# Math Bootcamp

UC San Diego

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# Today's Roadmap



Quadratic Forms

Unconstrained Optimization

Integration

Probability

# Quadratic Forms

Definition: A quadratic form on $\mathbb{R}^n$ is a real-valued function of the form:

$$Q(x_1, x_2, ..., x_n) = \sum_{i \leq j} a_{ij} x_i x_j$$

In which each term is a monomial of degree two.

Quadratic form $Q$ can be represented by a symmetric matrix $A$ so that:

$$Q(x) = x^T A x(t)$$

Example:

$$x_1^2 + x_2^2 = (x_1 \quad x_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The general quadratic form of one variable is $y = ax^2$. If $a > 0$, then $ax^2$ is always $\geq 0$ and the form is called positive definite ($x = 0$ is its global minimum).

If $a < 0$ then $ax^2$ is always $\leq 0$ and the form is called negative definite ($x = 0$ is its global maximum).

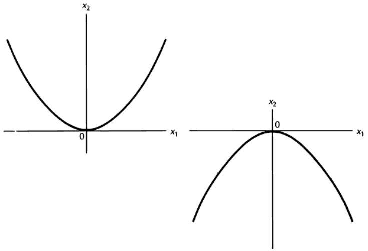
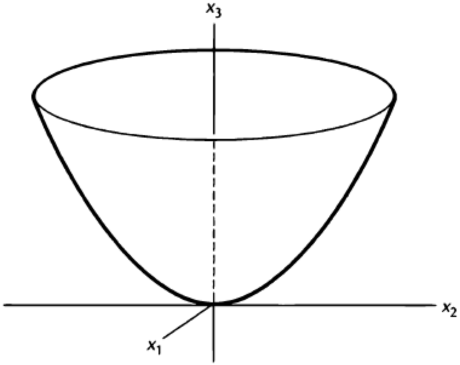



**Figure 16.1** *The functions of $f(x) = x^2$ and $f(x) = -x^2$.*

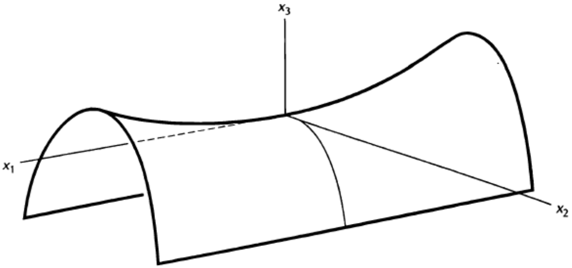
Two dimensions, $Q_1(x_1, x_2) = x_1^2 + x_2^2$ is always greater than zero at $(x_1, x_2) \neq (0, 0)$. So, we call $Q_1$ positive definite.




*Graph of the positive definite form* $Q_1(x_1, x_2) = x_1^2 + x_2^2$.

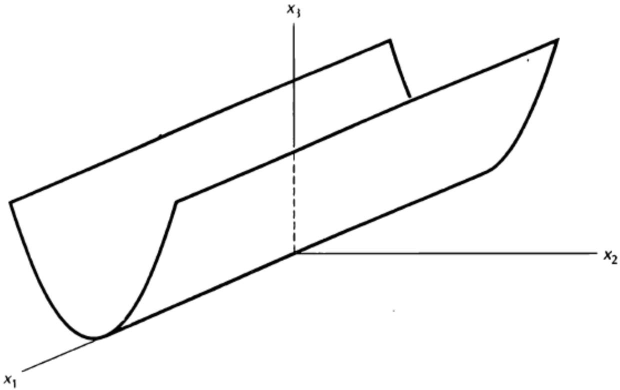
$$Q_3(x_1, x_2) = x_1^2 - x_2^2$$
$$Q_3(1, 0) = 1$$
$$Q_3(0, 1) = -1$$

are called indefinite.




*The graph of the indefinite form* $Q_3(x_1, x_2) = x_1^2 - x_2^2$.

