
EQUIVARIANT LOCALISATION AND FIXED-POINT THEOREMS

HODGE CLUB TALK NOTES

12th March 2021

ABSTRACT

1 Introduction

Often in mathematics, we are tasked with the problem of evaluating an integral over some space, that is trying to evaluate

$$\int_M \omega$$

for some space M . Depending on the form ω , this integral can be related to finding the volume, to finding topological or enumerative invariants, integrating characteristic classes, or computing partition functions of physical systems. Such computations can be difficult, and there are many ways that we can tackle the integral. Two methods are particularly fruitful - that of *localisation* and *symmetry*.

1.1 Symmetry and Localisation

By symmetry, we mean that we have a group G acting on M , and by identifying orbits we reduce the problem to that over a smaller space, M/G . Such an approach comes up in symplectic reduction, gauge theory, and integrable systems.

By localisation, informally this means that we relate global calculations to ones that are local. The Poincaré-Hopf theorem is an example of this, which relates the Euler characteristic of a compact manifold M to the sum of the indices of the zeros of a vector field on it:

$$\chi(M) = \int_M e(TM) = \sum_{\mathbf{V}(p)=0} \text{Ind}(\mathbf{V}).$$

Unsurprisingly, it is often easier to consider a finite set of points rather than the global space M . The notion of localisation in algebra is a similar notion, in which we consider a single point (a prime) at a time.

Symmetry and localisation synergise through the Atiyah-Bott fixed-point theorem; in the situation that we have a smooth manifold M together with the action of a compact connected Lie group G , then the integral on M localises on the (isolated) fixed-point set $M^G := F \subseteq M$ of the G -action. If $i : F \hookrightarrow M$ is the inclusion, and $e(\nu_p)$ is the Euler class of the normal bundle ν_p to the fixed-point $p \in F$, then

$$\int_M \omega = \sum_{p \in F} \int_p \frac{i^* \omega}{e(\nu_p)}.$$

What I want to do in this talk is discuss some interesting cases when the fixed-point formula can be applied, and investigate how to use it in these examples. Consider the following geometric sum:

$$\begin{aligned}
\sum_{k=0}^{10000} q^k &= 1 + q + q^2 + \dots + q^{10001} \\
&= \left(\frac{1-q}{1-q} \right) \cdot (1 + q + q^2 + \dots + q^{10001}) \\
&= \frac{1 - q^{10001}}{1 - q} \\
&= \frac{1}{1-q} + \frac{1 - q^{10000}}{1 - q^{-1}}.
\end{aligned}$$

To evaluate the left-hand side, we need to know the value of each term at the 10001 integral points inside of the closed interval $[0, 10000]$, whereas the right-hand side only needs the two terms to be evaluated. So we can say that this sum *localises* at the end points.

2 Equivariant Cohomology

Let G be a compact Lie group acting on a topological space M . If G acts freely on M , then the quotient space M/G is usually as nice as the space M is itself; for instance, if M is a manifold then so is M/G .

The idea behind an equivariant cohomology group, $H_G^*(M)$, is that the equivariant cohomology groups of M should just be the cohomology groups of M/G :

$$H_G^*(M) = H^*(M/G), \quad \text{when the action is free.}$$

For example, if G acts on itself by left multiplication, then

$$H_G^*(G) = H^*(\text{pt}).$$

However, if the action is not free, then the space M/G might not behave very nicely from a cohomological point of view. Then the idea is that $H_G^*(M)$ should be the “correct” substitute for $H^*(M/G)$.

2.1 Classifying Bundles

As cohomology is unchanged under homotopy equivalence, our guiding idea is that the equivariant cohomology of M should be the ordinary cohomology of M^*/G , where M^* is some topological space homotopy equivalent to M and on which G acts freely. The standard way of constructing such a space is to take it to be the product $M^* = M \times E$, where E is some contractible space on which G acts freely. Then the equivariant cohomology groups of M are defined by the recipe

$$H_G^*(M) := H^*((M \times E)/G).$$

Note that if G acts freely on M then the projection

$$(M \times E)/G \longrightarrow M/G$$

is a fibration with typical fibre E . Then as E is contractible, we get that

$$H_G^*(M) = H^*((M \times E)/G) = H^*(M/G),$$

so we arrive at the same situation if G acts freely on M .

2.2 The Cartan Model

Let M be an n -dimensional manifold acted on by a Lie group G with Lie algebra \mathfrak{g} . A G -equivariant differential form on M is defined to be a polynomial map $\alpha : \mathfrak{g} \rightarrow \Omega(M)$ such that

$$\alpha(gX) = g \cdot \alpha(X), \quad \text{for } g \in G.$$

Let $\mathbb{C}[\mathfrak{g}]$ denote the algebra of \mathbb{C} -valued polynomial functions on \mathfrak{g} . Then we can view the tensor product

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega(M),$$

as the algebra of polynomial maps from \mathfrak{g} to Ω . The group G acts on an element $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ by the formula¹

$$(g \cdot \alpha)(X) := g \cdot (\alpha(g^{-1} \cdot X)), \quad \text{for all } g \in G, \text{ and } X \in \mathfrak{g}.$$

Let $\Omega^G(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega(M))^G$ be the subalgebra of G -invariant elements; an element $\alpha \in \Omega^G(M)$ thus satisfies $\alpha(g \cdot X) = g \cdot \alpha(X)$, hence is an equivariant differential form. Equip $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ with the following \mathbb{Z} -grading,

$$\deg(P \otimes \alpha) := 2 \cdot \deg(P) + \deg(\alpha),$$

for the polynomial $P \in \mathbb{C}[\mathfrak{g}]$, and $\alpha \in \Omega(M)$. Define the *equivariant exterior differential*, or *Cartan differential*, d_G by

$$(d_G \alpha)(X) := (d - \iota_{X_M})\alpha(X),$$

where $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ is the usual de Rham differential, X_M is the fundamental vector field of $X \in \mathfrak{g}$ on M , and $\iota_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ is the contraction of X on a differential form.

