
HYPERTORIC MANIFOLDS

GENERAL NOTES

ABSTRACT

Preprint on toric hyperkähler manifolds.

1 Toric Hyperkähler Manifolds

1.1 Symplectic Quotients, [1]

Fix the standard Euclidean bilinear form on \mathbb{C}^n ,

$$g(z, w) = \sum_{i=1}^n (\Re(z_i)\Re(w_i) + \Im(z_i)\Im(w_i)).$$

The corresponding Kähler form is

$$\omega(z, w) = g(iz, w) = \sum_{i=1}^n (\Re(z_i)\Im(w_i) - \Im(z_i)\Re(w_i)).$$

Let $A = [u_1, \dots, u_n]$ be a $(d \times n)$ -matrix whose $(d \times d)$ -minors are relatively prime. Choose now an $n \times (n - d)$ -matrix $B = [b_1, \dots, b_n]^T$ that makes the following sequence exact:

$$\{0\} \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow \{0\}.$$

The choice of B is equivalent to choosing a basis in $\ker(A)$.

1.2 Hyperkähler Quotients

Let \mathbb{H} be the quaternions, the 4-dimensional \mathbb{R} -vector space with basis $\{1, i, j, k\}$ equipped with an associative algebra structure defined by

$$i^2 = j^2 = k^2 = ijk = -1.$$

Left-multiplication by i (respectively j and k) define the following respective complex structures on \mathbb{H} ,

$$I, J, K : \mathbb{H} \longrightarrow \mathbb{H}; \quad I^2 = J^2 = K^2 = IJK = -\text{Id}_{\mathbb{H}}.$$

Equipping \mathbb{H} with the flat metric g arising from the standard Euclidean scalar-product on $\mathbb{H} \cong \mathbb{R}^4$, with $\{1, i, j, k\}$ providing an orthonormal basis. This is called a *hyperkähler metric* since it is a Kähler metric with respect to each individual complex structure, I , J , and K . This also means that the so-called *Kähler forms*, given by

$\omega_I(X, Y) = g(IX, Y), \quad \omega_J(X, Y) = g(JX, Y), \quad \omega_K(X, Y) = g(KX, Y), \quad \text{for tangent vectors } X, Y,$
are closed differential 2-forms.

A special orthogonal transformation with respect to this metric is said to *preserve the hyperkähler structure* if it commutes with all three complex structures, I , J , and K ; or equivalently, it preserves the Kähler forms, ω_I, ω_J , and ω_K . The group of such transformations, the *unitary symplectic group* $\text{Sp}(1)$, is generated by the right-multiplication action by the unit quaternions.

A maximal abelian subgroup $T_{\mathbb{R}}^1 \cong \text{U}(1) \subset \text{Sp}(1)$ is then specified by a choice of unit quaternion, and we break the I, J, K symmetry by choosing a maximal torus, generated by right-multiplication by the unit quaternion i . Hence $\text{U}(1)$ acts on \mathbb{H} from the right by sending

$$\xi \mapsto \xi \exp(ti), \quad \exp(ti) \in \text{U}(1) \subset \mathbb{R} \oplus \mathbb{R}i \cong \mathbb{C}.$$

The moment map for this action $\mu_1 : \mathbb{H} \rightarrow \mathbb{R}$ with respect to the symplectic form ω_1 is then given by

$$\mu_1(x + yi + uj + vk) = \mu_1((x + yi) + (v - ui)k) = \frac{1}{2}(x^2 + y^2 - u^2 - v^2).$$

1.3

Proposition 1.1 ([2]). *Suppose that α and $(\alpha, 0)$ are regular values for μ and μ_{HK} , respectively. Then the cotangent bundle T^*X is isomorphic to an open subset of M , and is dense if it is non-empty.*

Proof. Let $Y = \{ (z, w) \in \mu_{\mathbb{C}}^{-1}(0)^{\text{st}} \mid z \in (\mathbb{C}^n)^{\text{st}} \}$, where z is semi-stable with respect to α for the $G_{\mathbb{C}}$ -action on \mathbb{C}^n , so that we have $X \cong (\mathbb{C}^n)^{\text{st}}/G_{\mathbb{C}}$. Let $[z] \in X$ be the representative of $z \in (\mathbb{C}^n)^{\text{st}}$. The tangent space $T_{[z]}X$ is equal to the quotient of $T_z\mathbb{C}^n$ by the tangent space to the $G_{\mathbb{C}}$ -orbit through z ,

$$T_{[z]}X = T_z\mathbb{C}^n / T_{[z]}(G_{\mathbb{C}} \cdot z).$$

Therefore,

$$T_{[z]}^*X \cong \{ w \in T_z^*\mathbb{C}^n \mid w(\hat{v}_z) = 0, \text{ for all } v \in \mathfrak{g}_{\mathbb{C}} \} = \{ w \in (\mathbb{C}^n)^* \mid \mu_{\mathbb{C}}(z, w) = 0 \}.$$