A quadratic form like $Q_5(x_1, x_2) = -(x_1 + x_2)^2$, which is never positive but can be zero at points other than origin, is called negative semidefinite.




*The graph of the positive semidefinite form $Q_4(x_1, x_2) = (x_1 + x_2)^2$.*

**Definition** Let $A$ be an $n \times n$ symmetric matrix, then $A$ is:

($a$) **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R^n}$,
($b$) **positive semidefinite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R^n}$,
($c$) **negative definite** if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R^n}$,
($d$) **negative semidefinite** if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$ in $\mathbf{R^n}$, and
($e$) **indefinite** if $\mathbf{x}^T A\mathbf{x} > 0$ for some $\mathbf{x}$ in $\mathbf{R^n}$ and $< 0$ for some other $\mathbf{x}$ in $\mathbf{R^n}$.

**Remark** A matrix that is positive (negative) definite is automatically positive (negative) semidefinite. Otherwise, every symmetric matrix falls into one of the above five categories.

# Principal Minor of $A$ matrix

Definition: Let $A$ be an $n \times n$ matrix. A $k \times k$ submatrix of $A$ formed by deleting $n-k$ columns, say $i_1$, $i_2$,..., $i_{n-k}$ and the same $n-k$ rows, rows $i_1$, $i_2$,..., $i_{n-k}$, form $A$ is called a $kth$ order principal submatrix of $A$. The determinant of $a$ $k \times k$ principal submatrix is called $a$ $kth$ order principal minor of $A$.

Find the principals minors of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

**Definition** Let $A$ be an $n \times n$ matrix. The $k$th order principal submatrix of $A$ obtained by deleting the *last* $n - k$ rows and the *last* $n - k$ columns from $A$ is called the $k$th order **leading principal submatrix** of $A$. Its determinant is called the $k$th order **leading principal minor** of $A$. We will denote the $k$th order leading principal submatrix by $A_k$ and the corresponding leading principal minor by $|A_k|$.

**Theorem 16.1** Let $A$ be an $n \times n$ symmetric matrix. Then,

(*a*) $A$ is positive definite if and only if all its $n$ leading principal minors are (strictly) positive.

(*b*) $A$ is negative definite if and only if its $n$ leading principal minors alternate in sign as follows:

$$|A_1| < 0, \quad |A_2| > 0, \quad |A_3| < 0, \quad \text{etc.}$$

The $k$th order leading principal minor should have the same sign as $(-1)^k$.

(*c*) If some $k$th order leading principal minor of $A$ (or some pair of them) is nonzero but does not fit either of the above two sign patterns, then $A$ is indefinite. This case occurs when $A$ has a *negative* $k$th order leading principal minor for an *even* integer $k$ or when $A$ has a negative $k$th order leading principal minor and a positive $\ell$th order leading principal minor for two distinct *odd* integers $k$ and $\ell$.

**Your turn!** Classify the following matrix:

$$\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$$

# Unconstrained Optimization

## Unconstrained Optimization:

The definitions of a maximum and minimum for a function of several variables are the same as the corresponding definitions for a function of one variable. Let $F: U \to \mathbf{R}^1$ be a real-valued function of $n$ variables, whose domain $U$ is a subset of $\mathbf{R}^n$.

(1) A point $\mathbf{x}^* \in U$ is a **max** of $F$ on $U$ if $F(\mathbf{x}^*) \geq F(\mathbf{x})$ for all $\mathbf{x} \in U$.

(2) $\mathbf{x}^* \in U$ is a **strict max** if $\mathbf{x}^*$ is a max and $F(\mathbf{x}^*) > F(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{x}^*$ in $U$.

(3) $\mathbf{x}^* \in U$ is a **local (or relative) max** of $F$ if there is a ball $B_r(\mathbf{x}^*)$ about $\mathbf{x}^*$ such that $F(\mathbf{x}^*) \geq F(\mathbf{x})$ for all $\mathbf{x} \in B_r(\mathbf{x}^*) \cap U$.

