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Math Bootcamp

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Functions

Polynomials

Properties of the Functions

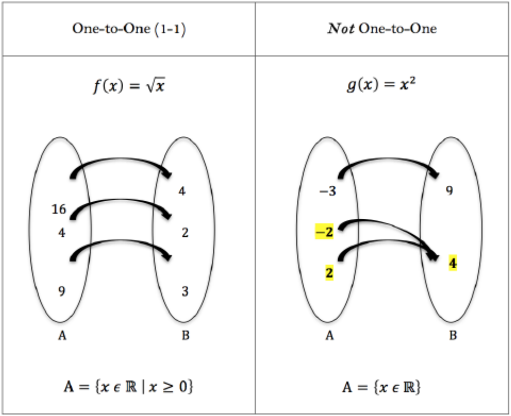
Derivative: Definition

Monotonicity

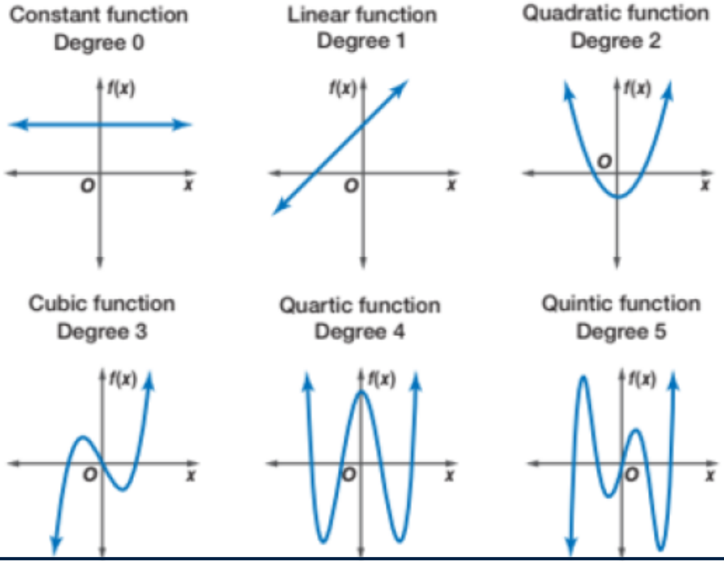
Concavity

Introduction

A function is a relation that associates each element $x \in X$, the *domain* of the function, to a single element $y \in Y$ (possibly the same set), the *co-domain* of the function.



Examples


Constant function
Degree 0
Linear function
Degree 1
Quadratic function
Degree 2
Cubic function
Degree 3
Quartic function
Degree 4
Quintic function
Degree 5
f(x)
O
x

Examples:

1.

$$f(x) = x + 1 \qquad f(2) = 2 + 1 = 3$$

2.

$$f(x) = \frac{1}{x+1} \qquad f(1) = \frac{1}{1+1} = \frac{1}{2}$$

Linear function

The simplest possible functions are the polynomials of degree 0: the constant functions $f(x) = b$. Since such functions assign the same number b to every real number x, they are too simple to be interesting. The simplest *interesting* functions are the polynomials of degree one: functions f of the form

$$f(x) = mx + b.$$

Such functions are called **linear functions** because they are precisely the functions whose graphs are straight lines, as will now be demonstrated.

Examples:

- $f(x) = 2x + 1$
- $f(x) = -x - 10$

Definition Let (x_0, y_0) and (x_1, y_1) be arbitrary points on a line ℓ. The ratio

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

is called the **slope** of line ℓ. The analysis in Figure 2.6 shows that the slope of ℓ is independent of the two points chosen on ℓ. The same analysis shows that two lines are **parallel** if and only if they have the same slope.

Example The slope of the line joining the points (4, 6) and (0, 7) is

$$m = \frac{7-6}{0-4} = -\frac{1}{4}.$$

This line slopes downward at an angle just less than the horizontal. The slope of the line joining (4, 0) and (0, 1) is also $-1/4$; so these two lines are parallel.

Theorem The line whose slope is m and whose y-intercept is the point $(0, b)$ has the equation $y = mx + b$.

If, instead, we are given two points on the line, say (x_0, y_0) and (x_1, y_1), we can use these two points to compute the slope m of the line:

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

Example Let x denote the temperature in degrees Centigrade and let y denote the temperature in degrees Fahrenheit. We know that x and y are linearly related, that 0^o Centigrade or 32^o Fahrenheit is the freezing temperature of water and that 100^o Centigrade or 212^o Fahrenheit is the boiling temperature of water. To find the equation which relates degrees Fahrenheit to degrees Centigrade, we find the equation of the line through the points (0, 32) and (100, 212).

Domain

Give a function f, the set of number x at which $f(x)$ is defined is called the domain of f.

