1 The cdf as a random variable

Let X be a continuous-valued random variable with cumulative distribution function $P_X(x) = Pr(X \leq x)$. Show that the random variable $Y = P_X(X)$ is uniformly distributed over $[0, 1]$. You can assume that P_X is invertible on $[0, 1]$.

Solution: Definition 2.2.1 in the textbook states: The cdf, denoted $P(x)$, of $X \in \mathbb{R}$ is the probability $Pr(X < x)$.

We have that $P_Y(y) = Pr(Y \leq y)$. Inserting for Y gives us

$$\begin{aligned} P_Y(y) &= Pr(P_X(X) \leq y) \\ &= Pr(X \leq P_X^{-1}(y)) \end{aligned}$$

This then gives us y by taking the inverse: $P_X(P_X^{-1}(y)) = y$. We then know that Y is uniformly distributed over $[0, 1]$, as it has a cdf that is linear in y.

2. Some results regarding the poisson distribution

a) N has Param λ. Show that its prob gen. func. is $e^{\lambda(t-1)}$

Poisson has $P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$$G(t) = \sum_{n=-\infty}^{\infty} P(x_n) t^{x_n} = \sum_{n=-\infty}^{\infty} \frac{\lambda^{x_n} e^{-\lambda}}{x_n!} t^{x_n}$$

Outcome space is $\mathbb{N} = \{0, 1, 2, 3 \dots\}$

$\Downarrow$

$$G(t) = \sum_{n=0}^{\infty} \frac{\lambda^{x_n} e^{-\lambda}}{x_n!} t^{x_n}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} t^x = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x t^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda t)^x}{x!} = e^{-\lambda} \cdot e^{\lambda t}$$

$$= e^{\lambda(t-1)}$$

b) show:
Binom has PGF $(1-r+rt)^n$

$M \sim \text{Binomial}(r; n)$

The binom is the sum of n i.i.d. Bernoulli dists. The Bern has a PGF of $1-r+rt$.
Just like in Ex. 2.7., we get a PGF of a sum of distributions: $G_S = (1-r+rt)\cdot(1-r+rt)\cdot \ldots \cdot (\ldots$

$= \underline{\underline{(1-r+rt)^n}}$

c) Binom where $n \to \infty$, $nr = \lambda$. What happens to PGF?

Binomial $\to$ Poisson?

$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

$\cancel{(1-r+rt)^n} \to (1+(+\frac{nrt}{n})$

$= (1+(r\frac{\lambda}{n}))^\infty$

$= \; (?)$

$= e^{\lambda t} \; (?)$

$(1+r(t-1))^n \Rightarrow \left(1+\frac{nr(t-1)}{n}\right)^n$

$\lim_{n\to\infty} \left(1+\frac{\lambda(t-1)}{n}\right)^\infty = \underline{\underline{e^{\lambda(t-1)}}}$

d) Use PGF to show that the dist $N = N_1 + N_2$ is Poiss dist. w. param $\lambda = \lambda_1 + \lambda_2$

$G_1(t) = \exp(\lambda_1(t-1))$

$G_2(t) = \exp(\lambda_2(t-1))$

$$\begin{aligned} G_S(t) &= \exp(\lambda_1(t-1))\exp(\lambda_2(t-1)) \\ &= \exp(\lambda_1(t-1) + \lambda_2(t-1)) \\ &= \exp((\lambda_1 + \lambda_2)(t-1)) \end{aligned}$$

which is (the PGF of) a Poiss. dist. w. param. $\lambda = \lambda_1 + \lambda_2$.

Convolution is tricky.

