First Year Annual Report

Benjamin Charles William Brown

First-Year Report
Graduate School of Mathematics

University of Edinburgh

August 2019

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2.1 Toric Varieties

Theorem 1 (Atiyah, Guillemin-Sternberg). *Let (M, ω) be a compact connected symplectic manifold, and let T^n be a torus that acts on M in a Hamiltonian manner. Now consider the moment map $\mu : M \to \mathfrak{t}^n$ for this T^n-action, then we have the following:*

- *the level sets $\mu^{-1}(c)$ are connected, for each $c \in \mathbb{R}^n$;*
- *the image $\mu(M)$ is convex;*
- *the image $\mu(M)$ is the convex hull of the images of the fixed points of the action.*

The image $\mu(M) \subset \mathfrak{t}^n$ of the moment map is called the *moment polytope* of the T^n-action.

Example 2. Consider the complex projective plane $\mathbb{CP}^2$ with the Fubini-Study Kähler form $(\mathbb{CP}^2, \omega_{FS})$. We can let the torus T^2 act on $\mathbb{CP}^2$ by

$$(t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2], \qquad \text{for } (t_1, t_2) \in T^2.$$

which has the following moment map $\mu : \mathbb{CP}^2 \to (\mathfrak{t}^2)^*$ as

$$\mu([z_0 : z_1 : z_2]) = \frac{1}{2}\left(\frac{|z_1|^2}{\|z\|^2}, \frac{|z_2|^2}{\|z\|^2}\right).$$

There are three fixed points of the T^2-action, namely

$$[1 : 0 : 0], \qquad [0 : 1 : 0], \qquad [0 : 0 : 1],$$

which get mapped to

$$(0,0), \qquad (\tfrac{1}{2},0), \qquad (0,\tfrac{1}{2}),$$

respectively. Hence the moment polytope $\mu(\mathbb{CP}^2) \subset (\mathfrak{t}^2) \cong \mathbb{R}^2$ is the right-angled triangle in the positive quadrant in $\mathbb{R}^2$.

2.2 The Delzant Construction

Definition 3. A *symplectic toric manifold* is a compact connected symplectic manifold (M^{2n},ω) with an effective Hamiltonian action of a torus T^n of dimension equal to half the dimension of the manifold

$$\dim T^n = \frac{1}{2}\dim M^{2n},$$

and with a choice of corresponding moment map $\mu : M \to (\mathfrak{t}^n)^*$.

Definition 4. A *Delzant polytope* Δ in $\mathbb{R}^n$ is a polytope satisfying:

- simplicity: there are n edges meeting at each vertex;
- rationality: each edge that meets a vertex p is of the form $p + tu_i$, with $t_i \geq 0$ and $u_i \in \mathbb{Z}^n$;
- smoothness: for each vertex, the corresponding $u_1,\dots,u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

It turns out that the moment polytope of a symplectic toric manifold is Delzant.

Lemma 5. *For any symplectic toric manifold (M,ω), its moment polytope is Delzant.*

So this shows that any toric symplectic manifold has, as the image of its moment map, a Delzant polytope associated to it.

2.2.1 Symplectic Construction

Theorem 6 (Delzant, [**?**])**.** *Toric manifolds are classified by Delzant polytopes. More specifically, the bijective correspondence between these two sets is given by the moment* *map:*

$$\frac{\{\textit{toric manifolds}\}}{\{T^n\textit{-equivariant symplectomorphisms}\}} \longleftrightarrow \frac{\{\textit{Delzant polytopes}\}}{\{\textit{translations}\}}$$

$$(M_\Delta^{2n}, \omega_\Delta, T^n, \mu) \longleftrightarrow \mu(M_\Delta) = \Delta.$$

We do not give a proof of this theorem, but we shall outline the steps involved in the construction of a toric symplectic manifold when given a convex Delzant polytope to introduce the notation used, before investigating two examples.

Let Δ be a convex polytope in $\mathbb{R}^n$ with N facets, which satisfy the Delzant condition. For each facet F_j of Δ, where $j = 1, \ldots, N$, let $v_j \in \mathbb{Z}^n$ be the primitive inward-pointing normal vector to F_j. Define a projection $\pi_* : \mathbb{R}^N \to \mathbb{R}^n$ by sending the j-th basis vector $e_j \in \mathbb{R}^N$ to $v_j \in \mathbb{R}^n$:

$$\pi_* : \mathbb{R}^N \to \mathbb{R}^n : \ e_j \mapsto v_j.$$

Since Δ is Delzant, the vectors v_j span $\mathbb{R}^n$, and further they form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$ (prove?). So π_* is surjective from $\mathbb{Z}^N$ onto $\mathbb{Z}^n$ and induces a map between tori:

$$\pi : \mathbb{R}^N/\mathbb{Z}^N \to \mathbb{R}^n/\mathbb{Z}^n.$$

Set $K := \ker(\pi)$ and $\mathfrak{k} := \ker(\pi_*)$, so that K is a subtorus of T^N with inclusion homomorphism $\imath : K \hookrightarrow T^N$ and $\imath_* : \mathfrak{k} \to \mathfrak{t}^N$ is the Lie algebra of K. Then we get two exact sequences

$$1 \longrightarrow K \xrightarrow{i_*} T^N \xrightarrow{\pi_*} T^n \longrightarrow 1,$$

$$0 \longrightarrow \mathfrak{k} \xrightarrow{i_*} \mathfrak{t}^N \xrightarrow{\pi_*} \mathfrak{t}^n \longrightarrow 0,$$

the second of which induces the dual exact sequence

$$0 \longleftarrow (\mathfrak{k})^* \xleftarrow[i^*]{} (\mathfrak{t}^N)^* \xleftarrow[\pi^*]{} (\mathfrak{t}^n)^* \longleftarrow 0.$$

With the primitive normal vectors v_j, we can write the polytope Δ as

$$\Delta = \{x \in \mathbb{R}^n \ : \ \langle x, \ v_j \rangle \geq \lambda_j, \ 1 \ \leq j \leq N\}$$

for some real numbers λ_j. (assuming the λ_j are integers $\implies$ prequantisable see mark hamilton).

