Hypertoric Manifolds

General Notes

Abstract

Preprint on toric hyperkähler manifolds.

1 Toric Hyperkähler Manifolds

1.1 Symplectic Quotients, [1]

Fix the standard Euclidean bilinear form on $\mathbb{C}^n$,

$$g(z,w) = \sum_{i=1}^{n} \left(\Re(z_i)\Re(w_i) + \Im(z_i)\Im(w_i)\right).$$

The corresponding Kähler form is

$$\omega(z,w) = g(iz,w) = \sum_{i=1}^{n} \left(\Re(z_i)\Im(w_i) - \Im(z_i)\Re(w_i)\right).$$

Let $A = [u_1, \ldots, u_n]$ be a $(d \times n)$-matrix whose $(d \times d)$-minors are relatively prime. Choose now an $n \times (n-d)$-matrix $B = [b_1, \ldots, b_n]^T$ that makes the following sequence exact:

$$\{0\} \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow \{0\}.$$

The choice of B is equivalent to choosing a basis in $\ker(A)$.

1.2 Hyperkähler Quotients

Let $\mathbb{H}$ be the quaternions, the 4-dimensional $\mathbb{R}$-vector space with basis $\{1, i, j, k\}$ equipped with an associative algebra structure defined by

$$i^2 = j^2 = k^2 = ijk = -1.$$

Left-multiplication by i (respectively j and k) define the following respective complex structures on $\mathbb{H}$,

$$I, J, K : \mathbb{H} \longrightarrow \mathbb{H}; \qquad I^2 = J^2 = K^2 = IJK = -\operatorname{Id}_{\mathbb{H}}.$$

Equipping $\mathbb{H}$ with the flat metric g arising from the standard Euclidean scalar-product on $\mathbb{H} \cong \mathbb{R}^4$, with $\{1, i, j, k\}$ providing an orthonormal basis. This is called a *hyperkähler metric* since it is a Kähler metric with respect to each individual complex structure, I, J, and K. This also means that the so-called *Kähler forms*, given by

$$\omega_I(X,Y) = g(IX,Y), \qquad \omega_J(X,Y) = g(JX,Y), \qquad \omega_K(KX,Y) = g(KX,Y), \qquad \text{for tangent vectors } X, Y,$$

are closed differential 2-forms.

A special orthogonal transformation with respect to this metric is said to *preserve the hyperkähler structure* if it commutes with all three complex structures, I, J, and K; or equivalently, it preserves the Kähler forms, ω_I, ω_J, and ω_K. The group of such transformations, the *unitary symplectic group* $\mathrm{Sp}(1)$, is generated by the right-multiplication action by the unit quaternions.

A maximal abelian subgroup $T^1_{\mathbb{R}} \cong \mathrm{U}(1) \subset \mathrm{Sp}(1)$ is then specified by a choice of unit quaternion, and we break the I, J, K symmetry by choosing a maximal torus, generated by right-multiplication by the unit quaternion i. Hence $\mathrm{U}(1)$ acts on $\mathbb{H}$ from the right by sending

$$\xi \mapsto \xi \exp(ti), \qquad \exp(ti) \in \mathrm{U}(1) \subset \mathbb{R} \oplus \mathbb{R}i \cong \mathbb{C}.$$

The moment map for this action $\mu_1 : \mathbb{H} \to \mathbb{R}$ with respect to the symplectic form ω_1 is then given by

$$\mu_1(x + yi + uj + vk) = \mu_1\left((x + yi) + (v - ui)k\right) = \frac{1}{2}\left(x^2 + y^2 - u^2 - v^2\right).$$

1.3

Proposition 1.1 ([2]). *Suppose that α and $(\alpha, 0)$ are regular values for μ and μ_{HK}, respectively. Then the cotangent bundle T^*X is isomorphic to an open subset of M, and is dense if it is non-empty.*

Proof. Let $Y = \left\{ (z,w) \in \mu_{\mathbb{C}}^{-1}(0)^{\mathrm{st}} \;\middle|\; z \in (\mathbb{C}^n)^{\mathrm{st}} \right\}$, where z is semi-stable with respect to α for the $G_{\mathbb{C}}$-action on $\mathbb{C}^n$, so that we have $X \cong (\mathbb{C}^n)^{\mathrm{st}}/G_{\mathbb{C}}$. Let $[z] \in X$ be the representative of $z \in (\mathbb{C}^n)^{\mathrm{st}}$. The tangent space $T_{[z]}X$ is equal to the quotient of $T_z\mathbb{C}^n$ by the tangent space to the $G_{\mathbb{C}}$-orbit through z,

$$T_{[z]}X = T_z\mathbb{C}^n / T_{[z]}(G_{\mathbb{C}} \cdot z).$$

Therefore,

$$T^*_{[z]}X \cong \{ w \in T^*_z\mathbb{C}^n \mid w(\hat{v}_z) = 0, \text{ for all } v \in \mathfrak{g}_{\mathbb{C}} \} = \{ w \in (\mathbb{C}^n)^* \mid \mu_{\mathbb{C}}(z,w) = 0 \}.$$

