HYPERTORIC MANIFOLDS AND EQUIVARIANT LOCALISATION

BENJAMIN C. W. BROWN

1. Index Theory

1.1. **Non-Equivariant Index Formula.** For a holomorphic vector bundle $\mathcal{L}$ over a complex n-dimensional variety M, the *index* $\mathrm{ind}(\bar{\partial}, \mathcal{L})$w is defined as

$$\mathrm{ind}(\bar{\partial}, \mathcal{L}) := \sum_{k=0}^{n} (-1)^k \dim H^k(M; \mathcal{L}).$$

Viewing the index $\mathrm{ind}(\bar{\partial}, \mathcal{L})$ as the Euler characteristic $\chi(M, \mathcal{L})$ of the vector bundle $\mathcal{L}$, we can apply the Atiyah-Singer index theorem, which we state below, to express the index as an integral over M of the product of the Todd class $\mathrm{Td}(TM)$ of the tangent bundle $TM \to M$ over M, and the Chern character $\mathrm{Ch}(\mathcal{L}) := \exp(c_1(\mathcal{L}))$ of $\mathcal{L}$, where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$.

Theorem 1.1 (Atiyah-Singer Index Theorem, [**?**])**.** *Let M be a compact complex manifold, $\mathcal{L}$ a holomorphic vector bundle over M. Let*

$$\mathrm{Td}(TM) = \prod \frac{x_i}{1 - e^{-x_i}}$$

be the Todd class of the complex vector bundle $TM \to M$, where the x_i are the Chern roots of TM. Then the Euler characteristic $\chi(M, \mathcal{L})$ of the sheaf of germs of holomorphic sections of $\mathcal{L}$ is given by

$$\chi(M, \mathcal{L}) = \int_M \mathrm{Td}(M) \cdot \mathrm{Ch}(\mathcal{L}).$$

Example. Let $M = \mathbb{CP}^1$ and let $\mathcal{L}$ be the line bundle $\mathcal{O}(k)$ for some positive integer k. If $\langle \xi \rangle = H^2(M; \mathbb{Z})$, *i.e.* ξ is the generator of $H^2(\mathbb{CP}^1; \mathbb{Z})$, then $c_1(\mathcal{L}) = k\xi$, and thus the Chern character of $\mathcal{L}$ is

$$\mathrm{Ch}(\mathcal{L}) = e^{c_1(\mathcal{L})} = \sum_{j=0}^{\infty} (k\xi)^j = 1 + k\xi$$

Date: March 6, 2021.

(the higher powers of ξ vanish since $\dim_{\mathbb{C}} M = 1$).

For n-dimensional complex projective space $\mathbb{CP}^n$, both the total Chern class

$$c(\mathbb{CP}^n) := c(T\mathbb{CP}^n) := 1 + c_1 + c_2 + c_3 + \dots,$$

and the Todd class $\mathrm{Td}(T\mathbb{CP}^n)$ for the tangent bundle $T\mathbb{CP}^n \to \mathbb{CP}^n$, can be calculated using the exact Euler sequence, along with the multiplicativity of

$$\{0\} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus(n+1)} \longrightarrow T\mathbb{CP}^n \longrightarrow \{0\},$$

the total Chern class and the Todd class,

$$c(\mathcal{F} \oplus \mathcal{G}) = c(\mathcal{F}) \cdot c(\mathcal{G}), \qquad \mathrm{Td}(\mathcal{F} \oplus \mathcal{G}) = \mathrm{Td}(\mathcal{F}) \cdot \mathrm{Td}(\mathcal{G}),$$

which yields

$$c(\mathbb{CP}^n) = c(T\mathbb{CP}^n \oplus \mathcal{O}) = c(\mathcal{O}(1)^{\oplus(n+1)}) = (1+\xi)^{n+1},$$

and

$$\mathrm{Td}(T\mathbb{CP}^n) = \mathrm{Td}(T\mathbb{CP}^n \oplus \mathcal{O}) = \mathrm{Td}(\mathcal{O}(1)^{\oplus(n+1)}) = \mathrm{Td}\left(\mathcal{O}(1)\right)^{n+1} = \left(\frac{\xi}{1-e^{-\xi}}\right)^{n+1}.$$

This expression can be expanded as a formal power series which, for $n = 1$ in our example with the complex projective line $\mathbb{CP}^1$, gets us

$$c(\mathbb{CP}^1) = (1+\xi)^2 = 1 + 2\xi, \qquad \mathrm{Td}(T\mathbb{CP}^1) = 1 + \tfrac{1}{2}c_1(T\mathbb{CP}^1) = 1 + \xi.$$

Finally, applying the Atiyah-Singer index theorem 1.1, we have

$$\chi(\mathbb{CP}^1, \mathcal{L}) = \int_{\mathbb{CP}^1} \mathrm{Td}(\mathbb{CP}^1) \cdot \mathrm{Ch}(\mathcal{L}) = \int_{\mathbb{CP}^1} (1+\xi) \cdot (1+k\xi) = \int_{\mathbb{CP}^1} 1 + (k+1)\xi = k+1.$$

