TCP2_LatticePoints_WeightMultiplicities

March 9, 2021

[1]:
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from sympy import *
from sympy.vector import Vector
from sympy.vector import CoordSys3D
import IPython.display as disp

N = CoordSys3D('N')

t, k, a = symbols( 't k a' )
# init_printing(use_unicode=True)
init_printing(use_latex='mathjax')

# Define the vector which is not parallel to any edge vector, which will tend␣
↪to zero:

Phi = t*(N.i + 2*N.j)

# Set the fixed points of the action; P denotes those that belong
# to the core, and Q those that come from the cut extended core:

def P12(k,a):
    return Vector.zero

def P23(k,a):
    return k*N.i

def P13(k,a):
    return k*N.j

def Q12_1(k,a):
    return -a*N.j

def Q12_2(k,a):
    return -a*N.i

def Q23_2(k,a):
    return (k+a)*N.i
```

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def Q23_3(k,a):
    return (k+a)*N.i - a*N.j

def Q13_1(k,a):
    return (k+a)*N.j

def Q13_3(k,a):
    return -a*N.i + (k+a)*N.j

# Basis for the edge/weight vectors for the points

v1 = N.i

v2 = N.j

# Define the term which is summed over each fixed point,
# representing the character for the representation

def f(P, edge1, edge2):
    return exp( Phi.dot(P) ) / ( (1 - exp( Phi.dot(edge1) ) ) * ( 1 - exp( Phi.
 ↪dot(edge2) ) ) )

def g(P, edge1, edge2, edge3, edge4):
    return exp( Phi.dot(P) ) / ( (1 - exp( Phi.dot(edge1) ) ) * ( 1 - exp( Phi.
 ↪dot(edge2) ) ) * ( 1 - exp( Phi.dot(edge3) ) ) * ( 1 - exp( Phi.dot(edge4) )␣
 ↪) )

# For each of the right-angled triangles:

def Delta1(k,a):
    return f(P23(k,a), -v1, -v1 + v2 ) * ( f(0*v1 + 0*v2, v1, v1 - v2) +␣
 ↪f(a*v1, -v1, -v2) + f(a*v1 - a*v2, v2, -v1 + v2) )

def Delta2(k,a):
    return f(P13(k,a), v1 - v2, -v2 ) * ( f(0*v1 + 0*v2, v2, -v1 + v2) +␣
 ↪f(a*v2, -v1, -v2) + f(-a*v1 + a*v2, v1, v1 - v2) )

def Delta3(k,a):
    return f(P12(k,a), v1 , v2 ) * ( f(0*v1 + 0*v2, -v1, -v2) + f(-a*v1, v1, v1␣
 ↪- v2) + f(-a*v2, v2, -v1 + v2) )

def Sum(k,a):
    return Delta1(k,a) + Delta2(k,a) + Delta3(k,a)
```

[2]:

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# Take the limit as t -> 0 to get the Euler characteristic:

def EulerCalculation(k,a):
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    return limit(Sum(k,a),t,0)

# Euler characteristic from Index Theorem:
def Euler(k,a):
    return simplify( Rational(1,4) * ( (a+1)*(a+2)*(k+a+1)*(k+a+2) ) )

# Lattice point count:
def LatticePoints(k,a):
    return simplify(Rational(1,2)*(k+1)*(k+2) + 3*a*(k+a+1))

# Different of the a-level and the (a-1)-level of the cut for the Euler
 ↪characteristic:
def CutDiff(k,a):
    return factor(Euler(k,a) - Euler(k,a-1))
```

[3]:
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disp.display(factor(EulerCalculation(k,a))) # Letting t -> 0  to obtain the
 ↪Euler characteristic (n.b. degree-0 term in a is equal to lattice point
 ↪count for CP2's polytope)
disp.display(LatticePoints(k,a))
disp.display(CutDiff(k,a))
```

$$\frac{(a+1)(a+2)(a+k+1)(a+k+2)}{4}$$

$$3a(a+k+1)+\frac{(k+1)(k+2)}{2}$$

$$\frac{(a+1)(a+k+1)(2a+k+2)}{2}$$

0.1 Summarising:

$$\chi(M_{\leq a}, \mathcal{L}^k_{\leq a}) = \frac{(a+1)(a+2)}{2} \cdot \frac{(k+a+1)(k+a+2)}{2},$$
$$\#\{\mathbb{Z}^2 \cap \mathcal{P}\} = \frac{(k+1)(k+2)}{2} + 3a(k+a+1).$$

