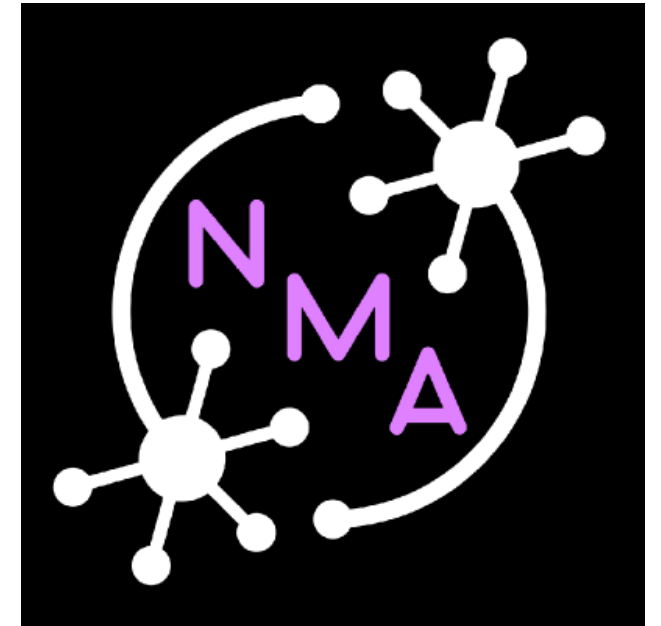


Intro to (linear) dynamical systems

Neuromatch academy study group

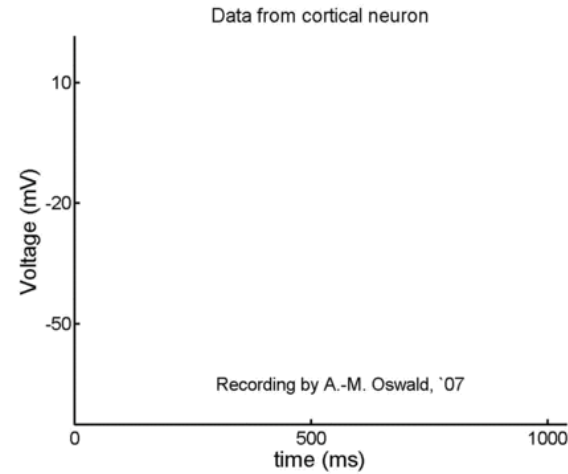
Adapted from <http://www.neuromatchacademy.org/syllabus/>

Benedetta Mariani



The dynamic brain

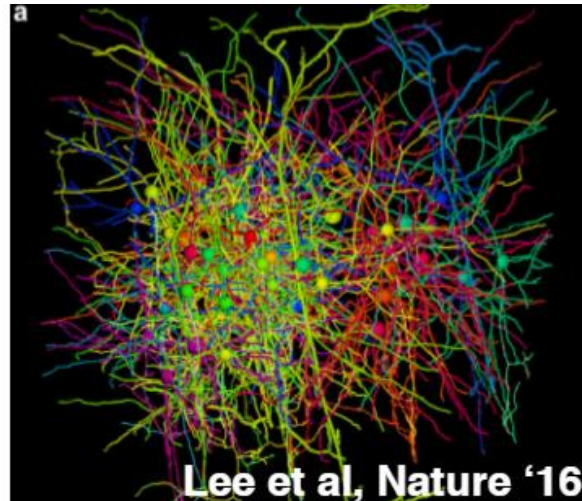
Single neuron scale



Connectivities on which the dynamics unfolds...