(need more here)

Now set $\nu = \imath(-\lambda) \in \mathfrak{k}^*$. As the sequence (label) is exact, $(\imath^*)^{-1}(0) = \mathrm{Im}(\pi^*)$, whence $(\imath^*)(\nu) = \mathrm{Im}(\pi^* - \lambda)$. Since $\imath^*$ is a linear map between vector spaces, $(\imath^*)^{-1}(\nu)$ is an affine subspace of $\mathbb{R}^N$. The intersection of this affine subspace with $\mathbb{R}^N_+$ can be identified with the polytope Δ.

The torus T^N acts on $\mathbb{C}^N$ in the standard way with the diagonal multiplication action, which is Hamiltonian with moment map

$$\phi : \mathbb{C}^N \to (\mathfrak{t}^N)^* : (z_1, \ldots, z_N) \mapsto \left(\pi|z_1|^2, \ldots, \pi|z_N|^2\right).$$

The inclusion $\imath : K \hookrightarrow T^N$ induces a Hamiltonian action of K on $\mathbb{C}^N$ as a subtorus, with moment map

$$\mu : \imath^* \circ \phi : \mathbb{C}^N \to \mathfrak{k}^*.$$

Let $M = \mu^{-1}/K$, then as the action of T^N on $\mathbb{C}^N$ commutes with the action of K it descends to a Hamiltonian action on the quotient M. It is not effective however, but the quotient torus $T^n = T^N/K$ does act effectively on M. Delzant's theorem is then the statement that M equipped with this action is a smooth toric manifold, with moment polytope Δ.

2.2.2 Complex Construction

Since the symplectic form ω_Δ on M_Δ coincides with that of ω_0 on $\mathbb{C}^d$ when both are pulled back to Z, we have the following:

Corollary 7. *The symplectic manifold* $(M_\Delta, \omega_\Delta)$ *constructed above has a natural Kähler structure.*

Remark 8. Let Δ be a Delzant polytope in $(\mathbb{R}^n)^*$ and with d facets. Let $v_i \in \mathbb{Z}^n$, $i = 1, \dots d$, be the primitive outward-pointing normal vectors to the facets of Δ. Then Δ can be described as n intersection of half-spaces

$$\Delta = \{x \in (\mathbb{R}^n)^* : \langle x, v_i \rangle \leq \lambda_i,\ i = 1, \dots, d\} \quad \text{for some } \lambda_i \in \mathbb{R}.$$

Example 9. From the $T^2 acts \mathbb{CP}^2$ example from before:

$$\begin{aligned}\Delta &= \left\{x \in (\mathbb{R}^2)^* : x_1 \leq 0,\ x_2 \leq 0,\ x_1 + x_2 \geq -\tfrac{1}{2}\right\} \\ &= \left\{x \in (\mathbb{R}^2)^* : \langle x, (1,0) \rangle \leq 0,\ \langle x, (0,1) \rangle \leq 0,\ \langle x, (-1,-1) \rangle \leq \tfrac{1}{2}\right\}\end{aligned}$$

2.3 Examples

In this section we shall apply the above theory to construct two examples of toric symplectic varieties, not just to illustrate the aforementioned theory, but also because these two examples will be necessary for further discussion on hypertoric varieties and quantisation later on in this report.

2.3.1 T^3 acting on $\mathbb{C}^3$

Consider the polytope Δ to be the triangle in $\mathbb{R}^2$ with vertices $(0,0), (0,m)$, and $(m,0)$, for $m \in \mathbb{Z}_+$. Here $N = 3$ and $n = 2$, and the three normal vectors to Δ are

$$v_1 = (0,1), \quad v_2 = (1,0), \quad v_3 = (-1,-1)$$

and λ is $(0,0,-m)$. Thus the map $\pi : \mathfrak{t}^3 \to \mathfrak{t}^2$ is represented by the matrix

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix},$$

or if the coordinates of $\mathfrak{t}^3$ are (x, y, z), then equivalently

$$\pi(x, y, z) = (y - z, x - z).$$

The kernel of this map is clearly

$$\ker \pi = \{(x, y, z) \in \mathfrak{t}^3 \ : \ x = y = z\} =: \mathfrak{k},$$

and $\mathfrak{k}$ can be identified with $\mathbb{R}$ by the inclusion $i : t \hookrightarrow (t, t, t)$. Exponentiating, the corresponding map on tori is

$$\pi : T^3 \to T^2 : \ (e^{2\pi ix}, e^{2\pi iy}, e^{2\pi z}) \mapsto \left(e^{2\pi i(y-z)}, e^{2\pi i(x-z)}\right),$$

with kernel $\ker \pi = (e^{2\pi it}, e^{2\pi it}, e^{2\pi it}) =: K$, which is S^1 embedded into T^3 as the diagonal subtorus.

For the dual sequence, the map π^* can be found by transposing the matrix for π_*,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix},$$

or equivalently if we represent the coordinates in $(\mathfrak{t}^2)^*$ by (a, b), then

$$\pi^*(a, b) = (a, b, -a - b).$$

Similarly, or since (image = kernel),

$$i^*(x, y, z) = x + y + z.$$

Recall that $\lambda = (0, 0, -m)$, so that $\nu = \imath^*(-\lambda) = \lambda$. Then the affine space $(\imath^*)^{-1}(\nu) \cap \mathbb{R}^3_+$ intersected with the positive orthant $\mathbb{R}^3_+$ is the space $\{(x, y, z) \in \mathbb{R}^3 \ : \ x + y + z = m\}$, which is a triangle (need more).

