GENERAL NOTES

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ABSTRACT

Rough general notes.

1 Geometry

1.1 Symplectic Geometry

1.1.1 The Differential of the Moment Map

Given a symplectic manifold with a Hamiltonian G-action, (M, ω, μ), where $\mu : M \to \mathfrak{g}^*$ is the moment map, the tangent map to μ at a point $p \in M$ is

$$d_p M : T_p M \longrightarrow \mathfrak{g}^*$$

which, by definition, is the transpose of

$$(d_p\mu)^T : \mathfrak{g} \longrightarrow T_p^* M; \qquad X \longmapsto (\imath_{X_M}\omega)_p \,.$$

In particular, its image $\operatorname{Im}(d_p\mu)$ is the annihilator in $\mathfrak{g}^*$ of $\ker\left((d_p\mu)^T\right)$, that is, the annihilator of

$$\{\, X \in \mathfrak{g} \mid (\imath_{X_M}\omega)_p = 0 \,\} = \{\, X \in \mathfrak{g} \mid X_M(p) = 0 \,\} = \mathfrak{g}_p.$$

Proposition 1.1. *The annihilator in $\mathfrak{g}^*$ of $\mathfrak{g}_p$ is the subspace $\operatorname{Im} d_p\mu$. Dually, the annihilator in $\mathfrak{g}$ of $\operatorname{Im} d_p\mu \subset \mathfrak{g}^*$ is $\mathfrak{g}_p$.*

Thus the rank of $d_p\mu$ is the dimension of the orbit of p. Thus we deduce, in particular:

Corollary 1.2. *The momentum mapping μ is a submersion at the point p if and only if the stabiliser subgroup G_p is discrete.*

Proposition 1.3. *The kernel $\ker d_p\mu$ is the orthogonal (for ω_p) of the tangent space to the orbit though p, that is*

$$\ker d_p\mu = (T(G \cdot p))^\circ \,.$$

Proof. $d_p\mu(Y)$ is zero if and only if $\langle d_p\mu(Y), X\rangle = 0$ for all vectors $X \in \mathfrak{g}$, i.e. if and only if $\omega_p(X_M(p), Y) = 0$ for all X, and that is, if and only if Y is orthogonal to the subspace generated by the fundamental vector fields. □

1.2 Principal Bundles

Definition 1.4. *Let G be a Lie group. A **principal G-bundle** is a smooth manifold P with a smooth right G-action*

$$P \times G \to P; \qquad (p, g) \longmapsto p \cdot g,$$

such that the action is free, the quotient space $B = P/G$ is a manifold, and the natural projection map $\pi : P \to B$ is a locally trivial fibration. If G is compact, then the first condition implies the second and third.

Denote by $T_pP \to T_{p\cdot g}P$; $v \mapsto v \cdot g$ for the induced action on the tangent bundle, and

$$p \cdot \xi := \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(t\xi) \in T_pM; \qquad p \in P,\, \xi \in \mathfrak{g},$$

for the infinitesimal action of the Lie algebra $\mathfrak{g}$.

Definition 1.5. *The **vertical tangent bundle** $V \subset TP$ has fibres*

$$V_p := \{\, p \cdot \xi \mid \xi \in \mathfrak{g} \,\} \,.$$

1.3 Quotient Manifolds

If we consider now a topological space M and its quotient M/G in the place of P and B, respectively, then the space of orbits -that is M/G - can be given the structure of a manifold with respect to the projection $\pi : M \to M/G$ is a smooth submersion, provided that G acts **freely** on M. As the action is free, non-zero vector fields generated by $\mathfrak{g}$ have no zeroes and so for each point $p \in M$ there is a subspace $V_pM \subset T_pM$, with $\dim V_pM \ \dim G$, spanned by the vector fields in $\mathfrak{g}$. This **vertical** space is the tangent space to the orbit of G through p. The tangent space to $\pi(p) \in M/G$ is then isomorphic to the quotient vector space T_pM/V_pM.