Proposition 2.1. *The Cartan differential d_G is closed on Ω_G^* , i.e. $d_G^2 = 0$.*

Proof. Let us compute d_G^2 on $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M)$. Some notation: set $\mathbb{C}[\mathfrak{g}] \cong \mathbb{C}[\phi^1, \dots, \phi^{\text{rk } \mathfrak{g}}]$, for a set of generators $\{\phi^\mu\}$ of \mathfrak{g} , then d_G can be written as

$$d_G = d \otimes 1 + \iota_\mu \otimes \phi^\mu.$$

Then the homotopy formula $\iota_\mu d + d\iota_\mu = \mathcal{L}_\mu$, where \mathcal{L}_μ is the Lie derivative of M along a vector field generated by V_μ , gets us that

$$\begin{aligned} d_G^2 &= (d \otimes 1 + \iota_\mu \otimes \phi^\mu)(d \otimes 1 + \iota_\mu \otimes \phi^\mu) \\ &= d^2 \otimes 1 + ((\iota_\mu d + d\iota_\mu) \otimes \phi^\mu) + (\iota_\mu \iota_\nu) \otimes (\phi^\mu \phi^\nu) \\ &= (\iota_\mu d + d\iota_\mu) \otimes \phi^\mu \\ &= \mathcal{L}_\mu \otimes \phi^\mu. \end{aligned}$$

TODO: FINISH! □

Corollary 2.2. *The space of equivariant differential forms $\Omega_G^*(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G$, equipped with the Cartan differential d_G forms a complex, called the **Cartan complex**:*

$$(\Omega_G^*(M), d_G) = ((\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G, d_G).$$

Definition 2.3. *The **equivariant cohomology** $H_G^*(M)$ of M is the cohomology of the Cartan complex, $(\Omega_G^*(M), d_G)$.*

¹ G acts on $\Omega(M)$ by the induced G -action on M , and on \mathfrak{g} by the adjoint action.

2.3 Equivariantly Closed Forms

In the Cartan model, an equivariant two-form

$$\tilde{\omega} \in \Omega_G^2(M) = ((\Omega_G^2(M) \otimes \mathbb{C}_0[\mathfrak{g}]) \oplus (\Omega^0(M) \otimes \mathbb{C}_1[\mathfrak{g}]))^G,$$

can be written as

$$\tilde{\omega} = \omega - \mu,$$

where $\omega \in \Omega^2(M)$ is a two-form invariant under G and $\mu \in (\Omega^0(M) \otimes \mathfrak{g})^G$ can be considered as a G -equivariant map,

$$\mu : \mathfrak{g} \longrightarrow \Omega^0(M) = \mathcal{C}^\infty(M),$$

from the Lie algebra, \mathfrak{g} , to the space of smooth functions on M . For each $\xi \in \mathfrak{g}$, $\mu(\xi)$ is a smooth function on M , and this function depends linearly on ξ . Therefore, for each $p \in M$, the value $\mu(\xi)(p)$ depends linearly on ξ , and thus we can think of μ as defining a map from M to the dual Lie algebra, \mathfrak{g}^* :

$$\mu : M \longrightarrow \mathfrak{g}^*; \quad \langle \mu(p), \xi \rangle := \mu(\xi)(p).$$

2.4 The Berline-Vergne-Atiyah-Bott Fixed Point Theorem

When a manifold has a torus action, the equivariant localisation formula is a powerful tool for doing calculations in **ordinary** cohomology, despite being formulated in **equivariant** cohomology.

Theorem 2.4 (Atiyah-Bott, Berline-Vergne Theorem). *Suppose an m -dimensional torus T acts on a compact oriented manifold M with fixed-point set $F := M^T$. If ϕ is an equivariant closed form on M and $i_F : F \hookrightarrow M$ is the inclusion map, then*

$$\int_M \phi = \int_F \frac{i_F^* \phi}{e_T(\nu_F)},$$

as elements of $H_T^*(\text{pt}) = \mathbb{R}[u_1, \dots, u_l]$. Here, ν_F is the normal bundle of F in M , and e_T is the T -equivariant Euler class.

3 Examples

3.1 Stationary Phase and Duistermaat-Heckman

Let M be a compact, oriented $2n$ -manifold, $f : M \rightarrow \mathbb{R}$ a function, and $\tau \in \Omega^{2n}(M)$.

TODO: SEE LORING TU'S BOOK.

3.2 Riemann-Roch-Hirzebruch Theorem

For $\mathcal{L} \rightarrow M$ a holomorphic line bundle over a complex manifold M . The Hirzebruch-Riemann-Roch Theorem then states that the Euler characteristic, $\chi(M; \mathcal{L})$, is equal to the characteristic number

$$\chi(M; \mathcal{L}) = \int_M e^{c_1(\mathcal{L})} \text{Td}(TM).$$

Here, $c_1(\mathcal{L})$ is the 1st Chern class of \mathcal{L} , and $\text{Td}(M)$ is the Todd class of the complex vector bundle $TM \rightarrow M$.