vector fields V_μ given above satisfy the half-flat* *condition*[1]

$$\frac{1}{2}\bar{\eta}^a_{\mu\nu}[V_\mu, V_\nu] = 0, \tag{13}$$

where [,] *is the Lie bracket for vector fields.*

Proof. Since the symplectic forms θ^a are closed, we have that

$$\begin{aligned} d\theta^a &= d\Big(\frac{1}{2}\bar{\eta}^a_{\mu\nu}\imath_{V_\mu}\imath_{V_\nu}\omega\Big) \\ &= \frac{1}{2}\bar{\eta}^a_{\mu\nu}d(\imath_{V_\mu}\imath_{V_\nu}\omega) \\ &= \frac{1}{2}\bar{\eta}^a_{\mu\nu}\imath_{[V_\mu,V_\nu]}\omega \\ &= \imath_{\frac{1}{2}\bar{\eta}^a_{\mu\nu}[V_\mu,V_\nu]}\omega = 0 \end{aligned}$$

where we have used the volume-preserving property of the V_μ, along with the identity

$$d(\imath_{V_\mu}\imath_{V_\nu}\omega) = \imath_{[V_\mu,V_\nu]}\omega + \imath_{V_\nu}\mathcal{L}_{V_\mu}\omega - \imath_{V_\mu}\mathcal{L}_{V_\nu}\omega + \imath_{V_\mu}\imath_{V_\nu}d\omega. \tag{14}$$

From the non-degeneracy of the volume form ω, it follows that

$$\frac{1}{2}\bar{\eta}^a_{\mu\nu}[V_\mu, V_\nu] = 0.$$

□

Definition 6. A 4-metric is said to be *half-flat* if its Riemann tensor is proportional to its dual.

Remark 7. A half-flat 4-metric induces a hyperkähler structure on the manifold, since half-flatness corresponds to the self-dual Weyl tensor vanishing, which is equivalent to the holonomy group of the manifold being equal to the compact symplectic group $Sp(1)$, which characterises hyperkähler structures by Berger's classification.

[1] A 4-metric is said to be *half-flat* if its Riemann tensor is proportional to its dual. Then, by the virtue of the Bianchi identity, a half-flat metric is necessarily Ricci flat [4].

We summarise the above results following [5].

Proposition 8. *Let $\Sigma^{(n)}$ be an n-dimensional manifold with corresponding volume form $\omega^{(n)}$, and consider the gauge Lie algebra $\mathfrak{sdiff}(\Sigma^{(n)})$ consisting of volume-preserving vector fields on $\Sigma^{(n)}$. The connections on Euclidean space $\mathbb{R}^n$ may be written explicitly as 1-forms valued in $\mathfrak{sdiff}(\Sigma^{(n)})$ as $A = A_\mu dx^\mu$ $(\mu = 0, 1, 2, 3)$. Then, if on $\Sigma^{(n)} \times \mathbb{R}^{4-n}$ we have that:*

1. *The A_μ are $\mathbb{R}^n$-invariant with respect to the coordinates $(x^0, ..., x^{n-1})$,*
2. *The covariant derivatives of the connection $D_\mu = \frac{\partial}{\partial x^\mu} + A_\mu$ satisfy the half-flat condition, namely*
$$\frac{1}{2}\bar{\eta}^a_{\mu\nu}[D_\mu, D_\nu] = 0,$$
3. *The A_μ $(0 \leq \mu \leq n-1)$ are linearly independent at each point of $\Sigma^{(n)}$.*

Then four vector fields V_μ may be defined on $\Sigma^{(n)} \times \mathbb{R}^{4-n}$ as follows:

$$V_\mu = \begin{cases} A_\mu, & \text{for } 0 \leq \mu \leq n-1, \\ D_\mu, & \text{for } n \leq \mu \leq 3. \end{cases}$$

These vector fields preserve the volume form $\omega = \omega^{(n)} \wedge ... \wedge dx^3$ and satisfy the half-flat condition I.B.. Hence, by the virtue of Remark 7, they induce a hyperkähler structure on $\Sigma^{(n)} \times \mathbb{R}^{4-n}$.

