
GEOMETRIC QUANTISATION OF HYPERTORIC MANIFOLDS BY SYMPLECTIC CUTTING

GENERAL NOTES

ABSTRACT

Lorem ipsum.

1 Introduction

Lorem ipsum.

2 Hyperkähler Reduction and Hyperkähler Analogues

2.1 Introduction and Definitions

A *hyperkähler manifold* is a Riemannian manifold (M, g) equipped with three orthogonal, parallel complex structures J_1, J_2, J_3 , satisfying the usual quaternion relations. These three complex structures give rise to three symplectic forms

$$\omega_1(v, w) = g(J_1 v, w), \quad \omega_2(v, w) = g(J_2 v, w), \quad \omega_3(v, w) = g(J_3 v, w),$$

so that each (g, J_i, ω_i) is in its own right a Kähler structure on M for $i = 1, 2, 3$. The complex-valued two-form $\omega_2 + \sqrt{-1}\omega_3$ is a closed, non-degenerate, and holomorphic two-form with respect to the complex structure J_1 . Thus any hyperkähler manifold can be considered as a *holomorphic symplectic* manifold with complex structure J_1 , real symplectic form $\omega_{\mathbb{R}} := \omega_1$, and holomorphic symplectic form $\omega_{\mathbb{C}} := \omega_2 + \sqrt{-1}\omega_3$.

An action of a Lie group G on a hyperkähler manifold M is called *hyperhamiltonian* if it is hamiltonian with respect to $\omega_{\mathbb{R}}$, and holomorphic hamiltonian with respect to $\omega_{\mathbb{C}}$, with a G -equivariant moment map

$$\mu_{HK} := \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} \longrightarrow \mathfrak{g}^* \oplus \mathfrak{g}_{\mathbb{C}}^*.$$

The following theorem describes the *hyperkähler quotient* construction, which is the quaternionic analogue of a Kähler quotient:

Theorem 2.1 ([1]). *Let M be a hyperkähler manifold equipped with a hyperhamiltonian action of a compact Lie group G , with moment maps μ_1, μ_2, μ_3 . Suppose that $\xi = \xi_{\mathbb{R}} \oplus \xi_{\mathbb{C}}$ is a central regular value for μ_{HK} , and that G acts freely on $\mu_{HK}^{-1}(\xi)/G$. Then there is a unique hyperkähler structure on the hyperkähler quotient $\mathfrak{M} = M \mathbin{/\!\!\!/\!\!/}_{\xi} G := \mu_{HK}^{-1}(\xi)/G$, with associated symplectic and holomorphic symplectic forms $\omega_{\mathbb{R}}^{\xi}$ and $\omega_{\mathbb{C}}^{\xi}$, such that $\omega_{\mathbb{R}}^{\xi}$ and $\omega_{\mathbb{C}}^{\xi}$ pull-back to the restrictions of $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$ on $\mu_{HK}^{-1}(\xi)$.*

In general, the action of G on $\mu_{HK}^{-1}(\xi)$ will not be free, but only locally free. In this situation, we would end up with a *hyperkähler orbifold*. However in the sequel, we shall only concern ourselves when the action is free, and that \mathfrak{M} is smooth, i.e. a manifold.

Let us specialise to the case when $M = T^*\mathbb{C}^n$, and let G act on $T^*\mathbb{C}^n$ with the induced action from a linear action of G on \mathbb{C}^n , with moment map $\mu : \mathbb{C}^n \rightarrow \mathfrak{g}^*$. We can identify \mathbb{H}^n with $T^*\mathbb{C}^n$ such that the complex structure J_1 on \mathbb{H}^n is given by right multiplication by i , and that J_1 corresponds to the natural complex structure on $T^*\mathbb{C}^n$. With this identification in mind, $T^*\mathbb{C}^n$ inherits a hyperkähler structure. The real symplectic form $\omega_{\mathbb{R}}$ is obtained from the sum of the pull-backs of the standard Kähler forms on \mathbb{C}^n and $(\mathbb{C}^n)^*$, and the holomorphic symplectic form $\omega_{\mathbb{C}}$ is $\omega_{\mathbb{C}} = d\eta$, where η is the canonical holomorphic one-form on $T^*\mathbb{C}^n$.

