ç

Math Bootcamp

UC San Diego

Umberto Mignozzetti

Today's Roadmap

Composite Function

Exponentials and Logarithms

Functions of Several Variables

Quadratic Forms

Let g and h two functions on R.

- The function built by applying g to any number x and then using h to the result $g(x)$ is called the *composition of functions* g and h.
- $f(x) = h(g(x))$ or $f(x) = (h \circ g)(x)$.
- The function f is called the composite of functions h and g.

Composite Function

If $g(x) = x^2$ and $h(x) = x + 4$:

- $(h \circ g)(x) = x^2 + 4$
- $(g \circ h)(x) = (x + 4)^2$
- **Important:** $(h \circ g)(x) \neq (g \circ h)(x)$

Let $g(x)$. If $h(x) = x^k$, then $(h \circ g)(x) = (g(x))^k$

Chain Rule

Let $f(x) = (h \circ g)(x) = h(g(x))$. Then:

$$f'(x) = h'(g(x))g'(x)$$

Example: Prove that $\dfrac{d}{dx}\left[g(x)^k\right] = k(g(x))^{k-1}g'(x)$.

Exercise

Your turn! Find the derivatives for:

1. $(h \circ g)(x)$, with $g(x) = x^2 + 4$ and $h(z) = 5z - 1$.
2. $(\varphi \circ \gamma)(\tau)$, with $\gamma(\tau) = \tau^3$ and $\varphi(\lambda) = \dfrac{\lambda - 1}{\lambda + 1}$

Definition: Exponential Function

For $a \in \mathbb{R}_+^*$ (positive real number), the exponential function is defined as:

$$f(x) = a^x$$

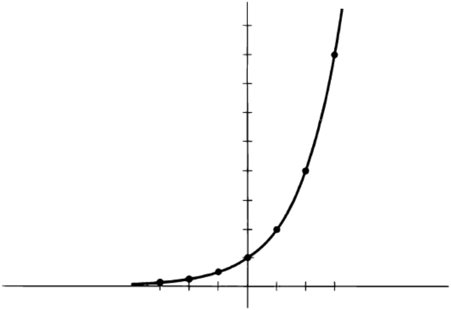
And a is called the base of the exp. function.

Examples:

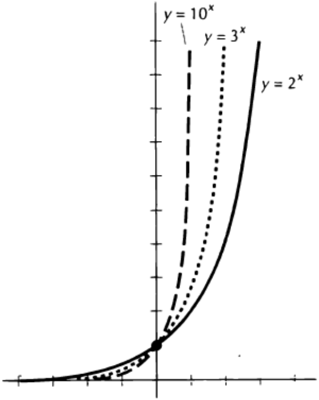
1. If x is a positive integer, a^x means "multiply a by itself x times."
2. If $x = 0$, $\quad a^0 = 1$, by definition.
3. If $x = \dfrac{1}{n}$, $\quad a^{\frac{1}{n}} = \sqrt[n]{a}$.
4. If $x = \dfrac{m}{n}$, $\quad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.

x	2^x
-3	$1/8$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4
3	8

(a) Function



(b) Graphic



The graphs of $f_1(x) = 2^x$, $f_2(x) = 3^x$, *and* $f_3(x) = 10^x$.

The graph of $y = b^x$ is a bit different of the base b lies between 0 and 1.

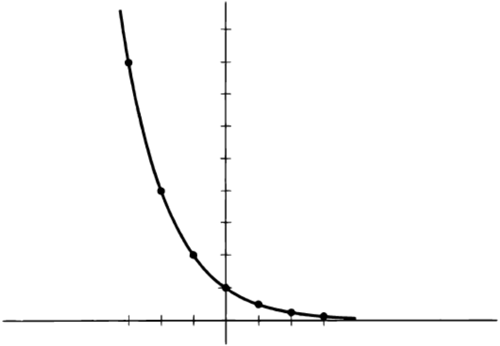
Consider $h(x) = \left(\frac{1}{2}\right)^x$ as an example.

$$h(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$$

This means that the graph of $h(x) = \left(\frac{1}{2}\right)^x$ is simply the reflection of the graph of $f(x) = 2^x$ in the y-axis.

x	$(1/2)^x$
−3	8
−2	4
−1	2
0	1
1	1/2
2	1/4
3	1/8

(c) Function



The graph of $y = (1/2)^x$.

