Sensor Fusion Ass 2

1.

a) $z = \Sigma^{-1/2}(x-\mu) = (\Sigma^{1/2})^{-1}(x-\mu) = (\sqrt{\Sigma})^{-1}(x-\mu) = \frac{(x-\mu)}{\sqrt{\Sigma}}$

$\Downarrow$

$z = \frac{x}{\sqrt{\Sigma}} - \frac{\mu}{\sqrt{\Sigma}} \Rightarrow E[z] = 0$

$Var(z) = Var\left(\frac{x}{\sqrt{\Sigma}} - \cancel{\frac{\mu}{\sqrt{\Sigma}}}\right) = Var\left(\frac{1}{\sqrt{\Sigma}}x\right) = \left(\frac{1}{\sqrt{\Sigma}}\right)^2 Var(x) = \frac{1}{\cancel{\Sigma}} \cdot \cancel{\Sigma} = \underline{I}$

$\Downarrow$

$\underline{z \sim \mathcal{N}(0, I)}$: white noise

b) $y_i = z_i^2$

Thm 2.5.1: $p_y(y) = \sum_i p_x(g_i^{-1}(y)) |\det(G_i^{-1}(y))|$

when $Y = g(x)$

In our case, this becomes

$p_{y_i}(y_i) = \sum_i p_{z_i}(g^{-1}(y_i)) |\det(G^{-1}(y_i))|$, where

$g^{-1}(y_i) = \pm\sqrt{y_i}$, $G^{-1}(y_i) = \frac{1}{2\sqrt{y_i}}$

$\Downarrow$

$p_{y_i}(y_i) = \frac{1}{\cancel{2}\sqrt{2\pi}\sqrt{y_i}} e^{-y_i/2} + \frac{1}{\cancel{2}\sqrt{2\pi}\sqrt{y_i}} e^{-y_i/2} = \underline{\frac{e^{-y_i/2}}{\sqrt{2\pi y_i}}}$

$\Downarrow$

$y_i \sim \underline{\chi^2(1)}$

c) $y = (x-\mu)^T \Sigma^{-1} (x-\mu) = z^T z = \sum z_i^2 = \sum y_i$

We now need the MGF of y, ex 2.8:

$M_Y(s) = \prod_i M_{Y_i}(s)$

$M_{Y_i} = \left(\frac{1}{1-2s}\right)^{1/2}$

$\Downarrow$

$M_Y(s) = \prod_i \left(\frac{1}{1-2s}\right)^{1/2} = \left(\frac{1}{1-2s}\right)^{n/2}$ when $i \in 0, 1, 2, \ldots, n$

Meaning $\underline{y \sim \chi^2(n)}$

2.

a) $E[z^c|x] = E[H^c x + v^c] = \underline{H^c x}$

$Var(z^c|x) = Var(H^c x + v^c | x) = Var(v^c) = \underline{R^c}$ rem. (*)

$\Rightarrow \underline{p(z^c|x) \sim \mathcal{N}(H^c x, R^c)}$

b) $p(x, z^c) = p(x \cap z^c) = p(x|z^c)p(x) = \mathcal{N}(H^c x, R^c)\mathcal{N}(\bar{x}, 0)$

using thm 3.3.1, we get

$$\mathcal{N}(z; \bar{z}; S)\mathcal{N}(x; \hat{x}; \hat{P})$$

which is true as

$\bar{z}^c = H^c \bar{x}$

$\hat{x} = \bar{x} + W(z^c - H^c \bar{x})$

$S = R^c + H^c \bar{P} H^{cT}$

$\hat{P} = (I - WH^c)\bar{P}$

$W = \bar{P} H^{cT} S^{-1}$

c) Thm 3.2.3

$p(z^c) = \mathcal{N}(z^c; \overset{(*)}{H^c \bar{x}}; \overset{(*)}{R^c + H^c \bar{P} H^{cT}})$