Example. Now we let $M = \mathbb{CP}^2$, and let $\mathcal{L} = \mathcal{O}(k)$ and $\langle \xi \rangle = H^2(M, \mathbb{Z})$ again as above. Now we have

$$c(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + k\xi + k^2\xi^2,$$

and

$$c(T\mathbb{CP}^2) = 1 + c_1 + c_2 = (1+\xi)^3 = 1 + 3\xi + 3\xi^2,$$

$$\mathrm{Td}(T\mathbb{CP}^2) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} = 1 + \frac{3}{2}\xi + \frac{9\xi^2 + 3\xi^2}{12} = 1 + \frac{3}{2}\xi + \xi^2.$$

Hence by the Atiyah-Bott index theorem 1.1,

$$\begin{aligned}\chi(M,\mathcal{L}) &= \int_M \mathrm{Td}(TM)\cdot \mathrm{Ch}(\mathcal{L}) = \int_M \left(1+\tfrac{3}{2}\xi+\xi^2\right)\cdot\left(1+k\xi+k^2\xi^2\right)\\ &= \int_M (k^2+\tfrac{3}{2}k+1)\xi^2 + O(\xi) = k^2+\tfrac{3}{2}k+1.\end{aligned}$$

Example. Let $M=\mathbb{CP}^3$, and let $\mathcal{L}$, ξ, etc. be as above. Then

$$\begin{aligned}\mathrm{Ch}(\mathcal{L}) &= 1+k\xi+(k\xi)^2+(k\xi)^3,\\ c(TM) &= (1+\xi)^4 = 1+4\xi+6\xi^2+4\xi^3,\\ \mathrm{Td}(TM) &= 1+\frac{c_1}{2}+\frac{c_1^2+c_2}{12}+\frac{c_1c_2}{24} = 1+2\xi+\frac{11}{6}\xi^2+\xi^3.\end{aligned}$$

Then by the Atiyah-Bott Index theorem 1.1,

$$\begin{aligned}\chi(M,\mathcal{L}) &= \int_M \mathrm{Td}(TM)\cdot \mathrm{Ch}(\mathcal{L}) = \int_M \left(1+2\xi+\frac{11}{6}\xi^2+\xi^3\right)\cdot\left(1+k\xi+k^2\xi^2+k^3\xi^3\right)\\ &= \int_M \left(k^3+2k^2+\frac{11}{6}k+1\right)\xi^3+O(\xi^2) =\end{aligned}$$

1.2. **Equivariant Index Theorems.**

1.2.1. *Equivariant Characteristic Classes.*

2. Compactifying the Hypertoric Variety via Symplectic Cutting

2.1. **Set-Up.** We can use the residual S^1-action to perform a symplectic cut of the toric hyperkähler manifold M in order to compactify it as follows: consider the product $M\times\mathbb{C}$, where now S^1 acts on $M\times\mathbb{C}$ as

$$e^{i\theta}\cdot\big([z,w],\xi\big) = \big([z,e^{i\theta}w],e^{i\theta}\xi\big),$$

which is a Hamiltonian action with associated moment map

$$\begin{aligned}\mu_{\text{cut}} &: M\times\mathbb{C}\longrightarrow \mathbb{R}_{\geqslant 0},\\ \mu_{\text{cut}}\big([z,w],\xi\big) &= \Phi[z,w]+\tfrac{1}{2}|\xi|^2-\epsilon,,\end{aligned}$$

where $\Phi: M\to\mathbb{R}_{\geqslant 0}$ is the moment map $\Phi[z,w]=\frac{1}{2}\|w\|^2$ for the residual S^1-action on M, and $\epsilon\in\mathbb{R}_{\geqslant 0}$.

Then we have

$$\begin{aligned}
\mu_{\text{cut}}^{-1}(0) &= \left\{([z,w],\xi) \in M \times \mathbb{C} : \|w\|^2 + |\xi|^2 = 2\epsilon\right\} \\
&= \left\{[z,w] \in M : \|w\|^2 = 2\epsilon\right\} \bigsqcup \left\{([z,w],\xi) \in M \times \mathbb{C} : |\xi| = \pm\sqrt{2\epsilon - \|w\|^2}\right\} \\
&= \left\{[z,w] \in M : \|w\|^2 = 2\epsilon\right\} \bigsqcup \left\{([z,w],\xi) \in M \times \mathbb{C} : \xi = e^{i\arg(\xi)}\sqrt{2\epsilon - \|w\|^2}\right\} \\
&= \Phi^{-1}(\epsilon) \bigsqcup (M \times S^1) \\
&=: \Sigma_1 \bigsqcup \Sigma_2,
\end{aligned}$$

where Σ_1 is just the level-set of Φ at the level ϵ in M, and $\Sigma_2 = M \times S^1$ is exhibited as a trivial S^1-bundle over Σ_2, using the globally defined section

$$M \to M \times S^1, \qquad [z,w] \longmapsto \left([z,w], e^{i\theta}\sqrt{2\epsilon - \|w\|^2}\right), \qquad e^{i\theta} \in S^1.$$

Finally, by taking the quotient of $\mu_{\text{cut}}^{-1}(0)$ by the S^1-action, we obtain the *symplectic cut of M at level ϵ*,

$$M_{\leqslant\epsilon} := \mu_{\text{cut}}^{-1}(0)/S^1 = \Sigma_1/S^1 \bigsqcup \Sigma_2/S^1,$$

where $\Sigma_1/S^1 = \Phi^{-1}(\epsilon)/S^1$ is just the symplectic reduction, and where Σ_2/S^1 is diffeomorphic to M for $\|w\|^2 < 2\epsilon$, which we denote by $M_{<\epsilon}$.

