Geometric Quantisation of Hypertoric Manifolds by Symplectic Cutting

General Notes

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Lorem ipsum.

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Lorem ipsum.

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3.1 Introduction and Definitions

A *hyperkähler manifold* is a Riemannian manifold (M, g) equipped with three orthogonal, parallel complex structures J_1, J_2, J_3, satisfying the usual quaternion relations. These three complex structures give rise to three symplectic forms

$$\omega_1(v,w) = g(J_1 v, w), \quad \omega_2(v,w) = (J_2 v, w), \quad \omega_3(v,w) = g(J_3 v, w),$$

so that each (g, J_i, w_i) is in its own right a Kähler structure on M for $i = 1, 2, 3$. The complex-valued two-form $\omega_2 + \sqrt{-1}\omega_3$ is a closed, non-degenerate, and holomorphic two-form with respect to the complex structure J_1. Thus any hyperkähler manifold can be considered as a *holomorphic symplectic* manifold with complex structure J_1, real symplectic form $\omega_{\mathbb{R}} := \omega_1$, and holomorphic symplectic form $\omega_{\mathbb{C}} := \omega_2 + \sqrt{-1}\omega_3$.

An action of a Lie group G on a hyperkähler manifold M is called *hyperhamiltonian* if it is hamiltonian with respect to $\omega_{\mathbb{R}}$, and holomorphic hamiltonian with respect to $\omega_{\mathbb{C}}$, with a G-equivariant moment map

$$\mu_{HK} := \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} \longrightarrow \mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}}.$$

The following theorem describes the *hyperkähler quotient* construction, which is the quaternionic analogue of a Kähler quotient:

Theorem 3.1 ([1]). *Let M be a hyperkähler manifold equipped with a hyperhamiltonian action of a compact Lie group G, with moment maps μ_1, μ_2, μ_3. Suppose that $\xi = \xi_{\mathbb{R}} \oplus \xi_{\mathbb{C}}$ is a central regular value for μ_{HK}, and that G acts freely on $\mu_{HK}^{-1}(\xi)/G$. Then there is a unique hyperkähler structure on the hyperkähler quotient $\mathfrak{M} = M /\!/\!/\!/_{\xi}\, G := \mu_{HK}^{-1}(\xi)/G$, with associated symplectic and holomorphic symplectic forms $\omega_{\mathbb{R}}^{\xi}$ and $\omega_{\mathbb{C}}^{\xi}$, such that $\omega_{\mathbb{R}}^{\xi}$ and $\omega_{\mathbb{C}}^{\xi}$ pull-back to the restrictions of $\omega_{\mathbb{R}}$ and $\omega_{\mathbb{C}}$ on $\mu_{HK}^{-1}(\xi)$.*

In general, the action of G on $\mu_{HK}^{-1}(\xi)$ will not be free, but only locally free. In this situation, we would end up with a *hyperkähler orbifold*. However in the sequel, we shall only concern ourselves when the action is free, and that $\mathfrak{M}$ is smooth, *i.e.* a manifold.

Let us specialise to the case when $M = T^*\mathbb{C}^n$, and let G act on $T^*\mathbb{C}^n$ with the induced action from a linear action of G on $\mathbb{C}^n$, with moment map $\mu : \mathbb{C}^n \to \mathfrak{g}^*$. We can identify $\mathbb{H}^n$ with $T^*\mathbb{C}^n$ such that the complex structure J_1 on $\mathbb{H}^n$ is given by right multiplication by i, and that J_1 corresponds to the natural complex structure on $T^*\mathbb{C}^n$. With this identification in mind, $T^*\mathbb{C}^n$ inherits a hyperkähler structure. The real symplectic form $\omega_\mathbb{R}$ is obtained from the sum of the pull-backs of the standard Kähler forms on $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$, and the holomorphic symplectic form $\omega_\mathbb{C}$ is $\omega_\mathbb{C} = d\eta$, where η is the canonical holomorphic one-form on $T^*\mathbb{C}^n$.

As G acts $\mathbb{H}^n$-linearly on $T^*\mathbb{C}^n \cong \mathbb{H}^n$ from the left, the action is hyperhamiltonian with moment map $\mu_{HK} = \mu_\mathbb{R} \oplus \mu_\mathbb{C}$, where

$$\mu_\mathbb{R}(z,w) = \mu(z) - \mu(w), \qquad \text{and} \qquad \mu_\mathbb{C}(z,w)(\hat{v}_z),$$

where $w \in T^*_z\mathbb{C}^n$, $v \in \mathfrak{g}_\mathbb{C}$, and $\hat{v}_z$ is the vector field in $T_z\mathbb{C}^n$ induced by v. For a central element $\alpha \in \mathfrak{g}^*$, we call the specialised hyperkähler quotient

$$\mathfrak{M} = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/_{(\alpha,0)}\, G := \big(\mu_\mathbb{R}^{-1}(\alpha) \cap \mu_\mathbb{C}^{-1}(0)\big)/G$$

the hyperkähler analogue of the corresponding Kähler quotient,

$$\mathfrak{X} = \mathbb{C}^n /\!\!/_\alpha\, G = \mu^{-1}(\alpha)/G.$$

We quote the following propositions without proof:

