# Math Bootcamp

Umberto Mignozzetti

# Today's Roadmap

This course portion intends to refresh/level your basic math knowledge. I divided it into five classes:

- Monday: Linear Algebra I
- Tuesday: Linear Algebra II
- Wednesday: Calculus I
- Thursday: Calculus II
- Friday: Calculus III + Probability

# How is it going to work?



Class organization:

- Spans from 9:00 to 12:00.
- 40-50 minutes of lecture.
- 10-15 minutes of exercises.
- 10 minutes break.

Exercises:

- Each class chunk has a few exercises.
- You have to get them done and hand them in by the end of the class.
- I will grade and return them by the next lecture.

There is no such thing as a bad question!

# Books and Resources

Books:

- Linear Algebra: Anton and Rorres, *Elementary Linear Algebra*.
- Calculus: Stewart, *Calculus*.
- Applied to Econ: Simon and Blume, *Math for Economists*.
- Applied for DS: `https://tinyurl.com/3wabr3j2`

Lectures:

- Microsoft FDS: `https://tinyurl.com/58jcey9w`
- Martin Osborne Math for Econ: `https://tinyurl.com/48p6sb78`

If you find something cool online, please let us know!

My name is Umberto Mignozzetti.

I am an Assistant Teaching Professor at the UCSD PoliSci Department.

I study comparative political economy focusing on improving welfare in developing economies.

Something interesting: This is my first UCSD class!

My email is `umbertomig@ucsd.edu`. Please let me know if we have any questions. I'll be glad to talk to you by Zoom or in person!

What is a linear equation?

### Definition: Linear Equation

A linear equation is an equation that can be represented as:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Where $x_i$ are the **unknowns** and $a_i$ and $b$ are **constants** (with at least some different than zero).

Examples of linear equations?

When we consider more than one equation simultaneously, we have a **system of linear equations**.

- Linear equations describe geometric objects such as lines and planes.
- Linear systems: advantage that we can calculate exact solutions to the equations.
- Solution the nonlinear system often cannot be calculate explicitly.
- Linear systems are among the most frequently studied in social sciences.

For a linear system, we are interested in the following three questions:

1. Does a solution exist?
2. How many solutions are there?
3. Is there an efficient algorithm that computes the solutions?

There are, essentially, three ways of solving such systems:

1. Substitution
2. Elimination of variables
3. Matrix methods (Gauss-Jordan)

Example:

$$x - 2y = 8$$

$$3x + y = 3$$

Example:

$$x - 2y = 8$$

$$3x + y = 3$$

**Solution:**

$$x = 8 + 2y,$$

$$3(8 + 2y) + y = 3,$$

$$7y = -21,$$

$$y = -3$$

$$x = 8 + 2(-3) = 2$$

Example:

$$x - 2y = 8 \tag{1}$$

$$3x + y = 3 \tag{2}$$

Example:

$$x - 2y = 8 \tag{1}$$

$$3x + y = 3 \tag{2}$$

**Solution:** Multiplying the equation 1 by $-3$ obtain $-3x + 6y = -24$. Adding this to 2, and the result is

$$x = 2 \qquad y = -3$$

**Your turn!** Find which equations are linear in $x_1$, $x_2$, and $x_3$:

(a) $x_1 + 5x_2 - \sqrt{2}\,x_3 = 1$

(b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$

(d) $x_1^{-2} + x_2 + 8x_3 = 5$

(e) $x_1^{3/5} - 2x_2 + x_3 = 4$

(f) $\pi x_1 - \sqrt{2}\,x_2 = 7^{1/3}$

Solve the following systems using the methods we have seen so far:

*a)* $\begin{aligned} 3x + 3y &= 4 \\ x - y &= 10; \end{aligned}$

*b)* $\begin{aligned} 4x + 2y - 3z &= 1 \\ 6x + 3y - 5z &= 0 \\ x + y + 2z &= 9; \end{aligned}$

*c)* $\begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0. \end{aligned}$

Another method that is more efficient is the Gauss-Jordan Elimination. We have to transform a system into a matrix to perform it:

$$x + 2y = 3$$

$$3x + 2y = 4$$

The coefficient matrix is

$$\mathrm{A} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

And the augmented matrix is:

$$A^* = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \end{pmatrix}$$

Find the augmented matrix for the following systems:

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$$

$$5x_3 + 10x_4 + 15x_6 = 5$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$$

And:

$$x_1 + x_2 + 2x_3 = 8$$

$$-x_1 - 2x_2 + 3x_3 = 1$$

$$3x_1 - 7x_2 + 4x_3 = 10$$

Elementary row operations are critical because, when done in the augmented matrix, they do not change the solution of the system.