Now let M be given a Riemannian metric g, and suppose G acts as isometries. We may define an **induced Riemannian metric** on M/G as follows: Let $H_pM \subset T_pM$ be the subspace of vectors orthogonal to V_pM, called the **horizontal space**. Then the derivative of π maps H_pM isomorphically to the tangent space of the quotient at $\pi(p)$. A tangent vector $X \in T_{\pi(p)}(G \cdot p)$ then has a unique horizontal lift $\tilde{X} \in H_pM \subset T_pM$, and we define an inner-product h on $T_{\pi(p)}(G \cdot p)$ by:

$$h : T_{\pi(p}(G \cdot p) \times T_{\pi(p}(G \cdot p) \longrightarrow \mathbb{R}; \qquad h(X, Y) := g(\tilde{X}, \tilde{Y}).$$

As G preserves the metric g, this is independent on the choice of point p in the orbit $\pi^{-1}(\pi(p))$.

This family of horizontal subspaces has an interpretation in terms of **connections**; the manifold M is a principal G-bundle over M/G, by definition of the free action of G. And the vector fields corresponding to a basis of the Lie algebra $\mathfrak{g}$ form a basis for V_pM at each point $p \in M$, thus the orthogonal projection from T_pM to V_pM defines a 1-form θ with values in $\mathfrak{g}$ and transforming under the adjoint representation of G, which determines a connection form for the principal bundle.

1.4 Symplectic Reduction

Now let (M, ω, μ) be a G-manifold with associated moment map $\mu : M \to \mathfrak{g}$, and consider the submanifold $N := \mu^{-1}(\mu(p)) \subseteq M$. Let Y be a tangent vector to N, then $d\mu(Y) = 0$, and thus

$$0 = d\mu^X(Y) = \omega(X, Y)$$

for all vector fields X generated by $\mathfrak{g}$. As the form ω is non-degenerate, this gives $\dim G$ independent equations for Y giving

$$\dim \ker d\mu = \dim M - \dim G,$$

so (if $\dim M = 2n$) N is a submanifold of dimension $2n - \dim G$.

Now suppose that there exists a $p \in M$ such that $\mu(p) = 0$, and let $N = \mu^{-1}(0)$. As G keeps the origin in $\mathfrak{g}^*$ fixed and μ is equivariant, G acts on the manifold N, and can form the quotient N/G which is a manifold of dimension $2n - 2\dim G$. It possesses a natural 2-form ρ, defined by

$$\rho(Y_1, Y_2) = \omega(\tilde{Y}_1, \tilde{Y}_2),$$

where $\tilde{Y}_i$ is any tangent vector on to N which projects to Y_i, a tangent vector to N/G. One can show that ρ is a well-defined 2-form on the quotient N/G, and moreover is a symplectic 2-form, making $(N/G, /w)$ a symplectic manifold.

More generally, if we take a point $x \in \mathfrak{g}^*$ which is not fixed by G, but has isotropy subgroup H, then $\mu^{-1}(x)/H$ has a symplectic structure in the same way.

[TODO: see Guillemin's 'Moment Maps and Combinatorial Invariants of T^n -Actions' for similar material]

2 Residue Theorems

Lemma 2.1 ([1]). *Let A be a graded commutative algebra over $\mathbb{C}$ and let $f = f(x)$ be a polynomial in x with coefficients in A. Then for indeterminants $z_1, \ldots, z_d$,*

$$\operatorname{Res}_x \frac{f(x)}{(x - z_1)\ldots(x - z_d)} = \sum_{i=1}^{d} \frac{f(z_i)}{\prod_{j\neq i}(z_i - z_j)}.$$

Proof. Decompose into simple fractions:

$$\frac{f(x)}{(x - z_1)\ldots(x - z_d)} = F(x) + \sum_{i=1}^{d} \frac{f(z_i)}{\prod_{j\neq i}(z_i - z_j)} \frac{1}{(x - z_i)}.$$

Here $F(x)$ is a polynomial term in x. $\square$

Let

$$h = \frac{f}{\prod_{j=1}^{d}(x - z_j)} \quad \text{and} \quad h_j = \frac{f(z_j)}{\prod_{r \neq j}(z_j - z_r)}, \quad \text{for all } j.$$