Examples:

- $f(x) = \dfrac{1}{x}$ is not defined at $x = 0$.
- $h(x) = \dfrac{1}{x^2 - 1}$ the domain is all x except $\{-1, 1\}$.
- $g(x) = \sqrt{x - 7}$ the domain is all $x \geq 7$.

If we were to consider our height above sea level, y, as a function of the amount of time we are walking, x, we would say that y is increasing as x is increasing while we are on that hill. In mathematics, we would say that when we are walking uphill, our function is an increasing function.

Definition: Increasing Function

A function f is increasing if:

$$x > y \text{ implies that } f(x) > f(y)$$

Analogously defined decreasing and we have:

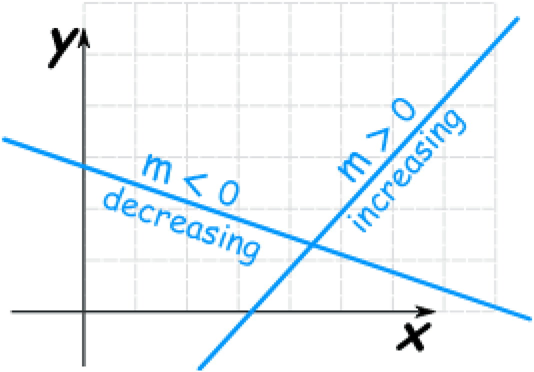
Definition: Decreasing Function

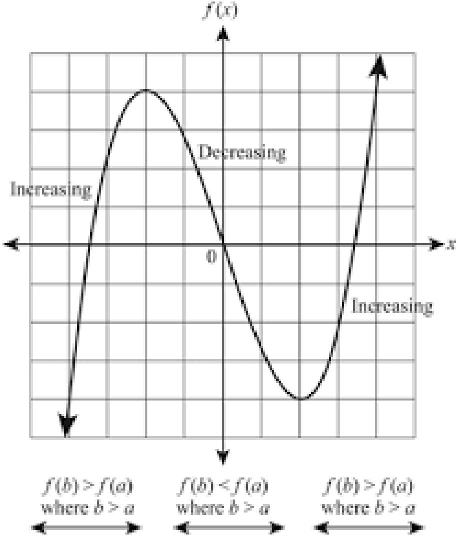
A function f is increasing if:

$$x > y \text{ implies that } f(x) < f(y)$$

In these cases, it is said strictly increasing or decreasing, because the inequality is strict.

Examples




f(x)
x
0
Increasing
Decreasing
Increasing
f(b) > f(a)
where b > a
f(b) < f(a)
where b > a
f(b) > f(a)
where b > a

Your turn! Find the formula for the linear function such that:

1. has slope 2 and y-intercept $(0, 3)$
2. has slope -3 and y-intercept $(0, 0)$

What is the domain of each of the following functions:

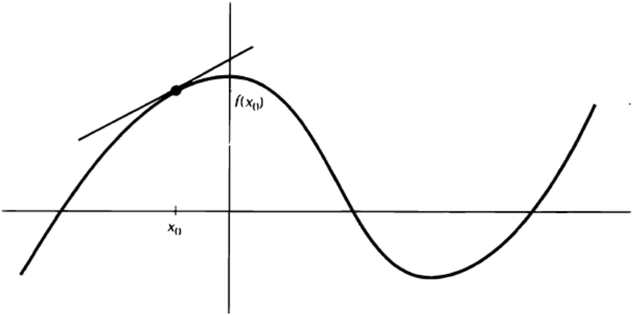
1. $y = \dfrac{1}{x-1}$
2. $y = \sqrt{1-x^2}$

Analyze the monotonicity of the following functions:

1. $f(x) = -x^7$
2. $f(x) = x^2$

we define the slope of a nonlinear function f at a point $(x_0, f(x_0))$ on its graph as the slope of the tangent line to the graph of f at that point. We call the slope of the tangent line to the graph of f at $(x_0, f(x_0))$ the **derivative** of f at x_0, and we write it as

$$f'(x_0) \qquad \text{or} \qquad \frac{df}{dx}(x_0).$$



Definition Let $(x_0, f(x_0))$ be a point on the graph of $y = f(x)$. The **derivative** of f at x_0, written

$$f'(x_0) \quad \text{or} \quad \frac{df}{dx}(x_0) \quad \text{or} \quad \frac{dy}{dx}(x_0),$$

is the slope of the tangent line to the graph of f at $(x_0, f(x_0))$. Analytically,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

if this limit exists. When this limit does exist, we say that the function f is **differentiable** at x_0 with derivative $f'(x_0)$.

Theorem For any positive integer k, the derivative of $f(x) = x^k$ at x_0 is $f'(x_0) = kx_0^{k-1}$.

