Math Bootcamp

UC San Diego

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Today's Roadmap

Inverse Matrix

Determinants

Vector Spaces

Eigenvectors and Eigenvalues

Applications

Definition: Inverse Matrix

Let A a matrix $n \times n$. The matrix B of $n \times n$ is an called the **inverse** for A if $AB = BA = I$ (where I is the Identity Matrix).

If $AB = I$ them B is a *right inverse*.

If $BA = I$ them B is a *left inverse*.

Example: Calculate the inverse of:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Important: These matrices generally are non-singular (and therefore invertible) if, and only if, $ad - bc$ differs from 0.

Inverse Matrix Applications

Inverse matrices are very useful. See the theorem below:

Theorem For any square matrix A, the following statements are equivalent:

(a) A is invertible.
(b) A has a right inverse.
(c) A has a left inverse.
(d) Every system $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.
(e) Every system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$.
(f) A is nonsingular.
(g) A has maximal rank n.

If A is nonsingular (invertible) them for $Ax = b$ $x = A^{-1}b$.

Solve the following system using the inverse of the matrix.

$$x - 2y = 8$$

$$3x + y = 3$$

Your turn! Compute letter (c) of the first set and letter (b) of the second set:

$$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad b)\ \begin{pmatrix} 4 & 5 \\ 2 & 4 \end{pmatrix}, \quad c)\ \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix},$$

$$d)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \quad e)\ \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix},$$

$$f)\ \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & -12 & 2 \end{pmatrix}.$$

Invert the coefficient matrix to solve the following systems of equations:

$$a)\ \begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 + x_2 &= 3; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 + x_2 \phantom{{}+6x_3} &= 4 \\ 6x_1 + 2x_2 + 6x_3 &= 20 \\ -4x_1 - 3x_2 + 9x_3 &= 3; \end{aligned}$$

What are determinants?

- Determinants: a number that determines whether a square-matrix is singular or not.
- Singular matrix: has no inverse.
- If the determinant is zero, then the matrix is singular.
- It also has an important role in optimization theory (but let's leave that for the next classes)

We define the Determinant computations inductively:

- Starting with a constant, we have: $det(a) = a$.
- Now let $A_{2\times 2}$. We define the determinant as:

$$det(A) = a_{11}det(a_{22}) - a_{12}det(a_{21})$$

- But what about higher dimension matrices?

We need a generalization of this technique.

Definition: Minor of a Matrix

Let $A_{n\times n}$. We define the matrix's minor M_{ij} as the determinant of the matrix obtained by deleting the ith row and the jth column.

Example:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Then:

$$M_{11} = det \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

What about M_{12} and M_{33}?

Definition: Cofactor

The cofactor C_{ij} is defined as:

$$Cij = (-1)^{i+j} M_{ij}$$

Examples:

$$C_{11} = (-1)^{1+1} M_{11} = M_{11}$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

And the determinant is defined as the sum of the cofactors along the row or column of our choice. This technique is called **cofactor expansion**.

Definition The **determinant** of a 3×3 matrix is given by

$$\begin{aligned}
\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\
&= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \\
&= a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \\
&\quad + a_{13} \cdot \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}.
\end{aligned}$$

Compute the determinant of A:

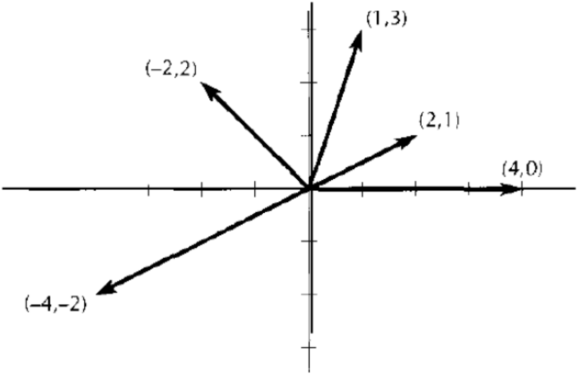
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

Your turn!
Compute the determinant of:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 1 \end{pmatrix}$$

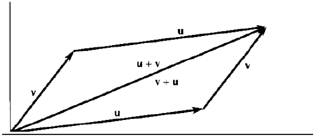
In Linear Algebra we frequently deal with vectors.

Vectors, for social scientists, are simply a collection of multidimensional data.