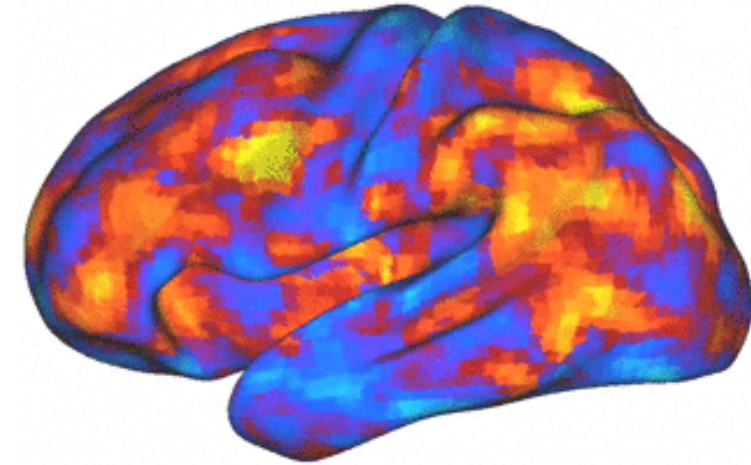


Oh et al. Nature '14

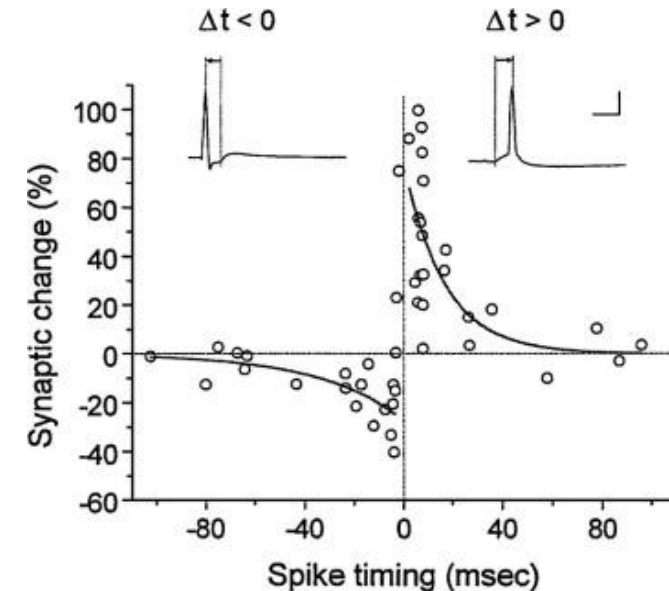


Lee et al, Nature '16

Whole brain scale

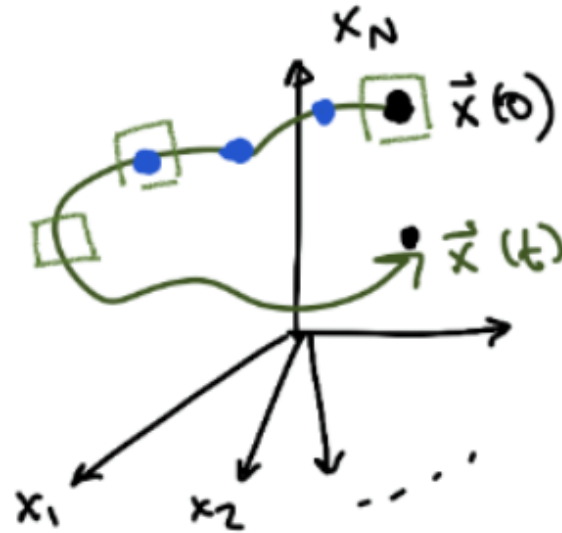


...the connectivity itself is dynamic!



Brain as a dynamical system

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \\ x_N(t) \end{pmatrix} = \bar{x}$$



Discrete or continuous?

- **Time**
 $x(t), x(t + \Delta t) \dots \Delta t \rightarrow 0?$
- **Space**

Dynamics = update rule:

$$x(t + \Delta t) = F(x(t))$$

COMPUTATION



$$x(t + \Delta t) = x(t) + f(x(t))\Delta t$$

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \approx \frac{dx}{dt} = f(x(t))$$



O. D. E.

Hodgkin and Huxley model: the dynamics of spikes generation

$$C \frac{dV}{dt} = -G_L(V - V_L) - G_{Na}m^3h(V - E_{Na}) - G_Kn^4(V - E_K) + I_e,$$

$$\frac{dm}{dt} = \alpha_m(V)(1 - m) - \beta_m(V)m,$$

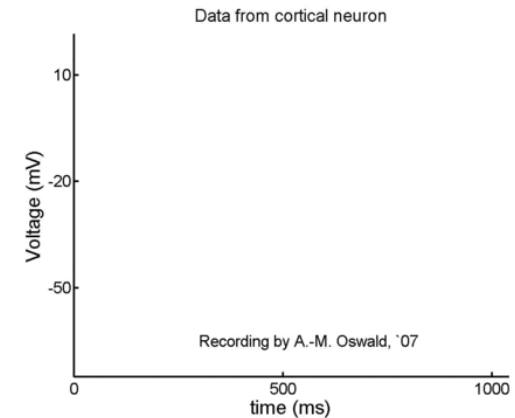
$$\frac{dh}{dt} = \alpha_h(V)(1 - h) - \beta_h(V)h,$$

$$\frac{dn}{dt} = \alpha_n(V)(1 - n) - \beta_n(V)n,$$

$$\alpha_m(V) = \frac{0.1(V + 40)}{1 - e^{-0.1(V+40)}}; \quad \beta_m(V) = 4e^{-0.0556(V+65)},$$

$$\alpha_h(V) = 0.07e^{-0.05(V+65)}; \quad \beta_h(V) = \frac{1}{1 + e^{-0.1(V+35)}},$$

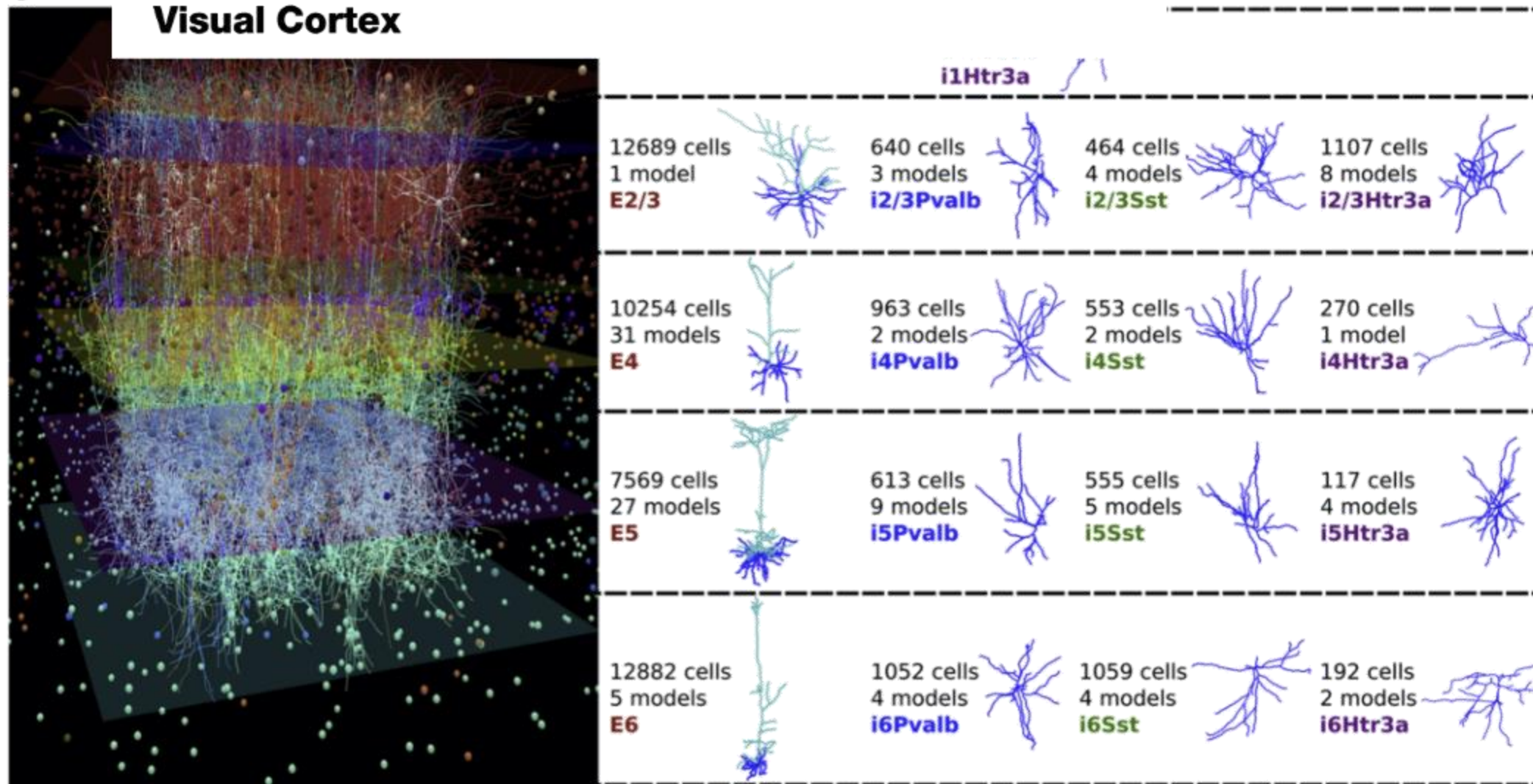
$$\alpha_n(V) = \frac{0.01(V + 55)}{1 - e^{-0.1(V+55)}}; \quad \beta_n(V) = 0.125e^{-0.0125(V+65)}.$$