They are:

1. Interchange two rows of a matrix.
2. Change a row by adding a multiple of another row.
3. Multiply each element in a row by the same number (scalar multiplication).

Examples?

### Definition: Row Echelon Form

A matrix row is said to have $k$ leading zeros if the first $k$ elements of the row are all zeros and the $(k+1)th$ element of the row is not zero. A matrix is in row echelon form if each row has more leading zeros than the preceding one.

Examples: $\begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

The reduced row echelon form is excellent for finding solutions for Systems of Linear Equations:

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve the following system using Gauss-Jordan:

$$x_1 + x_2 + 2x_3 = 8$$

$$-x_1 - 2x_2 + 3x_3 = 1$$

$$3x_1 - 7x_2 + 4x_3 = 10$$

Solve the system of equations $\begin{cases} -4x + 6y + 4z = 4 \\ 2x - y + z = 1. \end{cases}$

### Definition: Rank of a Matrix

The Rank of a matrix is the number of nonzero rows in its row echelon form.

Examples:

- $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$
- $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

Find the Rank of the following matrices:

$$a)\ \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \quad b)\ \begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad c)\ \begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix},$$

The following theorem allows classifying the solutions of a linear system of equations (homogeneous ($Ax = 0$) or not ($Ax = b$)) using the range of the coefficient matrix.

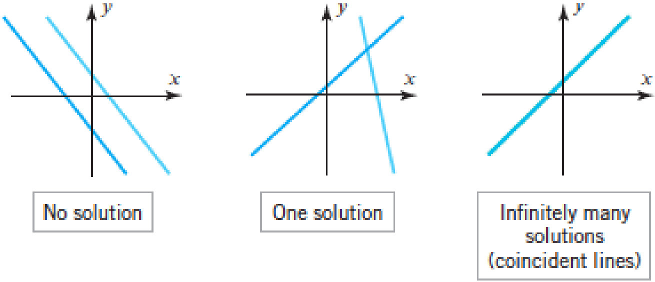
Consider the linear system of equations $A\mathbf{x} = \mathbf{b}$.

(*a*) If the number of equations < the number of unknowns, then:
- (*i*) $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions,
- (*ii*) for any given $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has 0 or infinitely many solutions, and
- (*iii*) if rank $A$ = number of equations, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for *every* $\mathbf{b}$.

(*b*) If the number of equations > the number of unknowns, then:
- (*i*) $A\mathbf{x} = \mathbf{0}$ has one or infinitely many solutions,
- (*ii*) for any given $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has 0, 1, or infinitely many solutions, and
- (*iii*) if rank $A$ = number of unknowns, $A\mathbf{x} = \mathbf{b}$ has 0 or 1 solution for *every* $\mathbf{b}$.

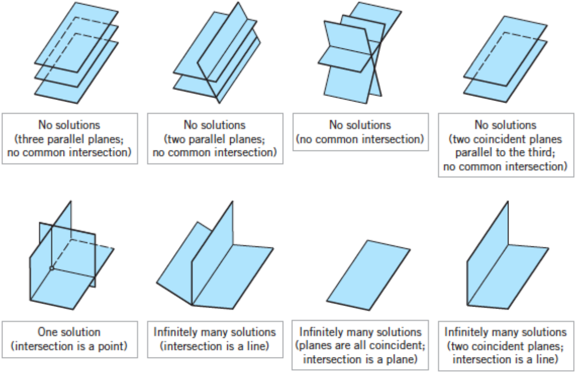
(*c*) If the number of equations = the number of unknowns, then:
- (*i*) $A\mathbf{x} = \mathbf{0}$ has one or infinitely many solutions,
- (*ii*) for any given $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has 0, 1, or infinitely many solutions, and
- (*iii*) if rank $A$ = number of unknowns = number of equations, $A\mathbf{x} = \mathbf{b}$ has exactly 1 solution for *every* $\mathbf{b}$.

A system with two unknowns is comprised of lines:




l.1

A system with three unknowns is comprised of planes:



**Your turn!** Solve the following systems:

$$a)\quad \begin{aligned} x - 3y + 6z &= -1 \\ 2x - 5y + 10z &= 0 \\ 3x - 8y + 17z &= 1; \end{aligned} \qquad b)\quad \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 12x_1 + 2x_2 - 3x_3 &= 5 \\ 3x_1 + 4x_2 + x_3 &= -4. \end{aligned}$$

Use Gauss-Jordan elimination to determine for what values of the parameter $k$ the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - kx_2 &= 1 \end{aligned}$$

has no solutions, one solution, and more than one solution.