Example 9 (Gibbons-Hawking Metric). Suppose $n = 1$ and that $\Sigma^{(1)} = \mathbb{R}$, *i.e.* the underlying space-time is $\mathbb{R}^4 = \{(\tau, x, y, z)\}$ with volume form $\omega = d\tau \wedge dx \wedge dy \wedge dz$. Let the four vector fields V_μ be given by

$$V_0 = \phi\frac{\partial}{\partial\tau}, \tag{15}$$
$$V_i = \frac{\partial}{\partial x^i} + \psi_i\frac{\partial}{\partial\tau}, \tag{16}$$

where ϕ and ψ_i $(i = 1, 2, 3)$ are smooth functions. For the V_μ to be volume-preserving, ϕ and ψ_i must be independent of τ. Moreover for the half-flat condition to be satisfied, we require that

$$\frac{1}{2}\bar{\eta}^a_{\mu\nu}[V_\mu, V_\nu] = 0 \implies \begin{cases} [V_0, V_1] + [V_2, V_3] = 0 \\ [V_0, V_2] + [V_3, V_1] = 0 \\ [V_0, V_3] + [V_1, V_2] = 0 \end{cases} \implies \begin{cases} \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi_3}{\partial y} - \dfrac{\partial \psi_2}{\partial z}, \\ \dfrac{\partial \phi}{\partial y} = \dfrac{\partial \psi_1}{\partial z} - \dfrac{\partial \psi_3}{\partial x}, \\ \dfrac{\partial \phi}{\partial z} = \dfrac{\partial \psi_2}{\partial z} - \dfrac{\partial \psi_3}{\partial y}. \end{cases} \tag{17}$$

Setting $\underline{\psi} \equiv (\psi_1, \psi_2, \psi_3)$, or $\psi \equiv \Sigma_{i=1}^3 \psi_i dx^i$, then 17 is equivalent to the condition that

$$\underline{\nabla}\phi = \underline{\nabla} \times \underline{\psi} \quad \textit{i.e.} \text{ that } \quad \underset{3}{*}\, d\phi = d\psi, \tag{18}$$

where $*_3$ is the Hodge duality operator acting on $\mathbb{R}^3 = \{(x, y, z)\}$. Equation 18 is known as the *Bogomolny equations* or the *monopole equations* [], and implies that ϕ is harmonic. This set up corresponds to the Gibbons-Hawking ansatz used to create the Gibbons-Hawking multi-centre hyperkähler metric

$$h = \phi^{-1}(d\tau + \psi)^2 + \phi(dx^2 + dy^2 + dz^2) \tag{19}$$

with a triholomorphic Killing vector $\frac{\partial}{\partial \tau}$, since the coefficients of h are independent of τ [6].

Remark 10. One may recover the canonical folded hyperkähler metric 1 from Example I.B. by choosing $\phi = z$, so that $*_3 d\phi = *_3 dz = dx \wedge dy = d\psi$.

Example 11. Suppose that $n = 3$, *i.e.* we consider the manifold $\Sigma^{(3)} \times \mathbb{R} = \{(x, y, z, \tau)\}$ with the $\mathfrak{sdiff}(\Sigma^{(3)})$-valued 1-forms A_μ independent of x, y, z. Then I.B. reduces to Nahm's equations

$$\frac{\partial V_a}{\partial \tau} + \frac{1}{2}\epsilon_{abc}[V_b, V_c] = 0. \tag{20}$$

We can then use the V_a to define three complex symplectic structures on the product manifold $\Sigma^{(3)} \times \mathbb{R}$ following [7]:

Proposition 12. *Let α be the volume form on $\Sigma^{(3)}$. Then given three time-dependent vector fields V_a $(a = 1, 2, 3)$ on $\Sigma^{(3)}$ which satisfy Nahm's equations 20 and are volume preserving on $\Sigma^{(3)}$,* i.e. $\mathcal{L}_{V_a}\alpha = 0$, *we can construct three complex symplectic structures on the product* *manifold* $\Sigma^{(3)} \times \mathbb{R}$.

Proof. For brevity, write $\mathcal{M} = \Sigma^{(3)} \times \mathbb{R}$. For each time τ, let $\epsilon_1, \epsilon_2, \epsilon_3$ be the basis of 1-forms dual to the V_a. Then, for some non-vanishing real function f on $\Sigma^{(3)}$ that $\alpha = f\epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3$ for the volume form on $\Sigma^{(3)}$. Define two 2-forms on $\mathcal{M}$ by

$$\theta^1 = f(d\tau \wedge \epsilon_1 + \epsilon_2 \wedge \epsilon_3), \tag{21a}$$

$$\theta^2 = f(d\tau \wedge \epsilon_2 + \epsilon_3 \wedge \epsilon_1). \tag{21b}$$