(d) Graphic

Important Properties:

Let $a \in \mathbb{R}_+^*$, we have:

1. $a^s a^r = a^{s+r}$
2. $a^{-r} = \dfrac{1}{a^r}$
3. $\dfrac{a^r}{a^s} = a^{r-s}$
4. $(a^n)^m = a^{nm}$
5. $a^0 = 1$

The number e is a mathematical constant that is the base of the natural logarithm: the unique number whose natural logarithm is equal to one.

Theorem As $n \to \infty$, the sequence $\left(1 + \frac{1}{n}\right)^n$ converges to a limit denoted by the symbol e. Furthermore,

$$\lim_{n\to\infty}\left(1 + \frac{k}{n}\right)^n = e^k.$$

If one deposits A dollars in an account which pays annual interest at rate r compounded continuously, then after t years the account will grow to Ae^{rt} dollars.

Consider a general exponential function $y = a^x$, with base $a > 1$. Such an exponential function is a strictly increasing function:

$$x_1 > x_2 \text{ implies } a^{x_1} > a^{x_2}$$

When $a < 1$, it is strictly decreasing.

The inverse of $z = a^y$, when the base a, is the **logarithm** with base a, and write:

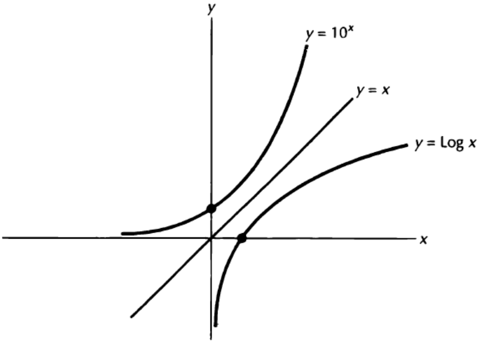
$$y = \log_a(z)$$

By definition, the logarithm of z is the power to which one must raise a to yield z.

$$a^{\log_a(z)} = z \text{ and } \log_a(a^z) = z$$

Examples:

1. $\log 10 = 1$ since $10^1 = 10$
2. $\log 1 = 0$ since $10^0 = 1$
3. $\log 100000 = 5$ since $10^5 = 100000$



The graph of $y = \text{Log}\, x$ is the reflection of the graph of $y = 10^x$ across the diagonal $\{y = x\}$.

Base e Logarithms

The inverse of e^x is called the natural logarithm function and is written as $\ln x$. Formally,

$$\ln x = y \text{ if and only if } e^y = x$$

Examples:

1. $\ln e = 1$ since $e^1 = e$
2. $\ln 1 = 0$ since $e^0 = 1$
3. $\ln 40 = 3.688...$ since $e^{3.688...} = 40$

Logarithms: Properties

Properties:

1. $\log(r \cdot s) = \log r + \log s$
2. $\log(\frac{1}{s}) = -\log s$
3. $\log(\frac{r}{s}) = \log r - \log s$
4. $\log r^s = s \log r$
5. $\log 1 = 0$

Solve the following equations:

1. $e^{5x} = 10$
2. $\ln x^2 = 5$
3. $2e^{6x} = 18$

Theorem The functions e^x and $\ln x$ are continuous functions on their domains and have continuous derivatives of every order. Their first derivatives are given by

$$a)\quad (e^x)' = e^x,$$

$$b)\quad (\ln x)' = \frac{1}{x}.$$

If $u(x)$ is a differentiable function, then

$$c)\quad \left(e^{u(x)}\right)' = \left(e^{u(x)}\right) \cdot u'(x),$$

$$d)\quad (\ln u(x))' = \frac{u'(x)}{u(x)} \qquad \text{if } u(x) > 0.$$

Your turn! Compute the following derivatives:

1. $\left(e^{5x}\right)'$
2. $\left(\ln x^2\right)'$
3. $\left(e^x \ln(x)\right)'$

Study the properties of the density function of the standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Definition: A function from a set A to a set B is a rule that assigns to each object in A one and only one object in B.

