Question 1

1

2

3 As shown in class $M^*(G) = M(G^*)$, and so we draw a geometric representation of figure 2's cycle matroid. Here a maximal tree has 4 verticies so our bases are of size 4, so our geometric representation is in $\mathbb{R}^3$.

Question 2

1 We perform the following operations: $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 4R_1$, $R_3 \to R_3 + 4R_2$. $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, $R_3 \to 3R_3$. Following this we obtain the following matrix:

$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 \end{pmatrix} \end{array}$$

2 This is a rank 3 matroid, so we draw its geometric representation in the plane. Initially, we see that a, b, c is independent, and so is $\{d, e, f\}$. Using the determinate trick from class, it is easy to determine that $\{a, b, d\}, \{b, c, f\}, \{a, c, e\}$ are all dependent, as their corresponding matrices are the 1×1, $[0]$ matrix. Similar checks will show that all other 3-element sets (not containing g) are independent. Finally, we see that $\{f, g, a\}$ is dependent, and all other matricies we might look at have zero determinate. Hence, our geometric representation is:

3 From class, we know that if $M = M([I|A])$, then $M^* = M([I|A^T])$. Now in reduced representation form, the rows are labeled by $\{a, b, c\}$, and columns by $\{d, e, f, g\}$. So, A^T is labeled by $\{a, b, c\}$, and the identity matrix by $\{d, e, f, g\}$.

$$\begin{array}{c} \begin{array}{ccccccc} d & e & f & g & a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 & 3 \end{pmatrix} \end{array}$$

3 Our geometric representation will be in $\mathbb{R}^3$

Question 3

Let M be a matroid on ground set E with independent sets $\mathcal{I}$, and $e \in E$ not be a loop in M. Consider the set $A = \{I \subseteq E - e : I \cup e \in \mathcal{I}\}$. We will show that A satisfies the independent set axioms.

I1 Firstly, note that e is not a loop, and so $e \in \mathcal{I}$. Then $\emptyset$ has $\emptyset \cup e = \{e\} \in \mathcal{I}$ and so $\emptyset \in A$.

I2 Now, suppose that $X \in A$ and $Y \subseteq X$. Then $Y \cup e \subseteq X \cup e \in \mathcal{I}$, and since $\mathcal{I}$ is a family of independent sets, $Y \cup e \in \mathcal{I}$ too. Hence $Y \in A$.

I3 Suppose $X, Y \in A$ with $|X| < |Y|$. Then $X \cup e, Y \cup e \in \mathcal{I}$ with $|X \cup e| < |Y \cup e|$. So by applying I3 to $\mathcal{I}$, we can find some $a \in (Y \cup e) - (X \cup e)$ such that $X \cup e \cup a \in \mathcal{I}$. However, note that $a \neq e$, and clearly $a \notin X$. So, we must have $a \in Y - X$ and thus $X \cup a \cup e \in \mathcal{I}$ implies that $X \cup a \in A$ as required.

Hence A satisfies all required properties, showing M/e is a matroid.

Question 4

a We expect $r(M/a) = r(M) - 1$, so our geometric representation will be in $\mathbb{R}^2$. Now $\{f, e, h, g, i\}$ is a hyperplane, and so we project onto this plane.

b Again, we use $\{f, e, h, g, i\}$ as our hyperplane

c We start with the geometric representation of M/b, and then delete c, e and any redundant edges. In particular, the lines $\{d, f, a, c\}$ and $\{i, e, c\}$ become redundant.

d For a contraction on f, we choose $\{g, e, c, a\}$ as our hyperplane.

Question 5

It suffices to show that for a sparse paving matroid, M, and $e \in E(M)$, that M/e and $M\backslash e$ are sparse paving. But in fact, we showed that sparse paving matroids are closed under duality, and in class we saw that $M/e = (M^*\backslash e)^*$, so actually all we need to do is show that $M\backslash e$ is sparse paving.

Suppose first that e is not a loop. Now the independent sets of $M\backslash e$ are $\{C \in \mathcal{C}(M) : e \notin C\}$. Now $r(M\backslash e) = r(M) = r$, so if C, C' are circuits of size r in $M\backslash e$, then they are circuits of size r in M. Hence by assumption $|C \cap C'| < r - 1$. Thus $M\backslash e$ is sparse paving in this case.

In the second case, when e is a loop, we have that $\{e\}$ is a circuit. Then the circuits of $M\backslash e$ are simply all circuits except this one. So again, the exact same reasoning will show that $M\backslash e$ is sparse paving

TODO: more detail?

Question 6

$i \implies ii$ Suppose that $e \in E(M)$ is a coloop, and further that for some $X \subseteq E(M)$ that $e \in cl(X)$. Then $r(X \cup e) = r(X)$.

Let $B_X \subseteq X$ be a maximal independent set contained in X. B_X must extend to a basis, B (it is an independent set), and since e is a coloop, it must be in that basis. S o, if $e \notin B_x$, then $B_X \cup e$ is independent, and so $r(X) + 1 = r(B_X \cup e) \leq r(X \cup e) = r(X)$, a contraction.

$ii \implies iii$ Suppose that for every $X \subseteq E(M)$ with $e \in cl(X)$, that $e \in X$. Now consider $X = E(M) - e$. $e \notin X$, and so by assumption, $e \notin cl(X)$ But now, $X \subseteq cl(X) \subseteq E(M) - e = X$, and so X is a flat. It remains to consider the rank of X. We have $r(M) = r(X \cup e) \neq r(X)$ by assumption.

Therefore as shown in class, we must have $r(X \cup e) = r(X) + 1$, and so $r(X) = r(M) - 1$, showing that X is a hyperplane.

$iii \implies i$ If $E(M) - e$ is a hyperplane, then by definition e is a coloop.**IS THIS CORRECT?**

Question 6

As seen in class, the Fano matroid serves as a valid example here. However, It is slightly easier to consider the matriod seen in figure..., which is evidently transversal, as seen in figure...

. When we contract along e, we obtain the following matroid, which we know is not transversal by the previous assignment. For the sake of completion I attach an argument below.