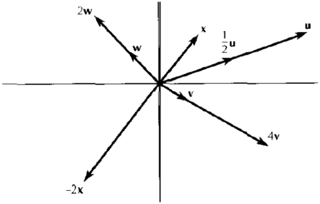


The algebra of vectors is mostly done component-wise.

- Sum?
- Scalar multiplication?



$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$



Definition: Length of a Vector

Let the vector $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n})$. The length of $\mathbf{x}$ is defined as:

$$||\mathbf{x}|| = \sqrt{\mathbf{x_1^2} + \cdots + \mathbf{x_n^2}}$$

Compute the length of vector $\mathbf{x} = (\mathbf{1}, \mathbf{2}, \mathbf{3})$

Definition: Unit Vector

Vector with length equal to 1.

We have sum and scalar product. Is there a way to *multiply two vectors*?

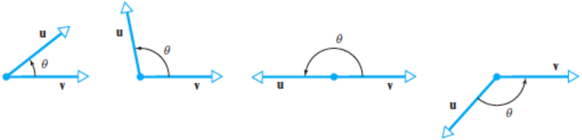
Definition: Inner Product (dot product)

Let $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n})$ and $\mathbf{v} = (\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n})$ vectors in $\mathbb{R}^n$. The euclidean inner product of $\mathbf{u}$ and $\mathbf{v}$, written as $\mathbf{u} \cdot \mathbf{v}$, is:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u_1 v_1} + \mathbf{u_2 v_2} + \cdots + \mathbf{u_n v_n}$$

Example: Let $\mathbf{u} = (\mathbf{4}, -\mathbf{1}, \mathbf{2})$ and $\mathbf{v} = (\mathbf{6}, \mathbf{3}, -\mathbf{4})$. Then,
$\mathbf{u} \cdot \mathbf{v} = \mathbf{13}$

A few facts about inner products:



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \le \theta \le \pi$.

DEFINITION 3 If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the ***dot product*** (also called the ***Euclidean inner product***) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta \tag{12}$$

If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then we define $\mathbf{u} \cdot \mathbf{v}$ to be 0.

Question: what does it mean for a vector to be perpendicular (orthogonal)?

A few facts about inner products:

Theorem Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be arbitrary vectors in $\mathbf{R}^{\mathbf{n}}$ and let r be an arbitrary scalar. Then,

(a) $\mathbf{u}\ \mathbf{v} = \mathbf{v}\ \mathbf{u}$,

(b) $\mathbf{u} \cdot (\mathbf{v} + w) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,

(c) $\mathbf{u}\ (r\mathbf{v}) = r(\mathbf{u}\ \mathbf{v}) = (r\mathbf{u})\ \mathbf{v}$,

(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$,

(e) $\mathbf{u}\ \mathbf{u} = 0$ implies $\mathbf{u} = 0$, and

(f) $(\mathbf{u} + \mathbf{v})\ (\mathbf{u} + \mathbf{v}) = \mathbf{u}\ \mathbf{u} + 2(\mathbf{u}\ \mathbf{v}) + \mathbf{v}\ \mathbf{v}$.

Your turn! Find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{u}$, $\mathbf{v} \cdot \mathbf{v}$, $||\mathbf{u}||$, and $||\mathbf{v}||$ for:

(a) $\mathbf{u} = (3, 1, 4)$, $\mathbf{v} = (2, 2, -4)$

(b) $\mathbf{u} = (1, 1, 4, 6)$, $\mathbf{v} = (2, -2, 3, -2)$

Eigenvectors and Eigenvalues are important to change the representation of the problem we are working on.

- r is eigenvalues of A if, and only if, $A - rI$ is singular ($det(A - rI) = 0$).
- $det(A - rI)$ is call characteristic polynomial of A.
- r is eigenvalues of A if and only if is the root of the characteristic polynomial of A.

Example:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A - rI = \begin{pmatrix} a - r & b \\ c & d - r \end{pmatrix}$$

$$det(A - rI) = r^2 - (a + d)r + (ad - bc)$$

When r is a eigenvalue of A, an nonzero vector $\mathbf{v}$ such as

$$(A - rI)\mathbf{v} = \mathbf{0}$$

Then $\mathbf{v}$ is called an of eigenvector of A corresponding to eigenvalue r.

