Question 1

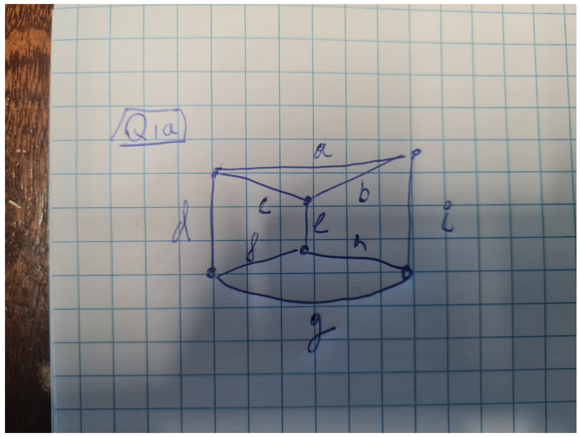

Figure 1: Question 1 a

1

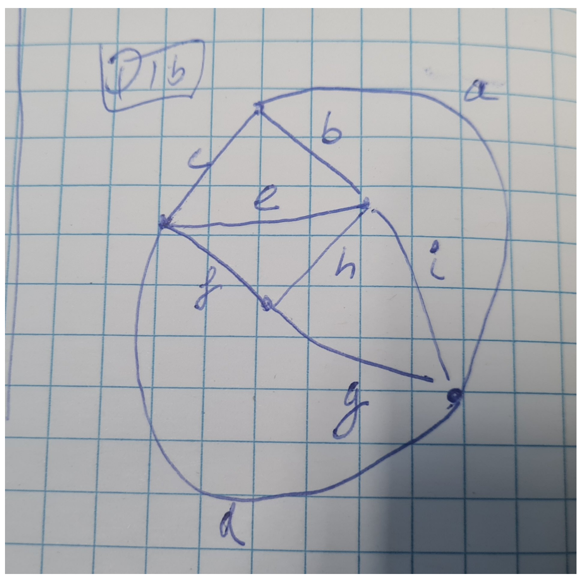

Figure 2: Question 1 b

2

3 As shown in class $M^*(G) = M(G^*)$, and so we draw a geometric representation of figure 2's cycle matroid. Here a maximal tree has 4 verticies so our bases are of size 4, so our geometric representation is in $\mathbb{R}^3$.

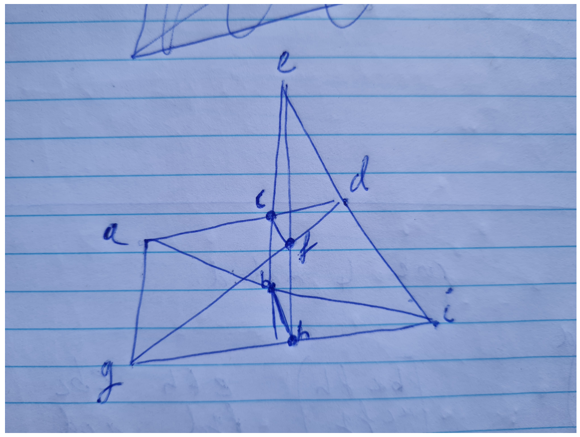

Figure 3: Question 1 c

First, notice that $\{f,h,b,c\},\{f,h,d,i\},\{d,i,b,c\}$ are all coplanar, and furthermore, we have $\{e,f,h\},\{e,d,i\}$ and $\{e,b,c\}$ all colinear. These elements must therefore form some sort of triangular pyramid. To see how a,g fit into the picture, note that we have $\{a,g,c,f\},\{a,g,b,h\},\{f,c,b,h\}$ all coplanar. Furthermore, $\{d,f,g\},\{d,c,a\}$ are colinear, as are $\{i,b,a\},\{i,h,g\}$. So these elements do not form a pyramid, and instead form a weird 3d shape (plz see picture). Furthermore, we observe that e,a and e,g are involved in no common cycles of length ≤ 4. Putting this together we obtain the following geometric representation.