$p(x|z^c) = \mathcal{N}(x; \mu_{x|z^c}, P_{x|z^c})$ where

$$\mu_{x|y_{z^c}} = \bar{x} + W(z^c - H^c \bar{x}) + \bar{P} H^{cT}(R^c + H^c \bar{P} H^{cT})^{-1}(z^c - H^c \bar{x})$$

$$P_{x|y_{z^c}} = P_{xx} - P_{xy} P_{yy}^{-1} P_{xy}^T = (I - WH^c)\bar{P} - \bar{P} H^{cT}(R^c + H^c \bar{P} H^{cT})^{-1}(\bar{P} H^{cT})^T$$

d) Yes, we apply the same method to find

$p(x|z^r) = \mathcal{N}(x; \mu_{x|z^r}; P_{x|z^r})$

$$\mu_{x|z^r} = \bar{x} + W(z^r - H^r \bar{x}) + \bar{P} H^{rT}(R^r + H^r \bar{P} H^{rT})^{-1}(z^r - H^r \bar{x})$$

$x^+ = Fx + w$

$$p(x^+) = \mathcal{N}(x^+; F\bar{x}; \overset{(*)}{w + F\bar{P}F^T})$$

e) MMSE: $\hat{x} = E[x|z^c] = \underline{\mu_{x|z^c}}$

MAP : $\hat{x} = \text{argmax}(p_{x|z^c}) \underset{\text{gauss}}{=} E[x|z^c] = \underline{\mu_{x|z^c}}$

f) See fig. 1

g) As we can see from fig 2, the distributions are identical, and conditioning order is arbitrary.

h.) See fig. 3, 4.

3. $\mathcal{N}^{-1}(x; a; B)\mathcal{N}^{-1}(y; Cx, D)$ (6)

a) thm. 3.4.1: $p(x,y) \propto \exp\left(\begin{bmatrix}\eta_a^T & \eta_b^T\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} - \frac{1}{2}\begin{bmatrix}x^T & y^T\end{bmatrix}\begin{bmatrix}\Lambda_{xx} & \Lambda_{xy}\\ \Lambda_{xy}^T & \Lambda_{yy}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\right)$

$$\mathcal{N}(x; \mu, P) = \exp\left(a + \eta^T x - \frac{1}{2}x^T \Lambda x\right) \quad (3.17)$$

$\Downarrow$ ~~insert 3.20~~

~~$= \exp\left(-(1/2)(n\ln(2\pi) - 2\ln|\Lambda| + \eta^T\Lambda^{-1}\eta) + (\Lambda\mu)^T x - \frac{1}{2}x^T P^{-1} x\right)$~~

~~$\Downarrow \ln$~~

~~$-\frac{1}{2}(n\ln(2\pi) - 2\ln|P^{-1}| + \ldots$~~

$\Downarrow \ln$

$a + \eta^T x - \frac{1}{2}x^T \Lambda x$

$=$ (insert 3.20)

$-\frac{1}{2}(n\ln(2\pi) - 2\ln|\Lambda| + \eta^T\Lambda^{-1}\eta) + (\Lambda\mu)^T x - \frac{1}{2}x^T P^{-1} x$

$= -\frac{1}{2}(n\ln(2\pi) - 2\ln|P^{-1}| + (\Lambda\mu)^T(P^{-1})^{-1}(\Lambda\mu)) + (P^{-1}\mu)^T x - \frac{1}{2}x^T P^{-1} x$

(let $a = P^{-1}\mu = \Lambda\mu$ & $B = P^{-1} = \Lambda$)

$= -\frac{1}{2}(n\ln(2\pi)) + \frac{1}{2}\ln|B|^{4} - \frac{1}{2}a^T B^{-1} a + a^T x - \frac{1}{2}x^T B x$

Subtracting the terms constant in x and y, and then using the given identity, we arrive at (7).