Then, by letting $[z] \in X$ vary, we have

$$T^*X \cong \left\{ (z,w) \;\middle|\; z \in (\mathbb{C}^n)^{\mathrm{st}} \text{ and } \mu_{\mathbb{C}}(z,w) = 0 \right\} / G_{\mathbb{C}} = Y/G_{\mathbb{C}}.$$

As each z-coordinate in Y is semi-stable, Y is an open subset of $\mu_{\mathbb{C}}^{-1}(0)$, and is dense if non-empty. $\square$

2 Cotangent Spaces to Extended Core Components

Let $M_\lambda = \left(\mu_{\mathbb{R}}^{-1}(\lambda) \cap \mu_{\mathbb{C}}^{-1}(0)\right)/K$ be a toric hyperkähler manifold. Define

$$\mathbb{C}_A := \left\{ (z_i, w_i) \in \mathbb{C}^{2n} \;\middle|\; w_i = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \right\} \cong \mathbb{C}^n \subset \mathbb{H}^n.$$

Lemma 2.1 ([3]). *Let M_λ be a toric hyperkähler manifold. If $\mathcal{E}_A$ is non-empty, then its holomorphic cotangent bundle $T^*\mathcal{E}_A$ is contained in M_λ as an open subset.*

Fix a subset $A \subset \{ 1, \dots, n \}$, and define

$$(x_i^{(A)}, y_i^{(A)}) := \begin{cases} (z_i, w_i), & \text{if } i \in A, \\ (w_i, -z_i), & \text{if } i \notin A. \end{cases}$$

Then $x^{(A)} = (x_1^{(A)}, \dots, x_n^{(A)})$ is a point in the vector space $\mathbb{C}^n_A$, and $y^{(A)} = (y_1^{(A)}, \dots, y_n^{(A)})$ is a point in the dual space $(\mathbb{C}^n_A)^*$. That is, we identify the cotangent bundle $T^*\mathbb{C}^n_A$ with $\mathbb{H}^n$ as above.

2.1 Kähler Quotients

The Kähler quotient $X = \mu^{-1}(0)/N$ can be identified with the quotient of an open subset of $\mathbb{C}^n$ by the complexified torus $N^{\mathbb{C}}$ as follows: every orbit in $\mathbb{C}^n$ of $T^n_{\mathbb{C}}$ is of the form

$$\mathbb{C}^n_A = \{ (z_1, \dots, z_n) \mid z_i = 0 \text{ if } i \in A \},$$

for some subset $A \subset \{1, \dots, n\}$. If F is a face of Δ of codimension r, then F is defined by the intersection of r hyperplanes $\cap_{j=1}^r H_{i_j}$.

3 Symplectic Cutting

3.1 Compactifying the Extended Core

Let S^1 act on M by rotating the cotangent fibres, that is, for $\tau \in S^1$,

$$\tau \cdot [z; w] = [z; \tau w].$$

This S^1-action is Hamiltonian, with moment map

$$\Phi : M \longrightarrow (\mathbb{R})^*; \qquad [z : w] \longmapsto \frac{1}{2}\|w\|^2.$$

Let S^1_A denote the residual S^1-action on M restricted to the extended core component

$$\mathcal{E}_A = \{ [z_1 : \dots z_n; w_1, \dots, w_n] \mid w_0 = 0 \text{ if } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \}.$$

Now the *global* S^1-action does not act on the cotangent fibres of M as a subtorus of T^n, but it does when *restricted* to each component of the extended core, $\mathcal{E}_A$. Indeed,

$$\tau \cdot [z; w] = [z; \tau w] = [z_1 : \ldots : z_n; \tau w_1 : \ldots : \tau w_n] = [\tau_1 z_1 : \ldots : \tau_n z_n; \tau_1^{-1} w_1 : \ldots : \tau_n^{-1} w_n],$$

where

$$\tau_i := \begin{cases} \tau^{-1}, & \text{if } i \in A, \\ 1, & \text{if } i \notin A, \end{cases}$$

which shows that the S^1-action restricted to each individual $\mathcal{E}_A$ acts as a subtorus of the original torus T^n.