Question 2

1 We perform the following operations: $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 4R_1$, $R_3 \to R_3 + 4R_2$. $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, $R_3 \to 3R_3$. Following this we obtain the following matrix:

$$\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 \end{pmatrix} \end{array}$$

2 This is a rank 3 matroid, so we draw its geometric representation in the plane. Initially, we see that a,b,c is independent, and so is $\{d,e,f\}$. Using the determinate trick from class, it is easy to determine that $\{a,b,d\},\{b,c,f\},\{a,c,e\}$ are all dependent, as their corresponding matrices are the 1×1, $[0]$ matrix. Similar checks will show that all other 3-element sets (not containing g) are independent. Finally, we see that $\{f,g,a\}$ is dependent, and all other matricies we might look at have zero determinate. Hence, our geometric representation is:

3 From class, we know that if $M = M([I|A])$, then $M^* = M([I|A^T])$. Now in reduced representation form, the rows are labeled by $\{a,b,c\}$, and columns by $\{d,e,f,g\}$. So, A^T is labeled

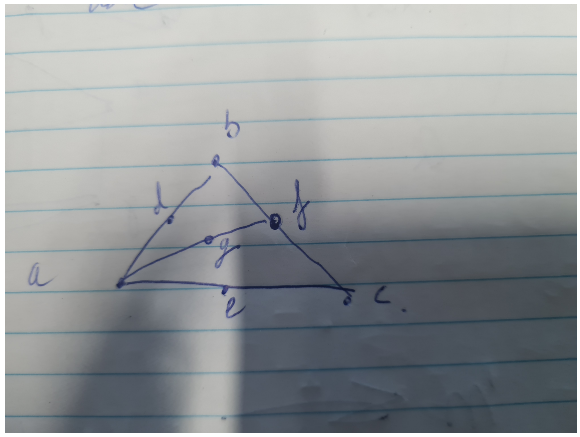

Figure 4: Question 2 b

Figure 5:

by $\{a, b, c\}$, and the identity matrix by $\{d, e, f, g\}$.

$$\begin{array}{c} \begin{array}{ccccccc} d & e & f & g & a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 & 3 \end{pmatrix} \end{array}$$

3 Our geometric representation will be in $\mathbb{R}^3$, and we immediately see that d, e, f, g are independent. Clearly, $\{a, d, e, g\}$ are coplanar, as are $\{b, d, f, g\}$ and $\{c, e, f, g\}$. In fact, removing any element from these sets gives us an independent set, so these are in fact circuits. However, $\{a, b, c, g\}$ is independent, as viewing it in reduced representation, we see that the determinate of the first 3 rows of a, b, c is non-zero. Therefore, these are the only circuits **CHECK!**, and our geometric representation is as follows:

Question 3

Let M be a matroid on ground set E with independent sets $\mathcal{I}$, and $e \in E$ not be a loop in M. Consider the set $A = \{I \subseteq E - e : I \cup e \in \mathcal{I}\}$. We will show that A satisfies the independent set axioms.

I1 Firstly, note that e is not a loop, and so $e \in \mathcal{I}$. Then $\emptyset$ has $\emptyset \cup e = \{e\} \in \mathcal{I}$ and so $\emptyset \in A$.

I2 Now, suppose that $X \in A$ and $Y \subseteq X$. Then $Y \cup e \subseteq X \cup e \in \mathcal{I}$, and since $\mathcal{I}$ is a family of independent sets, $Y \cup e \in \mathcal{I}$ too. Hence $Y \in A$.

I3 Suppose $X, Y \in A$ with $|X| < |Y|$. Then $X \cup e, Y \cup e \in \mathcal{I}$ with $|X \cup e| < |Y \cup e|$. So by applying I3 to $\mathcal{I}$, we can find some $a \in (Y \cup e) - (X \cup e)$ such that $X \cup e \cup a \in \mathcal{I}$. However, note that $a \neq e$, and clearly $a \notin X$. So, we must have $a \in Y - X$ and thus $X \cup a \cup e \in \mathcal{I}$ implies that $X \cup a \in A$ as required.

Hence A satisfies all required properties, showing M/e is a matroid.

Question 4

a We expect $r(M/a) = r(M) - 1$, so our geometric representation will be in $\mathbb{R}^2$. Now $\{f, e, h, g, i\}$ is a hyperplane, and so we project onto this plane.