Proposition 3.2. *Suppose that α and $(\alpha, 0)$ are regular values for μ and μ_{HK}, respectively. Then the cotangent bundle $T^*\mathfrak{X}$ is isomorphic to an open subset of $\mathfrak{M}$, and is dense if it is non-empty.*

3.2 The $\mathbb{C}^*$-Action and the Core of a Hyperkähler Analogue

Consider the action of $\mathbb{C}^*$ on $T^*\mathbb{C}^n$ given by

$$\hbar \cdot (z,w) = (z, \hbar w),$$

i.e. by scalar multiplication of the cotangent fibre. The holomorphic moment map $\mu_\mathbb{C} : T^*\mathbb{C}^n \to \mathfrak{g}^*_\mathbb{C}$ is $\mathbb{C}^*$-equivariant with respect to the scalar action on $\mathfrak{g}^*_\mathbb{C}$, and hence the $\mathbb{C}^*$-action descends to $\mu_\mathbb{C}^{-1}(0)$. Further, this $\mathbb{C}^*$-action commutes with the linear action of G on $\mathbb{C}^n$, and consequently the action of $\mathbb{C}^*$ is J_1-holomorphic on $\mathfrak{M} = \big(\mu_\mathbb{R}^{-1}(\alpha) \cap \mu_\mathbb{C}^{-1}(0)\big)/G$. However, the $\mathbb{C}^*$-action *does not* preserve the holomorphic symplectic form nor the hyperkähler structure on $\mathfrak{M}$; rather it scales $\mu_\mathbb{C}$ with "homogeneity one", *i.e.* $\hbar^*\omega_\mathbb{C} = \hbar\omega_\mathbb{C}$ for any $\hbar \in \mathbb{C}^*$.

Given that $\mathfrak{M}$ is smooth, the action of the compact subgroup $S^1 \subset \mathbb{C}^*$ is hamiltonian with respect to the real symplectic two-form $\omega_\mathbb{R}$, with corresponding moment map $\Phi[z,w] = \frac{1}{2}\|w\|^2$. This map is a perfect Morse-Bott function, and its image is contained in $\mathbb{R}_{\geq 0}$. Further, we note that $\Phi^{-1}(0) = \mathfrak{X} \subset \mathfrak{M}$. The following proposition will be instrumental in the sequel, though again we quote it without proof:

Proposition 3.3. *If the original moment map for the G-action on $\mathbb{C}^n$, $\mu : \mathbb{C}^n \to \mathfrak{g}^*$, if proper, then so is the moment map for the S^1 action, $\Phi : \mathfrak{M} \to \mathbb{R}_{\geq 0}$.*

Next we shall define what is known as the *core* of a hyperkähler analogue, which will be essential in describing the fixed points of the $\mathbb{C}^*$-action of $\mathfrak{M}$.

Definition 3.4. *Suppose that $\mathfrak{M}$ is smooth and Φ is proper. The* core *$\mathcal{L} \subset \mathfrak{M}$ of the hypertoric variety is defined to be the union of the $\mathbb{C}^*$ orbits whose closures are compact.*

Let F be a connected component of $\mathfrak{M}^{S^1} = \mathfrak{M}^{\mathbb{C}^*}$, and let U_F be the closure of the set of points $p \in \mathfrak{M}$ such that $\lim\limits_{\hbar \to \infty} \hbar \cdot p \in F$.

Proposition 3.5 ([?]; Proposition 2.8)**.** *The core $\mathcal{L} \subset \mathfrak{M}$ has the following properties:*

1. *$\mathcal{L}$ is an S^1-equivariant deformation retract of M;*
2. *U_F is isotropic with respect to the holomorphic symplectic form $\omega_\mathbb{C}$;*
3. *Provided that $\mathfrak{M}$ is smooth at F, then $\dim U_F = \frac{1}{2}\dim \mathfrak{M}$.*

4 Hypertoric Manifolds

4.1 Definition

In this section, we shall specialise further now to when a hyperkähler analogue $\mathfrak{M}$ is the analogue to a toric symplectic manifold $\mathfrak{X} = \mu^{-1}(\alpha)/N$, *i.e.* we replace the compact Lie group G with the torus $N = \ker(\pi : T^n \to T^d)$, using the same notation as in the second chapter.