Finally, using the moment map $\phi : \mathbb{C}^3 \to (\mathfrak{t}^3)^* \cong \mathbb{R}^3$ for the T^3-action on $\mathbb{C}^3$, we see that $\mu^{-1}(\nu) = \{z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = m/\pi\} \cong S^5$, and which the symplectic reduction by the diagonal action of $K \cong S^1$ is $M = \mu^{-1}(\nu)/K \cong \mathbb{CP}^2$.

(need more) Observe that the number of integral lattice points contained in Δ, including those on the boundary, is equal to

$$1 + 2 + \ldots + m + (m+1) = \frac{m(m+1)}{2}.$$

2.3.2 T^4 acting on $\mathbb{C}^4$

Now let Δ be the pyramid in $\mathbb{R}^3$ with vertices $(0,0,0),\ (m,0,0),\ (0,m,0),\ (0,0,m)$, where once again $m \in \mathbb{Z}_+$. In this case $N = 4$ and $n = 3$, and the primitive inward-pointing normal vectors to Δ are

$$v_1 = (0,0,1), \quad v_2 = (0,1,0), \quad v_3 = (1,0,0), \quad v_4 = (-1,-1,-1),$$

and λ is $(0,0,-m)$??.

With the same notation as before, this situation is almost identical to the previous example *mutatis mutandi*; one finds for $\imath^* : (\mathfrak{t}^4)^* \to \mathfrak{k}^* \cong \mathbb{R}$ that

$$\imath^*(w,x,y,z) = w + x + y + z,$$

Chapter 3

Geometric Quantisation

3.1 Geometric Quantisation

The mathematical procedure of quantisation associates to a symplectic manifold (M, ω) a Hilbert space $\mathcal{Q}(M)$. The motivation behind this procedure comes from physics at the quantum level, where the states of a physical system are represented by the rays in a Hilbert space $\mathcal{H}$ and the observables by a collection $\mathcal{O}$ of symmetric operators on $\mathcal{H}$, whilst on the other hand the classical state space is a symplectic manifold (M, ω) and the observables are smooth functions on M. As a question, quantisation asks whether given such (M, ω) is it possible to reconstruct $\mathcal{H}$ and $\mathcal{O}$, and as a procedure it represents the possible ways one may try to answer this question.

The procedure is governed by Dirac's general principals of quantum mechanics; that the canonical transformations of M generated by the classical observables should correspond to the unitary transformations of $\mathcal{H}$ generated by the quantum observables, and Poisson brackets of classical observables should correspond to commutators of quantum observables.

3.1.1 Pre-Quantisation

Consider a manifold M along with a closed two-form ω.

Definition 10. A *pre-quantisation line bundle* for (M, ω) is a complex line bundle $\mathcal{L}$ whose curvature class is the cohomology class $[\omega]$. Equivalently, the image of its first

Chern class $c_1(\mathcal{L})$ maps to $\frac{1}{2\pi}[\omega]$ under the natural inclusion homomorphism

$$i : H^2(M;\mathbb{Z}) \hookrightarrow H^2(M;\mathbb{R}).$$

Since complex line bundles are determined by $H^2(M;\mathbb{Z})$ via the map $\mathcal{L} \mapsto c_1(\mathcal{L}) \in H^2(M;\mathbb{Z})$, it follows that the manifold (M,ω) is pre-quantisable if and only if $\frac{1}{2\pi}[\omega]$ is integral, *i.e.* $\frac{1}{2\pi}[\omega] \in H^2(M;\mathbb{Z})$ originally.

The statement that $\frac{1}{2\pi}[\omega]$ must be integral can be interpreted geometrically; it is the curvature form of a connection of a principal $U(1)$-bundle over M, whose first Chern class is $\frac{1}{2\pi}[\omega]$.

Definition 11. A *pre-quantisation* of (M,ω) is a Hermitian line bundle $(\mathcal{L},\langle\ ,\ \rangle)$ equipped with a Hermitian connection ∇ whose curvature is ω. Thus, if we are provided with a pre-quantisation $(\mathcal{L},\langle\ ,\ \rangle,\nabla)$ for a symplectic manifold (M,ω), then $\mathcal{L}$ is a pre-quantisation line bundle on (M,ω).

Provided that a pre-quantisation of (M,ω) exists, we say that M is *pre-quantisable*, and the *quantisation space* $\mathcal{Q}(M)$ of M is constructed from sections of the pre-quantisation line bundle $\mathcal{L} \to M$.

However the space of such sections is too big in that it does not satisfy the "minimality condition" of Dirac's axioms. Hence

Definition 12. A *polarisation* of M is an integrable sub-bundle F of the complexified tangent bundle $TM \otimes \mathbb{C}$ of M such that at each point $p \in M$, F_p is a complex Lagrangian subspace of the complex symplectic space $T_pM \otimes \mathbb{C}$. Given a polarisation of M, a local section $s : (U \subset M) \to \mathcal{L}$

3.2 Moduli Spaces of Vector Bundles over a Riemann Surface

Let M be a connect compact Riemann surface.