As G acts \mathbb{H}^n -linearly on $T^*\mathbb{C}^n \cong \mathbb{H}^n$ from the left, the action is hyperhamiltonian with moment map $\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}$, where

$$\mu_{\mathbb{R}}(z, w) = \mu(z) - \mu(w), \quad \text{and} \quad \mu_{\mathbb{C}}(z, w)(\hat{v}_z),$$

where $w \in T_z^*\mathbb{C}^n$, $v \in \mathfrak{g}_{\mathbb{C}}$, and \hat{v}_z is the vector field in $T_z\mathbb{C}^n$ induced by v . For a central element $\alpha \in \mathfrak{g}^*$, we call the specialised hyperkähler quotient

$$\mathfrak{M} = T^*\mathbb{C}^n \mathbin{/\!\!/}_{(\alpha, 0)} G := (\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))/G$$

the hyperkähler analogue of the corresponding Kähler quotient,

$$\mathfrak{X} = \mathbb{C}^n \mathbin{/\!\!/}_{\alpha} G = \mu^{-1}(\alpha)/G.$$

We quote the following propositions without proof:

Proposition 2.2. *Suppose that α and $(\alpha, 0)$ are regular values for μ and μ_{HK} , respectively. Then the cotangent bundle $T^*\mathfrak{X}$ is isomorphic to an open subset of \mathfrak{M} , and is dense if it is non-empty.*

2.2 The \mathbb{C}^* -Action and the Core of a Hyperkähler Analogue

Consider the action of \mathbb{C}^* on $T^*\mathbb{C}^n$ given by

$$\hbar \cdot (z, w) = (z, \hbar w),$$

i.e. by scalar multiplication of the cotangent fibre. The holomorphic moment map $\mu_{\mathbb{C}} : T^*\mathbb{C}^n \rightarrow \mathfrak{g}_{\mathbb{C}}^*$ is \mathbb{C}^* -equivariant with respect to the scalar action on $\mathfrak{g}_{\mathbb{C}}^*$, and hence the \mathbb{C}^* -action descends to $\mu_{\mathbb{C}}^{-1}(0)$. Further, this \mathbb{C}^* -action commutes with the linear action of G on \mathbb{C}^n , and consequently the action of \mathbb{C}^* is J_1 -holomorphic on $\mathfrak{M} = (\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))/G$. However, the \mathbb{C}^* -action *does not* preserve the holomorphic symplectic form nor the hyperkähler structure on \mathfrak{M} ; rather it scales $\mu_{\mathbb{C}}$ with “homogeneity one”, i.e. $\hbar^* \omega_{\mathbb{C}} = \hbar \omega_{\mathbb{C}}$ for any $\hbar \in \mathbb{C}^*$.

Given that \mathfrak{M} is smooth, the action of the compact subgroup $S^1 \subset \mathbb{C}^*$ is hamiltonian with respect to the real symplectic two-form $\omega_{\mathbb{R}}$, with corresponding moment map $\Phi[z, w] = \frac{1}{2}\|w\|^2$. This map is a perfect Morse-Bott function, and its image is contained in $\mathbb{R}_{\geq 0}$. Further, we note that $\Phi^{-1}(0) = \mathfrak{X} \subset \mathfrak{M}$. The following proposition will be instrumental in the sequel, though again we quote it without proof:

Proposition 2.3. *If the original moment map for the G -action on \mathbb{C}^n , $\mu : \mathbb{C}^n \rightarrow \mathfrak{g}^*$, is proper, then so is the moment map for the S^1 action, $\Phi : \mathfrak{M} \rightarrow \mathbb{R}_{\geq 0}$.*

Next we shall define what is known as the *core* of a hyperkähler analogue, which will be essential in describing the fixed points of the \mathbb{C}^* -action of \mathfrak{M} .

