Question 1

1. (a) $\{a, b, d\}$
   (b) $\{e, f, g, h\}$
   (c) $\{a, b, e, g, h\}$
2. This is a rank 4 matroid, and circuits are minimal non-independent sets. Any 5-element subset of a 6+-element set is a depenendent set, and so no 6+-element set can be a circuit. M has no copunctual points, so no two elements are depenendent. Hence M has no 2-element circuits.
3. (a) $\{d, b, i, e\}$.
   (b) $\emptyset$
   (c) $\{a, b, c, d, e, f, g, h, i\}$
   (d) $\{d, g, i, f, e\}$ - any 4 element subset is coplanar, but d, g, i is independent.

Question 2

$(i) \implies (ii)$: If $\{e\} \in \mathcal{C}(M)$, then $\{e\} \notin \mathcal{I}(M)$. If $e \in B \in \mathcal{B}(M)$ for some basis, then $\{e\} \subseteq B \in \mathcal{I}(M)$, and so $\{e\} \in \mathcal{I}(M)$, a contradiction. Hence e is not in any basis.

$(ii) \implies (iii)$: Suppose that e is not in any basis of M. Then $\mathcal{I}(M) = \{Y \subseteq E : Y \subseteq B \in \mathcal{B}\}$. Since $x \notin B$ for any basis, and all independent sets are subsets of a basis, $x \notin Y$ for any $Y \in \mathcal{I}(M)$.

$(iii) \implies (i)$: Suppose e is in no independent set. Then $\{e\} \notin \mathcal{I}(M)$, so is depenendent. But then $\emptyset \in \mathcal{I}(M)$, so $\{e\}$ is a minimal depenendent set, and so is a circuit (hence e is a loop).

Question 3

By thm 1.7 it suffices to show that $\mathcal{I}$ satisfies I1-3.

- I1 $\emptyset \in I_1, I_2$. So $\emptyset = \emptyset \cup \emptyset \in \mathcal{I}$.
- I2 Let $Y = Y_1 \cup Y_2 \in \mathcal{I}$, and suppose $X \subseteq Y$. Now $X = X \backslash Y_2 \cup X \backslash Y_1$, since E_1, E_2 are disjoint. But $X \backslash Y_2 \subseteq Y \backslash Y_2 \subseteq Y_1 \in I_1$, and $X \backslash Y_1 \subseteq Y \backslash Y_1 \subseteq Y_2 \in I_2$. So $X \backslash Y_2 \in I_1$ and $X \backslash Y_1 \in I_2$, by applying I2 to each matroid. Thus $X = X \backslash Y_2 \cup X \backslash Y_1 \in \mathcal{I}$.
- I3 Let $X = X_1 \cup X_2, Y = Y_1 \cup Y_2 \subseteq \mathcal{I}$, with $|X| < |Y|$. Assume w.l.o.g. that $|Y_1| > |X_1|$ (We have at least one of this, or $|Y_2| > |X_2|$, so just swap the two if not). Then by the independence augmentation property of I_1, $\exists y \in Y_1 \backslash X_1$, such that $X_1 \cup y \in I_1$. Then we have found some $y \in Y_1 \backslash X_1 \subseteq Y \backslash X$, such that $X \cup y = (X_1 \cup y) \cup X_2 \in \mathcal{I}$. (Note again that E_1 and E_2 are disjoint).

Question 4

This is false. Consider $E = \{1,2,3,4,5,6\}$, and $C' = \{\{1,2,3,4\},\{1,2,5,6\}\}$, defining $B_{4,C'}$ as our bases (as per the next question). So our bases for this matroid are the 4-element sets not in C', and in particular, we see that $\{3,4,5,6\}$ is one of them. However, $\{1,2,3,4\}$ and $\{1,2,5,6\}$ are depenendent sets, and it is easy to see that any subset of either is contained in a basis set. So they are circuits.

Then we have $\{1,2,3,4\} \cap \{1,2,5,6\} \neq \emptyset$, and yet $\{1,2,3,4\}\Delta\{1,2,5,6\} = \{3,4\} \cup \{5,6\} = \{3,4,5,6\}$, which is a member of $B_{4,C'}$ and so not a circuit!