(4) $\mathbf{x}^* \in U$ is a **strict local max** of $F$ if there is a ball $B_r(\mathbf{x}^*)$ about $\mathbf{x}^*$ such that $F(\mathbf{x}^*) > F(\mathbf{x})$ for all $\mathbf{x} \neq \mathbf{x}^*$ in $B_r(\mathbf{x}^*) \cap U$.

- **Local Maximum:** A point $x^*$ is a local max there are no nearby points at which $F$ takes on a larger value. Of course, a max is always a local max.
- **Global Maximum:** If we want to emphasize that a point $x^*$ is a max of $F$ on the whole domain $U$, not just a local max, we call $x^*$ a global max or absolute max of $F$ on $U$.

**Theorem 17.1** Let $F: U \to \mathbf{R}^1$ be a $C^1$ function defined on a subset $U$ of $\mathbf{R}^n$. If $\mathbf{x}^*$ is a local max or min of $F$ in $U$ and if $\mathbf{x}^*$ is an interior point of $U$, then

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0 \quad \text{for } i = 1, \ldots, n. \tag{1}$$

Example:

Find the Gradient and the Hessian of

$$F(x, y) = x^3 - y^3 + 9xy$$

### Definition: Critical Point

We say that a $n-$vector $\mathbf{x}^* = (x_1, ..., x_n)$ is a **critical point** of a function $f(x_1, ..., x_n)$ if $\mathbf{x}^*$ satisfies:

$$\frac{\partial f}{\partial x_i}(x_1, ..., x_n) = 0 \quad \text{for } i = 1, 2, ..., n$$

This definition implies that $\nabla f(\mathbf{x}) = \mathbf{0}$.

$\nabla f(\mathbf{x}) = \mathbf{0}$ is called a *first order condition*, or a *necessary condition for optimization.*

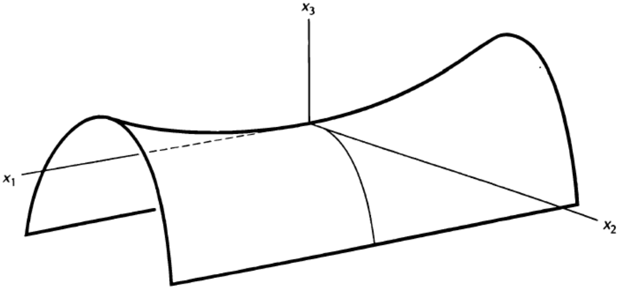
For finding whether a given point is indeed maximum or minimum, we need to analyze the Hessian matrix.

This will give us *sufficient conditions* (or *second-order conditions* for optimization).

**Theorem 17.2** Let $F: U \to \mathbf{R}^1$ be a $C^2$ function whose domain is an open set $U$ in $\mathbf{R}^n$. Suppose that $\mathbf{x}^*$ is a critical point of $F$ in that it satisfies (3).

(1) If the Hessian $D^2F(\mathbf{x}^*)$ is a negative definite symmetric matrix, then $\mathbf{x}^*$ is a strict local max of $F$;
(2) If the Hessian $D^2F(\mathbf{x}^*)$ is a positive definite symmetric matrix, then $\mathbf{x}^*$ is a strict local min of $F$.
(3) If $D^2F(\mathbf{x}^*)$ is indefinite, then $\mathbf{x}^*$ is neither a local max nor a local min of $F$.

**Definition**: A critical point $x^*$ of $F$ for which the Hessian $D^2F(x^*)$ is indefinite is called a saddle point of $F$. A saddle point $x^*$ is a min of $F$ in some directions and a max of $F$ in others directions.