Theorem Suppose that k is an arbitrary constant and that f and g are differentiable functions at $x = x_0$. Then,

$$
\begin{aligned}
&a)\quad (f \pm g)'(x_0) &&= f'(x_0) \pm g'(x_0),\\
&b)\quad (kf)'(x_0) &&= k(f'(x_0)),\\
&c)\quad (f \cdot g)'(x_0) &&= f'(x_0)g(x_0) + f(x_0)g'(x_0),\\
&d)\quad \left(\frac{f}{g}\right)'(x_0) &&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2},\\
&e)\quad ((f(x))^n)' &&= n(f(x))^{n-1} \cdot f'(x),\\
&f)\quad (x^k)' &&= kx^{k-1}.
\end{aligned}
$$

Use the theorem to calculate the derivative of the following functions:

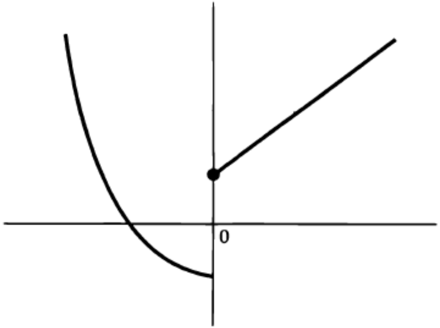
1. $f(x) = mx + b$
2. $f(x) = ax^2 + bx + c$
3. $f(x) = x^{100} + x + 1$
4. $f(x) = 3$
5. $f(x) = (x^2 + 2x + 3)(x^2 - 1)$

Definition: Continuous Functions

A function is continuous if its graph has no breaks.

Example: **All the elementary functions.**

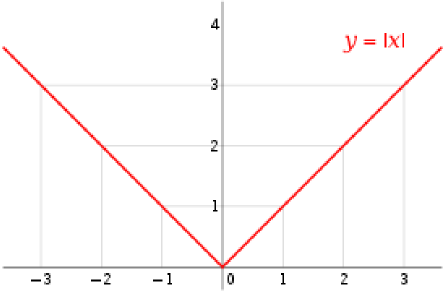
And an example of a discontinuous functions:



The function g is discontinuous at $x = 0$.

There are continuous but not differentiable (not derivable) functions at certain points of the domains. For example:

$$f(x) = |x|$$



Your turn! Exercise: Find the derivative of

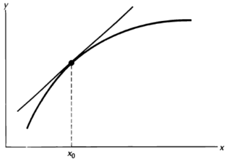
1. $f(x) = x^7 + 2x^3 + 5$
2. $f(x) = x^{-1} + 2x^{-3}$
3. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$
4. $f(x) = (x^3 + 2x^2)^5$

First Derivative

The first derivative describes the monotonicity of the function.

Theorem Suppose that the function f is continuously differentiable at x_0. Then,

(*a*) if $f'(x_0) > 0$, there is an open interval containing x_0 on which f is increasing, and
(*b*) if $f'(x_0) < 0$, there is an open interval containing x_0 on which f is decreasing.



If $f'(x_0) > 0$, the graph of f slopes upward.

Theorem Let f be a continuously differentiable function on domain $D \subset \mathbf{R}^1$

If $f' > 0$ on interval $(a, b) \subset D$, then f is increasing on (a, b).
If $f' < 0$ on interval $(a, b) \subset D$, then f is decreasing on (a, b).
If f is increasing on (a, b), then $f' \geq 0$ on (a, b).
If f is decreasing on (a, b), then $f' \leq 0$ on (a, b).

Example: Study the monotonicity of the following functions:

1. $f(x) = x^2 + 2x + 1$
2. $f(x) = 9x - 3x^3$

Definition: Convex (or Concave upward)

A function f is concave upward at the point $(c, f(c))$ if:

1. $f'(x)$ exists for c and for all x in some open interval containing c.
2. The point $(x, f(x))$ on the graph of f lies above the corresponding point on the graph of the tangent line to f at c.

- This is expressed by the inequality $f(x) < f(c) + f'(c)(x - c)$ for all x in some open interval containing c.
- Imagine holding a ruler along the tangent line through the point $(c, f(c))$. If the ruler supports the graph of f near $(c, f(c))$, then the graph of the function is concave upward.

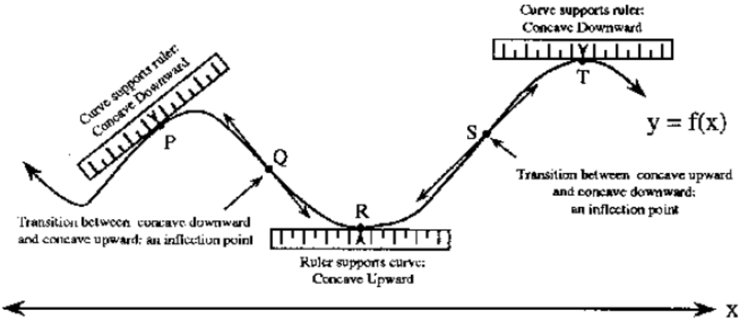
Definition: Concave (or Concave downward)

The graph of a function f is concave downward at the point $(c, f(c))$ if:

1. $f'(c)$ exists and if for all x in some open interval containing c.
2. The point $(x, f(x))$ on the graph of f lies below the corresponding point on the graph of the tangent line to f at c.