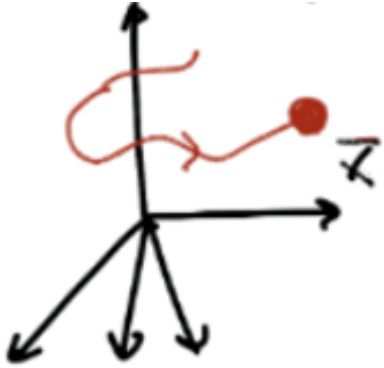
$$x(t + \Delta t) = x(t) + f(x(t))\Delta t$$

$$\frac{dx}{dt} = f(x(t))$$

Systematic Integration of Structural and Functional Data into Multi-scale Models of Mouse Primary Visual Cortex



Linear dynamical systems



$$\frac{d\vec{x}}{dt} = f(\vec{x})$$

$$f(\vec{x}) = 0$$

$$f(\vec{x})|_{x=\vec{x}} \approx 0 + A\vec{x} + \dots$$

$$\frac{d\vec{x}}{dt} \approx A\vec{x}$$

Where you go is the sum of your parts!

$$\vec{x}(t) = a_1(t) \vec{v}_1 + a_2(t) \vec{v}_2 + \dots$$

$$\frac{d\vec{x}}{dt} = A(a_1(t) \vec{v}_1 + a_2(t) \vec{v}_2 + \dots) = A a_1(t) \vec{v}_1 + A a_2(t) \vec{v}_2 + \dots$$

If $\{\vec{v}_j\}$ are eigenvectors of A , then $a_j(t) = e^{\lambda_j t}$

E.g., consider $x = a_1(t) \vec{v}_1$:

$$\frac{d\vec{x}}{dt} = A(a_1(t) \vec{v}_1) = a_1(t) \lambda_1 \vec{v}_1 = \frac{da_1(t)}{dt} \vec{v}_1$$

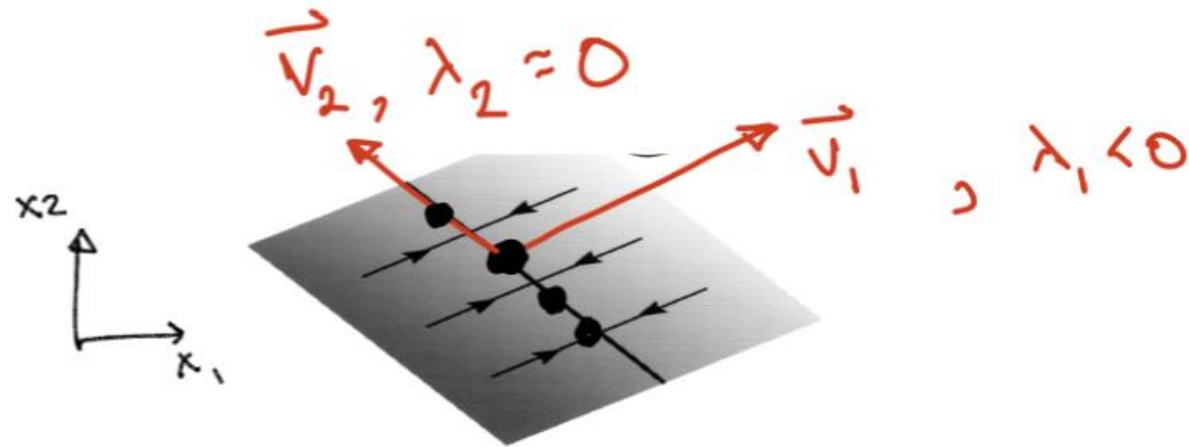
$$\frac{da_1(t)}{dt} = a_1(t) \lambda_1 \rightarrow a_1(t) = a_1(0) e^{\lambda_1 t}$$

Eigenvalues → timescales in neural networks

<https://doi.org/10.1073/pnas.93.23.13339>

How the brain keeps the eyes still

H. S. SEUNG



Eigenvalues → timescales in neural networks

[10.1037/0033-295x.108.3.550](https://doi.org/10.1037/0033-295x.108.3.550)