Lemma 2.2 ([1]). *$h \in A[x]$ if and only if $\mathrm{Res}_x(x^k h) = 0$, for all $k \geq 0$.*

Proof. From Lemma 2.1, we get that

$$\mathrm{Res}_x(x^k h) = \sum_{j=1}^{d} (z_j)^k h_j.$$

Then the condition that $\mathrm{Res}_x(x^k h) = 0$ for every $k = 1, \ldots, d$ can be written as

$$\begin{pmatrix} z_1^1 & \ldots & z_j^1 & \ldots & z_d^1 \\ \vdots & & \vdots & & \vdots \\ z_1^k & \ldots & z_j^k & \ldots & z_d^k \\ \vdots & & \vdots & & \vdots \\ z_1^d & \ldots & z_j^d & \ldots & z_d^d \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_j \\ \vdots \\ h_d \end{pmatrix} = 0.$$

As the corresponding Van der Monde determinant is non-zero, we deduce that $h_1 = \ldots = h_d = 0$, that is, $f(z_j) = 0$, for all $j = 1, \ldots, d$, from which we obtain that $h \in A[x]$. □

Theorem 2.3 ([2]). *Let V be an n-dimensional vector space over $\mathbb{C}$, and let τ_k be the standard representation of $\mathrm{GL}(V)$ on the k-th symmetric product, $S^k(V)$. Then for $z \in \mathbb{C}$ large and $B \in \mathrm{GL}(V)$,*

$$\det(z - B)^{-1} = z^{-n} \sum_{k=0}^{\infty} z^{-k} \, \mathrm{Tr}(\tau_k(B)).$$

Proof. Without loss of generality, assume that B is diagonalisable with eigenvalues, $\lambda_1, \ldots, \lambda_n$. The left-hand side then becomes

$$\det(z - B)^{-1} = z^{-n} \prod_{j=1}^{n} (1 - \lambda_j z^{-1})^{-1}.$$

Expanding each of the factors $(1 - \lambda_j z^{-1})^{-1}$ into a geometric series, we then get

$$z^{-n} \prod_{j=1}^{n} (1 - \lambda_j z^{-1})^{-1} = z^{-1} \left(\sum_{k=0}^{\infty} z^k t_k \right),$$

where

$$t_k = \sum_{|I|=k} \lambda_1^{i_1} \ldots \lambda_n^{i_n} = \mathrm{Tr}\left(\tau_k(B)\right).$$

□

Corollary 2.4 ([2]). *Let Γ be a contour about the origin containing the zeroes of $\det(z - B)$. Then*

$$\frac{1}{2\pi i} \int_{\Gamma} z^{n+k-1} \det(z - B)^{-1} dz = \mathrm{Tr}\left(\tau_k(B)\right).$$

3 Representation Theory

3.1 Symmetric Polynomials

Definition 3.1 ([3]). *A **representation** of $\Gamma = \mathrm{GL}(\mathbb{C}^n)$ (or Γ-**module**) is a pair (V, ρ), where V is a $\mathbb{C}$-vector space and*

$$\begin{aligned} \rho : \Gamma &\longrightarrow \mathrm{GL}(V), \\ A = (a_{ij})_{1 \le i,j \le n} &\longmapsto \rho(A) = (\rho_{kl}(A))_{1 \le k,l \le N} \end{aligned}$$

*is a group homomorphism. The **dimension** N of the representation (V, ρ) is the dimension of the vector space V. We say that (V, ρ) is a **polynomial representation** (of **degree** d) if the matrix entries $\rho_{kl}(A) = \rho_{kl}(a_{11}, a_{12}, \ldots, a_{nn})$ are polynomial functions (homogeneous of degree d).*

Example 1. *The d-th symmetric power representation: $V = S_d(\mathbb{C}^n)$ = the space of homogeneous polynomials of degree d in $x_1, x_2, \ldots, x_n$, ρ = action by linear substitution, $N = \binom{n+d-1}{d}$.*