Then, by letting $[z] \in X$ vary, we have

$$T^*X \cong \{ (z, w) \mid z \in (\mathbb{C}^n)^{\text{st}} \text{ and } \mu_{\mathbb{C}}(z, w) = 0 \} / G_{\mathbb{C}} = Y / G_{\mathbb{C}}.$$

As each z -coordinate in Y is semi-stable, Y is an open subset of $\mu_{\mathbb{C}}^{-1}(0)$, and is dense if non-empty. \square

2 Cotangent Spaces to Extended Core Components

Let $M_{\lambda} = (\mu_{\mathbb{R}}^{-1}(\lambda) \cap \mu_{\mathbb{C}}^{-1}(0)) / K$ be a toric hyperkähler manifold. Define

$$\mathbb{C}_A := \{ (z_i, w_i) \in \mathbb{C}^{2n} \mid w_i = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \} \cong \mathbb{C}^n \subset \mathbb{H}^n.$$

Lemma 2.1 ([3]). *Let M_{λ} be a toric hyperkähler manifold. If \mathcal{E}_A is non-empty, then its holomorphic cotangent bundle $T^*\mathcal{E}_A$ is contained in M_{λ} as an open subset.*

Fix a subset $A \subset \{1, \dots, n\}$, and define

$$(x_i^{(A)}, y_i^{(A)}) := \begin{cases} (z_i, w_i), & \text{if } i \in A, \\ (w_i, -z_i), & \text{if } i \notin A. \end{cases}$$

Then $x^{(A)} = (x_1^{(A)}, \dots, x_n^{(A)})$ is a point in the vector space \mathbb{C}_A^n , and $y^{(A)} = (y_1^{(A)}, \dots, y_n^{(A)})$ is a point in the dual space $(\mathbb{C}_A^n)^*$. That is, we identify the cotangent bundle $T^*\mathbb{C}_A^n$ with \mathbb{H}^n as above.

2.1 Kähler Quotients

The Kähler quotient $X = \mu^{-1}(0)/N$ can be identified with the quotient of an open subset of \mathbb{C}^n by the complexified torus $N^{\mathbb{C}}$ as follows: every orbit in \mathbb{C}^n of $T_{\mathbb{C}}^n$ is of the form

$$\mathbb{C}_A^n = \{ (z_1, \dots, z_n) \mid z_i = 0 \text{ if } i \in A \},$$

for some subset $A \subset \{1, \dots, n\}$. If F is a face of Δ of codimension r , then F is defined by the intersection of r hyperplanes $\cap_{j=1}^r H_{i_j}$.

3 Symplectic Cutting

3.1 Compactifying the Extended Core

Let S^1 act on M by rotating the cotangent fibres, that is, for $\tau \in S^1$,

$$\tau \cdot [z; w] = [z; \tau w].$$

This S^1 -action is Hamiltonian, with moment map

$$\Phi : M \longrightarrow (\mathbb{R})^*; \quad [z : w] \longmapsto \frac{1}{2} \|w\|^2.$$

Let S_A^1 denote the residual S^1 -action on M restricted to the extended core component

$$\mathcal{E}_A = \{ [z_1 : \dots : z_n; w_1, \dots, w_n] \mid w_0 = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \}.$$

Now the *global* S^1 -action does not act on the cotangent fibres of M as a subtorus of T^n , but it does when *restricted* to each component of the extended core, \mathcal{E}_A . Indeed,

$$\tau \cdot [z; w] = [z; \tau w] = [z_1 : \dots : z_n; \tau w_1 : \dots : \tau w_n] = [\tau_1 z_1 : \dots : \tau_n z_n; \tau_1^{-1} w_1 : \dots : \tau_n^{-1} w_n],$$

where

$$\tau_i := \begin{cases} \tau^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A, \end{cases}$$

which shows that the S^1 -action restricted to each individual \mathcal{E}_A acts as a subtorus of the original torus T^n .

Denote by S_A^1 the image of S^1 in T^n when considered as a subtorus restricted to each individual \mathcal{E}_A , and let $j_A : S^1 \hookrightarrow T^n$ be the respective inclusion homomorphism, so we have $S_A^1 := j_A(S^1) \triangleleft T^n$.