0.2 Conjecture:

$$H^0(M_{\leq a}; \mathcal{L}^k_{\leq a}) \cong \bigoplus_{m \leq a} H^0(M; \mathcal{L}^k)_m$$

$$\implies H^0(M; \mathcal{L}^k)_m \cong \frac{H^0(M_{\leq a}; \mathcal{L}^k_{\leq a})}{H^0(M_{\leq (a-1)}; \mathcal{L}^k_{\leq (a-1)})},$$

$$h^0(M; \mathcal{L}^k)_m = h^0(M_{\leq a}; \mathcal{L}^k_{\leq a}) - h^0(M_{\leq (a-1)}; \mathcal{L}^k_{\leq (a-1)}).$$

0.3 An Idea:

Since

$$\chi\left(M_{\leq a}, \mathcal{L}^k_{\leq a}\right) = \frac{(a+1)(a+2)}{2} \cdot \frac{(k+a+1)(k+a+2)}{2}$$

$$= \chi\left(\mathbb{CP}^2; \mathcal{O}(a)\right) \cdot \chi\left(\mathbb{CP}^2; \mathcal{O}(k+a)\right),$$

does

$$H^0(M_{\leq a}; \mathcal{L}^k_{\leq a}) \cong \bigoplus_{m \leq a} \{\text{"}k\text{-weight monomials for } K\text{"}\} \otimes \{\text{"}m\text{-weight monomials for } S^1\text{"}\}?$$

Or more importantly,

$$H^0(M; \mathcal{L}^k)_a \cong \{\text{"}k\text{-weight monomials for } K\text{"}\} \otimes \{\text{"}a\text{-weight monomials for } S^1\text{"}\}?$$

For example,

0.3.1 a = 0:

$$\{z_1^l z_2^m z_3^n : l+m+n = k\}, \qquad \text{for } \Delta_{\{1,2,3\}} = F_1 \cap F_2 \cap F_3.$$

0.3.2 a = 1:

$$\{z_1^l z_2^m w_3 : l+m-1 = k\}, \qquad \text{for } \Delta_{\{1,2\}} = F_1 \cap F_2 \cap G_3.$$

$$\{z_1^l w_2 z_3^n : l-1+n = k\}, \qquad \text{for } \Delta_{\{1,3\}} = F_1 \cap G_2 \cap F_3.$$

$$\{w_1^1 z_2^m z_3^n : -1+m+n = k\}, \qquad \text{for } \Delta_{\{2,3\}} = G_1 \cap F_2 \cap F_3.$$

0.3.3 a = 2:

$$\{z_1^l z_2^m w_3^2 : l+m-2=k\}, \qquad \text{for } \Delta_{\{1,2\}} = F_1 \cap F_2 \cap G_3.$$

$$\{z_1^l w_2^2 z_3^n : l-2+n=k\}, \qquad \text{for } \Delta_{\{1,3\}} = F_1 \cap G_2 \cap F_3.$$

$$\{w_1^2 z_2^m z_3^n : -2+m+n=k\}, \qquad \text{for } \Delta_{\{2,3\}} = G_1 \cap F_2 \cap F_3.$$

$$\{z_1^l w_2^m w_3^n : m+n=2; \quad l=k+2\}, \qquad \text{for } \Delta_{\{1\}} = F_1 \cap G_2 \cap G_3.$$

$$\{w_1^l z_2^m w_3^n : l+n=2; \quad m=k+2\}, \qquad \text{for } \Delta_{\{2\}} = G_1 \cap F_2 \cap G_3.$$

$$\{w_1^l w_2^m z_3^n : l+m=2; \quad n=k+2\}, \qquad \text{for } \Delta_{\{3\}} = G_1 \cap G_2 \cap F_3.$$

