Intro to (linear) dynamical systems

Neuromatch academy study group

Adapted from http://www.neuromatchacademy.org/syllabus/

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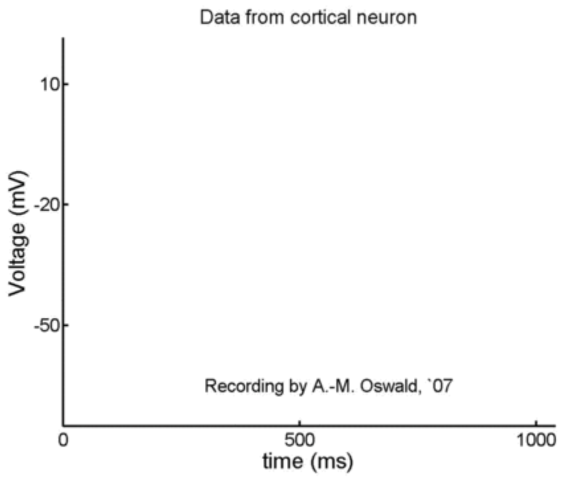
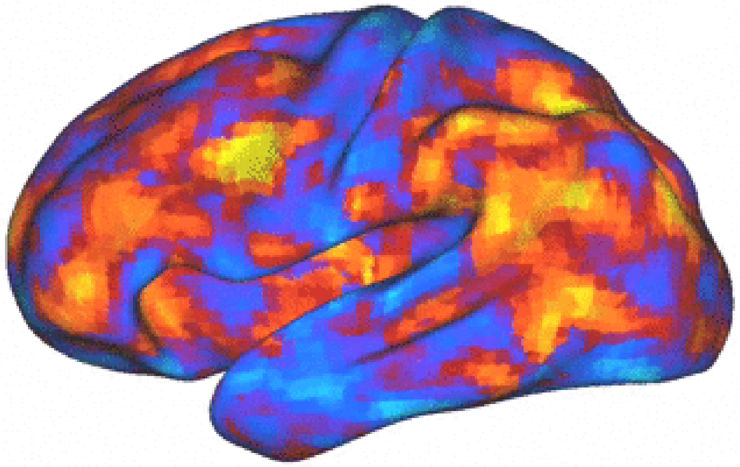
The dynamic brain