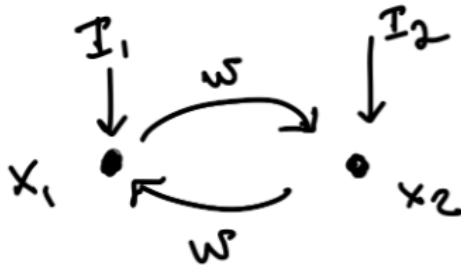
The Time Course of Perceptual Choice: The Leaky, Competing Accumulator Model

Marius Usher
Birkbeck College, University of London

James L. McClelland
Carnegie Mellon University and the Center for
the Neural Basis of Cognition

If w is large

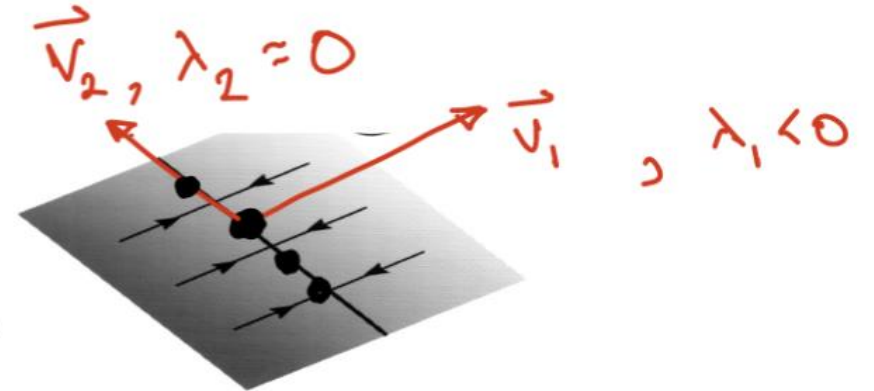
<https://doi.org/10.1146/annurev.neuro.29.051605.113038>



$$\frac{dx_1}{dt} = -kx_1 - wx_2 + I_1$$

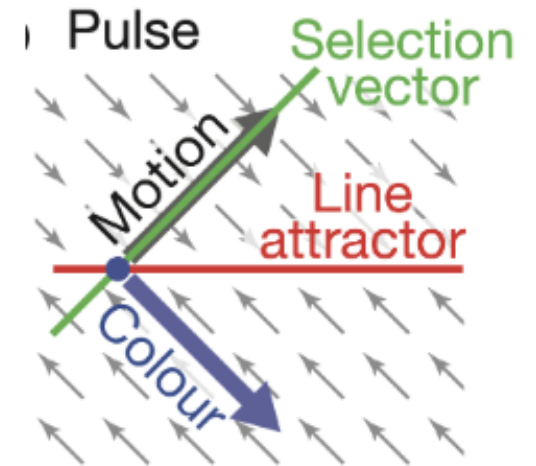
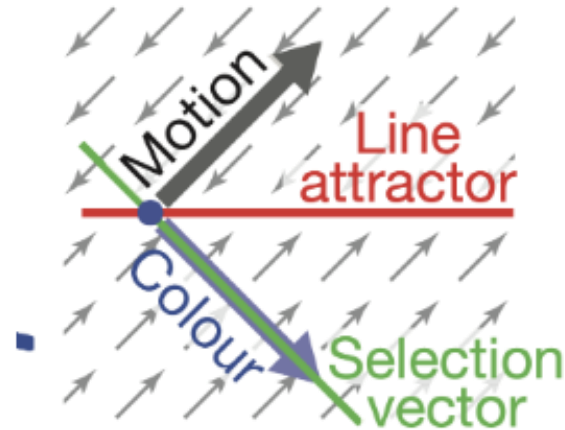
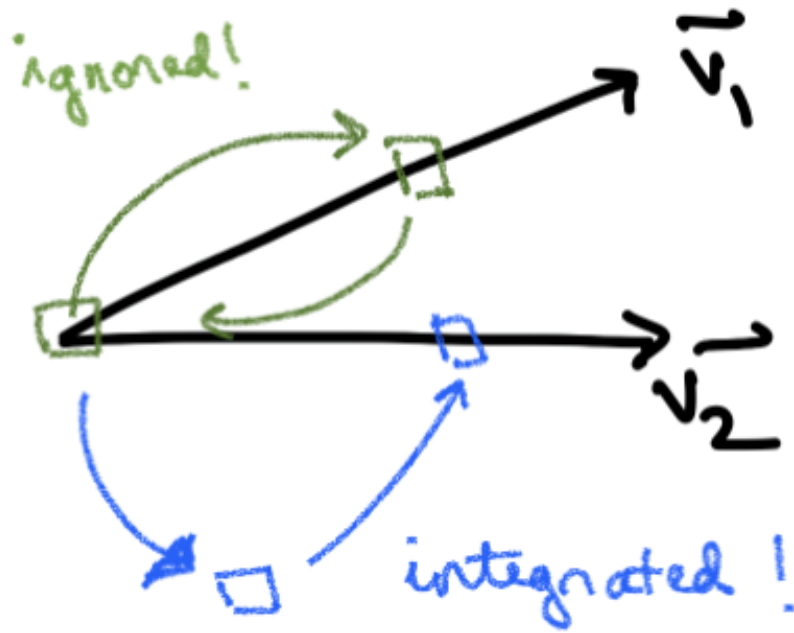
$$\frac{dx_2}{dt} = -kx_2 - wx_1 + I_2$$