3.3 The Verlinde Formula

Chapter 4

Toric Hyperkähler Manifolds

4.1 Hyperkähler Reduction

The quaternionic vector space $\mathbb{H}^N$ is a flat hyperkähler manifold with complex structures J_1, J_2, J_3 given by right multiplication by i, j, k respectively. The real torus T^N acts on $\mathbb{H}^N$ by left diagonal multiplication and preserves the hyperkähler structure. We may choose one complex structure, say J_2, and identify $\mathbb{H}^N$ with $\mathbb{C}^N \times \mathbb{C}^N$, so that the action of T^N can be written as

$$t \cdot (z, w) = (t \cdot z, t^{-1} \cdot w).$$

Alternatively, choosing the complex structure J_1 we may identify $\mathbb{H}^N$ with the cotangent bundle $T^*\mathbb{C}^N$, with the natural torus action induced from that on $\mathbb{C}^N$.

The three moment maps μ_1, μ_2, μ_3 corresponding to the complex structures may be written as

$$\mu_1(z, w) = \frac{1}{2} \sum_{k=1}^{N} \left(|z_k|^2 - |w_k|^2\right) e_k + c_1,$$
$$(\mu_2 + \sqrt{-1}\mu_3)(z, w) = \sum_{k=1}^{N} (z_k w_k) e_k + c_2 + \sqrt{-1} c_3,$$

for arbitrary constants $c_1, c_2, c_3 \in \mathbb{R}^N$, and where the e_k, $k = 1, \ldots, N$, are the standard basis vectors in $(\mathfrak{t}^N)^* \cong \mathbb{R}^N$. Observe now that in this case, unlike in the Delzant construction previously used for toric manifolds, that the hyperkähler moment map $\mu = (\mu_1, \mu_2, \mu_3)$ is surjective onto $\mathbb{R}^{3N}$.

Now let $u_i = \imath^*(e_i)$, $i = 1, \dots N$ define a subtorus K of T^N as $K = \ker(\pi)$, assuming like before that the vectors u_i are $\mathbb{Z}$-valued, primitive, and generate $\mathbb{R}^n$. The moment maps for the action of K are

$$\mu_1(z,w) = \frac{1}{2}\sum_{k=1}^{N}\left(|z_k|^2 - |w_k|^2\right)\alpha_k + c_1,$$
$$(\mu_2 + \sqrt{-1}\mu_3)(z,w) = \sum_{k=1}^{N}(z_k w_k)\alpha_k + c_2 + \sqrt{-1}c_3,$$

where $\alpha_k = \imath^*(e_k)$, and where the constants c_j are of the form

$$c_j = \sum_{k=1}^{N}\lambda_k^{(j)}\alpha_k, \qquad (j = 1, 2, 3),$$

for $\lambda_k^{(j)} \in \mathbb{R}$. As in (ref BD), we shall adopt the notation

$$\lambda_k = (\lambda_k^{(1)}, \lambda_k^{(2)}, \lambda_k^{(3)}), \quad (k = 1, \dots, N),$$

and also denote the hyperkähler quotient $\mu^{-1}(0)/K$ corresponding to $\underline{u} = (u_1, \dots, u_N)$ and $\underline{\lambda} = (\lambda_1, \dots, \lambda_N)$ by $M(\underline{u}, \underline{\lambda})$, or even just by M.

The action on $T^n = T^N/K$ on $M(\underline{u}, \underline{\lambda})$ preserves the hyperkähler structure and gives rise to another hyperkähler moment map $\phi = (\phi_1, \phi_2, \phi_3) : M(\underline{u}, \underline{\lambda}) \to (\mathfrak{t}^n)^* \otimes \mathbb{R}^3 \cong \mathbb{R}^{3n}$. Finally, we will need to consider the following hyperplanes in $\mathbb{R}^n$

$$H_k^{(j)} = \{y \in \mathbb{R}^n : \langle y, u_k\rangle = \lambda_k^{(j)}\}, \quad (j = 1, 2, 3, \quad k = 1, \dots, N)$$

and the codimension 3 flats (affine subspaces) in $\mathbb{R}^{3n}$

$$H_k = H_k^{(1)} \times H_k^{(2)} \times H_k^{(3)}.$$

4.2 Hyperplane Arrangements

It is the flats H^k defined in the previous section that determine the corresponding hyperkähler reduction, instead of the intersection of half-spaces as in the toric case. The following two theorems determine the necessary and sufficient conditions for the hyperkähler quotient $\mu^{-1}(0)/K$ to be either smooth or an orbifold respectively; their proofs can be found in (BD).

Theorem 13. *Suppose we have primitive vectors $u_1, \ldots u_N \in \mathbb{Z}^n$ that generate $\mathbb{R}^n$, and elements $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^3$ such that the flats H_k are distinct. Then the hyperkähler quotient $M(\underline{u}, \underline{\lambda})$ is smooth if and only if every $n+1$ flats among the H_k have empty intersection, and whenever some n flats $H_{k_1}, \ldots H_{k_n}$ have non-empty intersection, then the set $\{u_{k_1}, \ldots, u_{k_n}\}$ forms a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*

Let $\mathfrak{t}^N \cong \mathbb{R}^N$ and $\mathfrak{t} \cong \mathbb{R}^n$ be real vector spaces of dimensions N and n respectively, with integer lattices $\mathfrak{t}^N_{\mathbb{Z}} \subset \mathfrak{t}^N$ and $\mathfrak{t}^n_{\mathbb{Z}} \subset \mathfrak{t}^n$. Let $\{e_1, \ldots, e_N\}$ be an integer basis for $\mathfrak{t}^N_{\mathbb{Z}}$ and $\{e^1, \ldots, e^N\}$ be the respective dual basis for $(\mathfrak{t}^N_{\mathbb{Z}})^*$. Now suppose that we are given N non-zero integer vectors $\{u_1, \ldots, u_N\}$ that belong to $\mathfrak{t}^n_{\mathbb{Z}}$, which space the space $\mathfrak{t}^n$ over the real numbers $\mathbb{R}$.