Definition 2.4. *Suppose that \mathfrak{M} is smooth and Φ is proper. The core $\mathcal{L} \subset \mathfrak{M}$ of the hypertoric variety is defined to be the union of the \mathbb{C}^* orbits whose closures are compact.*

Let F be a connected component of $\mathfrak{M}^{S^1} = \mathfrak{M}^{\mathbb{C}^*}$, and let U_F be the closure of the set of points $p \in \mathfrak{M}$ such that $\lim_{\hbar \rightarrow \infty} \hbar \cdot p \in F$.

Proposition 2.5 ([?]; Proposition 2.8). *The core $\mathcal{L} \subset \mathfrak{M}$ has the following properties:*

1. \mathcal{L} is an S^1 -equivariant deformation retract of M ;
2. U_F is isotropic with respect to the holomorphic symplectic form $\omega_{\mathbb{C}}$;
3. Provided that \mathfrak{M} is smooth at F , then $\dim U_F = \frac{1}{2} \dim \mathfrak{M}$.

3 Hypertoric Manifolds

3.1 Definition

In this section, we shall specialise further now to when a hyperkähler analogue \mathfrak{M} is the analogue to a toric symplectic manifold $\mathfrak{X} = \mu^{-1}(\alpha)/N$, i.e. we replace the compact Lie group G with the torus $N = \ker(\pi : T^n \rightarrow T^d)$, using the same notation as in the second chapter.

Recall the short exact sequence of tori:

$$1 \longrightarrow N \xhookrightarrow{i} T^n \xrightarrow{\pi} T^d \longrightarrow 1,$$

and extend the linear action of the torus N on \mathbb{C}^n to $T^*\mathbb{C}^n$. This action is trihamiltonian and we obtain the following hyperkähler moment map

$$\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : T^*\mathbb{C}^n \longrightarrow \mathfrak{n}^* \oplus \mathfrak{n}_{\mathbb{C}}^*,$$

where

$$\mu_{\mathbb{R}}(z, w) = i^* \left(\frac{1}{2} \sum_{i=1}^n (|z_i|^2 - |w_i|^2) \partial_i \right), \quad \text{and} \quad \mu_{\mathbb{C}}(z, w) = i_{\mathbb{C}}^* \left(\sum_{i=1}^n (z_i w_i) \partial_i \right).$$

Given an element $\alpha \in \mathfrak{n}^*$ with a corresponding lift $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{R}^n)^*$, the Kähler quotient

$$\mathfrak{X} = \mathbb{C}^n //_{\alpha} N = \mu^{-1}(\alpha)/N$$

is our usual toric symplectic manifold with residual T^d -action from before, and moreover its hyperkähler analogue

$$\mathfrak{M} = T^*\mathbb{C}^n \mathrel{\mathbin{\mkern-1.5mu/\!\!\!/}}_{(\alpha, 0)} N = (\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0))/N$$

is what we shall call a *hypertoric manifold*¹. The hypertoric manifold \mathfrak{M} also admits a residual action of the torus T^d , which is hyperhamiltonian with hyperkähler moment map

$$\phi_{HK} := \phi_{\mathbb{R}} \oplus \phi_{\mathbb{C}} : \mathfrak{M} \longrightarrow (\mathbb{R}^d)^* \oplus (\mathbb{C}^d)^*,$$

where

$$\begin{aligned} \phi_{\mathbb{R}}[z, w] &= \frac{1}{2} \sum_{i=1}^n (|z_i|^2 - |w_i|^2 - \lambda_i) \partial_i \in \ker(i^*) = (\mathbb{R}^d)^*, \\ \phi_{\mathbb{C}}[z, w] &= \sum_{i=1}^n (z_i w_i) \partial_i \in \ker(i_{\mathbb{C}}^*) = (\mathbb{C}^d)^*. \end{aligned}$$

