
EQUIVARIANT LOCALISATION AND FIXED-POINT THEOREMS

HODGE CLUB TALK NOTES

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ABSTRACT

Often in mathematics, we are tasked with the problem of evaluating an integral $\int_M \omega$ over some space M . Depending on the form ω , this integral can be related to calculating volume, finding topological or enumerative invariants, or integrating characteristic classes. Such computations are often difficult, but two notions can simplify them, which are *symmetry* and *localisation*.

By symmetry, we mean that we have a group G acting on M , and by identifying orbits we reduce the problem to that over a smaller space, M/G ; this comes up in symplectic reduction, gauge theory, and integrable systems. By localisation, this means that we reduce global calculations to local ones; for example, as in the Poincaré-Hopf theorem. Symmetry and localisation synergise together through the Atiyah-Bott-Berline-Vergne fixed-point formula; when we have a smooth manifold M together with the action of a compact, connected Lie group G , then the integral on M localises on the fixed-point set of the G -action.

In this talk, I want to introduce the equivariant cohomology of a G -manifold via the Cartan model - that is, equivariant de Rham theory - before going through some example calculations including the equivariant Riemann-Roch-Hirzebruch, the Duistermaat-Heckman, and the Lefschetz fixed-point theorems.

1 Introduction

Consider the following geometric sum:

$$\begin{aligned} \sum_{k=0}^{10000} q^k &= 1 + q + q^2 + \dots + q^{10001} \\ &= \left(\frac{1-q}{1-q} \right) \cdot (1 + q + q^2 + \dots + q^{10001}) \\ &= \frac{1 - q^{10001}}{1 - q} \\ &= \frac{1}{1-q} + \frac{1 - q^{10000}}{1 - q^{-1}}. \end{aligned}$$

To evaluate the left-hand side, we need to know the value of each term at the 10001 integral points inside of the closed interval $[0, 10000]$, whereas the right-hand side only needs the two terms to be evaluated. So we can say that this sum *localises* at the end points.

2 Equivariant Cohomology

Let G be a compact Lie group acting on a topological space M . If G acts freely on M , then the quotient space M/G is usually as nice as the space M is itself; for instance, if M is a manifold then so is M/G .

The idea behind an equivariant cohomology group, $H_G^*(M)$, is that the equivariant cohomology groups of M should just be the cohomology groups of M/G :

$$H_G^*(M) = H^*(M/G), \quad \text{when the action is free.}$$

For example, if G acts on itself by left multiplication, then

$$H_G^*(G) = H^*(\text{pt}).$$

However, if the action is not free, then the space M/G might not behave very nicely from a cohomological point of view. Then the idea is that $H_G^*(M)$ should be the “correct” substitute for $H^*(M/G)$.

2.1 Classifying Bundles

As cohomology is unchanged under homotopy equivalence, our guiding idea is that the equivariant cohomology of M should be the ordinary cohomology of M^*/G , where M^* is some topological space homotopy equivalent to M and on which G acts freely. The standard way of constructing such a space is to take it to be the product $M^* = M \times E$, where E is some contractible space on which G acts freely. Then the equivariant cohomology groups of M are defined by the recipe

$$H_G^*(M) := H^*((M \times E)/G).$$

Note that if G acts freely on M then the projection

$$(M \times E)/G \longrightarrow M/G$$

is a fibration with typical fibre E . Then as E is contractible, we get that

$$H_G^*(M) = H^*((M \times E)/G) = H^*(M/G),$$

so we arrive at the same situation if G acts freely on M .