In this case, we write $f : A \to B$.

The set A of elements on which f is defined is called the domain of the function f, the set B in which f its values is called the target or target space of f and $y = f(x)$ is the image of x under f.

Example: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$.

Domain of f: $\mathbb{R}^2$, target space: $\mathbb{R}$, image of f: $\mathbb{R}_+$

Example: The amount of money (z) currently in a savings account depends on how much was originally invested (A), what the annual interest rate (r) is, and how many times (n) a year interest is compounded, and how many years (t) since the original deposit.

The functional relationship between these variables is:

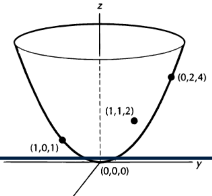
$$z = A\left(1 + \frac{r}{n}\right)^{nt}$$

Just as we need two dimensions to draw the graph of a function from $\mathbb{R}^1$ to $\mathbb{R}^1$, we need three dimensions to draw the graph of a function from $\mathbb{R}^2$ to $\mathbb{R}^1$.

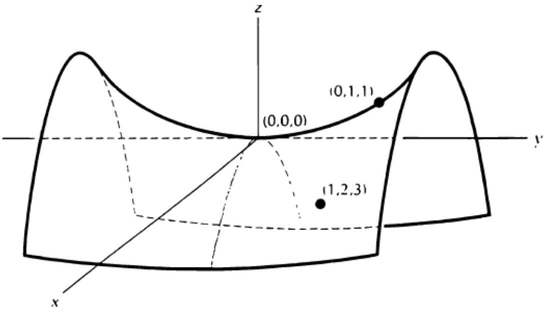
We will use (x, y, z) notation instead of (x_1, x_2, x_3) notation to describe the construction of these graphs.

For each value (x, y) in the domain, we evaluate f at (x, y) and mark the point $(x, y, f(x, y))$ in $\mathbb{R}^3$.

We have drawn the graph of $f(x, y) = x^2 + y^2$ and have labeled some points on the graph.



Now, the same for $f(x, y) = y^2 - x^2$:



Definition: Let $f : \mathbb{R}^n \to \mathbb{R}$. Then for each variable x_i at each point $x^0 = (x_1, x_2, ..., x_n)$ in the domain of f,

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ...x_n) - f(x_1, ..., x_n)}{h}$$

if this limit exists.

Only the *ith* variable changes; the others are treated as constants.

Examples: Compute all the partial derivatives of the following functions:

1. $ax^2 + bxy + cy^2$
2. ye^{x+y}
3. e^{x-y}

Compute the partial derivatives of the Cobb-Douglas production function $Q(x, y) = kx^a y^b$

We write the derivative of F at x^* as a column matrix:

$$\begin{pmatrix} \dfrac{\partial F}{\partial x_1}(x^*) \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial F}{\partial x_n}(x^*) \end{pmatrix}$$

We write it as $\nabla F(x^*)$ and call it the gradient or gradient vector of F at x^*.

The vital characteristic of the gradient vector is its length and direction.

Theorem Let $F: \mathbf{R}^n \to \mathbf{R}^1$ be a C^1 function. At any point $\mathbf{x}$ in the domain of F at which $\nabla F(\mathbf{x}) \neq \mathbf{0}$, the gradient vector $\nabla F(\mathbf{x})$ points at $\mathbf{x}$ into the direction in which F increases most rapidly.