2.2. **Restriction to the Extended Core Component, $\mathcal{E}_A$.** Since the residual circle S^1-action acts as a subgroup of the original torus T^n when restricted to each component $\mathcal{E}_A$ of the extended core $\mathcal{E}$, we can describe the resulting configuration of the hyperplane arrangement in $(\mathbb{R}^d)^*$ from taking the cut in a combinatorial way. For each component, let $j_A : \mathfrak{s}^1 \to \mathbb{R}^n$ be the derivative of the inclusion of S^1 into T^n on the Lie algebra level, that is

$$j_A(\xi) = (\xi_1, \ldots, \xi_n), \qquad \text{where } \xi_i = \begin{cases} -1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A, \end{cases}$$

so that its image in $\mathbb{R}^n$ generates a circle subgroup S^1 in T^n that depends on each component $\mathcal{E}_A$. Then the moment map for this restriction for the S^1-action is

$$\Phi[z,w] = j_A^* \circ \mu_{\mathbb{R}}[z,w] = \left\langle \mu_{\mathbb{R}}(z,w), \sum_{i\in A} \xi_i u_i \right\rangle,$$

and so from our above discussion of how we constructed the symplectic cut, the image in $(\mathbb{R}^d)^*$ of the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ is

$$\phi_{\mathbb{R}}(\Phi^{-1}(\epsilon)) = \left\{ y \in \Delta_A : \left\langle y, \sum_{i\in A} \xi_i u_i \right\rangle + \epsilon = 0 \right\} =: H_A$$

which introduces an inward-pointing half-space

$$F_A := \left\{ y \in \Delta_A : \left\langle y, \sum_{i \in A} \hbar_i u_i \right\rangle + \epsilon \geqslant 0 \right\}$$

such that the image of the extended core component $\mathcal{E}_A$ after being compactified is the original convex polytope Δ_A intersected with H_A. One can also see clearly that the symplectic quotient $\Phi^{-1}(\epsilon)/S^1$ has the restricted S^1-action as its stabiliser subgroup since, by definition of H_A, the moment map $\Phi|_{\mathcal{E}_A}$ equals the hyperplane H_A, *i.e.* $\Phi|_{\mathcal{E}_A}$ is constant along $\Phi^{-1}(\epsilon)/S^1$.

Example 1. *In our $M = T^*\mathbb{CP}^2$ example, for each component $\mathcal{E}_A$ of the extended core $\mathcal{E}$, we have:*

$$\begin{aligned}
\mathcal{E}_{123} &= \{[z_1, z_2, z_3; 0, 0, 0] \in M\}; & S^1_{123} &= \{(\tau, \tau, \tau) : \tau \in S^1\} < T^3,\\
\mathcal{E}_{12} &= \{[z_1, z_2, 0; 0, 0, w_3] \in M\}; & S^1_{12} &= \{(\tau, \tau, 1) : \tau \in S^1\} < T^3,\\
\mathcal{E}_{23} &= \{[0, z_2, z_3; w_1, 0, 0] \in M\}; & S^1_{23} &= \{(1, \tau, \tau) : \tau \in S^1\} < T^3,\\
\mathcal{E}_{13} &= \{[z_1, 0, z_3; 0, w_2, 0] \in M\}; & S^1_{13} &= \{(\tau, 1, \tau) : \tau \in S^1\} < T^3,\\
\mathcal{E}_{1} &= \{[z_1, 0, 0; 0, w_2, w_3] \in M\}; & S^1_{1} &= \{(\tau, 1, 1) : \tau \in S^1\} < T^3,\\
\mathcal{E}_{2} &= \{[0, z_2, 0; w_1, 0, w_3] \in M\}; & S^1_{2} &= \{(1, \tau, 1) : \tau \in S^1\} < T^3,\\
\mathcal{E}_{3} &= \{[0, 0, z_3; w_1, w_2, 0] \in M\}; & S^1_{3} &= \{(1, 1, \tau) : \tau \in S^1\} < T^3.
\end{aligned}$$

References

[1] Eugene Lerman and Susan Tolman. Hamiltonian torus actions on symplectic orbifolds and toric varieties. *Trans. Amer. Math. Soc.*, 349(10):4201–4230, 1997.

(Benjamin Brown) School of Mathematics and Maxwell Institute, The University of Edinburgh, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom

Email address: B.Brown@ed.ac.uk