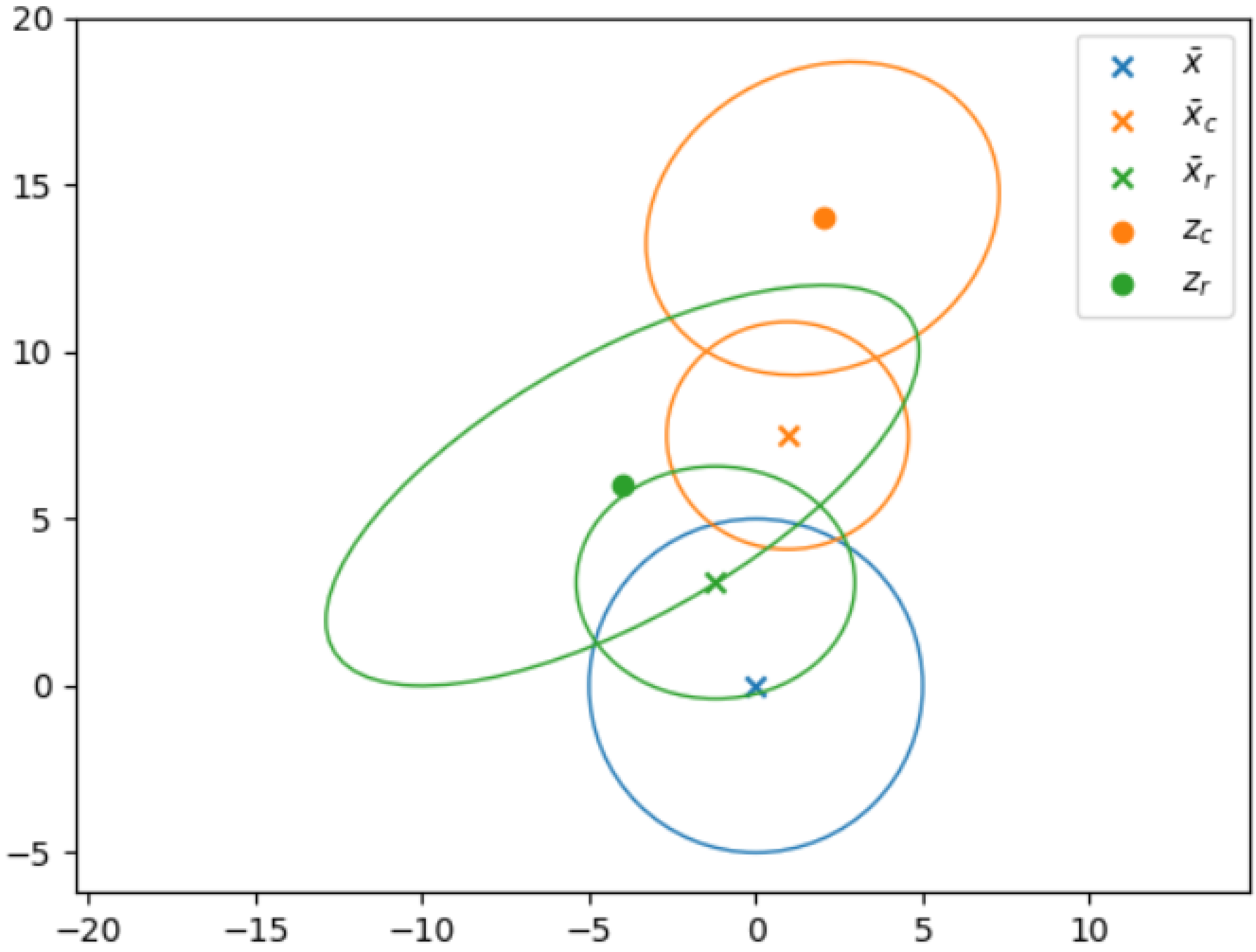
b) thm. 3.4.1

By inserting into (7), we see that the marg. dist. is (8.)