Recall the short exact sequence of tori:

$$1 \longrightarrow N \xhookrightarrow{i} T^n \xrightarrow{\pi} T^d \longrightarrow 1,$$

and extend the linear action of the torus N on $\mathbb{C}^n$ to $T^*\mathbb{C}^n$. This action is trihamiltonian and we obtain the following hyperkähler moment map

$$\mu_{HK} = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : T^*\mathbb{C}^n \longrightarrow \mathfrak{n}^* \oplus \mathfrak{n}^*_{\mathbb{C}},$$

where

$$\mu_{\mathbb{R}}(z,w) = i^*\bigg(\frac{1}{2}\sum_{i=1}^n (|z_i|^2 - |w_i|^2)\partial_i\bigg), \quad \text{and} \quad \mu_{\mathbb{C}}(z,w) = i^*_{\mathbb{C}}\bigg(\sum_{i=1}^n (z_i w_i)\partial_i\bigg).$$

Given an element $\alpha \in \mathfrak{n}^*$ with a corresponding lift $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^n)^*$, the Kähler quotient

$$\mathfrak{X} = \mathbb{C}^n /\!\!/_{\alpha} N = \mu^{-1}(\alpha)/N$$

is our usual toric symplectic manifold with residual T^d-action from before, and moreover its hyperkähler analogue

$$\mathfrak{M} = T^*\mathbb{C}^n /\!\!/\!\!/\!\!/_{(\alpha,0)} N = \big(\mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)\big)/N$$

is what we shall call a *hypertoric manifold*[1]. The hypertoric manifold $\mathfrak{M}$ also admits a residual action of the torus T^d, which is hyperhamiltonian with hyperkähler moment map

$$\phi_{HK} := \phi_{\mathbb{R}} \oplus \phi_{\mathbb{C}} : \mathfrak{M} \longrightarrow (\mathbb{R}^d)^* \oplus (\mathbb{C}^d)^*,$$

where

$$\phi_{\mathbb{R}}[z,w] = \frac{1}{2}\sum_{i=1}^n (|z_i|^2 - |w_i|^2 - \lambda_i)\partial_i \in \ker(i^*) = (\mathbb{R}^d)^*,$$

$$\phi_{\mathbb{C}}[z,w] = \sum_{i=1}^n (z_i w_i)\partial_i \in \ker(i^*_{\mathbb{C}}) = (\mathbb{C}^d)^*.$$

4.2 Hyperplane Arrangements

A fundamental difference between the toric manifold $\mathfrak{X}$ and the hypertoric manifold $\mathfrak{M}$ is that the hyperkähler moment map for $\mathfrak{M}$ is surjective, and that $\mathfrak{M}$ is non-compact. Despite this, we can still describe the image of the real moment map $\phi_{\mathbb{R}} : \mathfrak{M} \to (\mathbb{R}^d)^*$ combinatorially by means of a *hyperplane arrangement*. To describe this arrangement, recall that the map $\pi : \mathbb{R}^n \to \mathbb{R}^d$ was defined by $\pi(e_i) = u_i$, for $i = 1, \ldots, n$, where the u_i were the primitive, integral, inward-pointing normal vectors to the hyperplanes that determined our Delzant polytope. In the hypertoric case, they instead now describe a collection of *affine hyperplanes* $H_i \subset (\mathbb{R}^d)^*$ as follows: consider

$$H_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i = 0\},$$

so that the $u_i \in \mathbb{Z}^d$ is the normal vector to the hyperplane H_i. The hyperplane H_i divides $(\mathbb{R}^d)^*$ into two half-spaces

$$F_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \geq 0\},$$
$$G_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \leq 0\}.$$

Let

$$\Delta = \bigcap_{i=1}^n F_i = \{v \in (\mathbb{R}^d)^* : v \cdot u_i + \lambda_i \geq 0, \text{ for all } i = 1, \ldots, n\}$$

be the (possibly empty) polyhedron in $(\mathbb{R}^d)^*$ defined by the affine hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$. We note that choosing a different lift λ' of α corresponds combinatorially to translating the arrangement $\mathcal{A}$ inside of $(\mathbb{R}^d)^*$, and geometrically to shifting the Kähler and hyperkähler moment maps for the residual T^d-action by $\lambda' - \lambda \in \ker(i^*) = (\mathbb{R}^d)^*$.

We shall call that the arrangement $\mathcal{A}$ *simple* if every subset of m hyperplanes with non-empty intersection intersects with codimension m, and call $\mathcal{A}$ *smooth* if every collection of d linearly-independent vector $\{u_{i_1}, \ldots, u_{i_d}\}$ spans $(\mathbb{R}^d)^*$. The reason for this terminology is the following proposition.