4.3 The Core and Extended Core of a Hypertoric Variety

4.4 Symplectic Cutting

Let (M, ω) be a symplectic manifold with a Hamiltonian S^1-action whose moment map is $\Phi : M \to \mathbb{R}$. Now consider the product $M \times \mathbb{C}$ with the product symplectic structure and the S^1-action

$$e^{i\theta} \cdot [m, \xi] \longmapsto [e^{i\theta} \cdot m, e^{-i\theta}\xi],$$

and corresponding moment map

$$\Phi_{\text{cut}}[m, \xi] \longmapsto \Phi(m) - |\xi|^2.$$

The *symplectic cut* of M is then defined as $M_{\text{cut}} = \Phi^{-1}_{\text{cut}}(\epsilon)/S^1$, which is the symplectic quotient of $M \times \mathbb{C}$ at the level $\epsilon \in \mathbb{R}$. The construction fits into the following diagram

$$M \xleftarrow{\pi} \Phi^{-1}_{\text{cut}}(\epsilon) \xrightarrow{q} M_{\text{cut}},$$

where $\pi : \Phi^{-1}_{\text{cut}}(\epsilon) \to M$ is the projection $(m, \xi) \mapsto m$, and its image is

$$\{m \in M : \Phi(m) \geq \xi\}.$$

The map $q : \Phi_{\text{cut}}^{-1}(\epsilon) \to M_{\text{cut}}$ is the quotient map for the S^1-action.

4.5 Example: T^3 acting on $\mathbb{H}^3$

Let the torus T^3 act on $T^*\mathbb{C}^3$ by

$$(t_1, t_2, t_3) \cdot (z_1, z_2, z_3, w_1, w_2, w_3) = (t_1 z_1, t_2 t_2, t_3 z_3, t_1^{-1} w_1, t_2^{-1} w_2, t_3^{-1} w_3). \qquad (4.1)$$

The corresponding trihamiltonian moment maps are then

$$\mu = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : M \longrightarrow (\mathfrak{t}^3)^* \oplus (\mathfrak{t}^3_{\mathbb{C}})^*,$$
$$\mu_{\mathbb{R}}(z, w) = \frac{1}{2} \sum_{i=1}^{3} \left(|z_i|^2 - |w_i|^2 \right) \partial_i, \qquad \mu_{\mathbb{C}}(z, w) = \sum_{i=1}^{3} (z_i w_i) \partial_i.$$

Let us choose $u_1 = (1, 0), u_2 = (0, 1), u_3 = (-1, -1)$, then we have the short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{k} \xrightarrow{\imath} \mathfrak{t}^3 \xrightarrow{\pi} \mathfrak{t}^2 \longrightarrow 0$$

where $\pi(e_i) = u_i$ and $\mathfrak{k} := \ker(\pi)$. Now π is represented as the matrix

$$\pi = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

and so

$$\mathfrak{k} = \ker(\pi) = \{(x, y, z) \in \mathbb{R}^3 : x = z, y = z\} \cong \mathbb{R}.$$

Thus our map $\imath : \mathbb{R} \to \mathbb{R}^3$ is just

$$i(t) = (t, t, t),$$

with dual map (by transposing)

$$i^*(x, y, z) = x + y + z.$$

Choose $r = (0, 0, m) \in \mathbb{R}^3$ so that $i^*(r) = m$, then we can form the hyperkähler quotient as $M = \mu_{\mathbb{R}}^{-1}(m) \cap \mu_{\mathbb{C}}^{-1}(0)/K$, where $K = \exp(\mathfrak{k})$. The hyperkähler quotient now has a residual moment map for the torus $T^2 = T^3/K$, with moment maps

$$\bar{\mu} = \imath^* \circ \mu : M \longrightarrow (\mathfrak{t}^2)^* \oplus (\mathfrak{t}^2_{\mathbb{C}})^* = \ker(\imath^*) \oplus \ker(\imath^*_{\mathbb{C}}),$$

$$\bar{\mu}_{\mathbb{R}}(z,w) = \frac{1}{2}\sum_{i=1}^{3}\Big(|z_i|^2 - |w_i|^2 - r_i\Big)\partial_i, \qquad \bar{\mu}_{\mathbb{C}}(z,w) = \sum_{i=1}^{3}(z_i w_i)\partial_i.$$

Let

$$F_i = \{v \in \mathbb{R}^2 : v \cdot u_i + r_i \geq 0\},$$
$$G_i = \{v \in \mathbb{R}^2 : v \cdot u_i + r_i \leq 0\},$$
$$H_i = \{v \in \mathbb{R}^2 : v \cdot u_i + r_i - 0\},$$

which dissect $\mathbb{R}^2$ into convex polyhedra (some of which are non-compact):

(figure is below)