3.2 Hyperplane Arrangements

A fundamental difference between the toric manifold \mathfrak{X} and the hypertoric manifold \mathfrak{M} is that the hyperkähler moment map for \mathfrak{M} is surjective, and that \mathfrak{M} is non-compact. Despite this, we can still describe the image of the real moment map $\phi_{\mathbb{R}} : \mathfrak{M} \rightarrow (\mathbb{R}^d)^*$ combinatorially by means of a *hyperplane arrangement*. To describe this arrangement, recall that the map $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^d$ was defined by $\pi(e_i) = u_i$, for $i = 1, \dots, n$, where the u_i were the primitive, integral, inward-pointing normal vectors to the hyperplanes that determined our Delzant polytope. In the hypertoric case, they instead now describe a collection of *affine hyperplanes* $H_i \subset (\mathbb{R}^d)^*$ as follows: consider

$$H_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i = 0\},$$

so that the $u_i \in \mathbb{Z}^d$ is the normal vector to the hyperplane H_i . The hyperplane H_i divides $(\mathbb{R}^d)^*$ into two half-spaces

$$F_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \geq 0\},$$

$$G_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \leq 0\}.$$

Let

$$\Delta = \bigcap_{i=1}^n F_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \geq 0, \text{ for all } i = 1, \dots, n\}$$

be the (possibly empty) polyhedron in $(\mathbb{R}^d)^*$ defined by the affine hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$. We note that choosing a different lift λ' of α corresponds combinatorially to translating the arrangement \mathcal{A} inside of $(\mathbb{R}^d)^*$, and geometrically to shifting the Kähler and hyperkähler moment maps for the residual T^d -action by $\lambda' - \lambda \in \ker(i^*) = (\mathbb{R}^d)^*$.

We shall call that the arrangement \mathcal{A} *simple* if every subset of m hyperplanes with non-empty intersection intersects with codimension m , and call \mathcal{A} *smooth* if every collection of d linearly-independent vector $\{u_{i_1}, \dots, u_{i_d}\}$ spans $(\mathbb{R}^d)^*$. The reason for this terminology is the following proposition.

Proposition 3.1. *The hypertoric variety \mathfrak{M} is an orbifold if and only if \mathcal{A} is simple, and \mathfrak{M} is smooth if and only if \mathcal{A} is smooth.*

As we wish to restrict our attention to the case where \mathfrak{M} is a manifold, we shall assume in the sequel that \mathcal{A} is a smooth arrangement of hyperplanes.

¹More generally, \mathfrak{M} should be called a hypertoric variety, and only call \mathfrak{M} a manifold when it is smooth. However, we shall restrict our attention to the smooth case for simplicity.

3.3 The Core of a Hypertoric Manifold

The holomorphic moment map $\phi_{\mathbb{C}} : \mathfrak{M} \rightarrow (\mathbb{C}^d)^*$ is \mathbb{C}^* -equivariant with respect to the scalar action of \mathbb{C}^* on $(\mathbb{C}^d)^*$, hence both the core \mathcal{L} and the fixed-point set $M^{\mathbb{C}^*}$ will be contained in

$$\mathcal{E} := \phi_{\mathbb{C}}^{-1}(0) = \left\{ [z, w] \in \mathfrak{M} : z_i w_i = 0, 1 \leq i \leq n \right\}.$$

Definition 3.2. We shall call \mathcal{E} the extended core of \mathfrak{M} .

The restriction of $\phi_{\mathbb{R}}|_{\mathcal{E}} : \mathcal{E} \rightarrow (\mathbb{R}^d)^*$ is surjective from the defining equations, and further the extended core naturally breaks up into components

$$\mathcal{E}_A := \left\{ [z, w] \in \mathcal{E} : w_i = 0 \text{ for all } i \in A \text{ and } z_i = 0 \text{ for all } i \notin A \right\},$$

where $A \subseteq \{1, \dots, n\}$ is an indexing set. The hyperplanes $\{H_i\}_{i=1}^n$ divide $(\mathbb{R}^d)^*$ into a union of convex polyhedra

$$\Delta_A = \left(\bigcap_{i \in A} F_i \right) \cap \left(\bigcap_{i \notin A} G_i \right),$$

some of which may be empty. Note that $\mathcal{E}_{\emptyset} = \mathfrak{X}$ and that in general, each variety \mathcal{E}_A is a d -dimensional Kähler subvariety of \mathfrak{M} with an effective Hamiltonian T^d -action, so is itself a toric variety.

