HYPERTORIC MANIFOLDS

GENERAL NOTES

ABSTRACT

Notes on toric hyperkähler manifolds.

1 Toric Hyperkähler Manifolds

1.1 Symplectic Quotients, [1]

Fix the standard Euclidean bilinear form on $\mathbb{C}^n$,

$$g(z,w) = \sum_{i=1}^{n} (\Re(z_i)\Re(w_i) + \Im(z_i)\Im(w_i)\,.$$

The corresponding Kähler form is

$$\omega(z,w) = g(iz,w) = \sum_{i=1}^{n} (\Re(z_i)\Im(w_i) - \Im(z_i)\Re(w_i))\,.$$

Let $A = [u_1, \ldots, u_n]$ be a $(d \times n)$-matrix whose $(d \times d)$-minors are relatively prime. Choose now an $n \times (n-d)$-matrix $B = [b_1, \ldots, b_n]^T$ that makes the following sequence exact:

$$\{0\} \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow \{0\}.$$

The choice of B is equivalent to choosing a basis in $\ker(A)$.

1.2 Hyperkähler Quotients

Let $\mathbb{H}$ be the quaternions, the 4-dimensional $\mathbb{R}$-vector space with basis $\{1, i, j, k\}$ equipped with an associative algebra structure defined by

$$i^2 = j^2 = k^2 = ijk = -1.$$

Left-multiplication by i (respectively j and k) define the following respective complex structures on $\mathbb{H}$,

$$I, J, K : \mathbb{H} \longrightarrow \mathbb{H}; \qquad I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{\mathbb{H}}\,.$$

Equipping $\mathbb{H}$ with the flat metric g arising from the standard Euclidean scalar-product on $\mathbb{H} \cong \mathbb{R}^4$, with $\{1, i, j, k\}$ providing an orthonormal basis. This is called a *hyperkähler metric* since it is a Kähler metric with respect to each individual complex structure, I, J, and K. This also means that the so-called *Kähler forms*, given by

$$\omega_I(X,Y) = g(IX,Y), \qquad \omega_J(X,Y) = g(JX,Y), \qquad \omega_K(KX,Y) = g(KX,Y), \qquad \text{for tangent vectors } X, Y,$$

are closed differential 2-forms.

A special orthogonal transformation with respect to this metric is said to *preserve the hyperkähler structure* if it commutes with all three complex structures, I, J, and K; or equivalently, it preserves the Kähler forms, ω_I, ω_J, and ω_K. The group of such transformations, the *unitary symplectic group* $\mathrm{Sp}(1)$, is generated by the right-multiplication action by the unit quaterions.

2 Cotangent Spaces to Extended Core Components

Let $M_\lambda = \left(\mu_{\mathbb{R}}^{-1}(\lambda) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/K$ be a toric hyperkähler manifold. Define

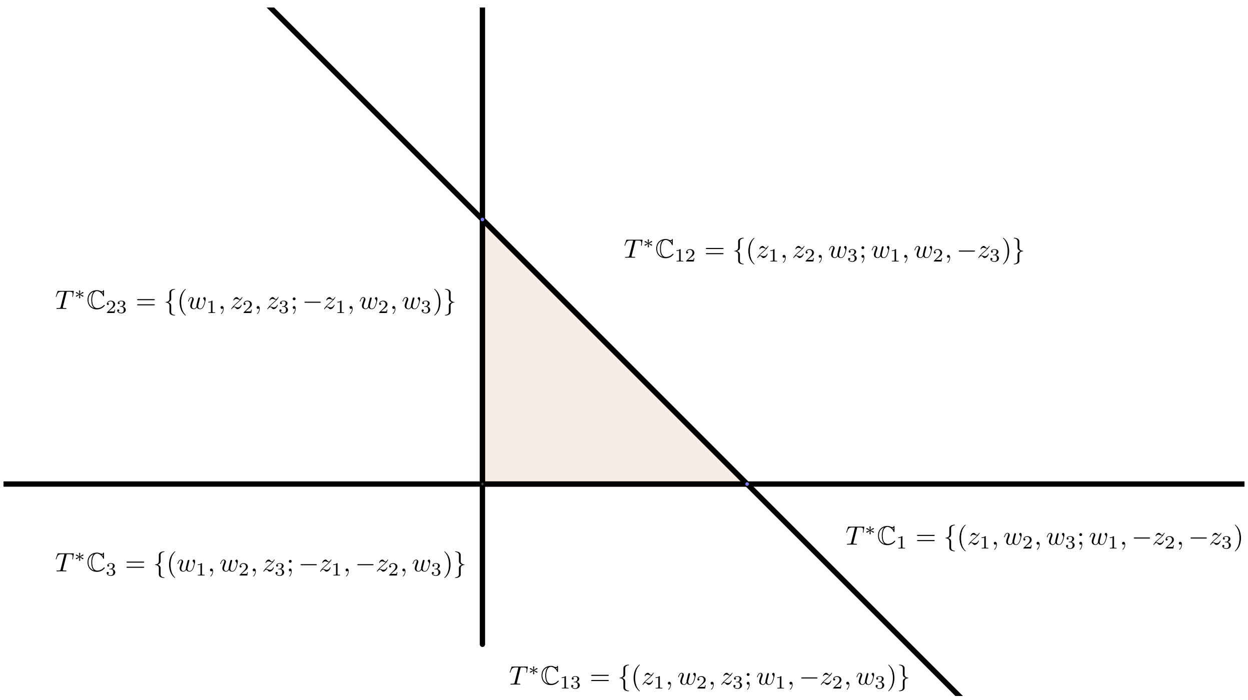
$$\mathbb{C}_A := \left\{ (z_i, w_i) \in \mathbb{C}^{2n} \;\middle|\; w_i = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \right\} \cong \mathbb{C}^n \subset \mathbb{H}^n.$$