If $w = k \rightarrow$



Non orthogonal eigenvectors

<https://doi.org/10.1038/nature12742>



Dynamics occurs that defies the eigenvalues: **Non normal dynamics**

- Longer timescales
- Growth when λ 's < 0

Where you go is the sum of your parts (2): response to time-dependent input

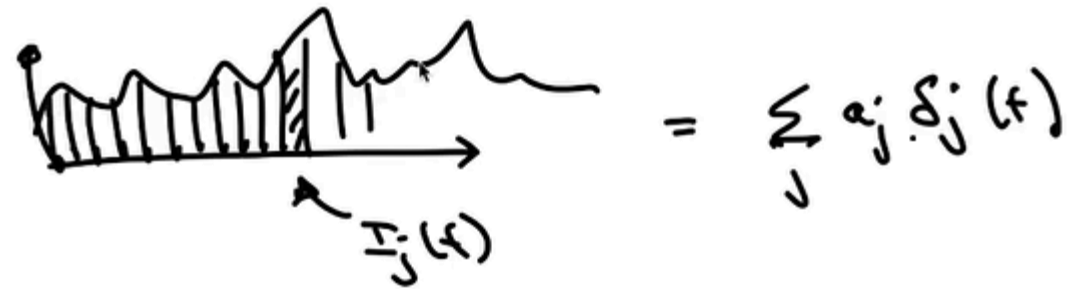
$$\frac{dx}{dt} = Ax + I(t)$$

$$\frac{dx_1}{dt} = Ax_1 + I_1(t)$$

$$\frac{dx_2}{dt} = Ax_2 + I_2(t)$$

$$x(t) = x_1(t) + x_2(t)$$

$$\frac{dx}{dt} = Ax + I_1(t) + I_2(t)$$



All I need to know to characterize response is to know the response to a unitary input (**impulse response**)

$$x(t) = \int d\tau h_A(t - \tau) I(\tau)$$

Where you go is the sum of your parts (2): response to time-dependent input

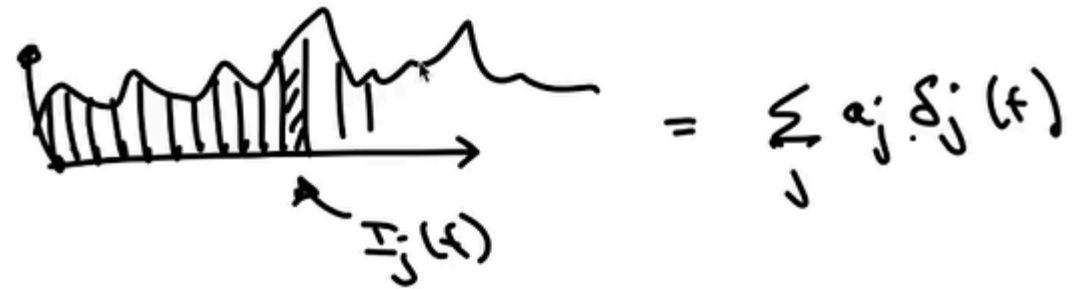
$$\frac{dx}{dt} = Ax + I(t)$$

$$\frac{dx_1}{dt} = Ax_1 + I_1(t)$$

$$\frac{dx_2}{dt} = Ax_2 + I_2(t)$$

$$x(t) = x_1(t) + x_2(t)$$

$$\frac{dx}{dt} = Ax + I_1(t) + I_2(t)$$


$$= \sum_j a_j \delta_j(t)$$

All I need to know to characterize response is to know the response to a unitary input (**impulse response**)

$$x(t) = \int d\tau h_A(t - \tau) I(\tau)$$

$$x(\omega) = h(\omega) I(\omega)$$

Networks of linear systems

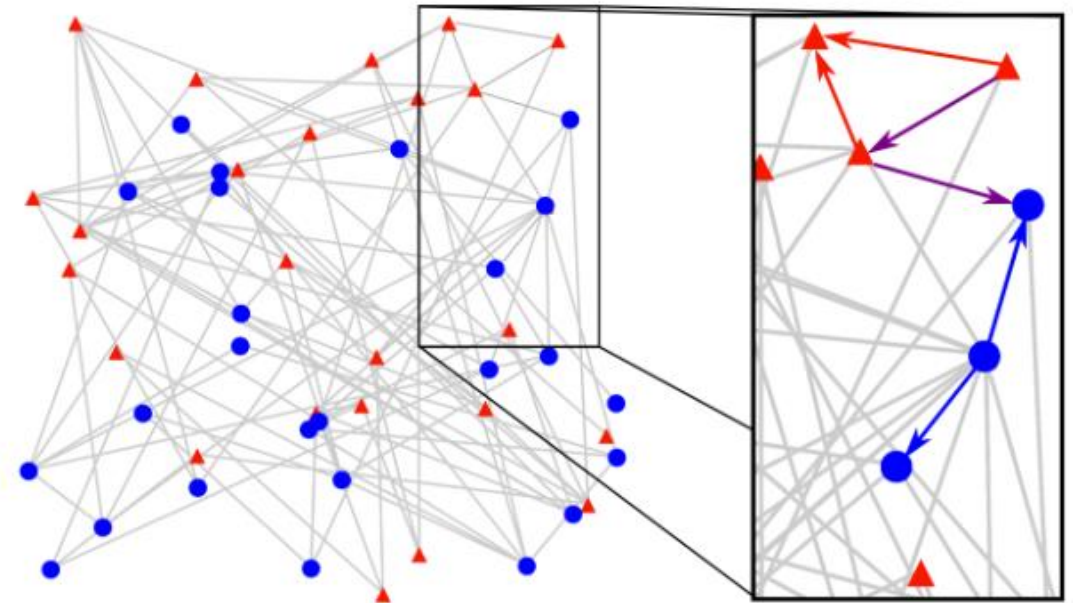
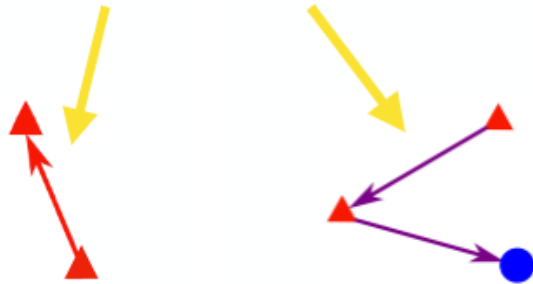


$$x_i(t) = \int d\tau h_A(t - \tau) \left[I(\tau) + \sum_{ij} W_{ij} x_j(t) \right]$$

$$\vec{x}(\omega) = h(\omega) [\vec{I}(\omega) + W \vec{x}(\omega)]$$

$$\vec{x}(\omega) = [Id - h(\omega)W]^{-1} h(\omega) \vec{I}(\omega)$$

$$\vec{x}(\omega) = [Id + hW + h^2W^2 + \dots] h(\omega) \vec{I}(\omega)$$



$$\text{dynamics} = f(W) = f(\text{motifs})$$

Stochastic linear dynamical systems

$$x(t) = \int d\tau h_A(t - \tau) I(\tau)$$



If I_t (and $x(0)$) are jointly gaussian then x is gaussian

Explicit formula for covariance!