Then $\theta_1^2 = \theta_2^2 = f dt \wedge \alpha$, and $\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0$ and so if θ_1, θ_2 are closed on $\mathcal{M}$, then we have a complex symplectic structure on $\mathcal{M}$. To this end, we apply the identity 14 to $d(\imath_{V_2}\imath_{V_3}\alpha)$ to yield

$$\begin{aligned} d(\imath_{V_2}\imath_{V_3}\alpha) &= \imath_{[V_2,V_3]}\alpha + \imath_{V_2}\mathcal{L}_{V_3}\alpha - \imath_{V_3}\mathcal{L}_{V_2}\alpha + \imath_{V_2}\imath_{V_3}d\alpha \\ &= \imath_{[V_2,V_3]}\alpha, \end{aligned}$$

since the vector fields are volume-preserving. Furthermore, we have that

$$\imath_{V_3}\alpha = f\epsilon_1 \wedge e_2, \qquad \imath_{V_2}\imath_{V_3}\alpha = f\epsilon_1, \qquad \imath_{V_1}\alpha = f\epsilon_2 \wedge \epsilon_3,$$

$$d(\imath_{V_1}\alpha) = \mathcal{L}_{V_1}\alpha - \imath_{V_1}d\alpha = 0,$$

and so $\imath_{V_1}\alpha$ is a closed 2-form. Temporarily let us write $\underline{d}$ for the exterior derivative on forms over $\mathcal{M}$, and d for the exterior derivative of forms over $\Sigma^{(3)}$ with time regarded as a parameter. In this notation,

$$\underline{d}\psi = d\psi + dt \wedge \frac{\partial \psi}{\partial \tau},$$

and so

$$\begin{aligned}
\underline{d}\theta_1 &= d\theta_1 + d\tau \wedge \frac{\partial \theta_1}{\partial \tau} \\
&= d(f\epsilon_2 \wedge \epsilon_3) + d\tau \wedge \left[\frac{\partial f}{\partial \tau} d\tau \wedge \epsilon_1 + \frac{\partial}{\partial \tau}(f\epsilon_2 \wedge e_3) \right] \\
&= d(\imath_{V_1}\alpha) + d\tau \wedge \left[d(f\epsilon_1) + \frac{\partial}{\partial \tau}(f\epsilon_2 \wedge e_3) \right] \\
&= 0 + d\tau \wedge \left[d(\imath_{V_2}\imath_{V_3}\alpha) + \frac{\partial}{\partial \tau}(\imath_{V_1}\alpha) \right],
\end{aligned}$$

where we have used the fact that $\imath_{V_1}\alpha$ is closed on $\Sigma^{(3)}$. Therefore θ_1 is closed on $\mathcal{M}$ if and only if

$$d(\imath_{V_2}\imath_{V_3}\alpha) + \frac{\partial}{\partial \tau}(\imath_{V_1}\alpha) = \imath_{[V_2,V_3]}\alpha + \imath_{\frac{\partial V_1}{\partial \tau}}\alpha = 0,$$

since α is time-independent. From the non-degeneracy of α, we conclude that θ_1 is closed on $\mathcal{M}$ if and only if $\frac{\partial V_1}{\partial \tau} + [V_2, V_3] = 0$, and the same argument for θ_2 proves that θ_2 is closed on $\mathcal{M}$ if and only if $\frac{\partial V_2}{\partial \tau} + [V_3, V_1] = 0$. Hence we have a complex symplectic structure on $\mathcal{M}$. □

Remark 13. If we define a third 2-form on $\mathcal{M}$ by $\theta_3 = f(d\tau \wedge \epsilon_3 + \epsilon_1 \wedge \epsilon_2)$, then θ_3 is closed on $\mathcal{M}$ if and only if $\frac{\partial V_3}{\partial \tau} + [V_1, V_2] = 0$. Therefore Nahm's equations 20 define three closed 2-forms on $\mathcal{M}$. By choosing the three almost complex structures given by 11 and Riemannian metric $g(V_\mu, V_\nu) = \delta_{\mu\nu}\omega(V_0, V_1, V_2, V_3)$, then the 2-forms are compatible with g and are actually Kähler 2-forms and g is a Hermitian metric - hence we have an almost hyperkähler structure on $\mathcal{M}$. By the virtue of Lemma 6.8 in [8], we actually have a hyperkähler structure on the manifold $\mathcal{M}$.