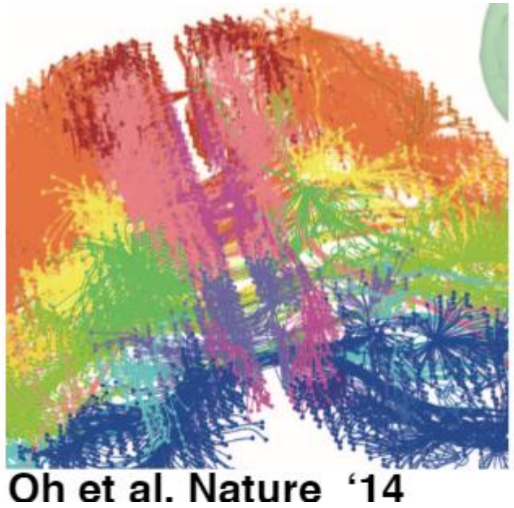
Single neuron scale

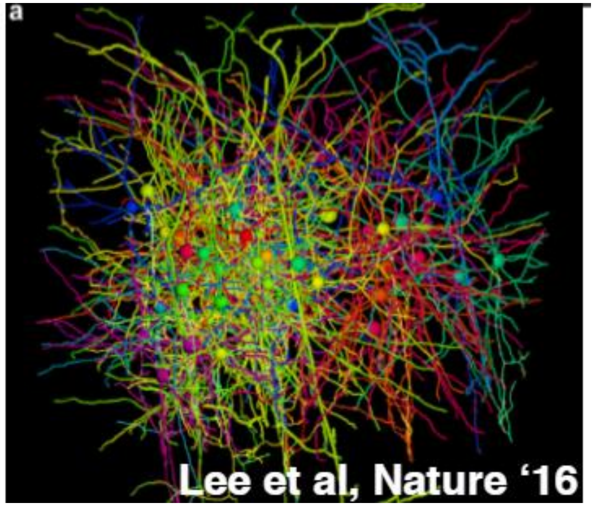


Whole brain scale

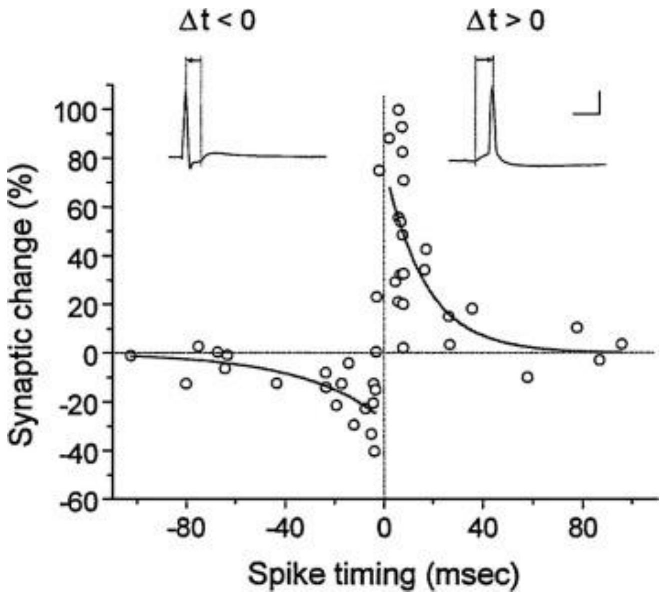


Connectivities on which the dynamics unfolds…



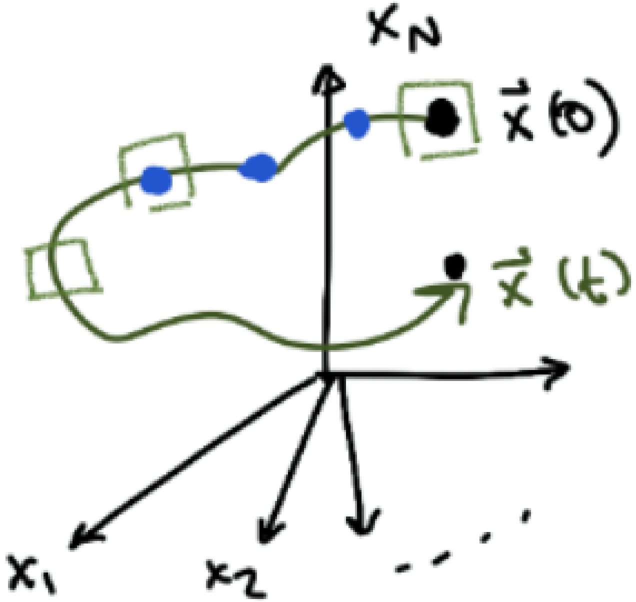


…the connectivity itself is dynamic!



Brain as a dynamical system

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \\ x_N(t) \end{pmatrix} = \overline{\boldsymbol{x}}$$



Discrete or continuous?

- **Time**

  $x(t), x(t+\Delta t) \ldots \Delta t \to 0?$
- **Space**

Dynamics = update rule:

$$x\,(t+\Delta t) = F\big(x(t)\big)$$

O. D. E.

COMPUTATION

$$x\,(t+\Delta t) = x(t) + f\big(x(t)\big)\Delta t$$

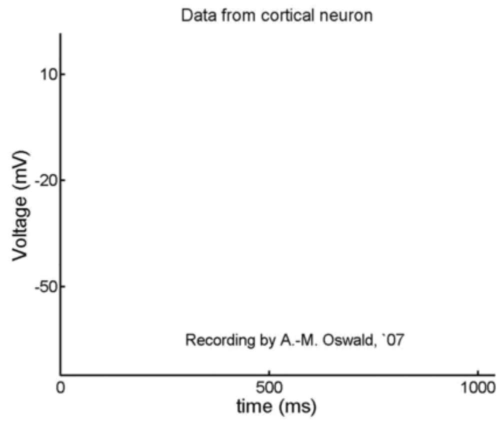
$$\frac{x\,(t+\Delta t) - x(t)}{\Delta t} \approx \frac{dx}{dt} = f\big(x(t)\big)$$