c) Using the same method as in c), we see that the cond. dist of $x|y$ is (9)

d) From (9), we insert vars. from 3.21 w. 3.10 and get

$$\hat{P}^{-1} = \bar{P}^{-1} + H^T R^{-1} H$$


$\bar{x}$
$\bar{x}_c$
$\bar{x}_r$
z_c
z_r
20
15
10
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0
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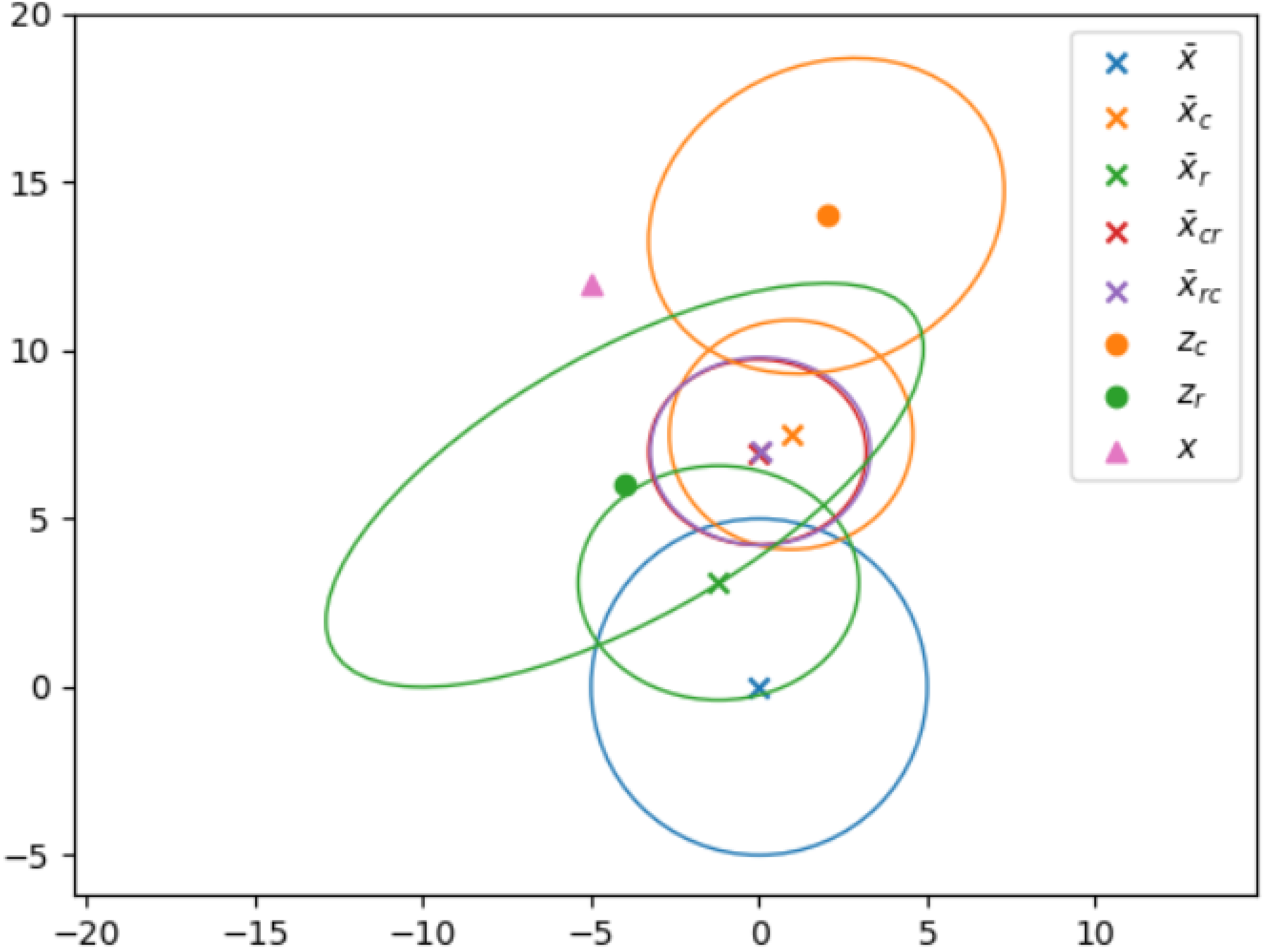
5
10