2.2 The Cartan Model

Let M be an n -dimensional manifold acted on by a Lie group G with Lie algebra \mathfrak{g} . A G -equivariant differential form on M is defined to be a polynomial map $\alpha : \mathfrak{g} \rightarrow \Omega(M)$ such that

$$\alpha(gX) = g \cdot \alpha(X), \quad \text{for } g \in G.$$

Let $\mathbb{C}[\mathfrak{g}]$ denote the algebra of \mathbb{C} -valued polynomial functions on \mathfrak{g} . Then we can view the tensor product

$$\mathbb{C}[\mathfrak{g}] \otimes \Omega(M),$$

as the algebra of polynomial maps from \mathfrak{g} to Ω . The group G acts on an element $\alpha \in \mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ by the formula¹

$$(g \cdot \alpha)(X) := g \cdot (\alpha(g^{-1} \cdot X)), \quad \text{for all } g \in G, \text{ and } X \in \mathfrak{g}.$$

Let $\Omega^G(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega(M))^G$ be the subalgebra of G -invariant elements; an element $\alpha \in \Omega^G(M)$ thus satisfies $\alpha(g \cdot X) = g \cdot \alpha(X)$, hence is an equivariant differential form. Equip $\mathbb{C}[\mathfrak{g}] \otimes \Omega(M)$ with the following \mathbb{Z} -grading,

$$\deg(P \otimes \alpha) := 2 \cdot \deg(P) + \deg(\alpha),$$

for the polynomial $P \in \mathbb{C}[\mathfrak{g}]$, and $\alpha \in \Omega(M)$. Define the *equivariant exterior differential*, or *Cartan differential*, d_G by

$$(d_G \alpha)(X) := (d - \iota_{X_M})\alpha(X),$$

where $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ is the usual de Rham differential, X_M is the fundamental vector field of $X \in \mathfrak{g}$ on M , and $\iota_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ is the contraction of X on a differential form.

¹ G acts on $\Omega(M)$ by the induced G -action on M , and on \mathfrak{g} by the adjoint action.

Proposition 2.1. *The Cartan differential d_G is closed on Ω_G^* , i.e. $d_G^2 = 0$.*

Proof. The derivations d and ι_v in $\Omega(M)$ are related to the **Lie derivative** \mathcal{L} , by means of the **homotopy formula**:

$$\mathcal{L}(v) := \left. \frac{d}{dt} \right|_{t=0} (e^{tv})^* = d \circ \iota_v + \iota_v \circ d.$$

Here e^{tv} is the flow in M after a time t of the velocity field equal to v .

Now for $X \in \mathfrak{g}$, if X_M represents the infinitesimal action of X in M , then

[TODO]

□

Corollary 2.2. *The space of equivariant differential forms $\Omega_G^*(M) = (\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G$, equipped with the Cartan differential d_G forms a complex, called the **Cartan complex**:*

$$(\Omega_G^*(M), d_G) = ((\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M))^G, d_G).$$

Definition 2.3. *The **equivariant cohomology** $H_G^*(M)$ of M is the cohomology of the Cartan complex, $(\Omega_G^*(M), d_G)$.*

2.3 Characteristic Classes

Let G and T be compact, connected Lie groups.

An ordinary characteristic class for a principal G -bundle on an n -dimensional manifold M is $[p(F_A)] \in H^{2n}(M)$, for a G -invariant degree n polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$. here F_A is the curvature of any connection A on the G -bundle.

To get a T -equivariant characteristic class for a principal G -bundle associated to a G -invariant, degree n polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$, we take $[p(F_{A,T})] \in H_T^{2n}(M)$, where now $F_{A,T}$ is the T -equivariant curvature of any T -equivariant connection A on the G -bundle.

Restricted to the T -fixed points M^T of M , the T -equivariant characteristic class associated to a polynomial $p \in \mathbb{R}[\mathfrak{g}]^G$ is

$$p(F_A + \epsilon^a \rho(T_a)).$$

TODO: EXPLAIN WHAT ϵ^a , etc. ARE!

In particular, when V is a representation of G and p is the Chern character of the vector bundle V , then, if M is a point, the equivariant Chern characters are just the ordinary characters of the space V as a G -module.

2.4 The Euler Class

Here, let $G = \text{SO}(2n)$ which preserves the Riemannian metric on an oriented real vector space V of dimension $\dim_{\mathbb{R}}(V) = 2n$.