$$\vec{x}(\omega) = [Id - h(\omega)W]^{-1}h(\omega)\vec{I}(\omega)$$

$$Cov(\vec{x}_i(\omega)\vec{x}_j(\omega)) \sim [Id - hW]^{-1}hCov(I)h[Id - hW]^{-T} = f(W)$$

Stochastic linear dynamical systems



$Cov = f(W)$, then:

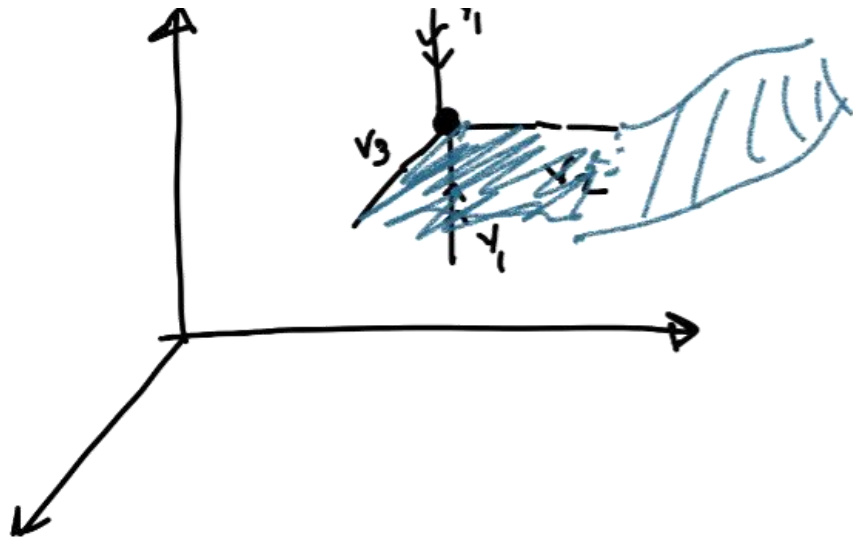
- $Cov = f(motifs)$
- $Dimension = f(W) = f(motifs)$
- $W \rightarrow W + \Delta W$

LEARNING

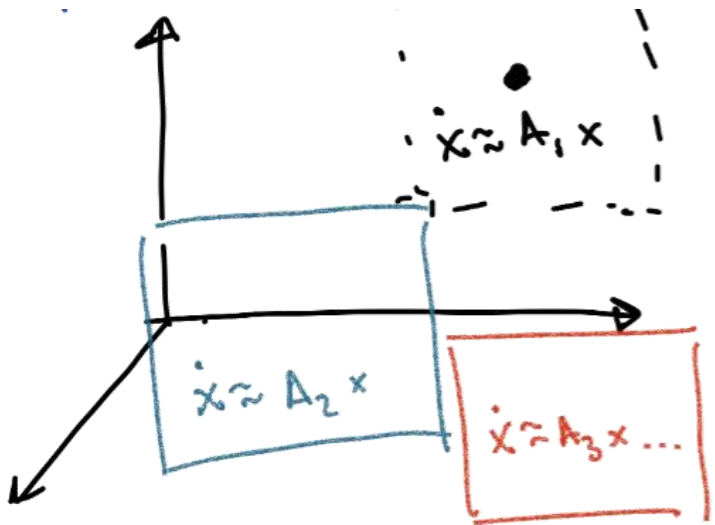
Say $\Delta W = f(Cov)$ (as in STDP),
then $\Delta W = f(Cov) = f(g(W))$

Closed form dynamical system also
for the connectivity!

Beyond linear dynamical systems



Stable manifold (low-D non linear dynamics)



Piecewise linear dynamical systems

Tutorial 1: deterministic 1 and 2D linear systems

- Simulate a 1D deterministic linear system

$$\frac{dx}{dt} = ax$$

Analytical solution:

$$x(t) = x(0)e^{at}$$

- If $a < 0$: exponential decay
- If $a = 0$: nothing changes
- If $a > 0$: exponential growth

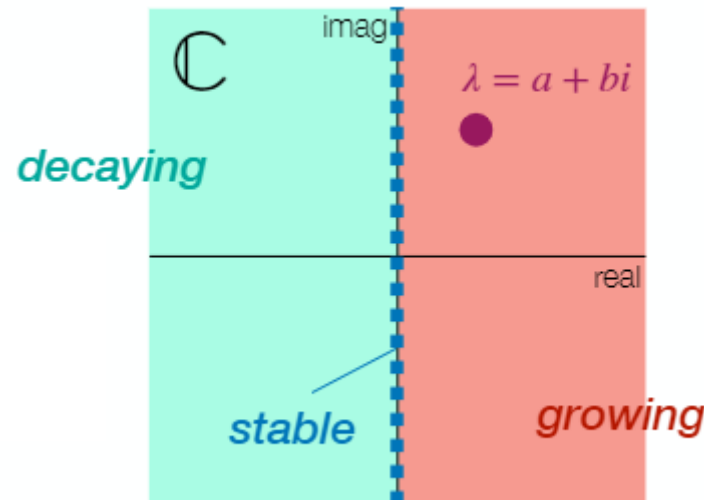


Tutorial 1: deterministic 1 and 2D linear systems

$$\frac{dx}{dt} = \lambda x$$

If λ is a complex number: $x(t) = x(0)e^{\lambda t} = x(0)e^{(a+ib)t} = x(0)e^{at}[\cos(bt) + i \sin(bt)]$