Lemma 2.1 ([2])**.** *Let M_λ be a toric hyperkähler manifold. If $\mathcal{E}_A$ is non-empty, then its holomorphic cotangent bundle $T^*\mathcal{E}_A$ is contained in M_λ as an open subset.*

Fix a subset $A \subset \{ 1, \dots, n \}$, and define

$$(x_i^{(A)}, y_i^{(A)}) := \begin{cases} (z_i, w_i), & \text{if } i \in A, \\ (w_i, -z_i), & \text{if } i \notin A. \end{cases}$$

Then $x^{(A)} = (x_1^{(A)}, \dots, x_n^{(A)})$ is a point in the vector space $\mathbb{C}^n_A$, and $y^{(A)} = (y_1^{(A)}, \dots, y_n^{(A)})$ is a point in the dual space $(\mathbb{C}^n_A)^*$. That is, we identify the cotangent bundle $T^*\mathbb{C}^n_A$ with $\mathbb{H}^n$ as above.



2.1 The Legendre Transform

In [2], Konno states that the proof of this lemma goes by an argument in [3], which itself is based on properties of the Legendre transform.

Lemma 2.2 (Section A1.3 of [3])**.** *Consider a smooth function of one variable*

$$f = f(x), \qquad -\infty < x < \infty.$$

Suppose that f is strictly convex ($f''(x) > 0$ for all x). Then the four conditions are equivalent:

1. *$f'(x_0) = 0$ at some point x_0.*
2. *f has a local minimum at some point x_0.*
3. *f has a unique local minimum.*
4. *$f(x)$ tends to $+\infty$ as x tends to $\pm\infty$.*

If f has any one (and hence all four) of the above properties, we will say that f is *stable*.

3 Symplectic Cutting

3.1 Compactifying the Extended Core

Let S^1 act on M by rotating the cotangent fibres, that is, for $\tau \in S^1$,

$$\tau \cdot [z; w] = [z; \tau w].$$

This S^1-action is Hamiltonian, with moment map

$$\Phi : M \longrightarrow (\mathbb{R})^*; \qquad [z : w] \longmapsto \frac{1}{2}\|w\|^2.$$

Let S^1_A denote the residual S^1-action on M restricted to the extended core component

$$\mathcal{E}_A = \{ [z_1 : \dots z_n; w_1, \dots, w_n] \mid w_0 = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \}.$$

Now the *global* S^1-action does not act on the cotangent fibres of M as a subtorus of T^n, but it does when *restricted* to each component of the extended core, $\mathcal{E}_A$. Indeed,

$$\tau \cdot [z; w] = [z; \tau w] = [z_1 : \dots : z_n; \tau w_1 : \dots : \tau w_n] = [\tau_1 z_1 : \dots : \tau_n z_n; \tau_1^{-1} w_1 : \dots : \tau_n^{-1} w_n],$$

where

$$\tau_i := \begin{cases} \tau^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A, \end{cases}$$

which shows that the S^1-action restricted to each individual $\mathcal{E}_A$ acts as a subtorus of the original torus T^n.

Denote by S^1_A the image of S^1 in T^n when considered as a subtorus restricted to each individual $\mathcal{E}_A$, and let $\jmath_A : S^1 \hookrightarrow T^n$ be the respective inclusion homomorphism, so we have $S^1_A := \jmath_A(S^1) \lhd T^n$.

On the Lie algebra level, we have that

$$(\jmath_A)_* : \mathrm{Lie}(S^1_A) \longrightarrow \mathfrak{t}^n; \qquad \xi \longmapsto (\xi_1, \dots, \xi_n),$$

where we analogously define

$$\xi_i := \begin{cases} -1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A. \end{cases}$$

Since S^1_A acts as the subtorus $\jmath_A(S^1)$ of T^n on each $\mathcal{E}_A$, the moment map $\Phi_A := \Phi\big|_{\mathcal{E}_A}$ for this action is given by composing $\mu_{\mathbb{R}}$ with the dual of the inclusion $(\jmath_A)_*$, so

$$\begin{aligned} \Phi_A[z, w] = (\jmath_A^* \circ \mu_{\mathbb{R}})\,[z; w] &= \jmath_A^* \left(\frac{1}{2} \sum_{i=1}^{n} \left(|z_i|^2 - |w_i|^2 \right) e^i \right) \\ &= -\frac{1}{2} \sum_{i \in A} |z_i|^2 \jmath_A^*(e^i) \\ &= \frac{1}{2} \sum_{i \notin A} |w_i|^2 \jmath_A^*(e^i) \\ &= \langle \mu_{\mathbb{R}}[z; w],\, \xi_A \rangle \\ &= \mu_{\mathbb{R}}^A[z; w], \end{aligned}$$

where $\xi_A = -\sum_{i \in A} \xi_i$, and $\mu_{\mathbb{R}}^A[z; w]$ is the component of $\mu_{\mathbb{R}}[z; w]$ in the ξ_A-direction.

References

[1] Tamás Hausel and Bernd Sturmfels. Toric hyperKähler varieties. *Documenta Mathematica*, 7:495–534, 2002.

[2] Hiroshi Konno. The topology of toric hyperKähler manifolds. In *Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999)*, volume 34 of *Adv. Stud. Pure Math.*, pages 173–184. Math. Soc. Japan, Tokyo, 2002.

[3] Victor Guillemin. *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, volume 122 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1994.