Definition 2.4. *Consider the following adjoint-invariant polynomial,*

$$\text{Pf} : \mathfrak{so}(2n; \mathbb{R}) \longrightarrow \mathbb{R},$$

*of degree n on the Lie algebra $\mathfrak{so}(2n; \mathbb{R})$, called the **Pfaffian**.*

The case that we shall be interested in is when we have the $(2n \times 2n)$ -antisymmetric matrix,

$$\text{Pf} \begin{pmatrix} 0 & \lambda_1 & \dots & \dots & 0 & 0 \\ -\lambda_1 & 0 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 & \lambda_n \\ 0 & 0 & \dots & \dots & -\lambda_n & 0 \end{pmatrix} = \lambda_1 \cdot \dots \cdot \lambda_n.$$

Definition 2.5. Let $P \rightarrow M$ be an $\mathrm{SO}(2n; \mathbb{R})$ -principal bundle over M . The **Euler characteristic class** of P , $e(P)$, is given by

$$e(P) := [\mathrm{Pf}(F)] \in H^{2n}(M; \mathbb{Z}).$$

Example 1. If M is an oriented, $2n$ -dimensional real manifold, then the **Euler characteristic** is given by

$$e(M) = \int_M e(TM) = \int_M \mathrm{Pf}(R_\nabla),$$

where R_∇ is the curvature form of the tangent bundle TM , equipped with the Levi-Civita connection.

To upgrade the Euler characteristic class e to a T -equivariant one e_T , where T is a torus acting on a manifold M with isolated fixed-point set M^T , we need to investigate the polynomial

$$\mathrm{Pf}(F_A + \epsilon^a \rho(T_a)).$$

For simplicity, let $T = S^1$. Then, for a point $p \in M^{S^1}$, the S^1 -action on $T_p M$ gives rise to an S^1 -representation,

$$\rho : S^1 \longrightarrow \mathrm{GL}(T_p M); \quad g \longmapsto l_{g,*},$$

where $l_{g,*} : T_p M \rightarrow T_{g \cdot p} M = T_p M$ is the differential of the action of $g \in S^1$ on $T_p M$.

As p is isolated, ρ decomposes into a direct sum of 2-dimensional irreducible representations,

$$T_p M \cong L^{m_1} \oplus \dots \oplus L^{m_n}.$$

Here, $L^m : S^1 \rightarrow \mathrm{GL}(2; \mathbb{R})$ is a representation of S^1 as m -fold rotations in \mathbb{R}^2 ,

$$L^m : g \longmapsto l_{g,*}; \quad L^m(e^{it}) = \begin{bmatrix} -m \sin(mt) & -m \cos(mt) \\ m \cos(mt) & -m \sin(mt) \end{bmatrix}.$$

2.5 Chern Classes

Now let $G = \mathrm{U}(n)$, then $\mathfrak{g} = \mathfrak{u}(n)$ can be identified with the space of matrices of the form iA , where $A = A^T$. Define the polynomial c_k , of degree k in A to be the coefficient of $(-1)^k \lambda^{n-k}$ in the characteristic polynomial of A :

$$\det(\lambda - A) = \lambda^n - c_1(A) \lambda^{n-1} + \dots + (-1)^n c_n(A).$$

In particular, $c_1(A) = \mathrm{Tr}(A)$ and $c_n(A) = \det(A)$. These polynomials are clearly adjoint invariant, thus the characteristic polynomial is.

The characteristic classes corresponding to the c_i for a complex vector bundle are called its **Chern classes**.

Remark 1. If we consider a complex vector bundle $V_{\mathbb{C}}$ of $\dim_{\mathbb{C}}(V) = n$ then, by forgetting the complex structure on $V_{\mathbb{C}}$, we get an oriented real vector bundle $V_{\mathbb{R}}$ of real dimension $\dim_{\mathbb{R}}(V) = 2n$.