Proposition 4.1. *The hypertoric variety $\mathfrak{M}$ is an orbifold if and only if $\mathcal{A}$ is simple, and $\mathfrak{M}$ is smooth if and only if $\mathcal{A}$ is smooth.*

As we wish to restrict our attention to the case where $\mathfrak{M}$ is a manifold, we shall assume in the sequel that $\mathcal{A}$ is a smooth arrangement of hyperplanes.

[1] More generally, $\mathfrak{M}$ should be called a hypertoric variety, and only call $\mathfrak{M}$ a manifold when it is smooth. However, we shall restrict our attention to the smooth case for simplicity.

4.3 The Core of a Hypertoric Manifold

The holomorphic moment map $\phi_{\mathbb{C}} : \mathfrak{M} \to (\mathbb{C}^d)^*$ is $\mathbb{C}^*$-equivariant with respect to the scalar action of $\mathbb{C}^*$ on $(\mathbb{C}^d)^*$, hence both the core $\mathcal{L}$ and the fixed-point set $M^{\mathbb{C}^*}$ will be contained in

$$\mathcal{E} := \phi_{\mathbb{C}}^{-1}(0) = \Big\{[z,w] \in \mathfrak{M} : z_i w_i = 0,\ 1 \le i \le n\Big\}.$$

Definition 4.2. *We shall call $\mathcal{E}$ the* extended core *of $\mathfrak{M}$.*

The restriction of $\phi_{\mathbb{R}}|_{\mathcal{E}} : \mathcal{E} \to (\mathbb{R}^d)^*$ is surjective from the defining equations, and further the extended core naturally breaks up into components

$$\mathcal{E}_A := \Big\{[z,w] \in \mathcal{E} : w_i = 0 \text{ for all } i \in A \text{ and } z_i = 0 \text{ for all } i \notin A\Big\},$$

where $A \subseteq \{1,\dots,n\}$ is an indexing set. The hyperplanes $\{H_i\}_{i=1}^n$ divide $(\mathbb{R}^d)^*$ into a union of convex polyhedra

$$\Delta_A = \Big(\bigcap_{i\in A} F_i\Big) \cap \Big(\bigcap_{i\notin A} G_i.\Big),$$

some of which may be empty. Note that $\mathcal{E}_\emptyset = \mathfrak{X}$ and that in general, each variety $\mathcal{E}_A$ is a d-dimensional Kähler subvariety of $\mathfrak{M}$ with an effective Hamiltonian T^d-action, so is itself a toric variety.

Lemma 4.3. *If $w_i = 0$ then $\mathrm{Im}(\phi_{\mathbb{R}}) \subseteq F_i$, and if $z_i = 0$ then $\mathrm{Im}(\phi_{\mathbb{R}}) \subseteq G_i$.*

Proof. Let $y \in (\mathbb{R}^d)^*$ be the image of the moment map $\phi_{\mathbb{R}}$ for a point $[z,w] \in \mathcal{E}$, then

$$y \cdot u_i + r_i = \mu_{\mathbb{R}}(z,w) \cdot e_i = \frac{1}{2}\Big(|z_i|^2 - |w_i|^2\Big),$$

and hence $y \ge 0$ if $i \in A$, and $y \le 0$ if $i \notin A$. □

Lemma 4.4. *The component $\mathcal{E}_A$ of the extended core is isomorphic to the toric variety corresponding to the polytope Δ_A.*

The S^1-action does not act as a subtorus of T^d on $\mathfrak{M}$ globally, but does when restricted to each individual component $\mathcal{E}_A$ of the extended core. Consider a component $\mathcal{E}_A \subset \mathcal{E}$, then for some $[z,w] \in \mathcal{E}_A$ and $\tau \in S^1$,

$$\tau \cdot [z,w] = [z,\tau w] = [\tau_1 z_1, \dots, \tau_n z_n | \tau_i^{-1} w_1, \dots, \tau_n^{-1} w_n], \qquad \text{where } \tau_i = \begin{cases} \tau^{-1} & \text{if } i \in A, \\ 1 & \text{if } i \notin A, \end{cases}$$

since for each pair (z_i, w_i), if $i \in A$ then $(\tau z_i, \tau^{-1} w_i) = (0, \tau w_i)$, and if $i \notin A$, then $(\tau z_i, \tau^{-1} w_i) = (z_i, 0)$.

Thus when restricting our attention to each individual component $\mathcal{E}_A$ of the extended core, the S^1-action acts on $\mathcal{E}_A$ via an inclusion homomorphism onto a one-parameter subgroup of T^n, which consequently then descends to a T^d-action after taking the quotient of T^n by K.