20
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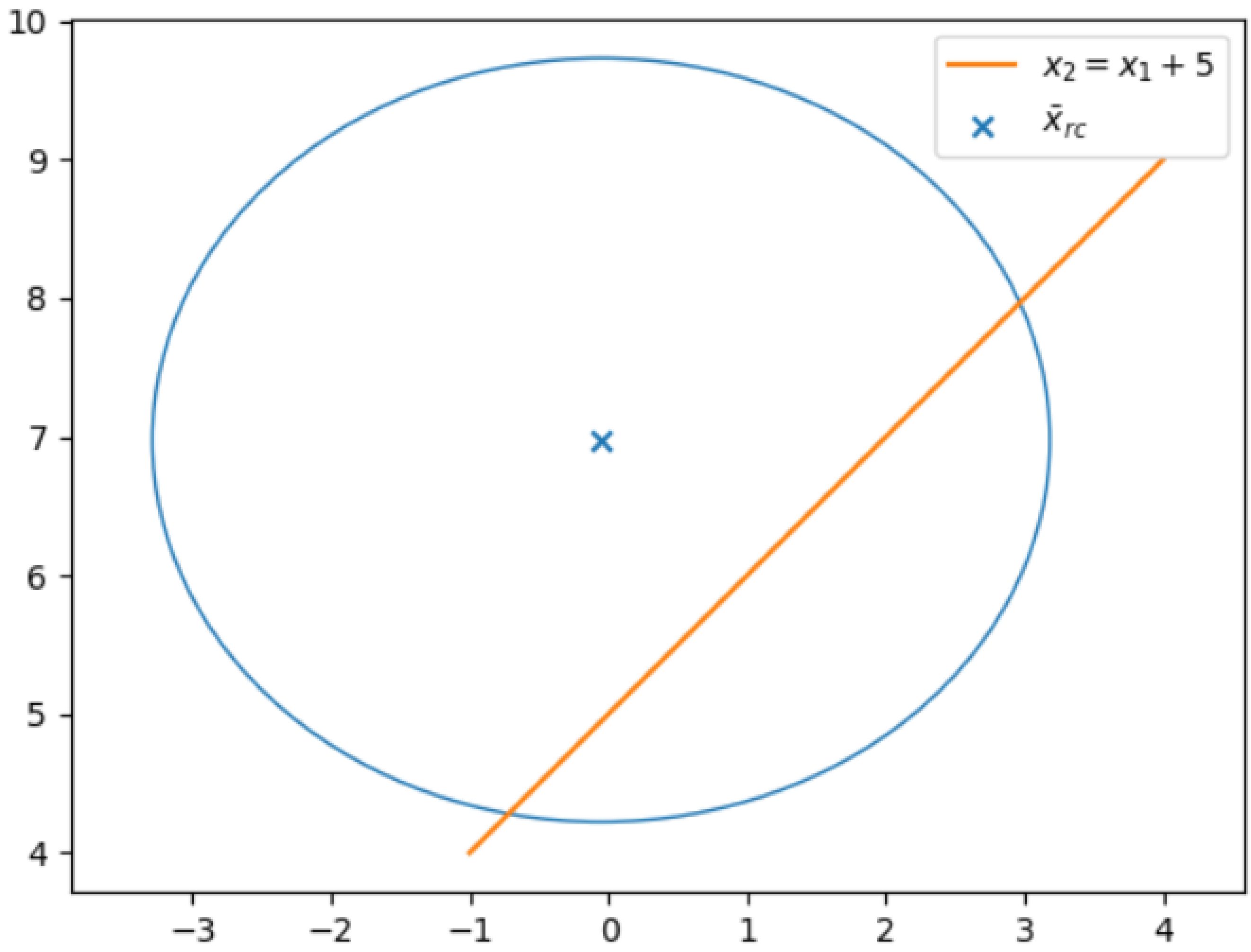
$\bar{x}$
$\bar{x}_c$
$\bar{x}_r$
$\bar{x}_{cr}$
$\bar{x}_{rc}$
z_c
z_r
x



10
9
8
7
6
5
4
−3
−2
−1
0
1
2
3
4
$x_2 = x_1 + 5$
$\bar{x}_{rc}$

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Probability that it is above x_2 = x_1 + 5 is 0.5425276506266974
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