$b \sim$ frequency of oscillation



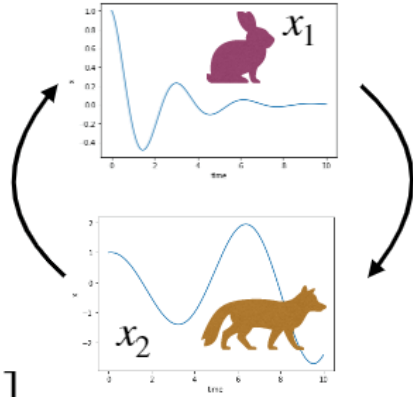
Tutorial 1: deterministic 1 and 2D linear systems

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

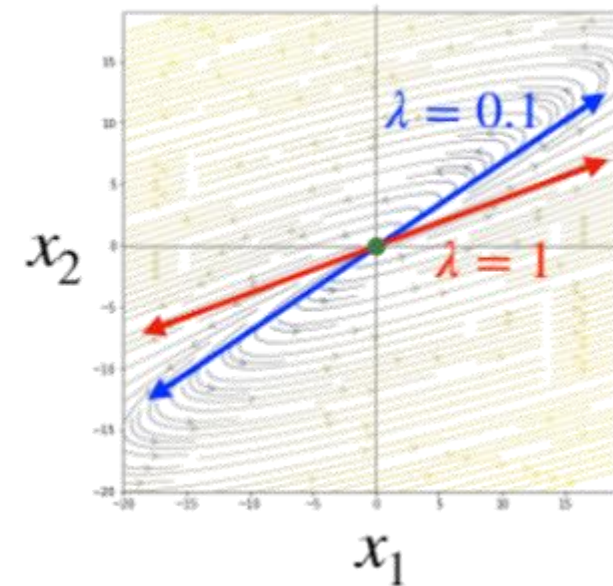
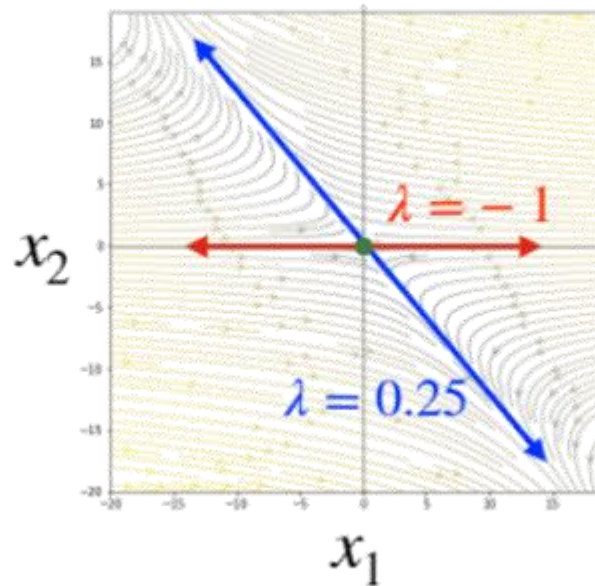
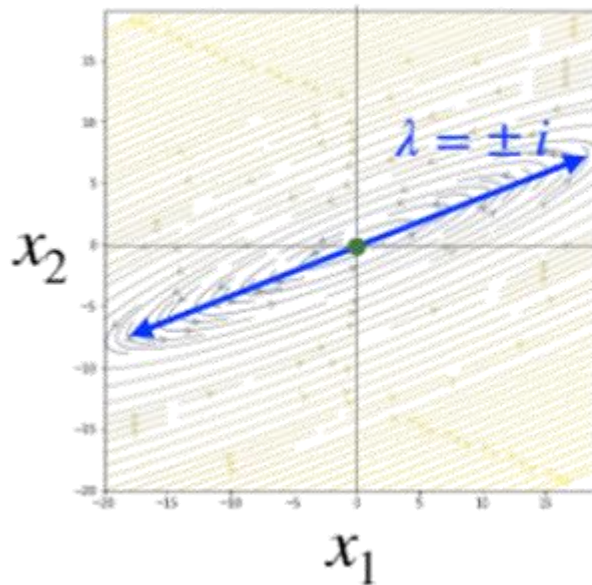


<https://www.youtube.com/watch?v=PFDu9oVAE-g>

$$\vec{\dot{x}} = A \vec{x}$$

$$\mathbf{x}(t) = \mathbf{v}_1 \mathbf{x}_0 \mathbf{v}_1^T e^{\lambda_1 t} + \mathbf{v}_2 \mathbf{x}_0 \mathbf{v}_2^T e^{\lambda_2 t}$$

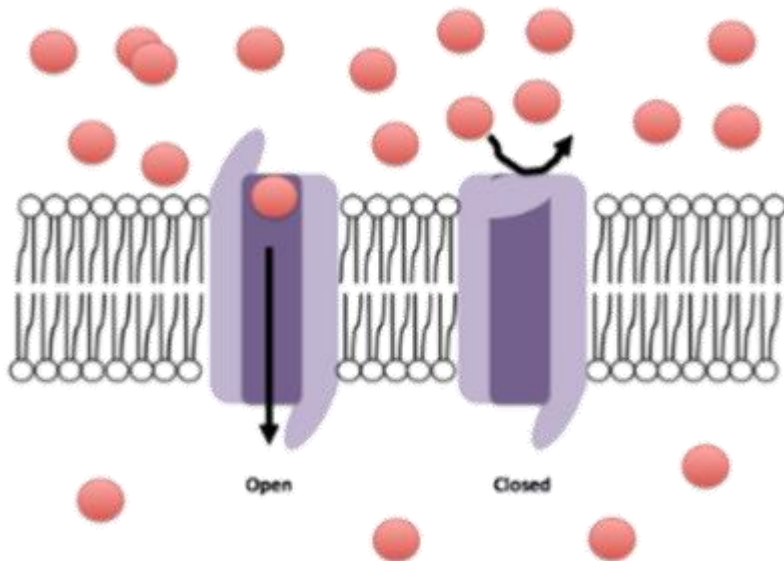
Behavior of the system depends on eigenvalues and eigenvectors of A