Lemma 3.3. If $w_i = 0$ then $\text{Im}(\phi_{\mathbb{R}}) \subseteq F_i$, and if $z_i = 0$ then $\text{Im}(\phi_{\mathbb{R}}) \subseteq G_i$.

Proof. Let $y \in (\mathbb{R}^d)^*$ be the image of the moment map $\phi_{\mathbb{R}}$ for a point $[z, w] \in \mathcal{E}$, then

$$y \cdot u_i + r_i = \mu_{\mathbb{R}}(z, w) \cdot e_i = \frac{1}{2}(|z_i|^2 - |w_i|^2),$$

and hence $y \geq 0$ if $i \in A$, and $y \leq 0$ if $i \notin A$. □

Lemma 3.4. The component \mathcal{E}_A of the extended core is isomorphic to the toric variety corresponding to the polytope Δ_A .

The S^1 -action does not act as a subtorus of T^d on \mathfrak{M} globally, but does when restricted to each individual component \mathcal{E}_A of the extended core. Consider a component $\mathcal{E}_A \subset \mathcal{E}$, then for some $[z, w] \in \mathcal{E}_A$ and $\tau \in S^1$,

$$\tau \cdot [z, w] = [z, \tau w] = [\tau_1 z_1, \dots, \tau_n z_n | \tau_1^{-1} w_1, \dots, \tau_n^{-1} w_n], \quad \text{where } \tau_i = \begin{cases} \tau^{-1} & \text{if } i \in A, \\ 1 & \text{if } i \notin A, \end{cases}$$

since for each pair (z_i, w_i) , if $i \in A$ then $(\tau z_i, \tau^{-1} w_i) = (0, \tau w_i)$, and if $i \notin A$, then $(\tau z_i, \tau^{-1} w_i) = (z_i, 0)$.

Thus when restricting our attention to each individual component \mathcal{E}_A of the extended core, the S^1 -action acts on \mathcal{E}_A via an inclusion homomorphism onto a one-parameter subgroup of T^n , which consequently then descends to a T^d -action after taking the quotient of T^n by K .

Let us denote the restricted action of the image of S^1 in T^d to \mathcal{E}_A by S_A^1 ,

$$S^1 \xhookrightarrow{j_A} T^n \xrightarrow{\pi} S_A^1 < T^d$$

$$\tau \longmapsto (\tau_1^{-1}, \dots, \tau_n^{-1}) \longmapsto v_A := \sum_{i \in A} u_i$$

where we have denoted the generator of the one-parameter subgroup S_A^1 in T^d by $v_A = \sum_{i \in A} u_i$.

Example 1. For $\mathfrak{M} = T^*\mathbb{CP}^2$,

Example 2. For \mathfrak{M} whose core consists of two \mathbb{CP}^2 intersecting at a point, so that the image of the real moment map is non-convex (todo)