Hodgkin and Huxley model: the dynamics of spikes generation

$$C\frac{dV}{dt} = -G_L(V - V_L) - G_{Na}m^3h(V - E_{Na}) - G_K n^4(V - E_K) + I_e,$$

$$\begin{aligned}
\frac{dm}{dt} &= \alpha_m(V)(1-m) - \beta_m(V)m, \\
\frac{dh}{dt} &= \alpha_h(V)(1-h) - \beta_h(V)h, \\
\frac{dn}{dt} &= \alpha_n(V)(1-n) - \beta_n(V)n,
\end{aligned}$$

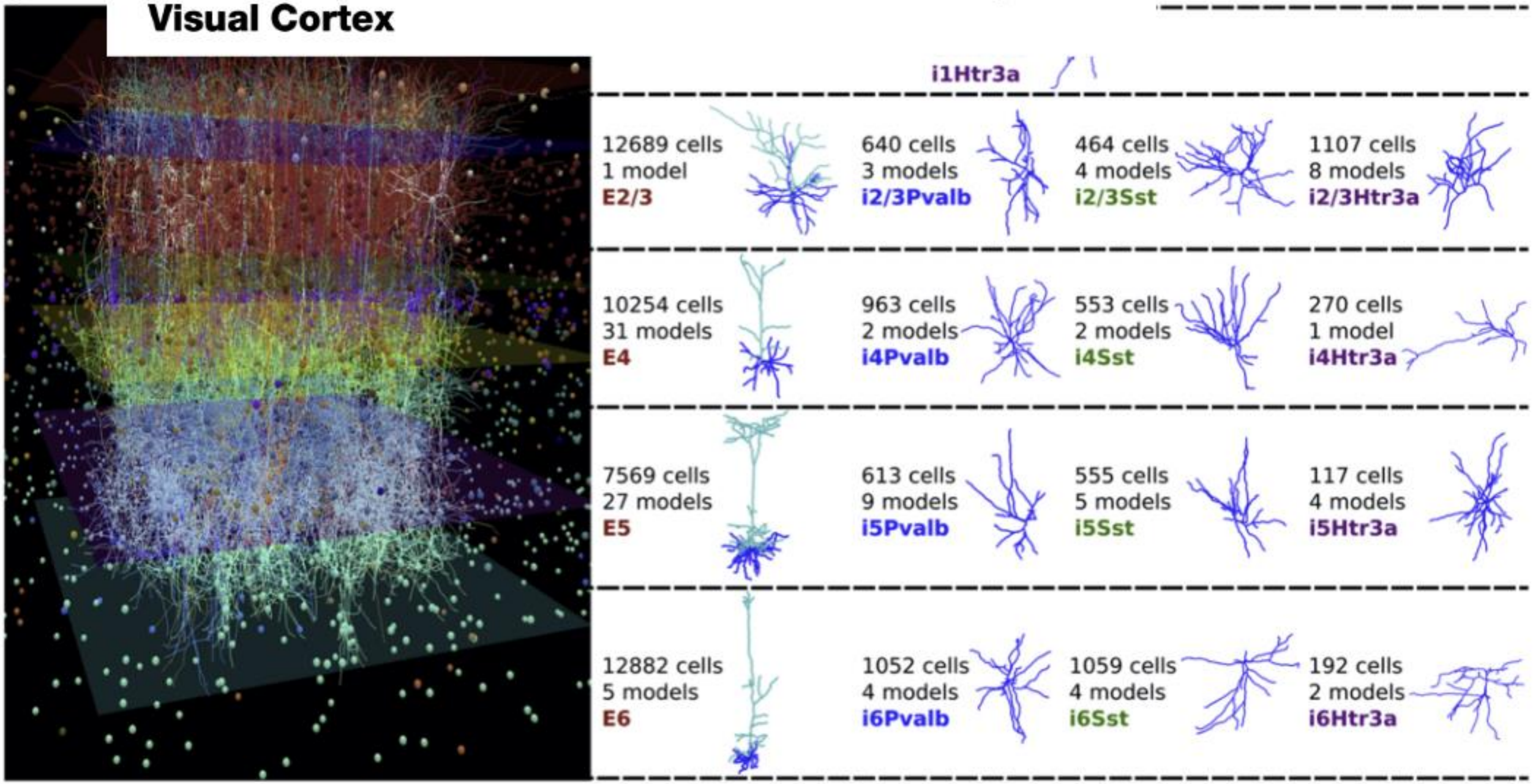
$$\begin{aligned}
\alpha_m(V) &= \frac{0.1(V+40)}{1-e^{-0.1(V+40)}}\;; \quad \beta_m(V) = 4e^{-0.0556(V+65)}, \\
\alpha_h(V) &= 0.07e^{-0.05(V+65)}\;; \quad \beta_h(V) = \frac{1}{1+e^{-0.1(V+35)}}, \\
\alpha_n(V) &= \frac{0.01(V+55)}{1-e^{-0.1(V+55)}}\;; \quad \beta_n(V) = 0.125e^{-0.0125(V+65)}.
\end{aligned}$$



