HYPERTORIC MANIFOLDS

GENERAL NOTES

ABSTRACT

Notes on toric hyperkähler manifolds.

1 Cotangent Spaces to Extended Core Components

Let $M_\lambda = \left(\mu_{\mathbb{R}}^{-1}(\lambda) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/K$ be a toric hyperkähler manifold. Define

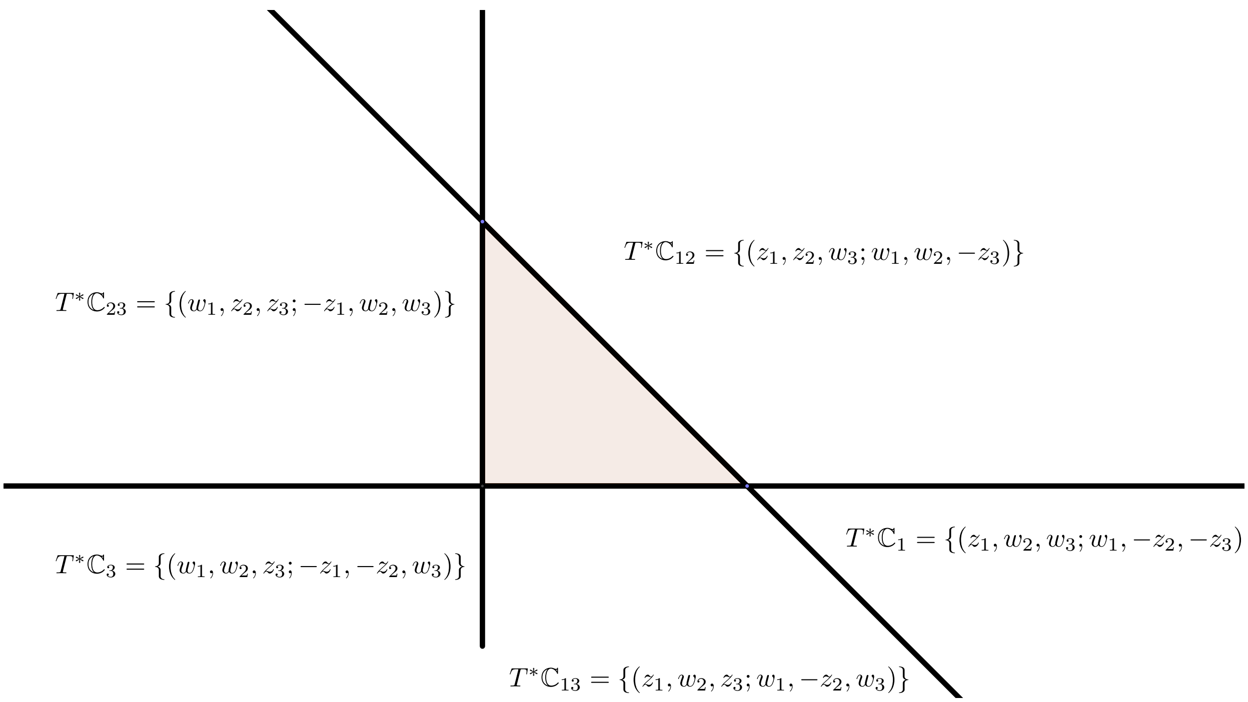
$$\mathbb{C}_A := \left\{ (z_i, w_i) \in \mathbb{C}^{2n} \;\middle|\; w_i = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \right\} \cong \mathbb{C}^n \subset \mathbb{H}^n.$$

Lemma 1.1 ([1]). *Let M_λ be a toric hyperkähler manifold. If $\mathcal{E}_A$ is non-empty, then its holomorphic cotangent bundle $T^*\mathcal{E}_A$ is contained in M_λ as an open subset.*

Fix a subset $A \subset \{ 1, \dots, n \}$, and define

$$(x_i^{(A)}, y_i^{(A)}) := \begin{cases} (z_i, w_i), & \text{if } i \in A, \\ (w_i, -z_i), & \text{if } i \notin A. \end{cases}$$

Then $x^{(A)} = (x_1^{(A)}, \dots, x_n^{(A)})$ is a point in the vector space $\mathbb{C}_A^n$, and $y^{(A)} = (y_1^{(A)}, \dots, y_n^{(A)})$ is a point in the dual space $(\mathbb{C}_A^n)^*$. That is, we identify the cotangent bundle $T^*\mathbb{C}_A^n$ with $\mathbb{H}^n$ as above.



1.1 The Legendre Transform

In [1], Konno states that the proof of this lemma goes by an argument in [2], which itself is based on properties of the Legendre transform.

Lemma 1.2 (Section A1.3 of [2])**.** *Consider a smooth function of one variable*

$$f = f(x), \qquad -\infty < x < \infty.$$

Suppose that f is strictly convex ($f''(x) > 0$ for all x). Then the four conditions are equivalent:

1. *$f'(x_0) = 0$ at some point x_0.*
2. *f has a local minimum at some point x_0.*
3. *f has a unique local minimum.*
4. *$f(x)$ tends to $+\infty$ as x tends to $\pm\infty$.*

If f has any one (and hence all four) of the above properties, we will say that f is *stable*.

2 Symplectic Cutting

2.1 Compactifying the Extended Core

Let S^1_A denote the residual S^1-action on M restricted to the extended core component $\mathcal{E}_A = \{\, [z_1 : \ldots z_n; w_1, \ldots, w_n] \mid w_0 = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \,\}$. Recall that the global S^1-action acts on M by rotating its cotangent fibres, that is

$$\tau \cdot [z; w] = [z; \tau w] = [z_1 : \ldots : z_n; \tau w_1 : \ldots : \tau w_n] = [\tau_1 z_1 : \ldots : \tau_n z_n; \tau_1^{-1} w_1 : \ldots : \tau_n^{-1} w_n],$$

where

$$\tau_i := \begin{cases} \tau^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A, \end{cases}$$

which shows that the S^1-action, when restricted to each individual $\mathcal{E}_A$, acts as a subtorus of the original torus T^n.

Denote by S^1_A the image of S^1 in T^n when considered as a subtorus when restricted to each individual $\mathcal{E}_A$, and let $\jmath_A : S^1 \hookrightarrow T^n$ denote the respective inclusion homomorphism.

On the Lie algebra level, we have that

$$\jmath_{A;*} : \mathrm{Lie}(S^1_A) \longrightarrow \mathfrak{t}^n; \qquad \xi \longmapsto (\xi_1, \ldots, \xi_n),$$

where analogously

$$\xi_i := \begin{cases} -1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A. \end{cases}$$

[TODO: S^1_A and $S^1_{A'}$-action (say) on adjacent Δ_A's coincide along the common hyperplane H_i; formalise this!]

References

[1] Hiroshi Konno. The topology of toric hyperKähler manifolds. In *Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999)*, volume 34 of *Adv. Stud. Pure Math.*, pages 173–184. Math. Soc. Japan, Tokyo, 2002.

[2] Victor Guillemin. *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces*, volume 122 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1994.