Introduction to Probability

Damien Benveniste

September 27, 2017

1 Discrete probability distribution

For an element x of all possible outcomes Ω, we define the probability mass function $f(x)$ such that

$$\begin{aligned} f(x) &\in [0,1]\, \forall x \in \Omega && (1) \\ \sum_{x\in\Omega} f(x) &= 1 && (2) \end{aligned}$$

The probability of an event $E \subseteq \Omega$ is defined as

$$P(E) = \sum_{x\in E} f(x) \tag{3}$$

2 Continuous probability distributions

For an element x of all possible outcomes Ω, we define the cumulative distribution function $F(x) = P(X \leq x)$ where X is a random variable such that

$$\lim_{x\to-\infty} F(x) = 0 \tag{4}$$

$$\lim_{x\to\infty} F(x) = 1 \tag{5}$$

If the derivative of $f(x) = \frac{dF(x)}{dx}$ exists, the probability of X to in an event $E \subseteq \Omega$ is

$$P(X \in E) = \int_{x\in E} f(x)dx \tag{6}$$

3 Law of large number

We have $\{X_1, \dots, X_n\} \sim D(\mu)$ n independent and identically distributed random variables with expected values μ. The Law of large number states that for

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{7}$$

we have

$$\lim_{n\to\infty} \bar{X}_n = \mu \tag{8}$$

4 Central limit theorem

We have $\{X_1, \ldots, X_n\} \sim D(\mu, \sigma^2)$ n independent and identically distributed random variables with expected values μ and variance σ^2. The central limit theorem states for

$$\bar{Z}_n = \frac{\sum_{i=1}^{n}(X_i - \mu)}{\sqrt{n}} \tag{9}$$

we have

$$\lim_{n\to\infty} \bar{Z}_n \sim \mathcal{N}(0, \sigma^2) \tag{10}$$

We have $X \sim \mathcal{N}(\mu, \sigma^2)$ if

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{11}$$

5 Non mutually exclusive events

$$P(A \cup B) = P(A) + P(B) - P(A \cup B) \tag{12}$$

6 Conditional Probability

The probability of an event A conditioned on an event B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{13}$$

The law of Total probability states that if Y and X are random variables

$$P(Y = y) = \sum_{x\in X} P(Y = y \cap X = x) = \sum_{x\in X} P(Y = y|X = x)P(X = x) \tag{14}$$

7 Bayes Theorem

For two events A and B

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{15}$$