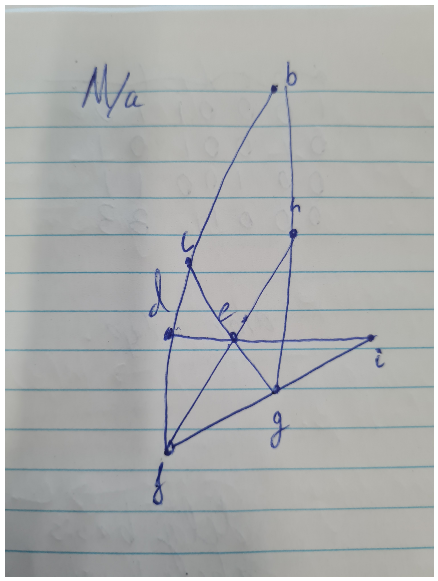

Figure 6: Question 4 A

b Again, we use $\{f, e, h, g, i\}$ as our hyperplane

c We start with the geometric representation of M/b, and then delete c, e and any redundant edges. In particular, the lines $\{d, f, a, c\}$ and $\{i, e, c\}$ become redundant.

d For a contraction on f, we choose $\{g, e, c, a\}$ as our hyperplane.

Question 5

It suffices to show that for a sparse paving matroid, M, and $e \in E(M)$, that M/e and $M \backslash e$ are sparse paving. But in fact, we showed that sparse paving matroids are closed under duality, and in class we saw that $M/e = (M^* \backslash e)^*$, so actually all we need to do is show that $M \backslash e$ is sparse paving.

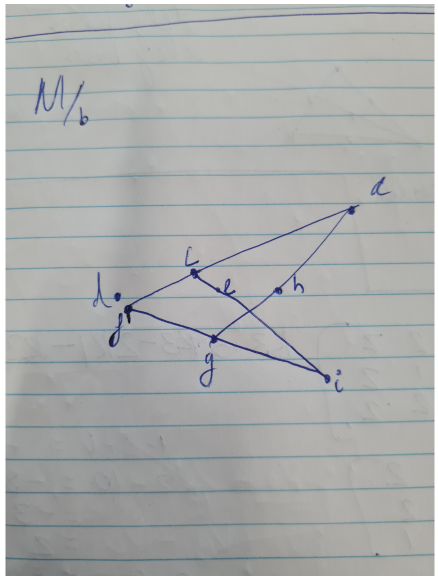

Figure 7: Question 4 B

Suppose first that e is not a loop. Now the independent sets of $M \backslash e$ are $\{C \in \mathcal{C}(M) : e \notin C\}$. Now $r(M \backslash e) = r(M) = r$, so if C, C' are circuits of size r in $M \backslash e$, then they are circuits of size r in M. Hence by assumption $|C \cap C'| < r - 1$. Thus $M \backslash e$ is sparse paving in this case.

In the second case, when e is a loop, we have that $\{e\}$ is a circuit. Then the circuits of $M \backslash e$ are simply all circuits except this one. So again, the exact same reasoning will show that $M \backslash e$ is sparse paving

TODO: more detail?

Question 6

$i \implies ii$ Suppose that $e \in E(M)$ is a coloop, and further that for some $X \subseteq E(M)$ that $e \in cl(X)$. Then $r(X \cup e) = r(X)$.

Let $B_X \subseteq X$ be a maximal independent set contained in X. B_X must extend to a basis, B (it is an independent set), and since e is a coloop, it must be in that basis. S o, if $e \notin B_x$, then $B_X \cup e$ is independent, and so $r(X) + 1 = r(B_X \cup e) \leq r(X \cup e) = r(X)$, a contraction.

$ii \implies iii$ Suppose that for every $X \subseteq E(M)$ with $e \in cl(X)$, that $e \in X$. Now consider $X = E(M) - e$. $e \notin X$, and so by assumption, $e \notin cl(X)$ But now, $X \subseteq cl(X) \subseteq E(M) - e = X$, and so X is a flat. It remains to consider the rank of X. We have $r(M) = r(X \cup e) \neq r(X)$ by assumption. Therefore as shown in class, we must have $r(X \cup e) = r(X) + 1$, and so $r(X) = r(M) - 1$, showing that X is a hyperplane.