Let us denote the restricted action of the image of S^1 in T^d to $\mathcal{E}_A$ by S^1_A,

$$\begin{tikzcd} S^1 \arrow[r, hook, "j_A"] & T^n \arrow[r, "\pi"] & S^1_A < T^d \\ \tau \arrow[r, mapsto] & (\tau_1^{-1}, \dots, \tau_n^{-1}) \arrow[r, mapsto] & v_A := \sum_{i\in A} u_i \end{tikzcd}$$

where we have denoted the generator of the one-parameter subgroup S^1_A in T^d by $v_A = \sum_{i\in A} u_i$.

Example 1. *For $\mathfrak{M} = T^*\mathbb{CP}^2$,*

Example 2. *For $\mathfrak{M}$ whose core consists of two $\mathbb{CP}^2$ intersecting at a point, so that the image of the real moment map is non-convex (todo)*

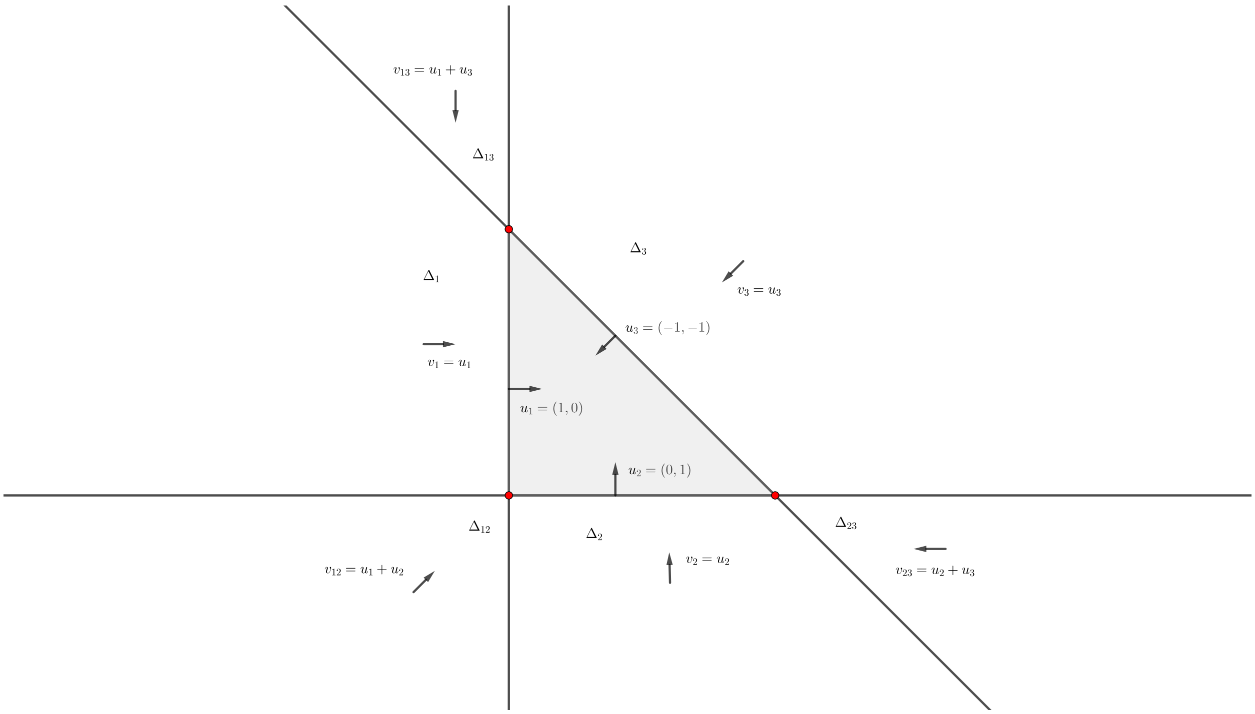

Figure 1: Combinatorics of the action of the residual S^1-action on the extended core $\mathcal{E}_A$ of $T^*\mathbb{CP}^2$, represented by each generator v_A of S^1_A in T^2.