3. Continuous time arrival Process

exp. dist. w. rate λ. $P_{\Delta T_i}(\Delta t_i) = \lambda e^{-\lambda(\Delta t_i)}$

$\Delta T_i = T_i - T_{i-1}$ for $i \geq 2$ and $\Delta T_1 = T_1$

a) Bayes' rule to show that $P_{T_1 | T_1 \geq t_0}(t_1) = \lambda e^{-\lambda(t_1 - t_0)}$ for $t_1 \geq t_0$ and 0 otherwise. Note: $P_{T_1}(t_1) = P_{\Delta T_1}(t_1) = \lambda e^{-\lambda(t_1)}$

Bayes' tells us: $P(t_1 | t_1 \geq 0) = \dfrac{P(t_1 \geq t_0 | T_1) P(t_1)}{P(t_0 \leq t_1)}$

$$P(t_1 \geq t_0 | t_1) = \begin{cases} 1 \text{ when } T_1 > t_0 \\ 0, \text{ else} \end{cases}$$

$P(t_1) = \lambda e^{-\lambda t_1}$, $P(t_0 \leq t_1) = \int_{t_0}^{\infty} \lambda e^{-\lambda t_1} dt_1$

$= e^{-\lambda t_0}$

$\Downarrow$

$$P(t_1 | t_1 \geq t_0) = \begin{cases} \dfrac{1 \cdot \lambda e^{-\lambda t_1}}{e^{-\lambda t_0}}, & t_1 \geq t_0 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \lambda e^{-\lambda(t_1 - t_0)}, & t_1 \geq t_0 \\ 0, & \text{else} \end{cases}$$

//

b) $T_0 = t_0$ now. What is the distribution of
$T_n - T_0 = \sum_{i=1}^{n} T_i - T_{i-1} = \sum_{i=1}^{n} \Delta T_i$ given
$T_1 > t_0$?

$$G(t) = \frac{1}{(1 - \frac{\Delta t}{n})}$$

We have the PGF $G_{T_n - T_0}(t) = \prod_{t_0}^{n} G(t) = \frac{1}{(1 - \Delta t / \lambda)^n}$

ex 2.8:

The sum of n exponentials becomes a gamma distributed random variable w. ~~[illegible]~~
$T \sim \Gamma(n, 1/\lambda)$ (?)

c) What is the prob. that boat $n+1$ did not arrive before $t > 0$ given boat n arrived at time $T_n = t_n < t$? That is, find
$Pr(T_{n+1} > t \mid T_n = t_n \in [0, t])$

Hint: $Pr(T_{n+1} > t \mid T_n = t_n \in [0,t]) =$

$$Pr(T_{n+1} - t_n > t - t_n \mid T_n = t_n \in [0, t]) =$$

$$Pr(\Delta T_{n+1} > t - t_n \mid T_n = t_n \in [0, t]) =$$

$$1 - \int_0^{t - t_n} \lambda e^{-\lambda \Delta t} \, d\Delta t = \cancel{1} - [-e^{-\lambda(t - t_n)} + \cancel{1}]$$

$$= \underline{e^{-\lambda(t - t_n)}}$$

d) Use the last two results to show that the dist. of no. of boats N to arrive between t_0 and t_1 given by

$P_N(n) = Pr(T_{n+1} > t \cap T_n \leq t \mid T_1 \geq t_0)$, is Poisson distributed.

$$= Pr(T_{n+1} > t \mid T_n \leq t / T_1 \geq t_0)$$

$$= Pr(T_{n+1} > t \mid T_n \leq t ; T_1 \geq t_0) \cdot Pr(T_n \leq t \mid T_1 \geq t_0)$$

$e^{-\lambda(t-t_0)}$ (see prev.)

$$Pr(\Delta T \leq t_n - t_0 / T_1 \geq t_0) = Pr(\Delta T \leq \Delta t)$$

$$= \int_{t_0}^{t_n - t_0} \left(\frac{\lambda^n \cdot \Delta t^{n-1}}{(n-1)!}\right) e^{-\lambda \Delta t} \, d\Delta t = \frac{\lambda^n \Delta t^n}{n!} e^{-\lambda(t_n - t_0)}$$

$\Downarrow$

$$\underline{P_N(n) = e^{-\lambda(t-t_0)} \frac{\lambda^n \Delta t^n}{n!}}$$ ← which is a Poisson distribution.