For example, for $d = 3, n = 2$, we have $S_3(\mathbb{C}^2)$ = binary cubics $= \mathrm{Span}\{x^3, x^2y, xy^2, y^3\} \cong \mathbb{C}^4$, and ρ is the group homomorphism

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}^3 & a_{11}^2 a_{12} & a_{11} a_{12}^2 & a_{12}^3 \\ 3a_{11}^2 & a_{11}^2 a_{22} + 2a_{11}a_{12}a_{21} & 2a_{11}a_{12}a_{22} + a_{12}a_{21}^2 & 3a_{12}^2 a_{22} \\ 3a_{11}^2 a_{21}^2 & 2a_{11}a_{21}a_{22} + a_{12}a_{21}^2 & a_{11}a_{22}^2 + 2a_{12}a_{21}a_{22} & 3a_{12}a_{22}^2 \\ a_{21}^3 & a_{21}^2 a_{22} & a_{21}a_{22}^2 & a_{22}^3 \end{pmatrix}.$$

4 Orbifolds

4.1 The Associated Inertia Orbifold

Definition 4.1. *Given an orbifold M, the **associated orbifold** $\hat{M}$ has charts $(\hat{V}, \Gamma, \psi)$ built as follows:*

For each chart $(\tilde{U}, \Gamma, \phi)$ of M, let

$$\hat{V} := \left\{ (u, g) \in \tilde{U} \times \Gamma \;\middle|\; g \cdot u = u \right\}.$$

Γ acts on $\hat{V}$ by

$$h \cdot (u, g) = (h \cdot u, hgh^{-1}).$$

Let $V := \hat{V}/\Gamma$ be the space of orbits with projection $\phi : \hat{V} \to V$. The orbifold charts $(\hat{V}, \Gamma, \phi)$ inherit the compatibility conditions from the $(\tilde{U}, \Gamma, \phi)$. As a set,

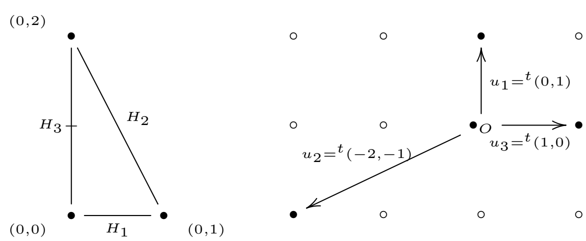
$$\hat{M} = \bigsqcup_{p \in M} \mathrm{Conj}(\Gamma_p),$$

where $\mathrm{Conj}(\Gamma_p)$ is the set of conjugacy classes in Γ_p.

4.2 From Simple Rational Polytopes

4.2.1 Weighted Projective Space $\mathbb{CP}^2_{(1,1,2)}$, from [?]

Consider the following polytope with facets H_i, facet labels all (implicitly) 1, and the corresponding primitive inward-pointing normal vectors u_i to the facets.



The polytope is given by

$$\Delta = \{ v \in \mathbb{R}^2 \mid \langle u_i, v \rangle \ge -\eta_i \ , \ i = 1, 2, 3 \}$$

where $(\eta_1, \eta_2, \eta_3) = (0, 2, 0)$. The corresponding matrix B is $\begin{pmatrix} 0 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ and A is $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Thus M is given by $|z_1|^2 + |z_2|^2 + 2|z_3|^2 = 2$ in
C^3 and $S = \{(t, t, t^2) \mid t \in U(1)\} \subset T^3 = U(1)^3$. The only elements g of S such that G_g is not empty are

$$(1,1,1) \quad (-1,-1,1)$$

5 Equivariant Localisation

5.1 Kawasaki-Riemann-Roch Formula for Orbifolds

5.1.1 C.f. '*On Riemann-Roch for Multiplicities*', [4]

Let $\mathcal{L} \to M$ be a G-equivariant Hermitian vector bundle over M, with fibre dimension n. Let $\mathcal{A}(M;\mathcal{L})$ be the $\mathcal{L}$-valued differential forms, and $\mathcal{A}_G(M;\mathcal{L})$ their equivariant counter-part. For each G-invariant Hermitian connection $\nabla : \mathcal{A}(M;\mathcal{L}) \to \mathcal{A}(M;\mathcal{L})$ the moment map $\mu \in \mathcal{A}_G(M; \mathrm{End}(\mathcal{L}))$ of Berline and Vergne [5] is defined by

$$\mu(\xi) \cdot \sigma := \xi \cdot \sigma - \nabla_{\xi_M} \cdot \sigma,$$

where $\sigma \mapsto \xi \cdot \sigma$ denote the representation of $\mathfrak{g}$ on the space of sections.