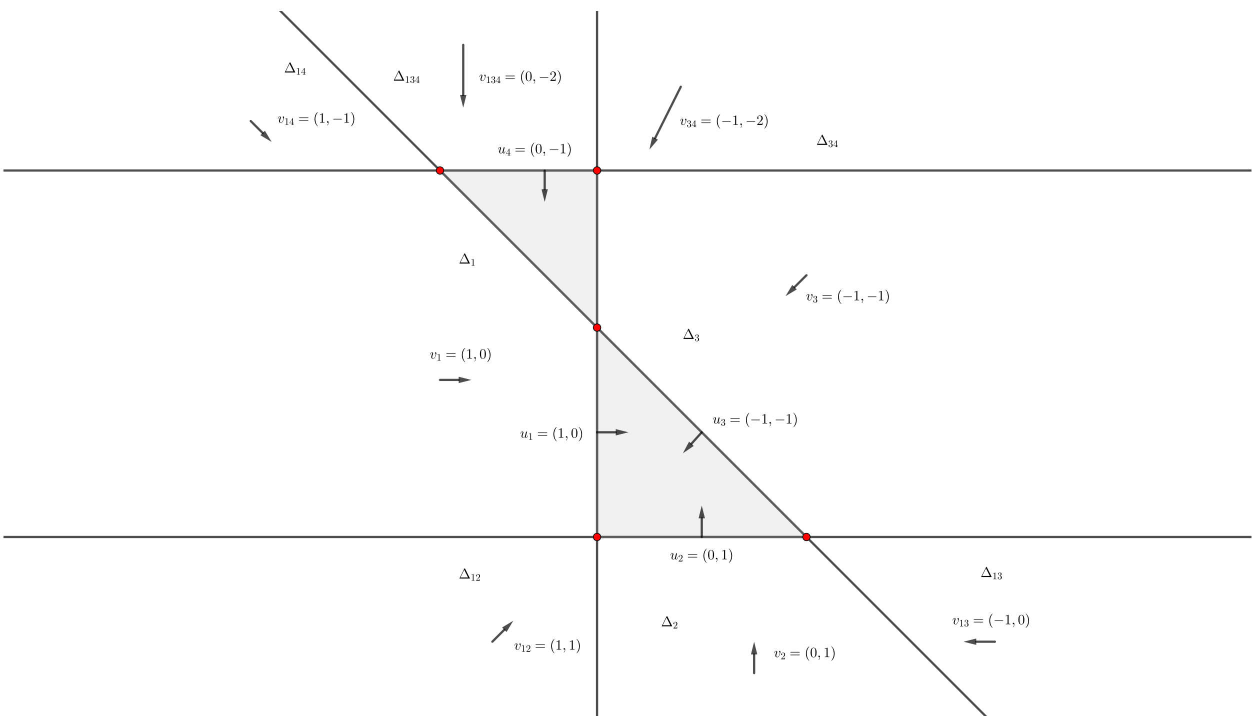

Figure 2: Combinatorics of the action of the residual S^1-action when the core consists of two non-convex components.

4.4 Compactification via Symplectic Cutting

We will use the S^1-action to perform a symplectic cut of the toric hyperkähler manifold $\mathfrak{M}$ to compactify it, which has the effect of bounding the $\|w\|^2$-norm component of the real moment map $\bar{\mu}_{\mathbb{R}}$ by above, and discarding the rest that lies above this bound. Consider the product $\mathfrak{M} \times \mathbb{C}$, and let S^1 act on $\mathfrak{M} \times \mathbb{C}$ via the diagonal product action, i.e. S^1 acts on M by rotating the cotangent fibre coordinates, and on $\mathbb{C}$ in the standard way:

$$e^{i\theta} \cdot \big([z,w],\xi\big) = \big(e^{i\theta} \cdot [z,w], e^{i\theta}\xi\big) = \big([z, e^{i\theta}w], e^{i\theta}\xi\big).$$

This action is Hamiltonian, and the corresponding moment map $\Phi : \mathfrak{M} \times \mathbb{C} \to \mathbb{R}_{\geq 0}$ for the S^1-action is

$$\Phi\big([z,w],\xi\big) = \phi[z,w] + |\xi|^2 = \|w\|^2 + |\xi|^2.$$

Then we have

$$\begin{aligned}
\Phi^{-1}(\epsilon) &= \big\{([z,w],\xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = \epsilon\big\} \\
&= \big\{[z,w] \in M : \|w\|^2 = \epsilon\big\} \bigsqcup \big\{([z,w],\xi) \in M \times \mathbb{C} : |\xi| = \pm\sqrt{\epsilon - \|w\|^2}\big\} \\
&= \big\{[z,w] \in M : \|w\|^2 = \epsilon\big\} \bigsqcup \big\{([z,w],\xi) \in M \times \mathbb{C} : \xi = e^{i\arg(\xi)}\sqrt{\epsilon - \|w\|^2}\big\} \\
&= \phi^{-1}(\epsilon) \bigsqcup \big(\mathfrak{M} \times S^1\big) \\
&=: \Sigma_1 \sqcup \Sigma_2,
\end{aligned}$$

where we denote the level-set $\phi^{-1}(\epsilon) \subseteq \mathfrak{M}$ by Σ_1, and $\Sigma_2 \cong \mathfrak{M} \times S^1$ is the trivial S^1-bundle over Σ_2 given by the globally defined section

$$\mathfrak{M} \to \mathfrak{M} \times S^1, \qquad [z,w] \longmapsto \big([z,w], e^{i\theta}\sqrt{\epsilon - \|w\|^2}\big), \qquad e^{i\theta} \in S^1.$$