Now consider $A \subseteq \{1,2,3\}$ as an indexing set; since the $\mathbb{C}^*$-action

$$\tau \cdot [z,w] = [z, \tau w]$$

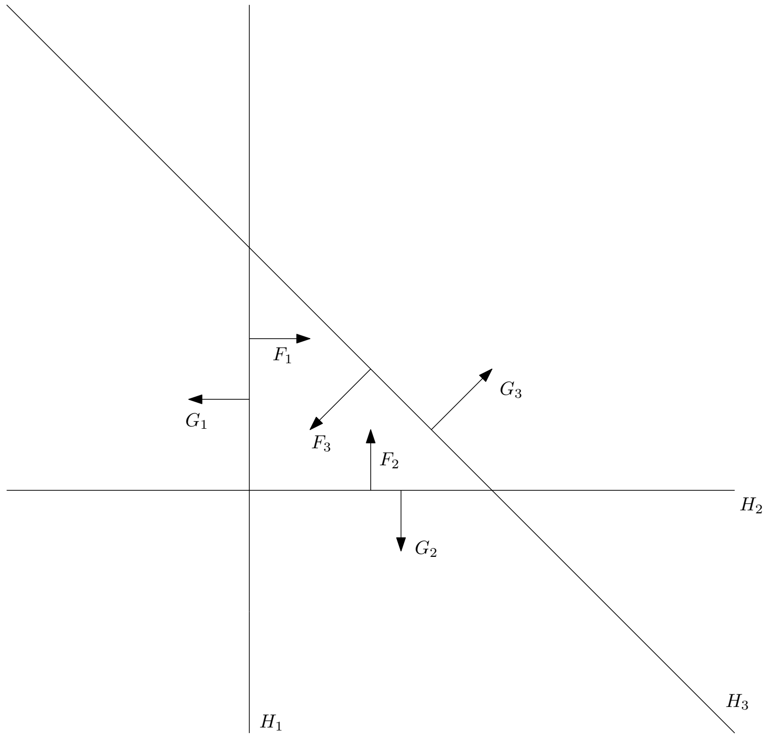
is $\mathbb{C}^*$-equivariant with respect to the moment map $\mu_{\mathbb{C}} : M \to \mathbb{C}^2$, it descends to the set $\mu_{\mathbb{C}}^{-1}(0)$ and hence the fixed point set $M^{\mathbb{C}^*}$ will be contained in $\mu_{\mathbb{C}}^{-1}(0)$. Define

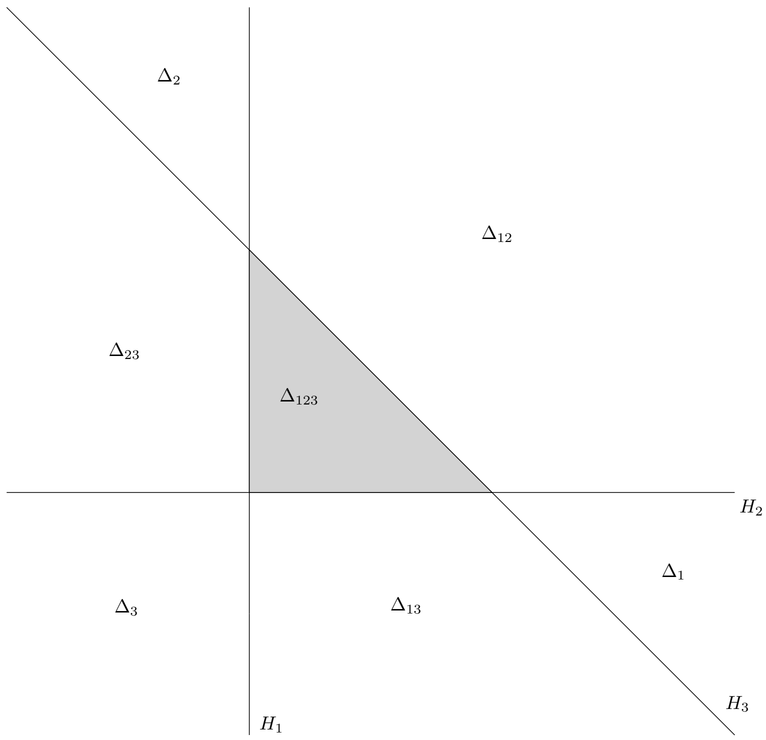
$$\mathcal{E}_A := \{[z,w] \in M : w_i = 0 \text{ for all } i \in A, \text{ and } z_i = 0 \text{ for all } i \notin A\}.$$

Then in setting

$$\Delta_A = \Big(\bigcap_{i \in A} F_i\Big) \cap \Big(\bigcap_{i \notin A} G_i\Big),$$

we have $\bar{\mu}_{\mathbb{R}}(\mathcal{E}_A) = \Delta_A$.





Now $\mathbb{C}^*$ does not act on M as a subtorus of $T^2_{\mathbb{C}}$, though it does when restricted to any single component of $\mathcal{E}_A$ of the extended core, with the action as a subtorus depending combinatorially on A.

For $[z, w] \in \mathcal{E}_A$ and $\tau \in \mathbb{C}^*$,

$$\tau \cdot [z, w] = [z, \tau w] = [\tau_1 z_1, \tau_2 z_2, \tau_3 z_3, \tau_1^{-1} w_1, \tau_2^{-1} w_2, \tau_3^{-1} w_3],$$

where $\tau_i = \tau^{-1}$ if $i \in A$ and $\tau_i = 1$ if $i \notin A$ (the subscripts are there emphasise that $\mathbb{C}^*$ is acting as a subtorus of $T^3_{\mathbb{C}}$). So the $\mathbb{C}^*$-action on each $\mathcal{E}_A$ is determined via the inclusion $\tau \mapsto (\tau_1, \tau_2, \tau_3)$. For example, when $A = \{1, 2\}$ the subtorus is $(\tau^{-1}, \tau^{-1}, 1) \subset T^3_{\mathbb{C}}$.