$$x\,(t+\,\Delta t) = x(t) + f\big(x(t)\big)\Delta t$$

$$\frac{dx}{dt} = f\big(x(t)\big)$$

Systematic Integration of Structural and Functional Data into Multi-scale Models of Mouse Primary Visual Cortex



Linear dynamical systems

$$\frac{d\vec{x}}{dt} = f(\vec{x})$$

$$f(\bar{x}) = 0$$

$$f(\vec{x})|_{x=\bar{x}} \approx 0 + A\vec{x} + \;\ldots$$

$$\frac{d\vec{x}}{dt} \approx A\vec{x}$$

Where you go is the sum of your parts!

$$\vec{x}\,(t) = a_1(t)\,\overrightarrow{v_1} + a_2(t)\,\overrightarrow{v_2} + \ldots$$

$$\frac{d\vec{x}}{dt} = A(a_1(t)\,\overrightarrow{v_1} + a_2(t)\,\overrightarrow{v_2} + \ldots) = A\,a_1(t)\,\overrightarrow{v_1} + A\,a_2(t)\,\overrightarrow{v_2} + \;\ldots$$

If $\{\overrightarrow{v_j}\}$ are eigenvectors of A, then $a_j(t) = e^{\lambda_j t}$

E.g., consider $x = \; a_1(t)\,\overrightarrow{v_1}$:

$$\frac{d\vec{x}}{dt} = \; A\,(a_1(t)\,\overrightarrow{v_1}) = a_1(t)\,\lambda_1\,\overrightarrow{v_1} = \frac{da_1(t)\,\overrightarrow{v_1}}{dt}$$

$$\frac{da_1(t)}{dt} = a_1(t)\,\lambda_1 \rightarrow a_1(t) = a_1(0)e^{\lambda_1 t}$$