Tutorial 2: Markovian dynamical system

A system is markovian if the **present** state determines the probability of transitions to the next state

Opening and closing of ion channels as random events



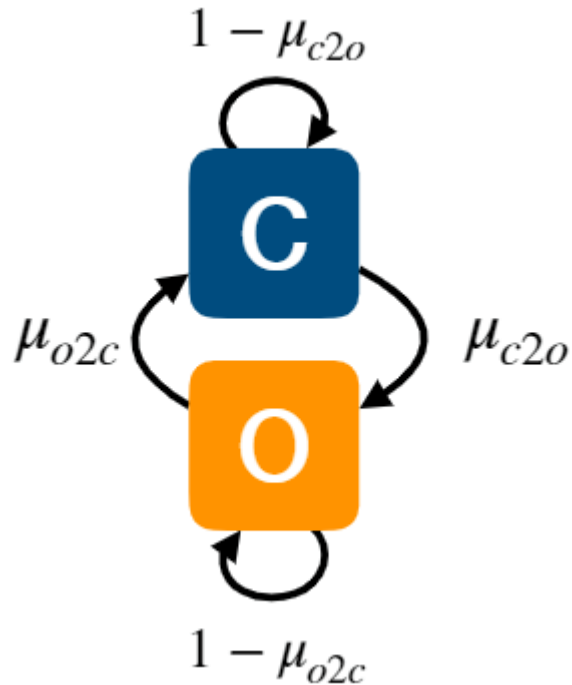
A telegraph process

Transition from open to close: $P(o2c)$

Transition from close to open: $P(c2o)$

Tutorial 2: Markovian dynamical system

Instead of keeping track of single channels and perform many simulations, now keep track of the probability state



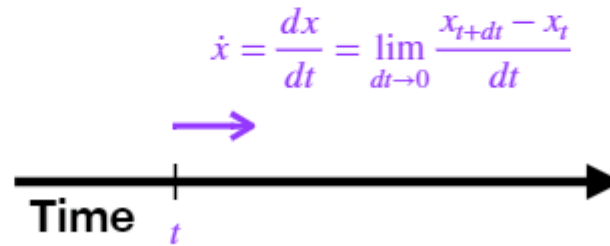
State vector at time k: probability of being in state O or C

$$\begin{bmatrix} C \\ O \end{bmatrix}_{k+1} = \mathbf{A} \begin{bmatrix} C \\ O \end{bmatrix}_k = \begin{bmatrix} 1 - \mu_{c2o} & \mu_{o2c} \\ \mu_{c2o} & 1 - \mu_{o2c} \end{bmatrix} \begin{bmatrix} C \\ O \end{bmatrix}_k$$

Tutorial 2: Markovian dynamical system

Continuous Time Formulation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

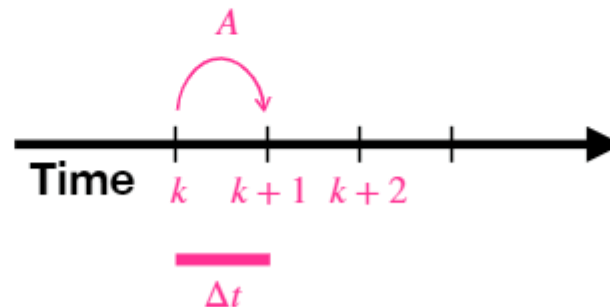


*Stable solution,
when things
don't change*

$$\mathbf{A} = 0$$

Discrete Time Formulation

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$



$$\mathbf{A} = 1$$

Tutorial 2: Markovian dynamical system

Continuous Time
Formulation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Discrete Time
Formulation

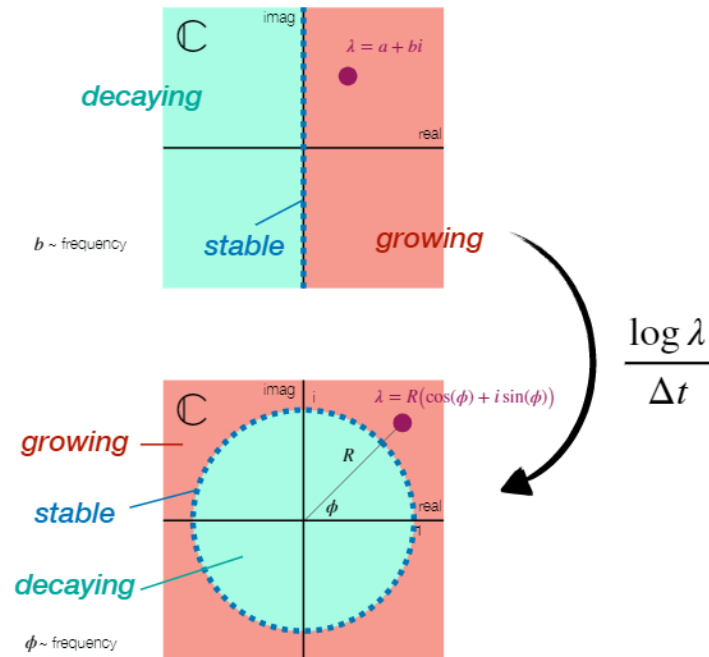
$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$

Stable solution,
when things
don't change

$$\mathbf{A} = 0$$

$$\mathbf{A} = 1$$

Eigenvalue
Spectrum of \mathbf{A}



- Identify the stable eigenvalue of \mathbf{A} and its corresponding eigenvector
- Compare the equilibrium distribution solutions with the eigenvalues and eigenvectors of \mathbf{A} matrix