Finally, taking the symplectic reduction of $\Phi^{-1}(\epsilon)$ with respect to the S^1-action, we obtain the *symplectic cut of* $\mathfrak{M}$ *at level-*ϵ,

$$M_{\leq\epsilon} := \Phi^{-1}(\epsilon)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where $\Sigma_1/S^1 \cong \phi^{-1}(\epsilon)/S^1$ is just the usual symplectic reduction, and where Σ_2/S^1 is diffeomorphic to $\mathfrak{M}$ for $\|w\|^2 < \epsilon$, which we will denote by $\mathfrak{M}_{<\epsilon}$.

4.5 The Combinatorics of the Cut Space, $\mathfrak{M}_{\leq\epsilon}$

Since the residual circle S^1-action acts as a subtorus S^1_A of the residual torus T^d on each component $\mathcal{E}_A$ of the extended core, the hyperplane arrangement determined in $(\mathfrak{t}^d)^*$ by the real moment map $\bar{\mu}_{\mathbb{R}}$ is compactified by dropping in half-spaces with an inwards-pointing normal vector, given by v_A when taking the cut.

Recall from the previous section that $j_A : S_1 \hookrightarrow T^n$ denoted the inclusion homomorphism of S^1 into the original torus T^n. If we let $j_{A,*} : \mathfrak{s}^1 \to \mathfrak{t}^n$ represent the differential of this inclusion, then

$$j_{A,*}(1) = \sum_{i \in A} e_i \in \mathfrak{t}^n,$$

and the generator $\exp(v_A)$ of the one-parameter subgroup S^1_A in T^d is

$$\exp(v_A) = \exp\left(\pi_* \circ j_{A,*}(1)\right),$$

or to be more concise,

$$S^1_A = \left\{ \exp\left(r \cdot \sum_{i \in A} u_i \right) \,\middle|\, r \in \mathbb{R} \right\}.$$

Then the moment map for the restricted S^1-action to $\mathcal{E}_A$ is

$$\phi_A[z,w] := \phi\big|_{\mathcal{E}_A}[z,w] = (j_A^* \circ \mu_{\mathbb{R}})[z,w] = \left\langle \bar{\mu}_{\mathbb{R}}[z,w], \sum_{i \in A} u_i \right\rangle,$$

where $j_A^* : (\mathfrak{t}^n)^* \to \mathbb{R}^*$ is the transposed differential of the inclusion, $j_{A,*}$.

As the S^1_A-action depends combinatorially on the component $\mathcal{E}_A$, the image of the real moment map in $(\mathfrak{t}^d)^*$ is compactified by inserting a half-space Z_A with inwards-pointing normal $v_A = \sum_{i \notin A} u_i$ determining the orientation, on each component Δ_A.

5 Symplectic Cutting and Hyperkähler Modifications

5.1 Universal Symplectic Cuts

Given compact connected Lie group K, we shall denote its left and right-actions on itself respectively by

$$\mathcal{L}_k(g) = kg, \quad \text{and} \quad \mathcal{R}_k(g) = gkk^{-1}.$$

Its Lie algebra $\mathfrak{k}$ will be identified with the left-invariant vector fields on K and, using polar decomposition as well as a bi-invariant metric on $\mathfrak{k}$, identify

$$TK \cong K \times \mathfrak{k}, \quad \text{and} \quad T^*K \cong K \times \mathfrak{k}^*.$$

Then the actions $\mathcal{L}$ and $\mathcal{R}$ on K both lift to T^*K which, in the above trivialisation, are given by

$$\mathcal{L}_k(g, \lambda) = (kg, \lambda), \quad \text{and} \quad \mathcal{R}_k(g, \lambda) = \left(gk^{-1}, \mathrm{Ad}_k^*(\lambda)\right).$$