Question 5

Let E be a finite set, and let r be an integer with $0 < r < |E|$. Let C' be a collection of r- element subsets of E such that if C_1 and C_2 are distinct members of C, then $|C_1 \cap C_2| < r - 1$. Let $B_{r,C'}$ be the family of r-element subsets of E that are not in C'; that is, $B_{r,C'} = B \subseteq E : |B| = r and B \notin C'$.

1. $(E, B_{r,C'})$ is a matroid: First B1 is trivial! Suppose $\mathcal{C}'$ contained every r-element subset of E. Then taking any two r-element subsets that differ by only one element, say C_1, C_2, we have that $|C_1 \cap C_2| = r - 1 \not< r - 1$. Hence at least one of the two subsets cannot be in C'. So some r-element subset is not in $\mathcal{C}'$, and is thus in $B_{r,C'}$ (and therefore it is non-empty).

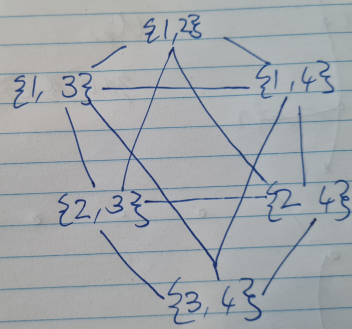
   For B2, take $B_1, B_2 \in B_{r,C'}$. Then take $x \in B_1 \backslash B_2$. Now say that for each $y \in B_2 \backslash B_1$, we have that $(B_1 \backslash x) \cup y \notin B_{r,C'}$, that is, $(B_1 \backslash x) \cup y \in C'$. For any distinct $y_1, y_2 \in B_2 \backslash B_1$, notice that we have $((B_1 \backslash x) \cup y_1) \cap ((B_1 \backslash x) \cup y_2) = B_1 \backslash x$. This has cardinality $r - 1$, which contradicts the property of C'. So, if there are multiple elements in $B_2 \backslash B_1$, at least one of the sets obtained by basis exchange is not in C', and is hence in $B_{r,C'}$. It remains to consider the case where $|B_2 \backslash B_1| = 1$. In this case, we have that $B_2 = (B_2 \cap B_1) \cup (B_2 \backslash B_1)$. Clearly these sets are disjoint, and thus we have $r = |B_2| = |B_2 \cap B_1| + |B_2 \backslash B_1|$. But then it follows from $|B_2 \backslash B_1| = 1$ that $|B_2 \cap B_1| = r - 1$.

   Thus, we see that $|B_1 \backslash B_2| = 1$, and therefore $B_1 \cap B_2 = B_1 \backslash \{x\} = B_2 \backslash \{y\}$. So finally, we have that $(B_1 \backslash x) \cup y = (B_2 \backslash y) \cup y = B_2 \notin C'$. Hence the basis exchange property holds, and so $(E, B_{r,C'})$ is a matroid!

2. Let $J(n, r)$ be the simple graph with r-element subsets of $1, ..., n$ as its verticies and 2 verticies adjacent iff the cardinality of their intersection is $r - 1$. Stable sets of this graph look like disjoint partitions of $\{1,2,3,4\}$. Since we connect two sets (in this case) iff they share 1 element in common, in order for two verticies to not be connected, they should not share any elements at all! So our stable sets are partitions of $\{1,2,3,4\}$ into 2 two-element subsets.

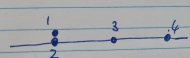
3. Clearly we have $U_{2,4}$, since this is the uniform matroid of rank 2. It remains to list matriods of the form $(\{1,2,3,4\}, B_{r,C'})$. In this case, we need $r = 2$, and C' is a collection of 2 element subsets such that for any distinct $C_1, C_2 \in C'$, we have $|C_1 \cap C_2| < 2 - 1$ (or in other words, equal to 0). So, we could have $C' = \{\{1,2\}\}$, or $\{\{1,2\},\{3,4\}\}$. Clearly any other choices of C' are isomorphic, we just relabel the elements in the sets. This relabeling will then produce the corresponding matriod. So, these are all the rank 2, sparse paving matroids on 4 elements.

$J(4,2)$



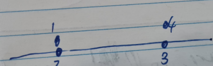
$C' = \{\{1,2\}\}$, $r = 2$.

1, 2 are copunctual!



$C' = \{\{1,2\},\{3,4\}\}$, $r = 2$

copunctual!



$U_{2,4}$ we are in $\mathbb{R}^1$ – any set of 3 elements are colinear!