By this correspondence, the Euler class $e(V)$ and the top Chern class $c_n(V)$ of V are related by

$$e(V_{\mathbb{R}}) = c_n(V_{\mathbb{C}}).$$

2.6 Equivariant Characteristic Classes

Definition 2.6. A *G-equivariant vector bundle* of a *G*-manifold *M* is a vector bundle $V \rightarrow M$ with an action of *G* on the total space *V* covering the action of *G* on *M*.

Definition 2.7. Let (M, ω) be a symplectic manifold, and suppose that a torus *T* acts on *M* preserving ω . The action is *Hamiltonian* if there exists a *moment map* $\mu : M \rightarrow \mathfrak{t}^*$, which satisfies

$$\iota_{\xi_M} \omega = d\langle \mu, \xi \rangle, \quad \text{for all } \xi \in \mathfrak{t}.$$

Here, ξ_M is the induced vector field on *M*.

Proposition 2.8. Let (M, ω, μ) be a symplectic manifold with a Hamiltonian action of a torus *T* and associated moment map $\mu : M \rightarrow \mathfrak{t}^*$. Set

$$\tilde{\omega} := \omega + \mu.$$

Then $\tilde{\omega}$ is a *T*-equivariantly closed two-form.

Proof. For $\tilde{\omega}$ to be equivariantly closed under d_T , we have

$$d_T \tilde{\omega} = 0 \iff (d - \iota_{\xi})(\omega + \mu^{\xi}) = d\omega - \iota_{\xi} \omega + d\mu^{\xi} - \iota_{\xi} \mu^{\xi} = -\iota_{\xi} \omega + d\mu^{\xi} = 0 \iff \iota_{\xi} \omega = d\mu^{\xi}.$$

□

So given an ordinary characteristic class $[\omega] \in H^2(M)$ and a moment map $\mu : M \rightarrow \mathfrak{g}^*$, we can elevate it to an equivariant characteristic class by the substitution

$$H^2(M) \ni [\omega] \mapsto [\omega_G] = [\omega + \mu] \in H_G^2(M).$$

Proposition 2.9. If *E* is a complex vector bundle with a *T*-action and $E \cong \bigoplus_j \mathcal{L}_j$, where \mathcal{L}_j are complex line bundles with *T*-action given by weights $\lambda_j : T \rightarrow U(1)$, then the equivariant Euler class of *E* is

$$e^T(E) = \prod_j c_1^T(\mathcal{L}_j).$$

In the Cartan model, this is represented by

$$e^T(E)(\xi) = \prod_j (F_j - \lambda_j)(\xi).$$

Example 2. If *T* acts on *M* and *F* is a component of M^T , then the normal bundle ν_F is a *T*-equivariant bundle over *V*. Assume that ν_F decomposes equivariantly as $\nu_F \cong \bigoplus_j \nu_{F,j}$ with weights $\lambda_{F,j} \in \mathfrak{t}^*$. Then the equivariant Euler class $e^T(\nu_F)$ is

$$e^T(\nu_F) = \prod_j (c_1(\nu_{F,j}) + \beta_{F,j}).$$

When the fixed-point set M^T consists of isolated fixed-points then, for $p \in M^T$,

$$e^T(\nu_p)$$

3 Equivariant Localisation

3.1 The Berline-Vergne-Atiyah-Bott Fixed Point Theorem

When a manifold has a torus action, the equivariant localisation formula is a powerful tool for doing calculations in **ordinary** cohomology, despite being formulated in **equivariant** cohomology.

Theorem 3.1 (Atiyah-Bott, Berline-Vergne Theorem). *Suppose an n -dimensional torus T acts on a compact oriented manifold M with fixed-point set $F := M^T$. If ϕ is an equivariant closed form on M and $i_F : F \hookrightarrow M$ is the inclusion map, then*

$$\int_M \phi = \sum_{F \subseteq M^T} \int_F \frac{i_F^* \phi}{e^T(\nu_F)},$$

as elements of $H_T^*(\text{pt}) = \mathbb{R}[u_1, \dots, u_n]$. Here, ν_F is the normal bundle of F in M , and e^T is the T -equivariant Euler class.

