Geometric Quantisation

Benjamin C. W. Brown
B.Brown@ed.ac.uk

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Classical Mechanics

- In classical mechanics, the space of all possible states of a system is given by *phase space*, M.
- "*State*" means not just the *position* but also the *momentum* (of a particle).

Example

For a particle of unit mass, energy function $E : M \to \mathbb{R}$:

$$E(x, p) = \frac{p^2}{2} + V(x).$$

$$\rightsquigarrow \frac{\partial E}{\partial x} = \frac{\partial V}{\partial x} = -\frac{\partial p}{\partial t}, \quad \text{and} \quad \frac{\partial E}{\partial p} = p = \frac{\partial x}{\partial t}.$$

Quantum Mechanics

- Still have phase space M and energy E, but replaces states with *wavefunctions*.
- Classical energy function $E : M \to \mathbb{R}$ now becomes an operator $\hat{E} : M \to M$.

Question

For a phase space M, what is $\mathcal{Q}(M)$?

- Phase space M represents the position and momentum of a system, with energy function $E : M \to \mathbb{R}$.
- Quantisation $\mathcal{Q}(M)$ should then represent the wavefunctions $\psi : M \to \mathbb{C}$, with operators $\hat{E} : M \to M$.

Baby Quantisation

- $M = \mathbb{R}^6 \cong \mathbb{C}^3$ via $(x_j, p_j) \rightsquigarrow z_j = p_j + ix_j \in \mathbb{C}^3$.
- Energy function and operator:

$$\phi(z) = \tfrac{\|z\|^2}{2} \rightsquigarrow \hat{\mathcal{Q}}(\phi) = \sum_{j=1}^{3} \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right).$$

- $k \in \mathbb{N}$, get level-set $E_k := \phi^{-1}(k) \cong S^5 \subseteq \mathbb{R}^6$.
- Each E_k has an S^1-symmetry:

$$e^{i\theta} \cdot (z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n) \rightsquigarrow \phi(e^{i\theta} \cdot z) = \phi(z) = k.$$

- "Identify" the S^1-symmetries of S^5, *i.e.* consider points in the same orbit as identical.

Fact

On E_k, eigenvectors for $\hat{\mathcal{Q}}(\phi)$ are *homogeneous* polynomials in 3 variables of *degree k*:

$$\mathcal{Q}(E_k) \cong \{z_1^{n_1} z_2^{n_2} z_3^{n_3} : n_1 + n_2 + n_3 = k\},$$

$$\dim \mathcal{Q}(E_k) = \#(k \cdot \Delta \cap \mathbb{Z}^2).$$

Half-Spaces and Hyperplanes

Where My Research Fits In

Before Edinburgh

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