Addition and subtraction of matrices are done element by element:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 5 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

**Note:** The matrix should have the same numbers of rows and columns (conform)!

Let $k \in \mathbb{R}$,

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

We define the matrix product $AB$ if, and only if:

The number of columns in $A$ = number of rows in $B$.

Example:

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}_{3\times 2} \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{2\times 2} = \begin{pmatrix} aA+bC & aB+bD \\ cA+dC & cB+dD \\ eA+fC & eB+fD \end{pmatrix}_{3\times 2}$$

**Note:** The product taken in the inverse order is not defined.

# Laws of Matrix Algebra

- Associative:

$$(A + B) + C = A + (B + C)$$

$$(AB)C = A(BC)$$

- Commutative

$$A + B = B + A$$

- Associative:

$$(A+B)+C = A+(B+C)$$

$$(AB)C = A(BC)$$

- Commutative

$$A+B = B+A$$

In general, the product is not available.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

- Distributive:

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

# Transpose



$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Properties:

$$(A+B)^T = A^T + B^T \qquad (AB)^T = B^T A^T$$

$$(A-B)^T = A^T - B^T$$

$$(A^T)^T = A$$

Example:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

**Square Matrix.** $k = n$, that is, equal number of rows and columns.

**Column Matrix.** $n = 1$, that is, one column. For example,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

**Row Matrix.** $k = 1$, that is, one row. For example,

$$\begin{pmatrix} 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 3 \end{pmatrix}.$$

**Diagonal Matrix.** $k = n$ and $a_{ij} = 0$ for $i \neq j$, that is, a square matrix in which all nondiagonal entries are 0. For example,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

**Upper-Triangular Matrix.** $a_{ij} = 0$ if $i > j$, that is, a matrix (usually square) in which all entries below the diagonal are 0. For example,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

**Lower-Triangular Matrix.** $a_{ij} = 0$ if $i < j$, that is, a matrix (usually square) in which all entries above the diagonal are 0. For example,

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

**Symmetric Matrix.** $A^T = A$, that is, $a_{ij} = a_{ji}$ for all $i, j$. These matrices are necessarily square. For example,

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$

**Idempotent Matrix.** A square matrix $B$ for which $B \cdot B = B$, such as $B = I$ or

$$\begin{pmatrix} 5 & -5 \\ 4 & -4 \end{pmatrix}.$$

**Permutation Matrix.** A square matrix of 0s and 1s in which each row and each column contains exactly one 1. For example,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Nonsingular Matrix.** A square matrix whose rank equals the number of its rows (or columns). When such a matrix arises as a coefficient matrix in a system of linear equations, the system has one and only one solution.

## Definition: Inverse Matrix

Let $A$ a matrix $n \times n$. The matrix $B$ of $n \times n$ is an called the **inverse** for $A$ if $AB = BA = I$ (where $I$ is the Identity Matrix).

If $AB = I$ them $B$ is a *right inverse*.

If $BA = I$ them $B$ is a *left inverse*.

Example: Calculate the inverse of:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Important: In general, these matrices are nonsingular (and therefore invertible) if and only if $ad - bc$ different from 0.

Inverse matrices are very useful. See the theorem below:

**Theorem** For any square matrix $A$, the following statements are equivalent:

(*a*) $A$ is invertible.
(*b*) $A$ has a right inverse.
(*c*) $A$ has a left inverse.
(*d*) Every system $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.
(*e*) Every system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$.
(*f*) $A$ is nonsingular.
(*g*) $A$ has maximal rank $n$.

If $A$ is nonsingular (invertible) them for $Ax = b$ $x = A^{-1}b$.

Solve the following system using the inverse of the matrix.

$$x - 2y = 8$$

$$3x + y = 3$$

**Your turn!** Find the inverse of the following matrices:

$$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad b)\ \begin{pmatrix} 4 & 5 \\ 2 & 4 \end{pmatrix}, \quad c)\ \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix},$$

$$d)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \quad e)\ \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix},$$

$$f)\ \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & -12 & 2 \end{pmatrix}.$$

Invert the coefficient matrix to solve the following systems of equations:

$$a)\ \begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 + x_2 &= 3; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 + x_2 \phantom{{}+6x_3} &= 4 \\ 6x_1 + 2x_2 + 6x_3 &= 20 \\ -4x_1 - 3x_2 + 9x_3 &= 3; \end{aligned}$$

Questions?

See you in the next class!