Find the eigenvalues and eigenvector of:

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$$

Solution:

$$det(A - rI) = det\begin{pmatrix} -1 - r & 3 \\ 2 & 0 - r \end{pmatrix} = r^2 + r - 6 = (r+3)(r-2)$$

The eigenvalues of A are the roots of the characteristic polynomial: -3 and 2.
For $r = -3$
$(A - (-3)I)\mathbf{v} = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$(3, -2)$ is eigenvector of A associated to -3 (eigenvalue).

Important facts:

Theorem Let A be a $k \times k$ matrix. Let $r_1, r_2, \ldots, r_k$ be eigenvalues of A, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ the corresponding eigenvectors. Form the matrix

$$P = [\mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_k]$$

whose columns are these k eigenvectors. If P is invertible, then

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}.$$

Conversely, if $P^{-1}AP$ is a diagonal matrix D, the columns of P must be eigenvectors of A and the diagonal entries of D must be eigenvalues of A.

Theorem Let $r_1, \ldots, r_h$ be h *distinct* eigenvalues of the $k \times k$ matrix A. Let $\mathbf{v}_1, \ldots, \mathbf{v}_h$ be corresponding eigenvectors. Then, $\mathbf{v}_1, \ldots, \mathbf{v}_h$ are linearly independent, that is, no one of them can be written as a linear combination of the others.

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$$

Solution:
The eigenvalues of A are the roots of the characteristic polynomial: -3 and 2.
$(3, -2)$ is eigenvector of A associated to 3 (eigenvalue).
$(1, 1)$ is eigenvector of A associated to -2 (eigenvalue).

$$D = P^{-1}AP \quad (A = PDP^{-1})$$

$$\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$$

This decomposition helps us calculate matrix powers without making the products using the following property:

$$A^n = PD^nP^{-1}$$

We know that D is a diagonal matrix, and its power is the power of the diagonal elements.
Example:

$$\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}^n = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} (-3)^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}^{-1}$$

There are three possibilities for the roots of the characteristic polynomial $p(r)$:

- k distinct, real roots.
- Some repeated roots.
- Some complex roots.

Definition: Trace

$Trace(A))$ is the sum of the diagonal elements of A.

And one important property of the eigenanalysis is:

Theorem Let A be a $k \times k$ matrix with eigenvalues $r_1, \ldots, r_k$. Then,

(a) $r_1 + r_2 + \cdots + r_k =$ trace of A, and
(b) $r_1 \cdot r_2 \cdots r_k = \det A$.

1.
$$A = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 3 & 0 & -1 \end{pmatrix}$$

$$Trace(A) = 4 + 3 - 1 = 6$$

Eigenvalues are $r_1 = 1, r_2 = 2$ and $r_3 = 3$,

$$r_1 + r_2 + r_3 = 6.$$

2.
$$\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$$

$$Trace(A) = 4 + 2 = 6$$

Eigenvalues are $r_1 = 3, r_2 = 3$, $r_1 + r_2 = 6$.

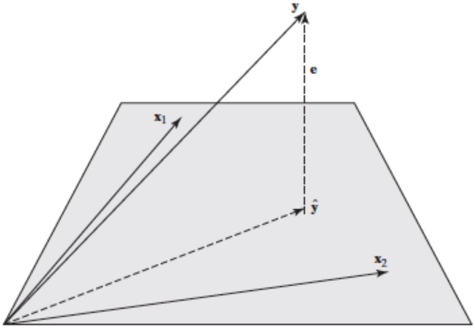
Your turn! Compute A^{10} for the matrix:

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

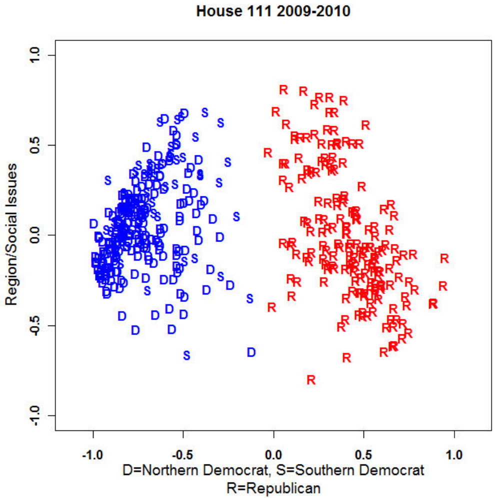
- Interpolation:
  Find a cubic polynomial whose graph passes through the points

$$(1, 3), \quad (2, -2), \quad (3, -5), \quad (4, 0)$$

- Regression Analysis



- Dimension Reduction



Questions?

See you in the next class!