Eigenvalues → timescales in neural networks

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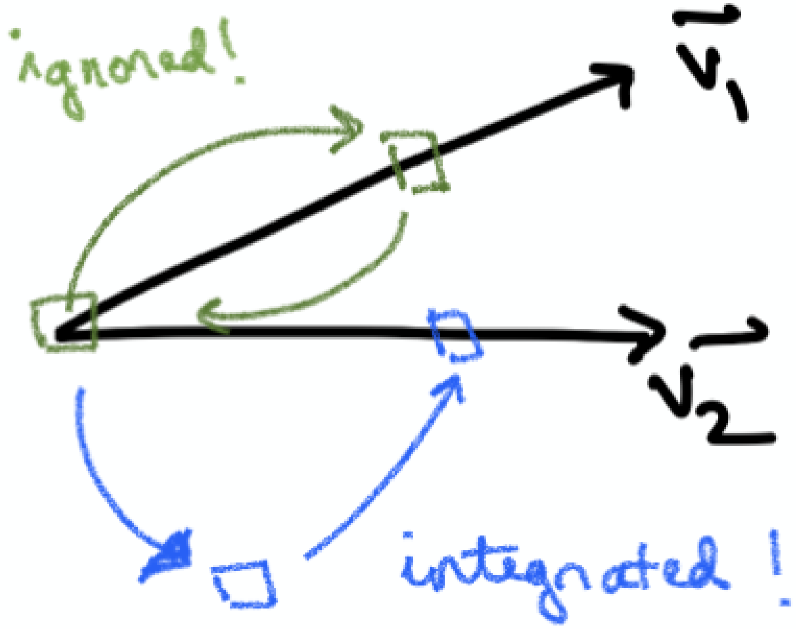
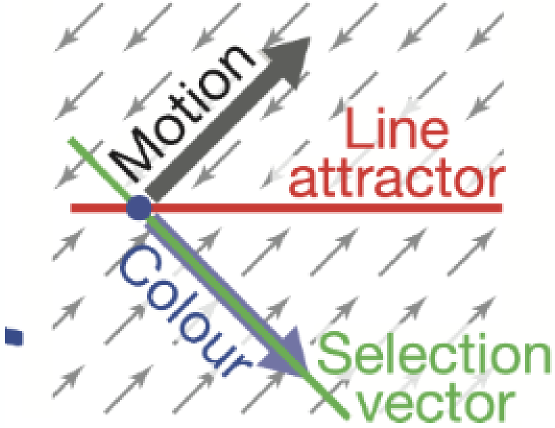
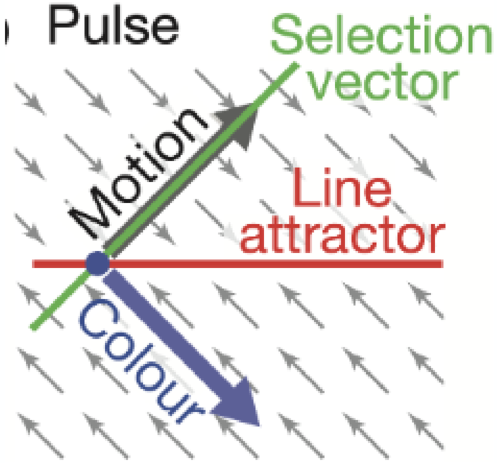
Non orthogonal eigenvectors