0.4 General a?

$$\{z_1^l z_2^m w_3^a : l+m-a=k\},$$

$$\{z_1^l w_2^a z_3^n : l-a+n=k\},$$

$$\{w_1^a z_2^m z_3^n : -a+m+n=k\},$$

$$\{z_1^l w_2^m w_3^n : m+n=a; \quad l=k+a\},$$

$$\{w_1^l z_2^m w_3^n : l+n=a; \quad m=k+a\},$$

$$\{w_1^l w_2^m z_3^n : l+m=a; \quad n=k+a\}.$$

[4]:
```
for a in range(0, 10):
    disp.display(LatticePoints(k,a))
```

$$\frac{(k+1)(k+2)}{2}$$

$$\frac{k^2}{2} + \frac{9k}{2} + 7$$

$$\frac{k^2}{2} + \frac{15k}{2} + 19$$

$$\frac{k^2}{2} + \frac{21k}{2} + 37$$

$$\frac{k^2}{2} + \frac{27k}{2} + 61$$

$$\frac{k^2}{2} + \frac{33k}{2} + 91$$

$$\frac{k^2}{2} + \frac{39k}{2} + 127$$

$$\frac{k^2}{2} + \frac{45k}{2} + 169$$

$$\frac{k^2}{2} + \frac{51k}{2} + 217$$

$$\frac{k^2}{2} + \frac{57k}{2} + 271$$

[5]:

```
for a in range(0, 10):
    disp.display(simplify(CutDiff(k,a)))
```

$$\frac{(k+1)(k+2)}{2}$$

$$(k+2)(k+4)$$

$$\frac{3(k+3)(k+6)}{2}$$

$$2(k+4)(k+8)$$

$$\frac{5(k+5)(k+10)}{2}$$

$$3(k+6)(k+12)$$

$$\frac{7(k+7)(k+14)}{2}$$

$$4(k+8)(k+16)$$

$$\frac{9(k+9)(k+18)}{2}$$

$$5(k+10)(k+20)$$

[23]:

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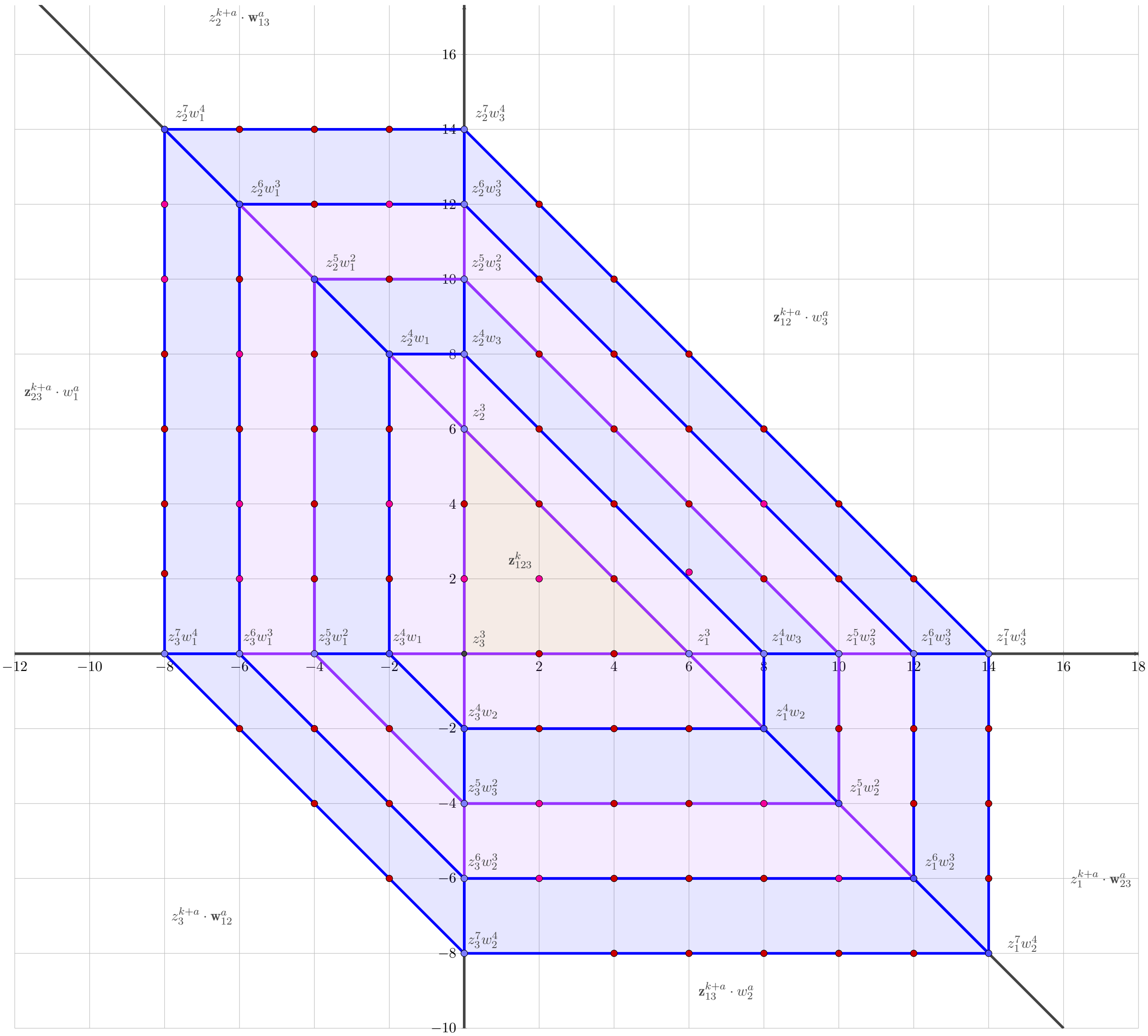
0.4.1 Cartoon

Notation:

$$\mathbf{z}_{123}^{k} := z_1^{k_1} z_2^{k_2} z_3^{k_3}, \qquad k_1 + k_2 + k_3 = k,$$
$$\mathbf{z}_{ij}^{k+a} := z_i^{k_i} z_j^{k_j}, \qquad k_i + k_j = k + a,$$
$$\mathbf{w}_{ij}^{k+a} := w_i^{m_i} w_j^{m_j}, \qquad m_i + m_j = k + a.$$



0.4.2 Calculations

From the Jupyter notebook, we have:

(k,a)	Cut Difference	$\#(\mathbb{Z}^2 \cap \mathcal{P})$	$\chi(M_{\leq a}; \mathcal{L}^k_{\leq a})$
$(k,0)$	$\frac{(k+1)(k+2)}{2}$	$\frac{(k+1)(k+2)}{2}$	
$(k,1)$	$(k+2)(k+4)$	$\frac{k^2}{2}+\frac{9k}{2}+7$	
$(k,2)$	$\frac{3(k+3)(k+6)}{2}$	$\frac{k^2}{2}+\frac{15k}{2}+19$	
$(k,3)$	$2(k+4)(k+8)$	$\frac{k^2}{2}+\frac{21k}{2}+37$	
$(k,4)$	$\frac{5(k+5)(k+10)}{2}$	$\frac{k^2}{2}+\frac{27k}{2}+61$	
$(k,5)$	$3(k+6)(k+12)$	$\frac{k^2}{2}+\frac{33k}{2}+91$	
$(k,6)$	$\frac{7(k+7)(k+14)}{2}$	$\frac{k^2}{2}+\frac{39k}{2}+127$	
$(k,7)$	$4(k+8)(k+16)$	$\frac{k^2}{2}+\frac{45k}{2}+169$	
$(k,8)$	$\frac{9(k+9)(k+18)}{2}$	$\frac{k^2}{2}+\frac{51k}{2}+217$	
$(k,9)$	$5(k+10)(k+20)$	$\frac{k^2}{2}+\frac{57k}{2}+271$	
$(k,10)$			