INDEX THEOREMS - NOTES

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1. Index Theory

1.1. **Non-Equivariant Index Formula.** For a holomorphic vector bundle $\mathcal{L}$ over a complex n-dimensional variety M, the *index* $\mathrm{ind}(\bar{\partial}, \mathcal{L})$w is defined as

$$\mathrm{ind}(\bar{\partial}, \mathcal{L}) := \sum_{k=0}^{n} (-1)^k \dim H^k(M; \mathcal{L}).$$

Viewing the index $\mathrm{ind}(\bar{\partial}, \mathcal{L})$ as the Euler characteristic $\chi(M, \mathcal{L})$ of the vector bundle $\mathcal{L}$, we can apply the Atiyah-Singer index theorem, which we state below, to express the index as an integral over M of the product of the Todd class $\mathrm{Td}(TM)$ of the tangent bundle $TM \to M$ over M, and the Chern character $\mathrm{Ch}(\mathcal{L}) := \exp(c_1(\mathcal{L}))$ of $\mathcal{L}$, where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$.

Theorem 1.1 (Atiyah-Singer Index Theorem, [1]). *Let M be a compact complex manifold, $\mathcal{L}$ a holomorphic vector bundle over M. Let*

$$\mathrm{Td}(TM) = \prod \frac{x_i}{1 - e^{-x_i}}$$

be the Todd class of the complex vector bundle $TM \to M$, where the x_i are the Chern roots of TM. Then the Euler characteristic $\chi(M, \mathcal{L})$ of the sheaf of germs of holomorphic sections of $\mathcal{L}$ is given by

$$\chi(M, \mathcal{L}) = \int_M \mathrm{Td}(M) \cdot \mathrm{Ch}(\mathcal{L}).$$

Example. Let $M = \mathbb{CP}^1$ and let $\mathcal{L}$ be the line bundle $\mathcal{O}(k)$ for some positive integer k. If $\langle \xi \rangle = H^2(M; \mathbb{Z})$, *i.e.* ξ is the generator of $H^2(\mathbb{CP}^1; \mathbb{Z})$, then $c_1(\mathcal{L}) = k\xi$, and thus the Chern character of $\mathcal{L}$ is

$$\mathrm{Ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = \sum_{j=0}^{\infty} (k\xi)^j = 1 + k\xi$$

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(the higher powers of ξ vanish since $\dim_{\mathbb{C}} M = 1$).

For n-dimensional complex projective space $\mathbb{CP}^n$, both the total Chern class

$$c(\mathbb{CP}^n) := c(T\mathbb{CP}^n) := 1 + c_1 + c_2 + c_3 + \dots,$$

and the Todd class $\mathrm{Td}(T\mathbb{CP}^n)$ for the tangent bundle $T\mathbb{CP}^n \to \mathbb{CP}^n$, can be calculated using the exact Euler sequence, along with the multiplicativity of

$$\{0\} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{CP}^n \longrightarrow \{0\},$$

the total Chern class and the Todd class,

$$c(\mathcal{F} \oplus \mathcal{G}) = c(\mathcal{F}) \cdot c(\mathcal{G}), \qquad \mathrm{Td}(\mathcal{F} \oplus \mathcal{G}) = \mathrm{Td}(\mathcal{F}) \cdot \mathrm{Td}(\mathcal{G}),$$

which yields

$$c(\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \mathcal{O}) = c(\mathcal{O}(1)^{\oplus(n+1)}) = (1+\xi)^{n+1},$$

and

$$\mathrm{Td}(T\mathbb{CP}^n) = \mathrm{Td}(T\mathbb{CP}^n \oplus \mathcal{O}) = \mathrm{Td}(\mathcal{O}(1)^{\oplus(n+1)}) = \mathrm{Td}\left(\mathcal{O}(1)\right)^{n+1} = \left(\frac{\xi}{1-e^{-\xi}}\right)^{n+1}.$$

This expression can be expanded as a formal power series which, for $n = 1$ in our example with the complex projective line $\mathbb{CP}^1$, gets us

$$c(\mathbb{CP}^1) = (1+\xi)^2 = 1 + 2\xi, \qquad \mathrm{Td}(T\mathbb{CP}^1) = 1 + \tfrac{1}{2} c_1(T\mathbb{CP}^1) = 1 + \xi.$$

Finally, applying the Atiyah-Singer index theorem 1.1, we have

$$\chi(\mathbb{CP}^1, \mathcal{L}) = \int_{\mathbb{CP}^1} \mathrm{Td}(\mathbb{CP}^1) \cdot \mathrm{Ch}(\mathcal{L}) = \int_{\mathbb{CP}^1} (1+\xi) \cdot (1+k\xi) = \int_{\mathbb{CP}^1} 1 + (k+1)\xi = k+1.$$

Example. Now we let $M = \mathbb{CP}^2$, and let $\mathcal{L} = \mathcal{O}(k)$ and $\langle \xi \rangle = H^2(M, \mathbb{Z})$ again as above. Now we have

$$c(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + k\xi + k^2\xi^2,$$

and

$$c(T\mathbb{CP}^2) = 1 + c_1 + c_2 = (1+\xi)^3 = 1 + 3\xi + 3\xi^2,$$

$$\mathrm{Td}(T\mathbb{CP}^2) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} = 1 + \frac{3}{2}\xi + \frac{9\xi^2 + 3\xi^2}{12} = 1 + \frac{3}{2}\xi + \xi^2.$$

Hence by the Atiyah-Bott index theorem 1.1,

$$\begin{aligned}\chi(M,\mathcal{L}) &= \int_M \mathrm{Td}(TM)\cdot \mathrm{Ch}(\mathcal{L}) = \int_M \left(1+\tfrac{3}{2}\xi+\xi^2\right)\cdot\left(1+k\xi+k^2\xi^2\right)\\ &= \int_M (k^2+\tfrac{3}{2}k+1)\xi^2+O(\xi) = k^2+\tfrac{3}{2}k+1.\end{aligned}$$

Example. Let $M=\mathbb{CP}^3$, and let $\mathcal{L}$, ξ, etc. be as above. Then

$$\begin{aligned}\mathrm{Ch}(\mathcal{L}) &= 1+k\xi+(k\xi)^2+(k\xi)^3,\\ c(TM) &= (1+\xi)^4 = 1+4\xi+6\xi^2+4\xi^3,\\ \mathrm{Td}(TM) &= 1+\frac{c_1}{2}+\frac{c_1^2+c_2}{12}+\frac{c_1c_2}{24} = 1+2\xi+\frac{11}{6}\xi^2+\xi^3.\end{aligned}$$

Then by the Atiyah-Bott Index theorem 1.1,

$$\begin{aligned}\chi(M,\mathcal{L}) &= \int_M \mathrm{Td}(TM)\cdot \mathrm{Ch}(\mathcal{L}) = \int_M \left(1+2\xi+\frac{11}{6}\xi^2+\xi^3\right)\cdot\left(1+k\xi+k^2\xi^2+k^3\xi^3\right)\\ &= \int_M \left(k^3+2k^2+\frac{11}{6}k+1\right)\xi^3+O(\xi^2) =\end{aligned}$$

1.2. **Equivariant Index Theorems.**

1.2.1. *Equivariant Characteristic Classes.*

References

[1] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

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