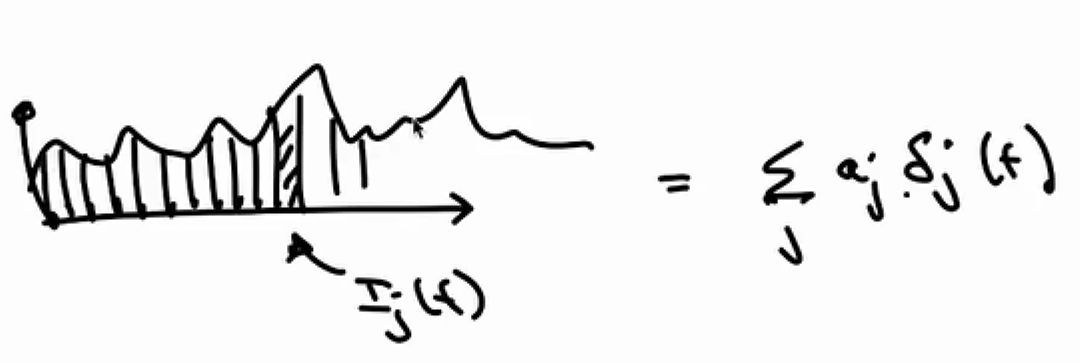


Non normal dynamics

Where you go is the sum of your parts (2): response to time-dependent input

$$\frac{dx}{dt} = Ax + I(t)$$

$$\frac{dx_1}{dt} = Ax_1 + I_1(t) \qquad \frac{dx_2}{dt} = Ax_2 + I_2(t)$$



$$x(t) = x_1(t) + x_2(t)$$

$$\frac{dx}{dt} = Ax + I_1(t) + I_2(t)$$

All I need to know to characterize response is to know the response to a unitary input (**impulse response**)

$$x(t) = \int d\tau h_A(t-\tau) I(\tau)$$

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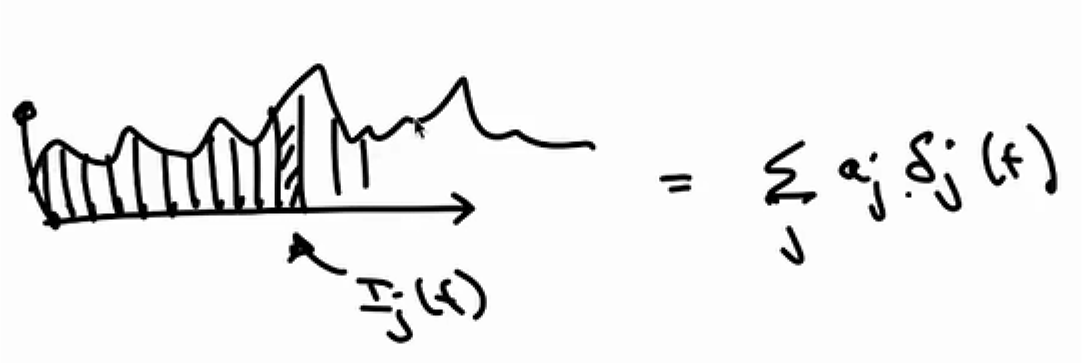

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$$x(t) = \int d\tau h_A(t-\tau) I(\tau) \qquad x(\omega) = h(\omega) I(\omega)$$