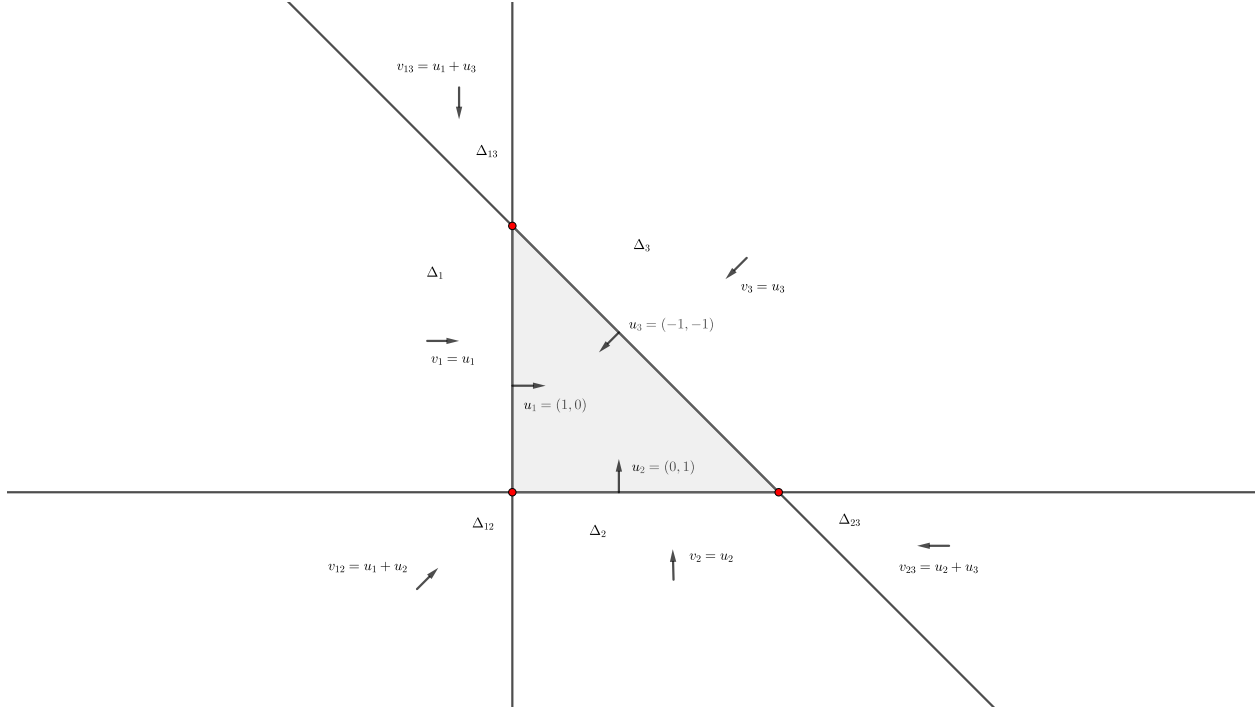


Figure 1: Combinatorics of the action of the residual S^1 -action on the extended core \mathcal{E}_A of $T^*\mathbb{CP}^2$, represented by each generator v_A of S^1_A in T^2 .

