Ben Durham
Multivariable Calculus

Homework 5/7/20

p.719 #50
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Problem #50

To find the distance between a point and a plane, one find the projection of a vector between the point and any point on the plane onto the unit normal vector of the plane. In this case, the normal vector is simply $\langle 2, 4, -1\rangle$ from the equation. Let A be any point on the plane.

$$\begin{aligned} A &= (0,0,1) \\ \overrightarrow{AP} &= \langle 2,-1,2\rangle \\ \hat{n} &= \frac{1}{\sqrt{21}}\langle 2,4,-1\rangle \\ \overrightarrow{AP}\cdot\hat{n} &= -\frac{2}{\sqrt{21}} \end{aligned}$$

Therefore, the distance between P and the given plane is $\frac{2}{\sqrt{21}}$

Problem #31

$$\begin{aligned} f(x,y) &= \sin(y^2 - xy) \\ \nabla\vec{f}(x,y) &= -y\cos(y^2-xy)\hat{\imath} + (2y-x)cos(y^2-xy)\hat{\jmath} \end{aligned}$$

Problem #40

The directional derivative is defined as the dot product of a given unit vector and the gradient.

$$\begin{aligned} f(x,y) &= x^3 - y^3 \\ \nabla\vec{f}(x,y) &= \langle 3x^2, 3y^2\rangle \\ \vec{u} &= \langle 1,-1\rangle \\ \hat{u} &= \frac{1}{\sqrt{2}}\langle 1,-1\rangle \\ \nabla\vec{f}(2,-1) &= \langle 12,-3\rangle \\ f_{\hat{u}}(2,-1) &= \nabla\vec{f}(2,-1)\cdot\hat{u} \\ f_{\hat{u}}(2,-1) &= \frac{15}{\sqrt{2}} \end{aligned}$$

Problem #48

One approach to finding a vector normal to a surface such as $z^2-2xyz=x^2+y^2$ at a given point $(1,2,-1)$ is to rewrite the surface as if it was the level curve of a four dimensional function. Let $f(x,y,z)$ be a function defined as follows

$$f(x,y,z)=z^2-2xyz-x^2-y^2$$

Now, since the gradient of a function is always perpendicular to its level curves, one need only find the gradient and evaluate it at the point in question.

$$\nabla\vec{f}(x,y,z)=(-2x-2yz)\hat{\imath}+(-2y-2xz)\hat{\jmath}+(2z-2xy)\hat{k}$$
$$\nabla\vec{f}(1,2,-1)=\langle 2,-2,-6\rangle$$

As such, $\vec{n}$ will be perpendicular to the surface

$$\vec{n}=\langle 2,-2,-6\rangle$$

Problem #65

$$f(x,y)=x^2e^{xy}$$

a. One can use a normal vector and a point to construct a plane normal to a given function. First, define a new function $g(x,y,z)$, which the given function will be a level curve of.

$$z=x^2e^{xy}$$
$$k=x^2e^{xy}-z$$
$$g(x,y,z)=x^2e^{xy}-z$$
$$\nabla\vec{g}(x,y,z)=(2xe^{xy}+x^2e^{xy})\hat{\imath}+x^3e^{xy}\hat{\jmath}-\hat{k}$$
$$\vec{n}=\nabla\vec{g}(1,0,1)=2\hat{\imath}+\hat{\jmath}-\hat{k}$$

By definition, the gradient of a function is perpendicular to the level curves of that function, so now this can be used along with the given point to construct a tangent plane.

$$2(x-1)+y-(z-1)=0$$

b. The plane from (a) can be used to create a linearization by solving for z

$$L(x,y)=z=2x+y-1$$

From here, one can just plug in the point $(1,0)$.

$$L(1,0)=1$$

c. By the definition of total differentials:

$$df=f_x(a,b)dx+f_y(a,b)dy$$

$$df=2dx+dy$$