On the Lie algebra level, we have that

$$(j_A)_* : \text{Lie}(S_A^1) \longrightarrow \mathfrak{t}^n; \quad \xi \longmapsto (\xi_1, \dots, \xi_n),$$

where we analogously define

$$\xi_i := \begin{cases} -1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A. \end{cases}$$

Since S_A^1 acts as the subtorus $j_A(S^1)$ of T^n on each \mathcal{E}_A , the moment map $\Phi_A := \Phi|_{\mathcal{E}_A}$ for this action is given by composing $\mu_{\mathbb{R}}$ with the dual of the inclusion $(j_A)_*$, so

$$\begin{aligned} \Phi_A[z, w] &= (j_A^* \circ \mu_{\mathbb{R}})[z; w] = j_A^* \left(\frac{1}{2} \sum_{i=1}^n (|z_i|^2 - |w_i|^2) e^i \right) \\ &= -\frac{1}{2} \sum_{i \in A} |z_i|^2 j_A^*(e^i) \\ &= \frac{1}{2} \sum_{i \notin A} |w_i|^2 j_A^*(e^i) \\ &= \langle \mu_{\mathbb{R}}[z; w], \xi_A \rangle \\ &= \mu_{\mathbb{R}}^A[z; w], \end{aligned}$$

where $\xi_A = -\sum_{i \in A} \xi_i$, and $\mu_{\mathbb{R}}^A[z; w]$ is the component of $\mu_{\mathbb{R}}[z; w]$ in the ξ_A -direction.

3.2 Moment Polyptychs

The holomorphic moment map $\bar{\mu}_{\mathbb{C}} : M \rightarrow (\mathfrak{t}_{\mathbb{C}}^d)^*$ is \mathbb{C}^* -equivariant with respect to the \mathbb{C}^* -scalar action on $(\mathfrak{t}_{\mathbb{C}}^d)^*$, hence both $M^{\mathbb{C}^*}$ and \mathcal{L} will be contained in

$$\mathcal{E} := \bar{\mu}_{\mathbb{C}}^{-1}(0) = \{ [z; w] \in M \mid z_i w_i = 0 \text{ for all } i = 1, \dots, n \},$$

which is called the **extended core** of M , and $\bar{\mu}_{\mathbb{R}}$ surjects onto \mathcal{E} . The extended core \mathcal{E} breaks into the components

$$\mathcal{E}_A := \{ [z; w] \in M \mid w_i = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \},$$

indexed by subsets $A \subseteq \{1, \dots, n\}$.

The hyperplanes $\{H_i\}_{i=1}^n$ divide $(\mathfrak{t}^d)^*$ into a union of convex, possibly empty, possibly unbounded, polyhedra

$$\Delta_A := \bigcap_{i \in A} F_i \cap \bigcap_{i \notin A} G_i,$$

where we recall that

$$\begin{aligned} F_i &= \{ v \in (\mathfrak{t}^d)^* \mid \langle v, u_i \rangle + r_i \geq 0 \}, \\ G_i &= \{ v \in (\mathfrak{t}^d)^* \mid \langle v, u_i \rangle + r_i \leq 0 \}, \\ H_i &= \{ v \in (\mathfrak{t}^d)^* \mid \langle v, u_i \rangle + r_i = 0 \} = F_i \cap G_i. \end{aligned}$$

Lemma 3.1 ([2]). *If $w_i = 0$, then $\bar{\mu}_{\mathbb{R}}[z; w] \in F_i$, whereas if $z_i = 0$, then $\bar{\mu}_{\mathbb{R}}[z; w] \in G_i$.*

Proof. Suppose that $\bar{\mu}_{\mathbb{R}}[z; w] = v \in (\mathfrak{t}^d)^*$. Then

$$\langle \bar{\mu}_{\mathbb{R}}[z; w], u_i \rangle \iff \langle \bar{\mu}_{\mathbb{R}}[z; w], u_i \rangle$$

□

Distinct S_A^1 actions have common zeros?

Work out this sequence of equivalences.

References

- [1] Tamás Hausel and Bernd Sturmfels. Toric hyperKähler varieties. *Documenta Mathematica*, 7:495–534, 2002.
- [2] Nicholas James Proudfoot. *Hyperkahler analogues of Kahler quotients*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)—University of California, Berkeley.
- [3] Hiroshi Konno. The topology of toric hyperKähler manifolds. In *Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999)*, volume 34 of *Adv. Stud. Pure Math.*, pages 173–184. Math. Soc. Japan, Tokyo, 2002.