Example 14 (Real Heaven Metric). Now we choose $n = 2$ *i.e.* consider $\Sigma^{(2)} \times \mathbb{R}^2 = \{(\tau, x, y, z)\}$ and a smooth function $\psi = \psi(x, y, z)$ independent of time τ. If we then choose the vector fields

$$V_0 = e^{\frac{\psi}{2}}\bigg(\partial_z\psi\cos(\tau/2)\frac{\partial}{\partial\tau} + \sin(\tau/2)\frac{\partial}{\partial z}\bigg), \tag{22a}$$

$$V_1 = e^{\frac{\psi}{2}}\bigg(- \partial_z\psi\sin(\tau/2)\frac{\partial}{\partial\tau} + \cos(\tau/2)\frac{\partial}{\partial z}\bigg), \tag{22b}$$

$$V_2 = \frac{\partial}{\partial x} + \partial_y\psi\frac{\partial}{\partial\tau}, \tag{22c}$$

$$V_3 = \frac{\partial}{\partial y} - \partial_x\psi\frac{\partial}{\partial\tau}, \tag{22d}$$

then for the V_μ to satisfy the half-flat condition I.B., the function ψ must satisfy the 3-dimensional continuum Toda equation[2] [9]

$$\frac{\partial^2}{\partial^2 z}(e^\psi) + \frac{\partial^2\psi}{\partial^2 y} + \frac{\partial^2\psi}{\partial^2 x} = 0. \tag{23}$$

This solution induces a hyperkähler metric with the Killing vector field $\frac{\partial}{\partial\tau}$, but is not triholomorphic. In the literature, this solution is known as the real heaven solution [10].

[2]Equivalently called the $SU(\infty)$ Toda equation in some literature.

Theorem 15 (Biquard, [2]). *Given the real analytic data $(\mathcal{S}, \beta_2, \beta_3)$, where $\mathcal{S}$ is a 3-manifold and β_2 and β_3 are closed 2-forms on $\mathcal{S}$, such that their kernels form a contact distribution, then there exists in a small neighbourhood $(-\epsilon, \epsilon) \times \mathcal{S}$ a unique folded hyperkähler metric such that $\imath^*\omega_2 = \beta_2$ and $\imath^*\omega_3 = \beta_3$. This metric satisfies the parity given in 3.*

Proof. A solution of the system of Nahm's equations for the vectors V_1, V_2, V_3 on $\mathcal{S}$, depending solely on τ, and preserve a fixed volume form α on $\mathcal{S}$ given by the system 20 gives rise to a hyperkähler metric.

Given $(\mathcal{S}, \beta_2, \beta_3)$ we take the basis of 1-forms $(\theta^1, \theta^2, \theta^3)$ which satisfy $d\theta^1 = \theta^2 \wedge \theta^3$, $\beta_2 = -\theta^1 \wedge \theta^3$, and $\beta_3 = \theta^1 \wedge \theta^2$, and (X_1, X_2, X_3) as the basis of vector fields dual to the 1-forms. Then the conditions $d\beta_2 = d\beta_3 = 0$ correspond to the fact that X_2 and X_3 both preserve the volume form α. We therefore solve the system of equations with the initial conditions

$$V_1(0) = 0, \qquad V_2(0) = X_2, \qquad V_3(0) = X_3. \tag{24}$$

For the given real analytic data, the theorem of Cauchy-Kowalevski produces a unique solution defined for a small enough τ.

We observe that $(-V_1(-\tau), V_2(-\tau), V_2(-\tau))$ is also a solution with the same initial conditions, and so V_1 is even whereas V_2, V_3 are odd, which implies the invariance under the involution 3 for the solution. Moreover, since $X_1 = -[X_2, X_3]$ we have that

$$V_1 = \tau X_1 + \mathcal{O}(\tau^3). \tag{25}$$

Hence we deduce that, for the behaviour of the metric (odd, positive for $\tau > 0$, negative for $\tau < 0$):

$$h = \tau(d\tau^2 + (\theta^2)^2 + (\theta^3)^2) + \tau^{-1}(\theta^1)^2 + \mathcal{O}(\tau^3)G(d\tau, \tau^{-1}\theta^1, \theta^2, \theta^3), \tag{26}$$

which gives us the three Kähler forms. Here, $G((e^i)) = \sum G_{ij}e^ie^j$ is a symmetric 2-tensor with smooth coefficient G_{ij}.

Reciprocally, given a real analytic hyperkähler metric with the behaviour 26, we calculate its Laplacian

$$\Delta = -\tau^{-1}(\partial_\tau^2 + \tau^2 X_1^2 + X_2^2 + X_3^2) + \dots \tag{27}$$

It then results immediately that we can resolve $\Delta y = 0$ in a neighbourhood of $\mathcal{S}$ with $y = \tau + \mathcal{O}(\tau^2)$. This solution is unique, and lets us reconstruct the unique vector fields V_a. $\square$

Background Theory

Work Done in Term 1

The results section is where you'll make a summary of your work done during Term 1. **It should occupy no more than one page.**

Aim for Term 2

The Plan of work to be done in Term 2 should occupy no more than one page. Again, it may be convenient to present it as a series of bullet points or as a Table. Provide estimated timescales for what you will do and try to be realistic.

References

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