$iii \implies i$ If $E(M) - e$ is a hyperplane, then by definition e is a coloop.**IS THIS CORRECT?**

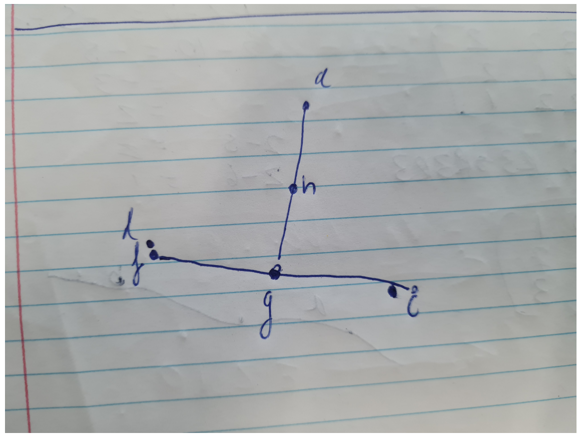

Figure 8: Question 4 C

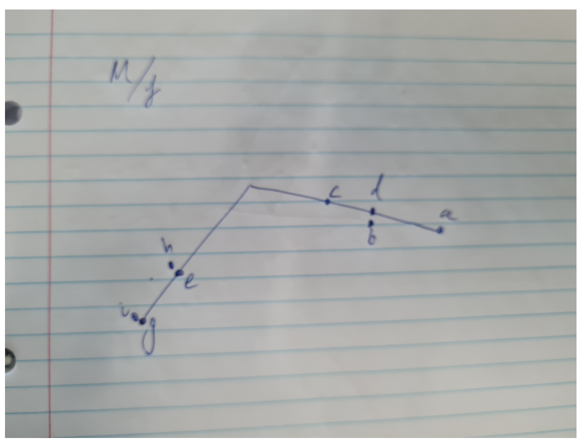

Figure 9: Question 4 D

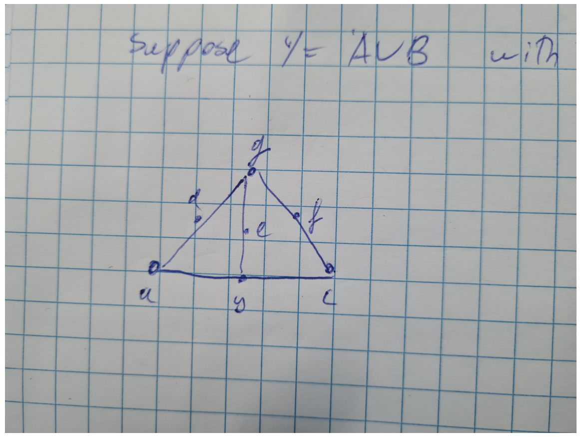

Figure 10: geometric representation of M

Question 6

As seen in class, the Fano matroid serves as a valid example here. However, It is slightly easier to consider the matriod, M (see figure), which is evidently transversal as seen in the next figure.

When we contract along g, we obtain the following matroid, M/e, seen in the figure. This is exactly the same matroid that I showed was not transversal in the previous assignment (I got full marks in that question so I hope I can just refer you to that assignment for an argument?)

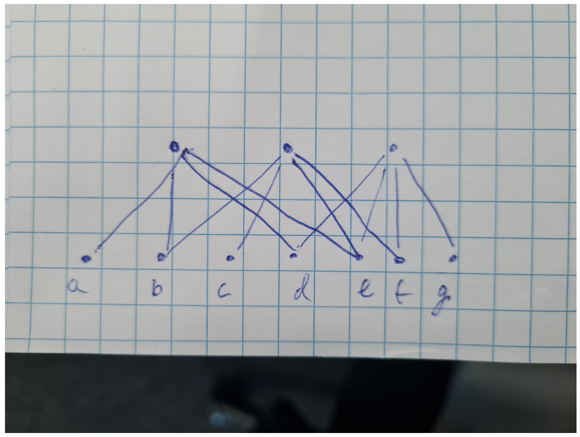

Figure 11: transversal representation of M

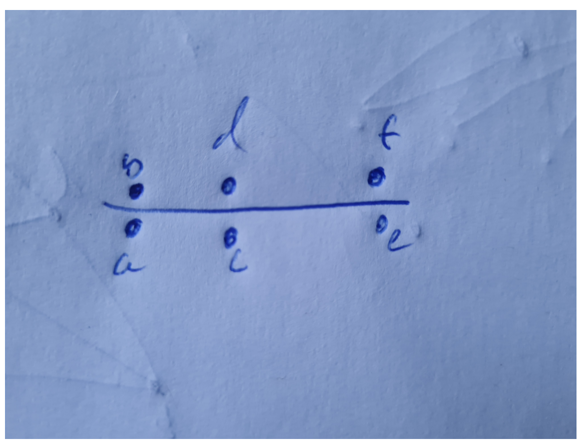

Figure 12: geometric representation of M/e, which we know is not transversal.