Geometrically, $\mu(\xi)$ is the vertical part (with respect to the connection ∇) of the fundamental vector field $\xi_{\mathcal{L}}$ on $\mathcal{L}$.

Let $F(\mathcal{L}) \in \mathcal{A}^2(M; \mathrm{End}(\mathcal{L}))$ denote the curvature of ∇; then the **equivariant curvature** $F_{\mathfrak{g}}(\mathcal{L};\xi)$ is defined by

$$F_{\mathfrak{g}}(\mathcal{L};\xi) := F(\mathcal{L}) + 2\pi i \mu(\xi),$$

and it satisfies the Bianchi identity with respect to the **equivariant covariant derivative**

$$\nabla_{\mathfrak{g}} := \nabla - 2\pi i \cdot \imath(\xi_m).$$

Suppose now that $A \mapsto f(A)$ is the germ of an $\mathrm{U}(n)$-invariant analytic function on $\mathfrak{u}(n)$; then $f(F_{\mathfrak{g}}) \in \mathcal{A}_G(M)$ is $d_{\mathfrak{g}}$-closed, and moreover one can show that choosing a different equivariant connection changes $f(F_{\mathfrak{g}})$ by a $d_{\mathfrak{g}}$-exact form. The corresponding cohomology classes are the **equivariant characteristic classes** of the bundle $\mathcal{L} \to M$.

If the action on M is locally free, one can choose ∇ in such a way that $\mu = 0$, which shows that the mapping $H^{\omega}_G(M) \to H(M/G)$ sends the equivariant characteristic classes of $\mathcal{L}$ to the usual characteristic classes of the orbifold bundle $\mathcal{L}/G$.

The following characteristic classes will play an important role:

- The **equivariant Chern character**, defined by

  $$\mathrm{Ch}_{\mathfrak{g}}(\mathcal{L};\xi) := \mathrm{Tr}\left(e^{\frac{i}{2\pi}F_{\mathfrak{g}}(\mathcal{L};\xi)}\right).$$

  In the setting of geometric quantisation, $\mathcal{L}$ is a line bundle, and for the equivariant curvature one has

  $$\frac{i}{2\pi} F_{\mathfrak{g}}(\mathcal{L};\xi) = \omega + 2\pi i \langle J, \xi\rangle,$$

  thus

  $$\mathrm{Ch}_{\mathfrak{g}}(\mathcal{L};\xi) = e^{\omega + 2\pi i \langle J, \xi\rangle}.$$

  More generally, if $g \in G$ acts trivially on the base M, one defines

  $$\mathrm{Ch}^g_{\mathfrak{g}}(\mathcal{L};\xi) = \mathrm{Tr}\left(\rho(g) e^{\frac{i}{2\pi}F_{\mathfrak{g}}(\mathcal{L};\xi)}\right),$$

  where $\rho(g) \in \Gamma(M; \mathrm{End}(\mathcal{L}))$ is the induced action of g on $\mathcal{L}$.

  In the line bundle case, this s simply $c_{\mathcal{L}}(g) \cdot \mathrm{Ch}_{\mathfrak{g}}(\mathcal{L};\xi)$, where $c_L(g) \in S^1$ is the action of g on the fibres.

- The **equivariant Todd class**,

$$\mathrm{Td}_{\mathfrak{g}}(\mathcal{V};\xi) := \det\left(\frac{\frac{i}{2\pi}F_{\mathfrak{g}}(\mathcal{V};\xi)}{\left(1-e^{-\frac{i}{2\pi}F_{\mathfrak{g}}(\mathcal{V};\xi)}\right)}\right).$$

  The Todd class of a complex manifold is defined as the Todd class of its tangent bundle.

