Geometric Quantisation

Benjamin C. W. Brown
B.Brown@ed.ac.uk

Women in STEM Society Talk

2nd February 2021

- In classical mechanics, the space of all possible states of a system is given by *phase space*, M.
- "*State*" means not just the *position* but also the *momentum* (of a particle).

Example

For a particle of unit mass, energy function $E : M \to \mathbb{R}$:

$$E(x, p) = \frac{p^2}{2} + V(x).$$

$$\rightsquigarrow \frac{\partial E}{\partial x} = \frac{\partial V}{\partial x} = -\frac{\partial p}{\partial t}, \quad \text{and} \quad \frac{\partial E}{\partial p} = p = \frac{\partial x}{\partial t}.$$

Quantum Mechanics

- Still have phase space M and energy E, but replaces states with *wavefunctions*.
- Classical energy function $E : M \to \mathbb{R}$ now becomes an operator $\hat{E} : M \to M$.

Example

State dynamics for a wavefunction $\psi : M \to \mathbb{C}$ are now described by solutions to *Schrödinger's equation*:

$$\frac{d\psi}{dt} = -\frac{i}{\hbar}\hat{E}\psi.$$

A Question

For a phase space M, what is $\mathcal{Q}(M)$?

- Phase space M represents the position and momentum of a system, with energy function $E : M \to \mathbb{R}$.
- Quantisation $\mathcal{Q}(M)$ should then represent the wavefunctions $\psi : M \to \mathbb{C}$, with operators $\hat{E} : M \to M$.

Fock Space

A good candidate for $\mathcal{Q}(M)$ is *Fock space*:

$$\mathcal{Q}(M) := \left\{ \Psi(z) = e^{-|z|^2}\psi(z) : \psi \text{ holomorphic}, \int_M |\Psi|^2 < \infty \right\}.$$

Quantisation in Action - Preamble

- $M = \mathbb{R}^{2n} \cong \mathbb{C}^2$ via $(x_k, p_k) \rightsquigarrow z_i = p_k + ix_k \in \mathbb{C}^n$.
- Consider the energy function:

$$E : M \to [0, \infty), \qquad E(z) = \frac{\|z\|^2}{2}.$$

- Associated operator is:

$$\mathcal{Q}(\phi) = \sum_{k=1}^{n} \left(q_k \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial q_k} \right), \qquad \text{(rotational vector field).}$$

- For $\lambda \in [0, \infty)$, energy level-set $E_\lambda := E^{-1}(\lambda) \cong S^{2n-1} \subseteq \mathbb{R}^{2n}$.
- Each E_λ has an S^1-symmetry,

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n) \rightsquigarrow E(e^{i\theta} \cdot z) = E(z) = \lambda.$$

- For $N \in \mathbb{N}$, can prove that a quantisation $\mathcal{Q}(E_N)$ exists.
- Eigenvalues for $\mathcal{Q}(\phi)$ acting on $\mathcal{Q}(E_N)$ can be identified with the set of *homogeneous* polynomials in n variables of *degree N*:

$$\mathcal{Q}(E_N) \cong \left\{ z_1^{k_1} \cdots z_n^{k_n} : \sum_{j=1}^{n} k_j = N \right\}.$$