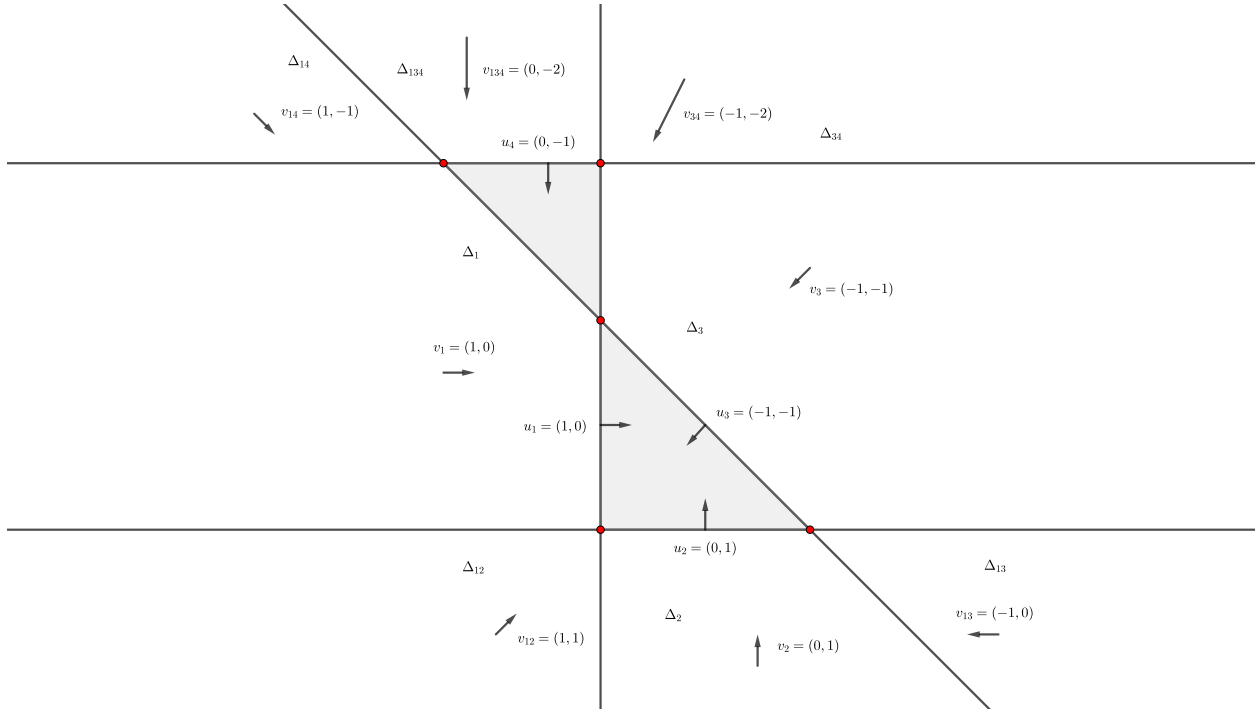


Figure 2: Combinatorics of the action of the residual S^1 -action when the core consists of two non-convex components.

4 Compactifying the Hypertoric Variety via Symplectic Cutting

We will use the S^1 -action to perform a symplectic cut of the toric hyperkähler manifold \mathfrak{M} to compactify it, which has the effect of bounding the $\|w\|^2$ norm component of $\bar{\mu}_{\mathbb{R}}$ by above, and discarding the rest that lies above this bound. Consider the product $\mathfrak{M} \times \mathbb{C}$, and let S^1 act on $\mathfrak{M} \times \mathbb{C}$ via the diagonal product action, i.e. S^1 acts on M by rotating the cotangent fibre coordinates, and on \mathbb{C} in the standard way:

$$e^{i\theta} \cdot ([z, w], \xi) = (e^{i\theta} \cdot [z, w], e^{i\theta} \xi) = ([z, e^{i\theta} w], e^{i\theta} \xi).$$

This action is Hamiltonian, and the corresponding moment map $\Phi : \mathfrak{M} \times \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$ for the S^1 -action is

$$\Phi([z, w], \xi) = \phi[z, w] + |\xi|^2 = \|w\|^2 + |\xi|^2.$$

Then we have

$$\begin{aligned} \Phi^{-1}(\epsilon) &= \{([z, w], \xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = \epsilon\} \\ &= \{[z, w] \in M : \|w\|^2 = \epsilon\} \sqcup \{([z, w], \xi) \in M \times \mathbb{C} : |\xi| = \pm \sqrt{\epsilon - \|w\|^2}\} \\ &= \{[z, w] \in M : \|w\|^2 = \epsilon\} \sqcup \{([z, w], \xi) \in M \times \mathbb{C} : \xi = e^{i \arg(\xi)} \sqrt{\epsilon - \|w\|^2}\} \\ &= \phi^{-1}(\epsilon) \sqcup (\mathfrak{M} \times S^1) \\ &=: \Sigma_1 \sqcup \Sigma_2, \end{aligned}$$

where we denote the level-set $\phi^{-1}(\epsilon) \subseteq \mathfrak{M}$ by Σ_1 , and $\Sigma_2 \cong \mathfrak{M} \times S^1$ is the trivial S^1 -bundle over Σ_2 given by the globally defined section

$$\mathfrak{M} \rightarrow \mathfrak{M} \times S^1, \quad [z, w] \mapsto ([z, w], e^{i\theta} \sqrt{\epsilon - \|w\|^2}), \quad e^{i\theta} \in S^1.$$

Finally, taking the symplectic reduction of $\Phi^{-1}(\epsilon)$ with respect to the S^1 -action, we obtain the *symplectic cut of \mathfrak{M} at level- ϵ* ,

$$M_{\leq \epsilon} := \Phi^{-1}(\epsilon)/S^1 = \Sigma_1/S^1 \sqcup \Sigma_2/S^1,$$

where $\Sigma_1/S^1 \cong \phi^{-1}(\epsilon)/S^1$ is just the usual symplectic reduction, and where Σ_2/S^1 is diffeomorphic to \mathfrak{M} for $\|w\|^2 < \epsilon$, which we will denote by $\mathfrak{M}_{< \epsilon}$.