In the case when the fixed-point set $M^T = \{p_i\}$ consists of isolated fixed-points, the localisation theorem simplifies greatly:

Theorem 3.2. *With the above hypotheses, if M^T consists of isolated fixed-points, then:*

$$\int_M \phi = \sum_{p \in M^T} \frac{\phi(p)}{\prod_i \lambda_{p,i}}.$$

4 Examples

4.1 Stationary Phase and Duistermaat-Heckman

Let M be a compact, oriented $2n$ -manifold, $f : M \rightarrow \mathbb{R}$ a function, and $\tau \in \Omega^{2n}(M)$.

TODO: SEE LORING TU'S BOOK.

4.2 The Index and Hirzebruch-Riemann-Roch Theorems

Theorem 4.1 (Hirzebruch-Riemann-Roch, (HRR)). *Let $\mathcal{L} \rightarrow M$ a holomorphic line bundle over a **complex projective algebraic variety** M . Then the Euler characteristic, $\chi(M; \mathcal{L})$, is equal to the characteristic number*

$$\chi(M; \mathcal{L}) = \int_M e^{c_1(\mathcal{L})} \text{Td}(TM).$$

Here, $c_1(\mathcal{L})$ is the 1st Chern class of \mathcal{L} , and $\text{Td}(M)$ is the Todd class of the complex vector bundle $TM \rightarrow M$.

Definition 4.2. The **Todd class** $\text{Td}(TM)$ of a complex tangent bundle (TM, J) is

$$\text{Td}(TM) := \prod_{j=1}^n \frac{c_1}{1 + e^{-c_1}},$$

where $c_1 := c_1(TM)$ is the 1st Chern class of the tangent bundle.

Example 3. Let $M = \mathbb{CP}^1$ and $\mathcal{L} = \mathcal{O}(k)$. Then $c_1(\mathcal{L}) = kH$, where H generates $H^2(\mathbb{CP}^1; \mathbb{Z})$. The Chern character of \mathcal{L} is

$$e^{c_1(\mathcal{L})} = 1 + kH.$$

The tangent bundle TM is a line bundle with 1st Chern class $c_1(TM) = 2H$, so its Todd class is

$$\text{Td}(c_1(TM)) = \frac{c_1(TM)}{1 + e^{-c_1(TM)}} = 1 + \frac{1}{2}c_1(TM) = 1 + H.$$

Thus, the Riemann-Roch integral is

$$\int_M (1 + kH)(1 + H) = \int_M (k+1)H + 1 = k + 1.$$

In particular, if $k = 10000$, then we get $\chi(\mathbb{CP}^1; \mathcal{L}) = 10001$, which we saw at the start of this talk.

Remark 2. The sections of \mathcal{L} can be identified with degree k polynomials in the two coordinate variables of \mathbb{CP}^1 , and hence the space of sections has dimension $k + 1$. The higher cohomology spaces vanish: $H^j(\mathbb{CP}^1; \mathcal{L}) = 0$ if $j \geq 1$ by Kodaira vanishing.

This integral formula was originally proven for complex projective algebraic varieties by Hirzebruch, but the Atiyah-Singer index theorem generalises it to include complex analytic manifolds by the following argument:

The characteristic number $\chi(M; \mathcal{L})$ also equals

$$\chi(M; \mathcal{L}) = \sum (-1)^k \dim H^k(M; \mathcal{L}).$$

Provided that M has a Hermitian structure and hence metric, then this metric gives rise to an inner-product on the vector bundles $\Lambda^{0,k} = \Lambda^{0,k} TM$. Moreover, provided \mathcal{L} also has a Hermitian structure, then there are Hermitian inner-products on the vector bundles $\mathcal{L} \otimes \Lambda^{0,k}$ also. These Hermitian products and the volume form give rise to differential operators,

$$\bar{\partial}^* : \Omega^{0,k}(M; \mathcal{L}) \longrightarrow \Omega^{0,k-1}(M; \mathcal{L}),$$

as the product gives rise to an adjoint to the Dolbeault operator, $\bar{\partial}$. By Hodge theory, one can show that

$$\text{ind}(\bar{\partial}) := \ker(\bar{\partial} + \bar{\partial}^*) - \text{coker}(\bar{\partial} + \bar{\partial}^*) = \sum (-1)^k \dim H^k(M; \mathcal{L}),$$

as virtual vector spaces.