Now consider the moment map for the $\mathbb{C}^*$-action on M,

$$\Phi : M \longrightarrow \mathbb{R}_{\geq 0}$$
$$\Phi[z, w] = \frac{1}{2}\big(|w_1|^2 + |w_2|^2 + |w_3|^2\big).$$

We will use it to symplectically cut the toric hyperkähler manifold M in order to compactify it as follows: consider the direct sum $M \times \mathbb{C}$, where now S^1 acts on $M \times \mathbb{C}$ as

$$e^{i\theta} \cdot \big([z, w], \xi\big) = \big([z, e^{i\theta}], e^{i\theta}\xi\big),$$

which is Hamiltonian with moment map

$$\mu_{\text{cut}} : M \times \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0},$$
$$\mu_{\text{cut}}\big([z, w], \xi\big) = \Phi[z, w] + \tfrac{1}{2}|\xi|^2.$$

Observe that $\mu_{\text{cut}}^{-1}(0) = \{\big([z, 0], 0\big) \in M \times \mathbb{C}\} = X$, the toric Kähler manifold obtained from the Delzant polytope cut out by the hyperplane configuration, which in this example is $\mathbb{CP}^2$.

Suppose now that $\epsilon \in \mathbb{R}_{\geq 0}$, then

$$\begin{aligned}
\mu_{\text{cut}}^{-1}(\epsilon) &= \big\{([z, w], \xi) \in M \times \mathbb{C} : \Phi[z, w] \leq \epsilon\big\} \\
&= \big\{([z, w], 0) \in M \times \{0\} : \Phi[z, w] = \epsilon\big\} \bigsqcup \big\{([z, w], \xi) \in M \times \mathbb{C} : \Phi[z, w] < \epsilon\big\} \\
&= \Phi^{-1}(\epsilon) \bigsqcup \big\{([z, w], e^{i\phi}) \in M \times S^1 : \xi = e^{i\phi}\sqrt{2\epsilon - \Phi[z, w]}\big\} \\
&= \Sigma_1 \sqcup \Sigma_2.
\end{aligned}$$

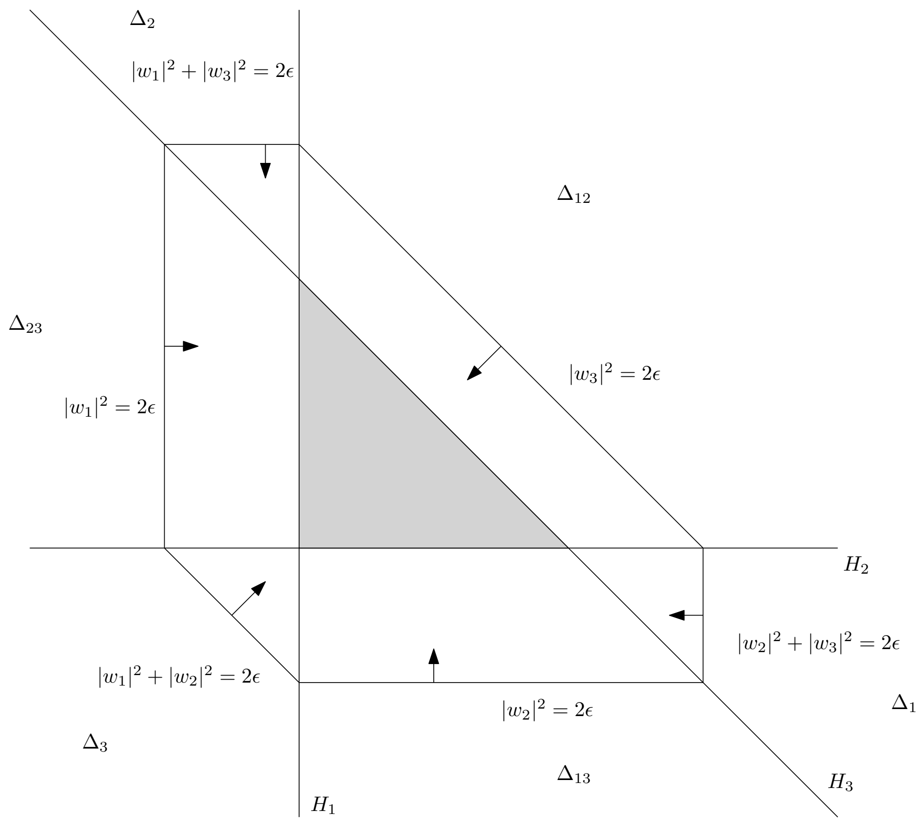
Now since there exists a unique S^1-orbit through a $\xi \in \mathbb{C}$ such that ξ is actually real, with modulus $|\xi| = \sqrt{2\epsilon - \Phi[z, w]}$, it follows that Σ_2/S^1 can be identified with $\big\{[z, w] \in M : \Phi[z, w] < \epsilon\big\}$. Moreover Σ_1/S^1 is just the symplectic reduction $\Phi^{-1}(\epsilon)/S^1$. Now these cut spaces still have the residual S^1-action from $\mathbb{C}^*$ since it can be thought of as acting on $M \times \mathbb{C}$ in the old sense on M (rotating the fibres), and trivially on $\mathbb{C}^*$, and this action commutes with the cutting action; in particular it remains Hamiltonian. Also note that the points of $M_{\mathrm{cut}} = M_{\epsilon>\Phi} \sqcup \Phi^{-1}(\epsilon)/S^1$ which have a non-trivial stabiliser for the residual S^1-action occur when

$$\big([z, e^{i\theta}w], \xi\big) = \big([z, e^{i\phi}w], e^{i\phi}\xi\big)$$

for some $e^{i\theta} \in S^1 \setminus \{1\}$ and $e^{i\phi} \in S^1$, and that these new fixed points lie precisely in the level-set $\{([z, w], 0) \in M \times \mathbb{C}\} = \Phi^{-1}(\epsilon)$.