**Theorem 17.3** Let $F: U \to \mathbf{R}^1$ be a $C^2$ function whose domain is an open set $U$ in $\mathbf{R}^1$. Suppose that

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0 \quad \text{for } i = 1, \ldots, n$$

and that the $n$ leading principal minors of $D^2F(\mathbf{x}^*)$ alternate in sign

$$|F_{x_1x_1}| < 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} \\ F_{x_1x_2} & F_{x_2x_2} \end{vmatrix} > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} & F_{x_3x_1} \\ F_{x_1x_2} & F_{x_2x_2} & F_{x_3x_2} \\ F_{x_1x_3} & F_{x_2x_3} & F_{x_3x_3} \end{vmatrix} < 0, \ldots$$

at $\mathbf{x}^*$. Then, $\mathbf{x}^*$ is a strict local max of $F$.

## Sufficient Conditions: Saddle Point

**Theorem 17.4** Let $F: U \to \mathbf{R}^1$ be a $C^2$ function whose domain is an open set $U$ in $\mathbf{R}^n$. Suppose that

$$\frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0 \quad \text{for } i = 1, \ldots, n$$

and that the $n$ leading principal minors of $D^2F(\mathbf{x}^*)$ are all positive:

$$|F_{x_1x_1}| > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} \\ F_{x_1x_2} & F_{x_2x_2} \end{vmatrix} > 0, \quad \begin{vmatrix} F_{x_1x_1} & F_{x_2x_1} & F_{x_3x_1} \\ F_{x_1x_2} & F_{x_2x_2} & F_{x_3x_2} \\ F_{x_1x_3} & F_{x_2x_3} & F_{x_3x_3} \end{vmatrix} > 0, \ldots$$

at $\mathbf{x}^*$. Then, $\mathbf{x}^*$ is a strict local min of $F$.

For each of the following functions defined on $\mathbb{R}^2$, find the critical points and classify these as local max, local min, saddle point, or can't tell:

- $x^2 - 2xy$
- $x^3 - y^3 + 9xy$

**Theorem 17.8** Let $F: U \to \mathbf{R}^1$ be a $C^2$ function whose domain is a convex open subset $U$ of $\mathbf{R}^n$.

(*a*) The following three conditions are equivalent:
  (*i*) $F$ is a concave function on $U$; and
  (*ii*) $F(\mathbf{y}) - F(\mathbf{x}) \leq DF(\mathbf{x})(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in U$; and
  (*iii*) $D^2F(\mathbf{x})$ is negative semidefinite for all $\mathbf{x} \in U$.

(*b*) The following three conditions are equivalent:
  (*i*) $F$ is a convex function on $U$; and
  (*ii*) $F(\mathbf{y}) - F(\mathbf{x}) \geq DF(\mathbf{x})(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in U$; and
  (*iii*) $D^2F(\mathbf{x})$ is positive semidefinite for all $\mathbf{x} \in U$.

(*c*) If $F$ is a concave function on $U$ and $DF(\mathbf{x}^*) = \mathbf{0}$ for some $\mathbf{x}^* \in U$, then $\mathbf{x}^*$ is a *global max* of $F$ on $U$.

(*d*) If $F$ is a convex function on $U$ and $DF(\mathbf{x}^*) = \mathbf{0}$ for some $\mathbf{x}^* \in U$, then $\mathbf{x}^*$ is a *global min* of $F$ on $U$.

If we are give $n$ data points in the plane $(x_1, y_1), ..., (x_n, y_n)$, and a straight line whose equation is $y = mx + b$, the we can write the aggregate distance from the $n$ points to the line as:

$$S = (mx_1 + b - y_1)^2 + ... + (mx_n + b - y_n)^2$$

Consider calculating $S$ for several different lines so that $S$ becomes a function of $m$ and $b$.

