

INDEX THEOREMS - NOTES

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1. INDEX THEORY

1.1. Non-Equivariant Index Formula. For a holomorphic vector bundle \mathcal{L} over a complex n -dimensional variety M , the *index* $\text{ind}(\bar{\partial}, \mathcal{L})$ is defined as

$$\text{ind}(\bar{\partial}, \mathcal{L}) := \sum_{k=0}^n (-1)^k \dim H^k(M; \mathcal{L}).$$

Viewing the index $\text{ind}(\bar{\partial}, \mathcal{L})$ as the Euler characteristic $\chi(M, \mathcal{L})$ of the vector bundle \mathcal{L} , we can apply the Atiyah-Singer index theorem, which we state below, to express the index as an integral over M of the product of the Todd class $\text{Td}(TM)$ of the tangent bundle $TM \rightarrow M$ over M , and the Chern character $\text{Ch}(\mathcal{L}) := \exp(c_1(\mathcal{L}))$ of \mathcal{L} , where $c_1(\mathcal{L})$ is the first Chern class of \mathcal{L} .

Theorem 1.1 (Atiyah-Singer Index Theorem, [1]). *Let M be a compact complex manifold, \mathcal{L} a holomorphic vector bundle over M . Let*

$$\text{Td}(TM) = \prod \frac{x_i}{1 - e^{-x_i}}$$

be the Todd class of the complex vector bundle $TM \rightarrow M$, where the x_i are the Chern roots of TM . Then the Euler characteristic $\chi(M, \mathcal{L})$ of the sheaf of germs of holomorphic sections of \mathcal{L} is given by

$$\chi(M, \mathcal{L}) = \int_M \text{Td}(M) \cdot \text{Ch}(\mathcal{L}).$$

Example. Let $M = \mathbb{CP}^1$ and let \mathcal{L} be the line bundle $\mathcal{O}(k)$ for some positive integer k . If $\langle \xi \rangle = H^2(M; \mathbb{Z})$, i.e. ξ is the generator of $H^2(\mathbb{CP}^1; \mathbb{Z})$, then $c_1(\mathcal{L}) = k\xi$, and thus the Chern character of \mathcal{L} is

$$\text{Ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = \sum_{j=0}^{\infty} (k\xi)^j = 1 + k\xi$$

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(the higher powers of ξ vanish since $\dim_{\mathbb{C}} M = 1$).

For n -dimensional complex projective space \mathbb{CP}^n , both the total Chern class

$$c(\mathbb{CP}^n) := c(T\mathbb{CP}^n) := 1 + c_1 + c_2 + c_3 + \dots,$$

and the Todd class $\text{Td}(T\mathbb{CP}^n)$ for the tangent bundle $T\mathbb{CP}^n \rightarrow \mathbb{CP}^n$, can be calculated using the exact Euler sequence, along with the multiplicativity of

$$\{0\} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{CP}^n \longrightarrow \{0\},$$

the total Chern class and the Todd class,

$$c(\mathcal{F} \oplus \mathcal{G}) = c(\mathcal{F}) \cdot c(\mathcal{G}), \quad \text{Td}(\mathcal{F} \oplus \mathcal{G}) = \text{Td}(\mathcal{F}) \cdot \text{Td}(\mathcal{G}),$$

which yields

$$c(\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \mathcal{O}) = c(\mathcal{O}(1)^{\oplus(n+1)}) = (1 + \xi)^{n+1},$$

and

$$\text{Td}(T\mathbb{CP}^n) = \text{Td}(T\mathbb{CP}^n \oplus \mathcal{O}) = \text{Td}(\mathcal{O}(1)^{\oplus(n+1)}) = \text{Td}(\mathcal{O}(1))^{n+1} = \left(\frac{\xi}{1 - e^{-\xi}} \right)^{n+1}.$$

This expression can be expanded as a formal power series which, for $n = 1$ in our example with the complex projective line \mathbb{CP}^1 , gets us

$$c(\mathbb{CP}^1) = (1 + \xi)^2 = 1 + 2\xi, \quad \text{Td}(T\mathbb{CP}^1) = 1 + \frac{1}{2}c_1(T\mathbb{CP}^1) = 1 + \xi.$$

Finally, applying the Atiyah-Singer index theorem 1.1, we have

$$\chi(\mathbb{CP}^1, \mathcal{L}) = \int_{\mathbb{CP}^1} \text{Td}(\mathbb{CP}^1) \cdot \text{Ch}(\mathcal{L}) = \int_{\mathbb{CP}^1} (1 + \xi) \cdot (1 + k\xi) = \int_{\mathbb{CP}^1} 1 + (k+1)\xi = k+1.$$

Example. Now we let $M = \mathbb{CP}^2$, and let $\mathcal{L} = \mathcal{O}(k)$ and $\langle \xi \rangle = H^2(M, \mathbb{Z})$ again as above. Now we have

$$c(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + k\xi + k^2\xi^2,$$

and

$$\begin{aligned} c(T\mathbb{CP}^2) &= 1 + c_1 + c_2 = (1 + \xi)^3 = 1 + 3\xi + 3\xi^2, \\ \text{Td}(T\mathbb{CP}^2) &= 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} = 1 + \frac{3}{2}\xi + \frac{9\xi^2 + 3\xi^2}{12} = 1 + \frac{3}{2}\xi + \xi^2. \end{aligned}$$

Hence by the Atiyah-Bott index theorem 1.1,

$$\begin{aligned}\chi(M, \mathcal{L}) &= \int_M \text{Td}(TM) \cdot \text{Ch}(\mathcal{L}) = \int_M (1 + \tfrac{3}{2}\xi + \xi^2) \cdot (1 + k\xi + k^2\xi^2) \\ &= \int_M (k^2 + \tfrac{3}{2}k + 1)\xi^2 + O(\xi) = k^2 + \tfrac{3}{2}k + 1.\end{aligned}$$

Example. Let $M = \mathbb{CP}^3$, and let \mathcal{L} , ξ , etc. be as above. Then

$$\begin{aligned}\text{Ch}(\mathcal{L}) &= 1 + k\xi + (k\xi)^2 + (k\xi)^3, \\ c(TM) &= (1 + \xi)^4 = 1 + 4\xi + 6\xi^2 + 4\xi^3, \\ \text{Td}(TM) &= 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1c_2}{24} = 1 + 2\xi + \frac{11}{6}\xi^2 + \xi^3.\end{aligned}$$

Then by the Atiyah-Bott Index theorem 1.1,

$$\begin{aligned}\chi(M, \mathcal{L}) &= \int_M \text{Td}(TM) \cdot \text{Ch}(\mathcal{L}) = \int_M \left(1 + 2\xi + \frac{11}{6}\xi^2 + \xi^3\right) \cdot (1 + k\xi + k^2\xi^2 + k^3\xi^3) \\ &= \int_M \left(k^3 + 2k^2 + \frac{11}{6}k + 1\right) \xi^3 + O(\xi^2) =\end{aligned}$$

1.2. Equivariant Index Theorems.

1.2.1. Equivariant Characteristic Classes.

REFERENCES

- [1] M. F. Atiyah and I. M. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87:546–604, 1968.

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