- The **equivariant Euler class**,

$$\chi_{\mathfrak{g}}(\mathcal{V};\xi) := \det\left(\frac{i}{2\pi}F_{\mathfrak{g}}(\mathcal{V};\xi)\right).$$

- The class

$$D^{g}_{\mathfrak{g}} := \det\left(\mathrm{Id} - \rho(g)^{-1}\cdot e^{-\frac{i}{2\pi}F_{\mathfrak{g}}(\mathcal{V};\xi)}\right),$$

  for $g \in G$ acting trivially on M.

The equivariant Euler class occurs in the Localisation Formula for Atiyah-Bott-Berline-Vergne:

Theorem 5.1. *Suppose that M is an orientable G-manifold, and $\sigma \in \mathcal{A}^{\omega}_{G}(M)$ is $d_{\mathfrak{g}}$-closed. Assume that $\xi \in \mathfrak{g}$ is in the domain of definition of $\sigma(\xi)$, and let F be the set of zeros of ξ_M. The connected components of F are then smooth submanifolds of even codimension, and the normal bundle N_F admits a Hermitian structure which is invariant under the flow of ξ_M. Choose orientations on F and M which are compatible with the corresponding orientation of N_F. Then*

$$\int_M \sigma(\xi) = \int_F \frac{\imath_F^{*}\sigma(\xi)}{\chi_{\mathfrak{g}}(N_F;\xi)},$$

where $\imath_F : F \hookrightarrow M$ denotes the embedding.

5.1.2 C.f. ‘*The Heat Kernel Lefschetxz Fixed-Point Formula*’, [6]

Lemma 5.2. *The linear transformations $\gamma_{L,x}$ and $\gamma_{N,x}$ in L_x and N_x, respectively, are diagonalisable. The respective eigenvalues $\lambda_{L,j}$ and $\lambda_{N,k}$ are constant when x varies in a connected component F of M^{γ}, the fixed-point set of γ in M. The corresponding eigenspaces $L_{j,x} \subset L_x$ and $N_{k,x} \subset N_x$, with x ranging over F, form smooth complex vector subbundles L_j and N_j of L and N, respectively.*

Proof. We may assume that γ_L belongs to a compact group K of transformations[1]. Now let H be the closure in K of the set of γ^p with $p \in \mathbb{Z}$. Then H is a compact abelian group of automorphisms of our structure, and $h(x) = x$ for each $x \in M^{\gamma}$, $h \in H$. We therefore have, for each $x \in M^{\gamma}$, the complex representations

$$h \longmapsto T_x h_M\big|_{N_x}$$

and

$$h \longmapsto h_{L,x}$$

of H in N_x and L_x, respectively.

□

[1] In Section 10.1 of [6], it is remarked that the isometry group of a fixed-point is compact.

$z_1^4w_3 \quad z_1^3z_2w_3 \quad z_1^2z_2^2w_3 \quad z_1z_2^3w_3 \quad z_2^4w_3$

$z_2^4w_1 \quad z_2^3z_3w_1 \quad z_2^2z_3^2w_1 \quad z_2z_3^3w_1 \quad z_3^4w_1$

$z_3^4w_2 \quad z_3^3z_1w_2 \quad z_3^2z_1^2w_2 \quad z_3z_1^3w_2 \quad z_1^4w_2$

6 Personal Calculations

6.1 Index Formulae

Recall from Lemma (2.1) and (2.2):

$$\mathrm{Res}_x \frac{f(x)}{(x-z_1)\dots(x-z_d)} = \sum_{i=1}^{d} \frac{f(z_i)}{\prod_{j\neq i}(z_i - z_j)}.$$

and

$$\frac{1}{2\pi i}\int_\Gamma z^{n+k-1}\det(z-B)^{-1}dz = \mathrm{Tr}\left(\tau_k(B)\right),$$

for Γ a contour circling the origin and the zeros of $\det(z-B)$.