4. Finding Posterior estimates of the number of boats in the region.

$$m = n_D + m_{fa} \quad \text{error, ind, Poisson}$$

$P_D \in (0,1)$ independently

Each detection is Bernoulli distributed w. param. $P_D \Rightarrow p(n_D|n) = \text{Binom}(n_D; P_D, n)$
$= \binom{n}{n_D} P_D^{n_D} (1-P_D)^{n-n_D}$. Let $n_U = n - n_D$, we have that $p(n_D, n_U|n) = p(n_D|n)$ since the binom. is a prob. unordered split into U and D.

$n \sim \text{Poisson}(\lambda)$

a) Show that the marginal dist. for n_D and $n_U = n - n_D$ can be written as the prod. of two indep. Poisson distributions $p(n_D, n_U) = p(n_D)p(n_U)$.

$$p(n_D, n) = p(n_D|n) \cdot p(n)$$
$$= \binom{n}{n_D} P_D^{n_D} (1-P_D)^{n-n_D} \cdot \frac{e^{-\lambda}\lambda^n}{n!}$$

which by writing out the xpression for NCR is

$$= \frac{\cancel{n!}}{n_D! \cdot \underbrace{(n-n_D)}_{n_U}!} \cdot P_D^{n_D} (1-P_D)^{\overbrace{n-n_D}^{n_U}} \frac{e^{-\lambda}\lambda^{n}}{\cancel{n!}}$$

$$= \frac{P_D^{n_D}(1-P_D)^{\cancel{n-n_D}\, n_U} \cdot e^{-\lambda} \cdot \lambda^{n \leftarrow n_D+n_U}}{n_D! \cdot n_U!} \qquad \lambda^{n_D}\cdot\lambda^{n_U}$$

$$= \left(\frac{P_D^{n_D}}{n_D!}\cdot\frac{(1-P_D)^{n_U}}{n_U!}\right)\cdot \overbrace{\lambda^{n_D+n_U}} \cdot e^{-\lambda+\overbrace{\lambda P_D-\lambda P_D}^{0}}$$

$$= \left(\frac{P_D^{n_D}\lambda^{n_D}}{n_D!} \cdot e^{-\lambda P_D}\right)\left(\frac{(1-P_D)^{n_U}\lambda^{n_U}}{n_U!} e^{-\lambda(1-P_D)}\right)$$

$$= \underline{P(n_D)P(n_U)} \quad (=P(n_D, n_U))$$

b) $m = n_D + m_{fa}$. Find $P(n_D|m)$

$m_{fa} \sim \text{Poiss}(\Lambda)$

$$P(n_D|m) = P(m|n_D)\frac{P(n_D)}{P(m)} \qquad (*)$$

$n_D \sim \text{Poiss}(\lambda P_D)$

The sum of two Poiss is also Poiss:

$$m = n_D + m_{fa} \sim \text{Poiss}(\lambda P_D + \Lambda)$$

$$\Downarrow$$

$$(*) \qquad P(n_D|m) = \frac{\Lambda^{m-n_D}}{(m-n_D)!}e^{-\Lambda}\left(\frac{\frac{(\lambda P_D)^{n_D}}{n_D!}e^{-\lambda P_D}}{\frac{(\Lambda+\lambda P_D)^m}{m!}e^{-(\Lambda+\lambda P_D)}}\right)$$

$$\Downarrow \text{ math}$$

$$\binom{m}{n_D}(1-r)^m r^{m-n_D}$$

(probably)

c)

MMSE estimator

$$\hat{n}_{D(MMSE)} = E[n_D|m]$$
$$= \sum_{n_D} P(n_D|m)\cdot n_D,$$

which can be found from the results of a)