Denote by S^1_A the image of S^1 in T^n when considered as a subtorus restricted to each individual $\mathcal{E}_A$, and let $\jmath_A : S^1 \hookrightarrow T^n$ be the respective inclusion homomorphism, so we have $S^1_A := \jmath_A(S^1) \lhd T^n$.

On the Lie algebra level, we have that

$$(\jmath_A)_* : \mathrm{Lie}(S^1_A) \longrightarrow \mathfrak{t}^n; \qquad \xi \longmapsto (\xi_1, \ldots, \xi_n),$$

where we analogously define

$$\xi_i := \begin{cases} -1, & \text{if } i \in A, \\ 0, & \text{if } i \notin A. \end{cases}$$

Since S^1_A acts as the subtorus $\jmath_A(S^1)$ of T^n on each $\mathcal{E}_A$, the moment map $\Phi_A := \Phi\big|_{\mathcal{E}_A}$ for this action is given by composing $\mu_\mathbb{R}$ with the dual of the inclusion $(\jmath_A)_*$, so

$$\begin{aligned} \Phi_A[z, w] = (\jmath_A^* \circ \mu_\mathbb{R})\,[z; w] &= \jmath_A^* \left(\frac{1}{2} \sum_{i=1}^n \left(|z_i|^2 - |w_i|^2 \right) e^i \right) \\ &= -\frac{1}{2} \sum_{i \in A} |z_i|^2 \jmath_A^*(e^i) \\ &= \frac{1}{2} \sum_{i \notin A} |w_i|^2 \jmath_A^*(e^i) \\ &= \langle \mu_\mathbb{R}[z; w],\, \xi_A \rangle \\ &= \mu_\mathbb{R}^A[z; w], \end{aligned}$$

where $\xi_A = -\sum_{i \in A} \xi_i$, and $\mu_\mathbb{R}^A[z; w]$ is the component of $\mu_\mathbb{R}[z; w]$ in the ξ_A-direction.

3.2 Moment Polyptychs

Distinct S^1_A actions have common zeros?

The holomorphic moment map $\bar{\mu}_\mathbb{C} : M \to (\mathfrak{t}^d_\mathbb{C})^*$ is $\mathbb{C}^*$-equivariant with respect to the $\mathbb{C}^*$-scalar action on $(\mathfrak{t}^d_\mathbb{C})^*$, hence both $M^{\mathbb{C}^*}$ and $\mathcal{L}$ will be contained in

$$\mathcal{E} := \bar{\mu}_\mathbb{C}^{-1}(0) = \left\{\, [z; w] \in M \mid z_i w_i = 0 \text{ for all } i = 1, \ldots, n \,\right\},$$

which is called the **extended core** of M, and $\bar{\mu}_\mathbb{R}$ surjects onto $\mathcal{E}$. The extended core $\mathcal{E}$ breaks into the components

$$\mathcal{E}_A := \left\{\, [z; w] \in M \mid w_i = 0 \text{ if i } i \in A, \text{ and } z_i = 0 \text{ if } i \notin A \,\right\},$$

indexed by subsets $A \subseteq \{1, \ldots, n\}$.

The hyperplanes $\{H_i\}_{i=1}^n$ divide $(\mathfrak{t}^d)^*$ into a union of convex, possibly empty, possibly unbounded, polyhedra

$$\Delta_A := \bigcap_{i \in A} F_i \cap \bigcap_{i \notin A} G_i,$$

where we recall that

$$\begin{aligned} F_i &= \left\{\, v \in (\mathfrak{t}^d)^* \;\middle|\; \langle v,\, u_i \rangle + r_i \geq 0 \,\right\}, \\ G_i &= \left\{\, v \in (\mathfrak{t}^d)^* \;\middle|\; \langle v,\, u_i \rangle + r_i \leq 0 \,\right\}, \\ H_i &= \left\{\, v \in (\mathfrak{t}^d)^* \;\middle|\; \langle v,\, u_i \rangle + r_i = 0 \,\right\} = F_i \cap G_i. \end{aligned}$$

Lemma 3.1 ([2]). *If $w_i = 0$, then $\bar{\mu}_\mathbb{R}[z; w] \in F_i$, whereas if $z_i = 0$, then $\bar{\mu}_\mathbb{R}[z; w] \in G_i$.*

Proof. Suppose that $\bar{\mu}_\mathbb{R}[z; w] = v \in (\mathfrak{t}^d)^*$. Then

Work out this sequence of equivalences.

$$\langle \bar{\mu}_\mathbb{R}[z; w],\, u_i \rangle \iff \langle \bar{\mu}_\mathbb{R}[z; w],\, u_i \rangle$$

□

References

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