Example We consider once again the production function $Q = 4K^{3/4}L^{1/4}$. Suppose again that the current input bundle is (10,000,625). If we want to know in what proportions we should add K and L to (10,000,625) to increase production most rapidly, we compute the gradient vector

$$\nabla F(10{,}000{,}625) = \begin{pmatrix} 1.5 \\ 8 \end{pmatrix}$$

Higher-Order Derivatives:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

Is called the $x_i x_j$-second order partial derivative of f. It is usually written as:

$$\frac{\partial^2 f}{\partial x_j \partial x_i}$$

The $x_i x_i - derivative$ is usually written as $\dfrac{\partial^2}{\partial x_i^2}$.

Terms of the form $\frac{\partial^2}{\partial x_i \partial x_j}$ with $i \neq j$ are called cross partial derivatives or mixed partial derivatives.

Useful theorem:

Theorem 14.5 Suppose that $y = f(x_1, \cdots, x_n)$ is C^2 on an open region J in R^n. Then, for all $\mathbf{x}$ in J and for each pair of indices i, j,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}).$$

Example: Consider a production function Q that depends on capital K and labor L. Find all the second derivatives of the production function for

$$Q = 4K^{\frac{3}{4}}L^{\frac{1}{4}}$$

The matrix with all the second derivatives is called **Hessian Matrix**. Here is an example of it:

$$D^2 f_{\mathbf{x}} = \begin{pmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{pmatrix}$$

In the previous example, write the Hessian Matrix for the production function.

Your turn! Compute the Gradient vector and the Hessian Matrix of the following functions:

1. $x^2 + 2xy - y^2$
2. ye^x
3. e^{2x+3y}

Definition: A quadratic form on $\mathbb{R}^n$ is a real-valued function of the form:

$$Q(x_1, x_2, ..., x_n) = \sum_{i \leq j} a_{ij} x_i x_j$$

In which each term is a monomial of degree two.

Quadratic form Q can be represented by a symmetric matrix A so that:

$$Q(x) = x^T A x(t)$$

Example:

$$x_1^2 + x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The general quadratic form of one variable is $y = ax^2$. If $a > 0$, then ax^2 is always ≥ 0 and the form is called positive definite ($x = 0$ is its global minimum).

If $a < 0$ then ax^2 is always ≤ 0 and the form is called negative definite ($x = 0$ is its global maximum).

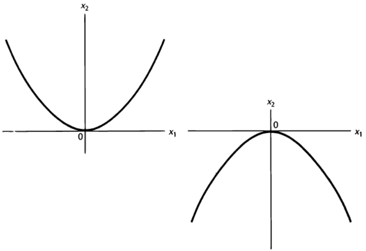
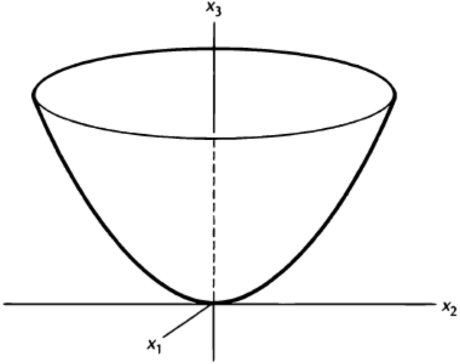


Figure 16.1 *The functions of $f(x) = x^2$ and $f(x) = -x^2$.*

Two dimensions, $Q_1(x_1, x_2) = x_1^2 + x_2^2$ is always greater than zero at $(x_1, x_2) \neq (0, 0)$. So, we call Q_1 positive definite.



