Title: Model for filament studies in toroidal geometry
Author: Ben Dudson
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Model

A finite-β electromagnetic isothermal model in toroidal geometry. Evolving quantities are the vorticity ω, electromagnetic potential $A_{||}$, electron density n and parallel momentum $\Gamma = m_i n v_{||i}$. A constant electron temperature T_e and ion temperature T_i are assumed, which can be separately specified. The magnetic field consists of an equilibrium field $\mathbf{B}_0$, and a time-evolving "poloidal" field so that the total field is:

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_0 + \nabla\times\left(A_{||}\hat{\mathbf{e}}_\phi\right) && (1) \\ &= \mathbf{B}_0 + \nabla\psi\times\nabla\phi && (2) \end{aligned}$$

where $\psi = RA_{||}$ is the poloidal flux, and $A_{||}$ is the parallel component of the vector potential. Note that this makes a large aspect-ratio approximation.

The equations in SI units are:

$$\begin{aligned} \frac{\partial\omega}{\partial t} + \left(\mathbf{v}_E + \mathbf{v}_{||i}\right)\cdot\nabla\omega &= \nabla_{||}\left(J_{||}\right) + \nabla\cdot\left(p\nabla\times\frac{\mathbf{b}}{B}\right) + \nu\nabla_\perp^2\omega && \text{(3a)} \\ \frac{\partial}{\partial t}\left[A_{||} - \frac{m_e}{e}v_{||e}\right] &= -\partial_{||}\phi + \frac{1}{n}\partial_{||}p_e - \frac{1}{en}\eta J_{||} && \text{(3b)} \\ \frac{\partial n}{\partial t} + \mathbf{v}_E\cdot\nabla n &= -\nabla_{||}\left(v_{||e}n\right) + \nabla\cdot\left(p_e\nabla\times\frac{\mathbf{b}}{B}\right) && \text{(3c)} \\ \frac{\partial\Gamma}{\partial t} + \mathbf{v}_E\cdot\nabla\Gamma &= -\nabla_{||}\left(v_{||i}\Gamma\right) - \nabla\cdot\left(\Gamma e T_i\nabla\times\frac{\mathbf{b}}{B}\right) - \partial_{||}p && \text{(3d)} \\ \omega &= \nabla\cdot\left[\frac{m_i n}{B_0^2}\left(\nabla_\perp\phi + \frac{\nabla_\perp p_i}{en}\right)\right] && \text{(3e)} \\ J_{||} &= -\frac{1}{\mu_0}\nabla_\perp^2 A_{||} && \text{(3f)} \end{aligned}$$

Where the notation is $\partial_{||} \equiv \mathbf{b}\cdot\nabla$ and $\nabla_{||}f \equiv \nabla\cdot(\mathbf{b}f) = B\partial_{||}\left(\frac{f}{B}\right)$. The total pressure is $p = p_e + p_i = en\left(T_e + T_i\right)$ and parallel momentum $\Gamma = m_i n v_{||i}$. The vector $\mathbf{b}_0 = \mathbf{e}_\phi$ is the "toroidal" magnetic field unit vector, and $\mathbf{b} = \mathbf{B}/B_0$ is the unit vector along the total magnetic field, assuming the poloidal magnetic field is small compared to the toroidal field. $\nabla_\perp = \nabla - \mathbf{b}_0\mathbf{b}_0\cdot\nabla$ is the component of the gradient in the poloidal plane.

The dissipation terms are the kinematic viscosity ν (units m^2/s) and resistivity η (units $\Omega\mathrm{m}$).

The magnetic drift term is written as

$$\nabla\cdot\left[p\nabla\times\frac{\mathbf{b}}{B}\right] = \nabla\times\frac{\mathbf{b}}{B}\cdot\nabla p \tag{4}$$

$$= \left(\nabla\frac{1}{B^2}\times\mathbf{B}+\frac{1}{B^2}\nabla\times\mathbf{B}\right)\cdot\nabla p \tag{5}$$

$$= -\frac{2}{B^3}\nabla B\times\mathbf{B}\cdot\nabla p \tag{6}$$

$$= \frac{2}{B}\mathbf{b}\times\nabla\log B\cdot\nabla p \tag{7}$$

which uses $\nabla\times\mathbf{B}\cdot\nabla p = 0$ which is valid in equilibrium since $\mathbf{J}\cdot\nabla p = 0$. Here we define the curvature operator

$$C\left(f\right) = \frac{2}{B}\mathbf{b}\times\nabla\log B\cdot\nabla f \tag{8}$$