$$S(m, b) = \sum_{i=1} (mx_1 + b - y_1)^2$$

**Your turn:** Find the critical points of

$$S(m, b) = \sum_{i=1} (mx_1 + b - y_1)^2$$

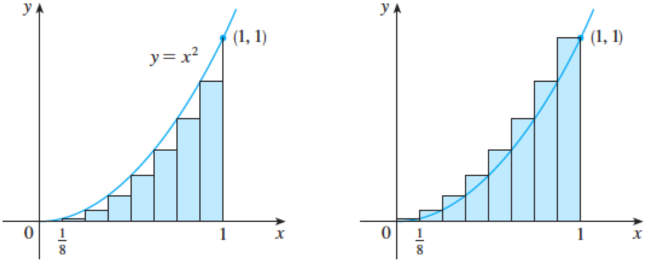
Then, using the critical point (which is the only minimum for this function), adjust a straight line to the following points by least squares $(0, 1), (-3, -5), (2.5, 6), (4, 9)$ and $(0.5, 0)$.

# Integration

- Integration is one of the two main operations of Calculus.
    1. Differentiation
    2. Integration
- Integration: inverse operation of differentiation.
- Idea: compute area, volume, and other concepts that arise by combining infinitesimal data.

Given a function $f$ of a real variable $x$, and an interval $[a, b] \in \mathbb{R}$, the definite integral is equal to:

$$\int_a^b f(x)dx$$




(a) Using left endpoints

(b) Using right endpoints

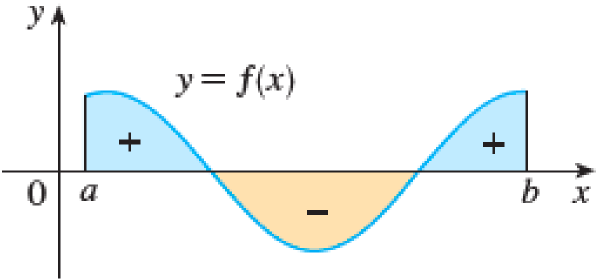



**FIGURE 4**

$\int_a^b f(x)\,dx$ is the net area.

- Integration, up to an additive constant, is the inverse of the operation of differentiation.
- For this reason, the term integral may also refer to the related notion of the antiderivative, a function F whose derivative is the given function $f$.

Indefinite integral:

$$F(x) = \int f(x)dx + C$$

# Indefinite Integrals

An indefinite integral, defined as the inverse of a derivative (antiderivative), is as follows:

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

The integral is the inverse of the derivative. The example below shows this:

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\frac{d}{dx}\left[\frac{x^3}{3} + C\right] = x^2$$

# Properties



$$\int cf(x)dx = c\int f(x)dx$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int kdx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^x dx = e^x + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

Fundamental Theorem of Calculus:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Properties:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

**Properties of the Integral**

1. $\int_a^b c\,dx = c(b-a)$, where $c$ is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where $c$ is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

# Substitution Rule

**4 The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Examples:

1. $\int \sqrt{2x+1}dx$
2. $\int_1^2 \dfrac{dx}{(3-5x)^2}$

For the indefinite integrals:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Example:

1. $\int t^2 e^t dt$

# Formula for integration by parts



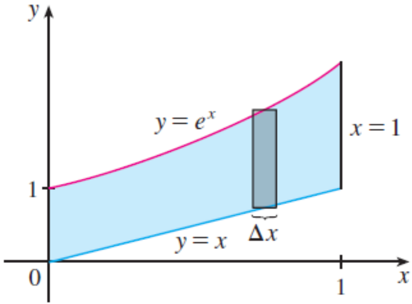
$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b g(x)f'(x)\,dx}$$

Exercise:

$$\int_0^1 (x^2+1)e^{-x}dx$$

We can use integrals to find areas of regions that lie between the graphs of two functions.

Find the region's area bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



**Your turn!** Compute:

1. $$\int \frac{4}{x^2} dx$$

2. $$\int (x^3 - 6x) dx$$

3. $$\int (2x^3 - 6x + e^x) dx$$

Sketch the region enclosed by the given curves and find its area $y = 12 - x^2$ and $y = x^2 - 6$.