4.1 The Combinatorics of the Cut Space, $\mathfrak{M}_{\leq \epsilon}$

Since the residual circle S^1 -action acts as a subtorus S_A^1 of the residual torus T^d on each component \mathcal{E}_A of the extended core, the hyperplane arrangement determined in $(\mathfrak{t}^d)^*$ by the real moment map $\bar{\mu}_{\mathbb{R}}$ is compactified by dropping in half-spaces with an inwards-pointing normal vector, given by v_A when taking the cut.

Recall from the previous section that $j_A : S_1 \hookrightarrow T^n$ denoted the inclusion homomorphism of S^1 into the big torus T^n . If we let $j_{A,*} : \mathfrak{s}^1 \rightarrow \mathfrak{t}^n$ represent the differential of this inclusion, then

$$j_{A,*}(1) = \sum_{i \in A} e_i \in \mathfrak{t}^n,$$

and the generator $\exp(v_A)$ of the one-parameter subgroup S_A^1 in T^d is

$$\exp(v_A) = \exp(\pi_* \circ j_{A,*}(1)),$$

or to be more concise,

$$S_A^1 = \left\{ \exp \left(r \cdot \sum_{i \in A} u_i \right) \mid r \in \mathbb{R} \right\}.$$

Then the moment map for this restriction for the S^1 -action is

$$\Phi[z, w] = j_A^* \circ \mu_{\mathbb{R}}[z, w] = \left\langle \mu_{\mathbb{R}}(z, w), \sum_{i \in A} \xi_i u_i \right\rangle,$$

and so from our above discussion of how we constructed the symplectic cut, the image in $(\mathbb{R}^d)^*$ of the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ is

$$\phi_{\mathbb{R}}(\Phi^{-1}(\epsilon)) = \left\{ y \in \Delta_A : \left\langle y, \sum_{i \in A} \xi_i u_i \right\rangle + \epsilon = 0 \right\} =: H_A$$

which introduces an inward-pointing half-space

$$F_A := \left\{ y \in \Delta_A : \left\langle y, \sum_{i \in A} \hbar_i u_i \right\rangle + \epsilon \geq 0 \right\}$$

such that the image of the extended core component \mathcal{E}_A after being compactified is the original convex polytope Δ_A intersected with H_A . One can also see clearly that the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ has the restricted S^1 -action as its stabiliser subgroup since, by definition of H_A , the moment map $\Phi|_{\mathcal{E}_A}$ equals the hyperplane H_A , i.e. $\Phi|_{\mathcal{E}_A}$ is constant along $\Phi^{-1}(\epsilon)/S^1$.

Remark 1. If we had used instead the following action for S^1

$$e^{i\theta} \cdot ([z, w], \xi) = ([z, e^{i\theta} w], e^{-i\theta} \xi)$$

with respective moment map

$$\mu_{cut}([z, w], \xi) = \frac{1}{2} \|w\|^2 - \frac{1}{2} |\xi|^2 - \epsilon,$$

and taken the cut, then the resulting then we would obtain the other “discarded half” $\mathfrak{M}_{>\epsilon}$ of the hypertoric manifold \mathfrak{M} along with the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ with the opposite orientation:

$$\mathfrak{M}_{\geq \epsilon} = \mathfrak{M}_{>\epsilon} \bigsqcup \left(-(\Phi^{-1}(\epsilon)/S^1) \right).$$

The component M_ϵ is non-compact however, so we focus on $M_{<\epsilon}$.

The following two figures show the resulting moment polytope after compactification, for the hypertoric varieties $T^*\mathbb{CP}^2$ and $T^*\mathbb{CP}^3$.

References

- [1] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.