We also use the notation

$$\left[f,g\right] = \frac{1}{B}\mathbf{b}\cdot\nabla f\cdot\nabla g$$

Normalised equations

Normalising to a reference density n_0 (in m^{-3}), temperature T_0 (in eV), magnetic field B_0 (in Tesla) with ion mass m_i (in kg), we define time and length scales:

$$\Omega_i = \frac{eB_0}{m_i} \qquad c_s = \sqrt{\frac{eT_0}{m_i}} \qquad \rho_s = c_s/\Omega_i \tag{9}$$

so that spatial and time derivatives are

$$\frac{\partial}{\partial\hat{t}} = \frac{1}{\Omega_i}\frac{\partial}{\partial t} \qquad \hat{\nabla} = \rho_s\nabla \tag{10}$$

i.e. a Bohm normalisation. We also get a reference beta value:

$$\beta = \frac{\mu_0 e n_0 T_0}{B_0^2} \tag{11}$$

The evolving variables are normalised as:

$$\hat{\omega} = \frac{\omega}{en_0} \qquad \hat{n} = \frac{n}{n_0} \qquad \hat{A}_{||} = \frac{A_{||}}{B_0\rho_s} \qquad \hat{\Gamma} = \frac{\Gamma}{n_0 c_s} \tag{12}$$

and auxilliary variables as

$$\hat{\phi} = \frac{\phi}{T_0} \qquad \hat{J}_{||} = \frac{J_{||}}{en_0 c_s} \tag{13}$$

with dissipation parameters

$$\hat{\nu} = \nu B_0/T_0 \qquad \hat{\eta} = \eta\frac{en_0}{B_0} \tag{14}$$

The electron and ion temperatures are fixed, and specified separately: $\hat{T}_e = T_e/T_0$ and $\hat{T}_i = T_i/T_0$ The normalised equations are therefore (omitting the hats):

$$
\begin{aligned}
\frac{\partial \omega}{\partial t} &= -\left[\phi, \omega\right] - \nabla_{||}\left(v_{||i}\omega\right) + \nabla_{||}J_{||} + C\left(p\right) + \nu\nabla_\perp^2\omega && \text{(15a)}\\
\frac{\partial A_{||}}{\partial t} &= -\partial_{||}\phi + T_e\partial_{||}n - \eta J_{||} && \text{(15b)}\\
\frac{\partial n}{\partial t} &= -\left[\phi, n\right] - \nabla_{||}\left(v_{||e}n\right) + C\left(nT_e\right) && \text{(15c)}\\
\frac{\partial \Gamma}{\partial t} &= -\left[\phi, \Gamma\right] - \nabla_{||}\left(v_{||i}\Gamma\right) - C\left(\Gamma T_i\right) - \partial_{||}p && \text{(15d)}\\
\omega &= \nabla_\perp^2\left(\phi + T_i n\right) && \text{(15e)}\\
J_{||} &= \frac{1}{\beta}\nabla_\perp^2 A_{||} && \text{(15f)}\\
& && \text{(15g)}
\end{aligned}
$$

where $p = \left(T_e + T_i\right)n$ and $v_{||i} = \Gamma/n$

If finite electron mass is included, then Ohm's law and auxillary equations become:

$$
\begin{aligned}
\frac{\partial A_{j||}}{\partial t} &= -\partial_{||}\phi + T_e\partial_{||}n - \eta J_{||} && \text{(16a)}\\
A_{j||} &= A_{||} - \frac{m_e}{m_i}v_{||e} && \text{(16b)}\\
n\left(A_{j||} + \frac{m_e}{m_i}v_{||i}\right) &= nA_{||} - \frac{m_e}{m_i}\frac{1}{\beta}\nabla_\perp^2 A_{||} && \text{(16c)}\\
J_{||} &= n\left[\frac{m_i}{m_e}\left(A_{j||} - A_{||}\right) + v_{||i}\right] && \text{(16d)}
\end{aligned}
$$

This removes a second derivative from the equations, which tends to amplify noise, but introduces an inversion to obtain $A_{||}$ from $A_{j||}$.

Numerical methods

All cross-field drifts ($E \times B$ and curvature) are implemented as 2^{nd}-order Arakawa brackets. The curvature operator is implemented as in equation 8 as an Arakawa bracket with $\log B$ as an input.

Parallel derivatives

Parallel derivatives use the Flux Coordinate Independent (FCI) method with Hermite spline interpolation for the starting (equilibrium) magnetic field.

Parallel divergences of the form $\nabla_{||}\left(v_{i||}f\right)$ are implemented in skew-symmetric form, which helps to suppress zig-zag instabilities and parasitic modes.

$$
\nabla_{||}\left(v_{i||}f\right) = \frac{1}{2}\left(\nabla_{||}\left(v_{i||}f\right) + f\nabla_{||}v_{||i} + v_{||i}\partial_{||}f\right) \tag{17}
$$

Dissipation

In the vacuum region the resistivity and viscosity is increased, and there is additional damping of the vorticity. This is done using a vacuum mask M, based on a tanh function:

$$M = \frac{1}{2}\left[1 - \tanh\left(\frac{n - n_{vac}}{\Delta n_{vac}}\right)\right]$$

which transitions from 0 in the plasma to 1 in the vacuum at a density n_{vac}, called `vacuum_density` in the input. Δn_{vac} is the range of density over which the transition occurs, called `vacuum_trans`. The viscosity and resistivity terms are modified to:

$$\frac{\partial \omega}{\partial t} = \nu\left(1 + MD\right)\nabla_\perp^2\omega \tag{18a}$$

$$\frac{\partial A_{||}}{\partial t} = -\eta\left(1 + MD\right)J_{||} \tag{18b}$$

where D is a factor multiplying the dissipation terms in the vacuum, called `vacuum_mult`. In addition there is a dissipation term in the vorticity equation:

$$\frac{\partial \omega}{\partial t} = -Md\omega \tag{19}$$

where d is called `vacuum_damp` in the input and represents a timescale (in units of Ω_{ci}^{-1} for vorticity to be damped in the vacuum region.