These actions are both Hamiltonian with respect to the canonical symplectic form on T^*K, with respective moment maps

$$\mu_{\mathcal{L}}(k, \lambda) = -\mathrm{Ad}_k^*(\lambda), \quad \text{and} \quad \mu_{\mathcal{R}}(k, \lambda) = \lambda.$$

$$\begin{aligned}
M_P &\cong (M \times \mathbb{C}^n) /\!\!/_{\alpha} \left(S^1\right)^n \\
&\cong \left(\left[(M \times T^*T) /\!\!/_0 T_{\mathcal{L}} \right] \times \mathbb{C}^n \right) /\!\!/_{\alpha} \left(S^1\right)^n \\
&\cong (M \times T^*T \times \mathbb{C}^n) /\!\!/_{(\alpha,0)} \left(S^1\right)^n \times T_{\mathcal{L}} \\
&\cong (M \times (T^*T)_P) /\!\!/_0 T_{\mathcal{L}},
\end{aligned}$$

where $\left(S^1\right)^n \times T_{\mathcal{L}}$ acts on $M \times T^*T \times \mathbb{C}^n$ by

$$(s, t) \cdot (m, k, \lambda, z) = (s \cdot m, tk\pi(s)^{-1}, \lambda, sz).$$

5.2 Universal Modifications

Lemma 5.1. *Let $(M, I, \omega, K_{\mathbb{C}}, \mu)$ be a complex-symplectic K-Hamiltonian manifold. Then the complex-symplectic quotient $(M \times T^*K_{\mathbb{C}}) /\!\!/_0 K_{\mathbb{C}}$ is isomorphic to M as complex-symplectic manifolds. Here, the complex-symplectic quotient $(M \times T^*K_{\mathbb{C}}) /\!\!/_0 K_{\mathbb{C}}$ is taken with respect to the diagonal $K_{\mathbb{C}}$-action, where $K_{\mathbb{C}}$ acts on $T^*K_{\mathbb{C}}$ from the left.*

Proof. The $K_{\mathbb{C}}$-action on $T^*K_{\mathbb{C}}$ is free and proper, and thus so is the diagonal action on the product $M \times T^*K_{\mathbb{C}}$, so we are in the position to apply the holomorphic Marsden-Weinstein reduction theorem. Identifying $T^*K_{\mathbb{C}} \cong K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^*$ via polar decomposition, the moment map for the diagonal action is

$$\Phi : M \times K_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}}^* \longrightarrow \mathfrak{k}_{\mathbb{C}}^*, \qquad \Phi(m, k, \lambda) = \mu(m) - \mathrm{Ad}_g^*(\lambda),$$

thus

$$(M \times T^*K_{\mathbb{C}}) /\!\!/_0 K_{\mathbb{C}} = \Phi^{-1}(0)/K_{\mathbb{C}} = \{\, (m, k, \lambda) \in M \times T^*K_{\mathbb{C}} \mid \mu(m) = \mathrm{Ad}_k^*(\lambda) \,\}/K_{\mathbb{C}},$$

with $K_{\mathbb{C}}$ acting as

$$g \cdot (m, k, \lambda) = (g \cdot m, gk, \mathrm{Ad}_g^*(\lambda)).$$

Now define $j : M \to \Phi^{-1}(0) \subset M \times T^*K_{\mathbb{C}}$ by $j(m) = (m, e, \mu(m))$, which descends to a biholomorphism $\bar{j} : M \to \Phi^{-1}(0)/K_{\mathbb{C}}$.

By the Marsden-Weinstein reduction theorem, there exists a unique complex-symplectic form η on $(M \times T^*K_{\mathbb{C}}) /\!\!/_0 K_{\mathbb{C}}$ such that, if ν denotes the canonical form on $T^*K_{\mathbb{C}}$, $i : \Phi^{-1}(0) \hookrightarrow M \times T^*K_{\mathbb{C}}$ denotes the inclusion, and $q : \Phi^{-1}(0) \to \Phi^{-1}(0)/K_{\mathbb{C}}$ denotes the quotient map, then $q^*\eta = i^*(\omega + \nu)$.

$$\begin{tikzcd}
(M, \omega) \arrow[r, "j"] \arrow[dr, dotted, "\bar{j}"'] & \left(\Phi^{-1}(0), i^*(\omega + \nu)\right) \arrow[d, "q"] \arrow[dr, hook, "i"] & \\
& (\Phi^{-1}(0)/K_{\mathbb{C}}, \eta) & (M \times T^*K_{\mathbb{C}}, \omega + \nu)
\end{tikzcd}$$

To prove that $\bar{j}$ is a symplectomorphism, first observe that the map $m \mapsto (e, \mu(m))$ sends M into the fibre $T_e^*K_{\mathbb{C}} \cong \mathfrak{k}_{\mathbb{C}}^*$ which is a Lagrangian submanifold, as any fibre of a cotangent bundle is. Also, $j^*i^*(\omega + \nu) = \omega$, and by the Marsden-Weinstein reduction theorem, we have $q^*\eta = i^*(\omega + \nu)$, implying that $\omega = j^*i^*(\omega + \nu) = j^*q^*\eta = (j \circ q)^*\eta$. However, $\bar{j} = q \circ j$, so $\bar{j}^*\eta = \omega$ and $\bar{j}$ is also a symplectomorphism.

□

References

[1] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.