4.3 The Equivariant Index Theorem

Suppose now that in addition to the hypotheses above, we now have a bundle automorphism γ of \mathcal{L} , which leaves all the given structures invariant. Then it induces an operator on $\Lambda^{0,*}$ which commutes with $\bar{\partial}$. Hence if γ comes from a representation of a G -action, we get a **virtual representation** on $\text{Ind}(\bar{\partial})$, by lifting the G -action to the sections of \mathcal{L} . Letting χ denote the character of this representation and letting $G = T$ be a torus, then we have:

Theorem 4.3 (The Equivariant Index Formula).

$$\chi(e^{it}) = \int_M e^{c_1^T(\mathcal{L})} \text{Td}^T(TM), \quad t \in \mathfrak{t}.$$

Example 4. In the case when the set M^T fixed points is isolated, we obtain the **equivariant Lefschetz fixed-point formula**:

$$\text{Ind}(\bar{\partial}; \mathcal{L})(t) = \sum_{p \in M^T} \frac{\text{Tr}_{\mathcal{L}_p}(t)}{\det_{T_p^{1,0}}(1 - t^{-1})}.$$

Fact 1. When the T -action on a symplectic manifold (M, ω) is Hamiltonian with moment map $\mu : M \rightarrow \mathfrak{t}^*$, the image of μ is a convex polytope, whose vertices are the images of the fixed-points of the T -action. The isotropy weights at each fixed-point $p \in M^T$ are given by the primitive edge vectors $\alpha_{p,j} \in \mathfrak{t}^*$, $j = 1 \dots, \dim_{\mathbb{C}} M$.

Example 5. Let $M = \mathbb{CP}^1$ and $G = U^1$ act on M as

$$t \cdot [z_0 : z_1] = [z_0 : tz_1],$$

and let $\mathcal{L} = \mathcal{O}(n)$ be a complex line bundle of degree n over \mathbb{CP}^1 . Then the moment map for this action $\mu : \mathbb{CP}^1 \rightarrow \mathbb{R}^*$ is

$$\mu([z_0 : z_1]) = n \cdot \frac{|z_1|^2}{|z_0|^2 + |z_1|^2},$$

with image $\mu(\mathbb{CP}^1) = [0, n]$.

The primitive edge vectors for the two fixed-points are

$$\alpha_{[0:1]} = +1, \quad \alpha_{[1:0]} = -1.$$

Letting the $U(1)$ action on the fibre of \mathcal{L} at $[0 : 1]$ be trivial, by the gluing relation for sections

$$s_{\infty} = z^{-n} s_0,$$

we find that the action on the fibre at \mathcal{L} at $[1 : 0]$ to be of weight $-n$.

Then

$$\text{Ind}(\bar{\partial}; \mathcal{L}, \mathbb{CP}^1)(t) = \frac{t^0}{1 - t^{-1}} + \frac{t^{-n}}{1 - t^{+1}} = \sum_{k=0}^n t^{-k}, \quad n \geq 0.$$

Example 6. Now for $M = \mathbb{CP}^2$ with T^2 -action

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2],$$

and moment map $\mu : M \rightarrow \mathbb{R}^2$ given by

$$\mu([z_0 : z_1 : z_2]) = \frac{n}{|z_0|^2 + |z_1|^2 + |z_2|^2} \cdot (|z_1|^2, |z_2|^2),$$

for the bundle $\mathcal{L} = \mathcal{O}(n)$, so its moment map image is the dilated 2-simplex,

$$\mu(M) = n \cdot \Delta = \{(x, y) \in \mathbb{R}_{\geq 0}^2 : x + y \leq n\}.$$