6.1.1 $\mathrm{Ind}(\mathbb{CP}^2, \mathcal{O}(k), T^2)$

$$\begin{aligned}\mathrm{Ind}(\mathbb{CP}^2, \mathcal{O}(k), T^2)(z_1,z_2) &= \frac{1}{(1-z_1)(1-z_2)} + \frac{z_1^k}{(1-z_1^{-1})(1-z_1^{-1}z_2)} + \frac{z_2^k}{(1-z_2^{-1})(1-z_2^{-1}z_1)} \\ &= \frac{1}{(1-z_1)(1-z_2)} + \frac{z_1^{k+2}}{(z_1-1)(z_1-z_2)} + \frac{z_2^{k+2}}{(z_2-1)(z_2-z_1)}\end{aligned}$$

6.2 Representation Theory of Polynomial Rings

6.2.1 $k=3$, $a=2$ Polyptych

Set

$$V := \mathbb{C}[z_1,z_2,z_3]$$

, the polynomial

$$\mathbb{C}$$

-algebra in three variables.

$a=0$ monomials:

$$\begin{array}{ccccccc} z_1^3 & & z_1^2z_2 & & z_1z_2^2 & & z_2^3 \\ & z_1^2z_3 & & z_1z_2z_3 & & z_2^2z_3 & \\ & & z_1z_3^2 & & z_2z_3^2 & & \\ & & & z_3^3 & & & \end{array}$$

So we have $S_3[V]$

$a=1$ monomials:

$a=2$ monomials:

$a=3$ monomials:

$$
\begin{array}{rcl}
\begin{pmatrix} z_1^5 & z_1^4 z_2 & z_1^3 z_2^2 & z_1^2 z_2^3 & z_1 z_2^4 & z_2^5 \end{pmatrix} & \times & w_3^2 \\
\begin{pmatrix} z_2^5 & z_2^4 z_3 & z_2^3 z_3^2 & z_2^2 z_3^3 & z_2 z_3^4 & z_3^5 \end{pmatrix} & \times & w_1^2 \\
\begin{pmatrix} z_3^5 & z_3^4 z_1 & z_3^3 z_1^2 & z_3^2 z_1^3 & z_3 z_1^4 & z_1^5 \end{pmatrix} & \times & w_2^2 \\
\begin{pmatrix} z_1^5 \end{pmatrix} & \times & w_2 w_3 \\
\begin{pmatrix} z_2^5 \end{pmatrix} & \times & w_1 w_3 \\
\begin{pmatrix} z_3^5 \end{pmatrix} & \times & w_1 w_2
\end{array}
$$

$$
\begin{array}{rcl}
\begin{pmatrix} z_1^6 & z_1^5 z_2 & z_1^4 z_2^2 & z_1^3 z_2^3 & z_1^2 z_2^4 & z_1 z_2^5 & z_2^6 \end{pmatrix} & \times & w_3^3 \\
\begin{pmatrix} z_2^6 & z_2^5 z_3 & z_2^4 z_3^2 & z_2^3 z_3^3 & z_2^2 z_3^4 & z_2 z_3^5 & z_3^6 \end{pmatrix} & \times & w_1^3 \\
\begin{pmatrix} z_3^6 & z_3^5 z_1 & z_3^4 z_1^2 & z_3^3 z_1^3 & z_3^2 z_1^4 & z_3 z_1^5 & z_1^6 \end{pmatrix} & \times & w_2^3 \\
\begin{pmatrix} z_1^6 \end{pmatrix} & \times & \begin{pmatrix} w_2^2 w_3 & w_2 w_3^2 \end{pmatrix} \\
\begin{pmatrix} z_2^6 \end{pmatrix} & \times & \begin{pmatrix} w_1^2 w_3 & w_1 w_3^2 \end{pmatrix} \\
\begin{pmatrix} z_3^6 \end{pmatrix} & \times & \begin{pmatrix} w_3^2 w_1 & w_3 w_1^2 \end{pmatrix}
\end{array}
$$

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