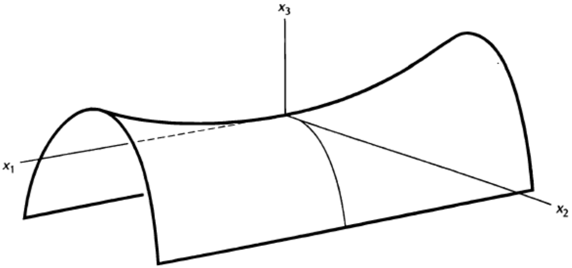
Graph of the positive definite form $Q_1(x_1, x_2) = x_1^2 + x_2^2$.

$$Q_3(x_1, x_2) = x_1^2 - x_2^2$$

$$Q_3(1, 0) = 1$$

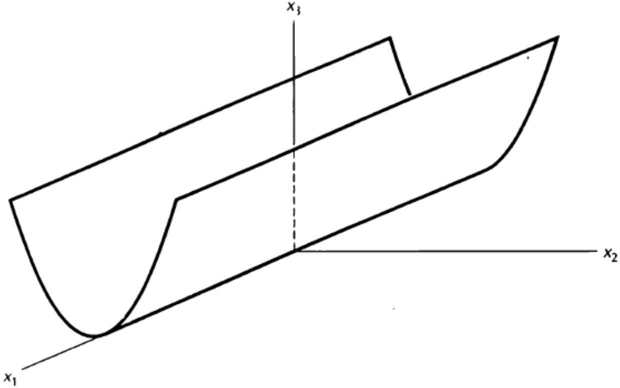
$$Q_3(0, 1) = -1$$

are called indefinite.



The graph of the indefinite form $Q_3(x_1, x_2) = x_1^2 - x_2^2$.

A quadratic form like $Q_5(x_1, x_2) = -(x_1 + x_2)^2$, which is never positive but can be zero at points other than origin, is called negative semidefinite.



The graph of the positive semidefinite form $Q_4(x_1, x_2) = (x_1 + x_2)^2$.

Definition Let A be an $n \times n$ symmetric matrix, then A is:

(*a*) **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R}^n$,
(*b*) **positive semidefinite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R}^n$,
(*c*) **negative definite** if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R}^n$,
(*d*) **negative semidefinite** if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R}^n$, and
(*e*) **indefinite** if $\mathbf{x}^T A\mathbf{x} > 0$ for some $\mathbf{x}$ in $\mathbf{R}^n$ and < 0 for some other $\mathbf{x}$ in $\mathbf{R}^n$.

Remark A matrix that is positive (negative) definite is automatically positive (negative) semidefinite. Otherwise, every symmetric matrix falls into one of the above five categories.

Definition: Let A be an $n \times n$ matrix. A $k \times k$ submatrix of A formed by deleting $n - k$ columns, say i_1, i_2,..., i_{n-k} and the same $n - k$ rows, rows i_1, i_2,..., i_{n-k}, form A is called a kth order principal submatrix of A. The determinant of a $k \times k$ principal submatrix is called a kth order principal minor of A.

Find the principals minors of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Definition Let A be an $n \times n$ matrix. The kth order principal submatrix of A obtained by deleting the *last* $n - k$ rows and the *last* $n - k$ columns from A is called the kth order **leading principal submatrix** of A. Its determinant is called the kth order **leading principal minor** of A. We will denote the kth order leading principal submatrix by A_k and the corresponding leading principal minor by $|A_k|$.

Theorem 16.1 Let A be an $n \times n$ symmetric matrix. Then,

(a) A is positive definite if and only if all its n leading principal minors are (strictly) positive.

(b) A is negative definite if and only if its n leading principal minors alternate in sign as follows:

$$|A_1| < 0, \quad |A_2| > 0, \quad |A_3| < 0, \quad \text{etc.}$$

The kth order leading principal minor should have the same sign as $(-1)^k$.

(c) If some kth order leading principal minor of A (or some pair of them) is nonzero but does not fit either of the above two sign patterns, then A is indefinite. This case occurs when A has a *negative* kth order leading principal minor for an *even* integer k or when A has a negative kth order leading principal minor and a positive ℓth order leading principal minor for two distinct *odd* integers k and ℓ.

Your turn! Classify the following matrix:

$$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$$

Questions?

See you in the next class!