On each component $\mathcal{E}_A$ of the extended core, the cut introduces a half-space with normal determined by the Lie algebra generator for the S^1-action, as well as how S^1 acts on $\mathcal{E}_A$ as a subgroup of the torus T^3, e.g. on $\mathcal{E}_{12}$ where S^1 acts through the inclusion as $\tau \mapsto (\tau^{-1}, \tau^{-1}, 1)$, the normal vector is $-1 \cdot u_1 + -1 \cdot u_2 + 0 \cdot u_3 = (-1, -1)$, hence the polyhedron Δ_{12} is cut via the intersection

$$\Delta_{12} \cap \{v \in \mathbb{R}^2 : v \cdot (-1, -1) \geq \epsilon\}.$$



Let us look at how the cutting procedure affects each component $\mathcal{E}_A$ of the extended core individually.

$$\mathcal{E}_A \cap (\mu_{\text{cut}}^{-1}(\epsilon)/S^1) = \mathcal{E}_A \cap \Big((\Sigma_1 \sqcup \Sigma_2)/S^1\Big) \cong (\mathcal{E}_A \cap \Sigma_2) \sqcup \Big(\mathcal{E}_A \cap (\Phi^{-1}(\epsilon)/S^1)\Big),$$

where

$$\mathcal{E}_A \cap (\Sigma_2/S^1) = \big\{[z,w] \in \mathcal{E}_A : \Phi[z,w] < \epsilon\big\} = \Big\{[z,w] \in \mathcal{E}_A : \tfrac{1}{2}\sum_{i \notin A} |w_i|^2 < \epsilon\Big\}$$

and

$$\begin{aligned}\mathcal{E}_A \cap (\Phi^{-1}(\epsilon)/S^1) &= \Big\{[z,w] \in \mathcal{E}_A : \tfrac{1}{2}\sum_{i \notin A} |w_i|^2 = \epsilon\Big\}/S^1 \\ &= \Big\{[z,w] \in \mathcal{E}_A : \tfrac{1}{2}\sum_{i \notin A} |w_i|^2 = \epsilon, \text{ and } w_i \sim e^{i\theta} w_i \text{ for each } i \notin A\Big\} =: H_A\end{aligned}$$

We shall want to investigate the nature of the half-spaces H_A that were introduced in the cutting procedure. First of all, recall that

$$\mathcal{E}_A = \Big\{[z_1, \ldots, z_n, w_1, \ldots, w_n] \in M : w_i = 0 \text{ if } i \in A \text{ and } z_i = 0 \text{ if } i \notin A\Big\},$$

and also that the residual S^1-action on M is given by

$$t \cdot [z,w] = [z_1, \ldots, z_n, tw_1, \ldots, tw_n].$$

Hence in introducing the symplectic cut, the quotient $\Phi^{-1}(\epsilon)/S^1$ introduces additional stabiliser subgroups N_A for each $\tilde{H}_A$, that act on $\tilde{H}_A$ through the inclusion into the original torus T^n.

Example: again when $A = \{1,2\}$, then

$$\mathcal{E}_{12} = \{[z,w] \in M : w_1 = 0 = w_2, \text{ and } z_3 = 0\} \cong \{[z_1, z_2, 0; 0, 0, w_3] \in M\} = F_1 \cap F_2 \cap G_3.$$

Then the stabiliser subgroup $N_{12} \cong S^1 \subset T^3$ for the residual S^1-action acting on the wall $\tilde{H}_{12}$ is $N_{12} \hookrightarrow T^3$, $t \mapsto (1,1,t)$, since $tw_3 \sim w_3$ as a coordinate of $\mathcal{E}_{12} \cap (\Phi^{-1}(\epsilon)/S^1)$. Observe also that this action is equivalent to that of $t \cdot [z_1, z_2, w_3] = [t^{-1}z_1, t^{-1}z_2, w_3]$, when acting on $\mathcal{E}_{12}$.

After this discussion, set $N_A := (t_i, t_i, \ldots, t_i) \subset T^n$, where $t_i = t^{-1}$ if $i \in A$ and $t_i = 1$ if $i \notin A$; or equivalently $N_A = (t_j^{-1}, \ldots, t_j^{-1})$ where $t_j = t^{-1}$ if $j \notin A$, and $t_j = 1$ if $j \in A$. Now set $\mathfrak{n}_A := \mathrm{Lie}(N_A)$, then $\mathfrak{n}_A$ is generated by the vectors $\sum_{i \notin A} u_i = -\sum_{i \in A} u_i$.

Example: $A = \{1, 2\}$, then $N_A = (1, 1, t)$ and $\mathfrak{n}_A = (0, 0, t)$, and also $\sum_{i \notin A} u_i = u_3 = (-1, -1)$. Thus

$$\tilde{H}_A = \{x \in \mathbb{R}^2 : \langle x, u_3 \rangle + \epsilon = 0\} = \{x \in \mathbb{R}^2 : x + y = \epsilon\}.$$

Since N_A acts as a subtorus of T^3, this gives rise to a convenient way to evaluate the image of the moment map $\Phi|_{\mathcal{E}_A} : \mathcal{E}_A \to \mathbb{R}$, namely

$$\Phi|_{\mathcal{E}_A}[z, w] = \left\langle \mu_{\mathbb{R}}(z, w), -\sum_{i \in A} u_i \right\rangle$$

The fixed point set of the residual S^1-action occurs when $\Phi|_{\mathcal{E}_A}$ is constant so that $d\Phi|_{\mathcal{E}_A} = 0$. Clearly this happens precisely on the locus $\Phi|_{\mathcal{E}_A}^{-1}(\epsilon)$ (we ignore the case when $\epsilon = 0$), and the T^d-stabiliser subgroup is generated by the Lie algebra spanned by the vectors u_k that determine the H_i and also $\tilde{H}_A$.

Bibliography