# Probability

Random Experiments are experiments that we cannot tell the result beforehand:

1. Picking a ball from a box containing 20 numbered balls, is a random experiment since the process can lead to one of the 20 possible outcomes.
2. Rolling a six-sided die and observing the number which appears on the uppermost face of the die. The result can be any of the numbers 1, 2, 3, ..., 6.
3. If we measure the distance between two points A and B, many times, under the same conditions, we expect to have the same result. This is therefore not a random experiment.

**Note**: Probability allows us to quantify the variability in the outcome of a random experiment. However, before we can introduce probability, it is necessary to specify the space of outcomes and the events on which it will be defined.

In statistics, the set of all possible outcomes of an experiment is called the **sample space** of the experiment, because it usually consists of the things that can happen when one takes a sample.

Sample spaces are usually denoted by the letter $S$.

Each outcome in a sample space is called an **element** or a member of the sample space, or simply a sample point.

**Example 1.5**

Define a sample space for each of the following experiments.

(a) The heights, in centimetres, of five children are 60, 65, 70, 45, 48. Select a child from this group of children, then measure and record the child's height.

(b) Select a number at random from the interval [0, 2] of real numbers. Record the value of the number selected.

**Solution**

(a) $S = \{60, 65, 70, 45, 48\}$.

(b) $S = \{x: 0 \leq x \leq 2$, where $x$ is a real number$\}$.

In some experiments, it is helpful to list the elements of the sample space systematically by means of a *tree diagram*. The following example illustrates the idea.

# Event



A subset of a sample space is called an **event**.

The empty set, , is a subset of S and S is also a subset of $S$.

and $S$ are therefore events. We call the impossible event and $S$ the certain event.

A subset of $S$ containing one element of $S$ is called a simple event.

# Example



Sample spaces and events, particularly relationships among events, are often depicted by means of Venn diagrams like those of Fig. 1.2. In each case, the sample space is represented by a rectangle, whereas events are represented by regions within the rectangle, usually by circles or parts of circles. The shaded regions of the four diagrams of Fig. 1.2 represent event $A$, the complement of $A$, the union of events $A$ and $B$, and the intersection of events $A$ and $B$.

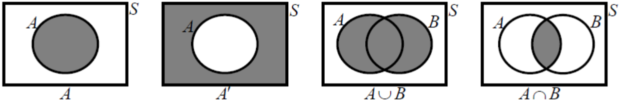



***Fig. 1.2**: Venn diagrams showing the complement, union and intersection*

**Mutually exclusive (or disjoint) events**: Any two events that cannot occur simultaneously, so that their intersection is the impossible event, are said to be mutually exclusive (or disjoint).

Operations with Events:

Since an event is a subset of a sample space, we can combine events to form new events, using the various set operations. The sample space is considered as the universal set. If $A$ and $B$ are two events defined on the same sample space, then:

(1) $A \cup B$ denotes the event "$A$ or $B$ or both". Thus the event $A \cup B$ occurs if either $A$ occurs or $B$ occurs or both $A$ and $B$ occur.

(2) $A \cap B$ denotes the event "both $A$ and $B$". Thus the event $A \cap B$ occurs if both $A$ and $B$ occur.

(3) $A'$ (or $\overline{A}$) denotes the event which occurs if and only if $A$ does not occur.

**Example 1.9**

If $A = \{x: 3 < x < 9\}$ and $B = \{y: 5 \leq y < 12\}$, then $A \cup B = \{z: 3 < z < 12\}$, and $A \cap B = \{y: 5 \leq y < 9\}$.

To operate efficiently with sets, we need to know a few facts about set theory:

**Theorem 1.1**

$(A \cup B)' = A' \cap B'.$

**Theorem 1.2**

$(A \cap B)' = A' \cup B'.$

**Definition 1.1**

If a trial of an experiment can result in $m$ mutually exclusive and equally likely outcomes, and if exactly $h$ of these outcomes correspond to an event $A$, then the probability of event $A$ is given by

$$P(A) = \frac{h}{m} = \frac{\text{number of ways that } A \text{ can occur}}{\text{number of ways the sample space } S \text{ can occur}}.$$

**Example 1.19**

A mixture of candies contains 6 mints, 4 toffees, and 3 chocolates. If a person makes a random selection of one of these candies, find the probability of getting

(a) a mint, (b) a toffee or a chocolate.