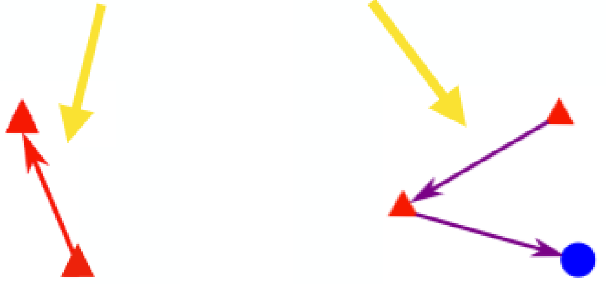
Networks of linear systems

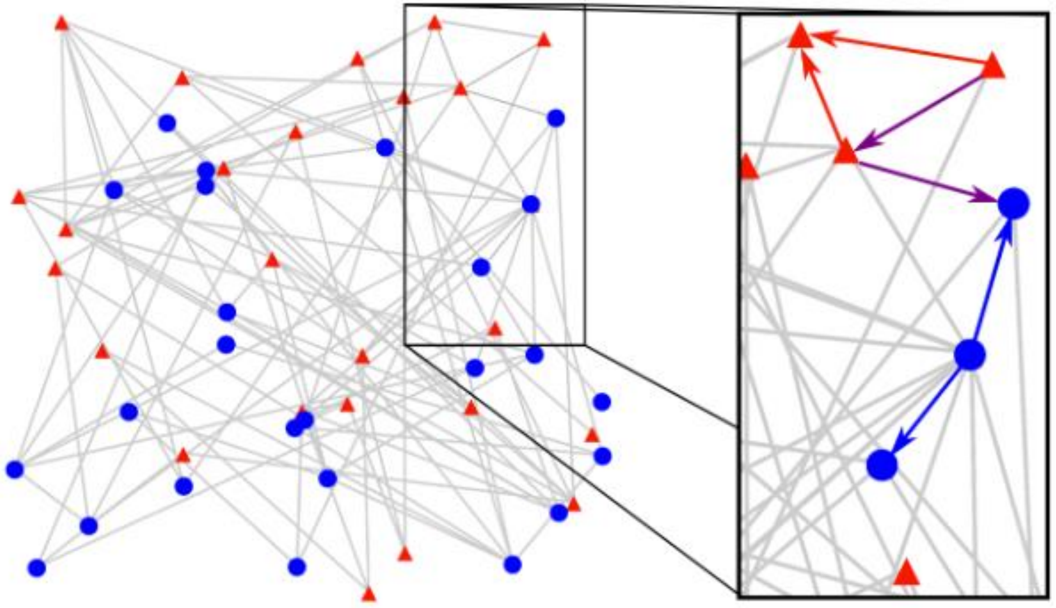
$$x_i(t) = \int d\tau h_A(t-\tau)\left[I(\tau) + \sum_{ij} W_{ij} x_j\,(t)\right]$$

$$\vec{x}(\omega) = h(\omega)[\vec{I}(\omega) + W\vec{x}(\omega)]$$

$$\vec{x}(\omega) = [Id\ - h(\omega)W]^{-1}h(\omega)\vec{I}(\omega)$$

$$\vec{x}(\omega) = [Id + hW + h^2W^2 + \cdots]h(\omega)\vec{I}(\omega)$$

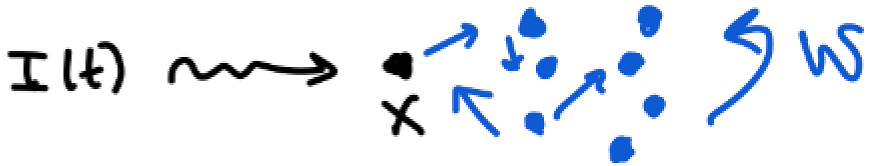




$$dynamics = f(W) = f(motifs)$$

Stochastic linear dynamical systems

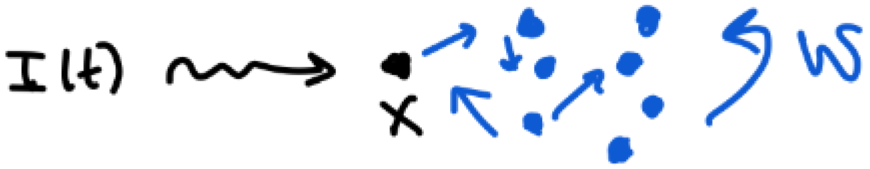
$$x(t) = \int d\tau h_A(t-\tau) I(\tau)$$



If I_i (and $x(0)$) are jointly gaussian (?) then x is gaussian

Explicit formula for covariance! $\vec{x}(\omega) = [Id \ - h(\omega)W]^{-1} h(\omega)\vec{I}(\omega)$

$$Cov\left(\overrightarrow{x_i}(\omega)\overrightarrow{x_j}(\omega)\right) \sim [Id \ - hW]^{-1} h Cov(I) h [Id \ - hW]^{-T} = f(W)$$



$Cov = f(W)$

- $Cov = f(motifs)$
- $Dimension = f(W) = f(motifs)$
- $W \rightarrow W + \Delta W$

LEARNING

Say $\Delta W = f(Cov)$ (as in SPTD),
then $\Delta W = f(Cov) = f(g(W))$

Closed form dynamical systems for the connectivity