- This is expressed by the inequality $f(x) > f(c) + f'(c)(x - c)$ for all x in some open interval containing c.
- Imagine holding a ruler along the tangent line through the point $(c, f(c))$. If the graph of f supports the ruler near $(c, f(c))$, then the graph of the function is concave downward.

Concavity


Curve supports ruler:
Concave Downward
Curve supports ruler:
Concave Downward
P
Q
R
S
T
y = f(x)
Transition between concave downward
and concave upward: an inflection point
Ruler supports curve:
Concave Upward
Transition between concave upward
and concave downward:
an inflection point
x

Another definition of concavity, but now with inequalities:

Definition: Convex and Concave Functions

Let $-\infty \leq a < b \leq \infty$, and let $\varphi\colon (a,b) \to \mathbb{R}$ be a function.

1. We say that φ is **convex** if

$$\varphi\big((1-\lambda)x + \lambda y\big) \;\leq\; (1-\lambda)\varphi(x) + \lambda\varphi(y)$$

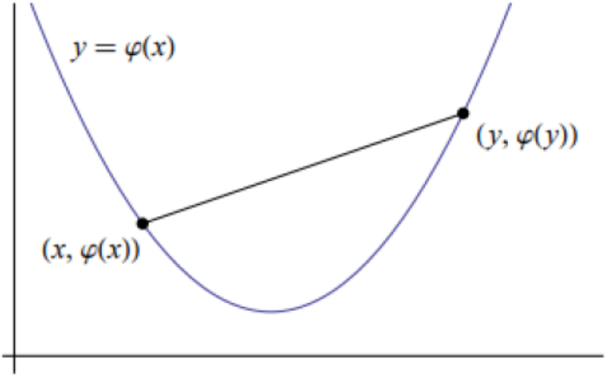
   for all $x, y \in (a,b)$ and $\lambda \in [0,1]$.

2. We say that φ is **concave** if

$$\varphi\big((1-\lambda)x + \lambda y\big) \;\geq\; (1-\lambda)\varphi(x) + \lambda\varphi(y)$$

   for all $x, y \in (a,b)$ and $\lambda \in [0,1]$.

The definition of concavity using inequalities resembles holding chords between points of the graph:

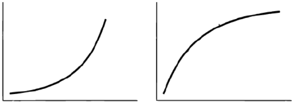


For a convex function, every chord lies above the graph.

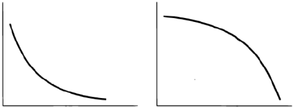
Theorem: Concavity

If the function f is C^2 (twice differentiable) at $x = c$, then:

- The graph of f is concave upward at $(c, f(c))$ if $f''(c) > 0$.
- The graph of f is concave downward if $f''(c) < 0$.



An increasing function can be concave up or concave down.

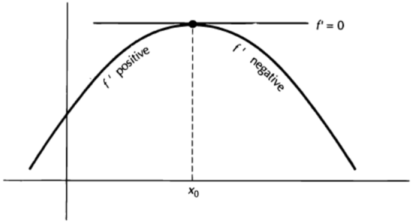


Example: study $f(x) = x^3 - 3x^2 + x - 2$.

Optimization

Suppose that c is a critical point at which $f'(c) = 0$. if $f''(x)$ exists in a neighborhood around c, then:

- f has a *relative maximum* value at c if $f''(c) < 0$.
- f has a *relative minimum* value at c if $f''(c) > 0$.
- And if $f''(c) = 0$, the test is not informative.



Optimization: Extreme Value Theorem

If we are looking for the global maximum of a C^1 function f with domain $I = [a, b]$, we need only:

(1) compute the critical points of f by solving $f'(x) = 0$ for x in (a, b),
(2) evaluate f at these critical points and at the endpoints a and b of its domain, and
(3) choose the point from among these that gives the largest value of f in step 2.

Find the maxima and minima of the following functions:

- $f(x) = x^3 - 3x^2 + x - 2$
- $f(x) = x^3 + 6x$

Your turn! Find the local maxima and minima of the following functions:

- $f(x) = x^4 - 4x^3 + 4x^2 + 4$
- $f(x) = x^2 + 1$ where $x \in [-2, 1]$

Suppose that x years after its founding in 1960, the association of X had a membership given by the function $f(x) = 2x^3 - 45x^2 + 300x + 500$. Between 1960 and 1980, what was its largest and smallest membership, and when were these two extremes realized?

Questions?

See you in the next class!