**Solution**

Let $M$, $T$, and $C$ represent the events that the person selects, respectively, a mint, toffee, or chocolate candy. The total number of candies is 13, all of which are equally likely to be selected.

(a) Since 6 of the 13 candies are mints,

$$P(M) = \tfrac{6}{13}.$$

(b) $P(T \cup C) = \dfrac{n(T \cup C)}{n(S)} = \dfrac{7}{13}.$

### Example 1.20

The following table shows 100 patients classified according to blood group and sex.

| | Blood group | | |
|---|---|---|---|
| | ***A*** | ***B*** | ***O*** |
| **Male** | 30 | 20 | 17 |
| **Female** | 15 | 10 | 8 |

If a patient is selected at random from this group, find the probability that the patient selected:
(a) is a male and has blood group $B$, (b) is a female and has blood group $A$.

### Solution

There are 100 ways in which we can select a patient from the 100 patients. Since the patient is selected at random, all the 100 ways of selecting a patient are equally likely.

(a) There are 20 males with blood group $B$. Therefore the probability that the patient selected is a male and has blood group $B$ is $\frac{20}{100} = 0.2$.

(b) There are 15 females with blood group $A$. Therefore the probability that the patient selected is a female and has blood group $A$ is $\frac{15}{100} = 0.15$.

# Axioms of Probability

In 1933, the axiomatic approach to probability was formalized by the Russian mathematician A. N. Kolmogorov (1964). The basis of this approach is embodied in three axioms from which a whole system of probability theory is derived. The three axioms are as follows.

**Axioms of probability**

Let $S$ be the sample space of an experiment and $P$ be a set function which assigns a number $P(A)$ to every $A \subset S$. Then the function $P(A)$ is said to be a probability function if it satisfies the following three axioms:

**Axiom 1:** $P(S) = 1$.

**Axiom 2:** $P(A) \geq 0$ for every event $A$.

**Axiom 3:** If $A$ and $B$ are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Theorem 1:

$$P(\emptyset) = 0$$

Proof: $S \cup \emptyset$ and $\emptyset$ are mutually exclusive

$$P(S) + P(\emptyset) = P(S)$$

them

$$P(\emptyset) = 0$$

Theorem 2:

$$P(A^{'}) = 1 - P(A)$$

Proof: $A \cup A^{'} = S$ and $A$ and $A^{'}$ are mutually exclusive

$$P(A) + P(A^{'}) = P(S) = 1$$

them

$$P(A^{'}) = 1 - P(A).$$

Theorem 3:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example:

**Example 1.22**

The probability that Akosua passes Mathematics is $\frac{2}{3}$, and the probability that she passes English is $\frac{4}{9}$. If the probability that she passes both courses is $\frac{1}{4}$, what is the probability that she passes at least one of the two courses?

**Solution**

Let $M$ denote the event "Akosua passes Mathematics" and $E$ the event "Akosua passes English". We wish to find $P(M \cup E)$. By the addition rule of probability, (see Theorem 1.10 on page 17),

$$P(M \cup E) = P(M) + P(E) - P(M \cap E) = \frac{2}{3} + \frac{4}{9} - \frac{1}{4} = \frac{31}{36}.$$

**Your turn!** Solve:

Example 1.20

The following table shows 100 patients classified according to blood group and sex.

| | Blood group | | |
|---|---|---|---|
| | *A* | *B* | *O* |
| **Male** | 30 | 20 | 17 |
| **Female** | 15 | 10 | 8 |

If a patient is selected at random from the 100 patients, find the probability that the patient selected:

1. Is a male or has blood group A.
2. Does not have blood group A.
3. Is a female or does not have blood group B.

Questions?

Thank you!!