Intro to (linear) dynamical systems

Neuromatch academy study group

Adapted from http://www.neuromatchacademy.org/syllabus/

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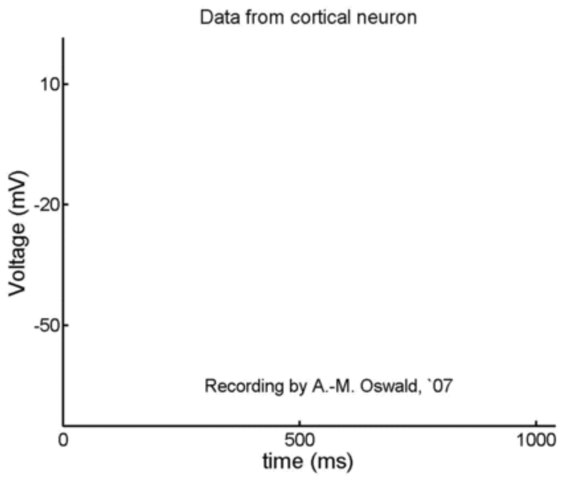
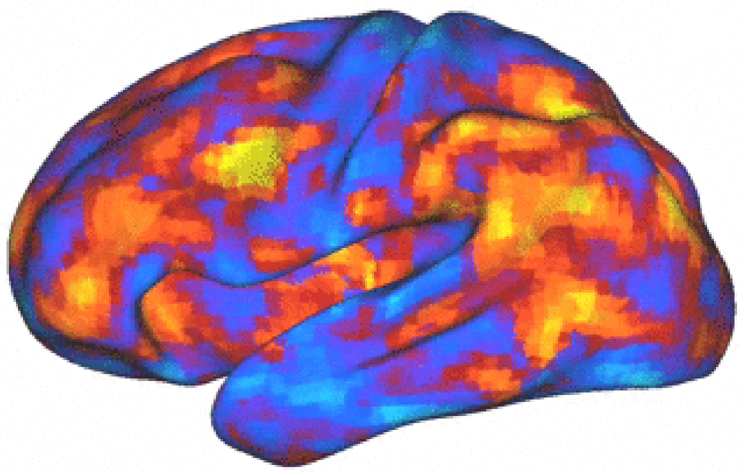
The dynamic brain

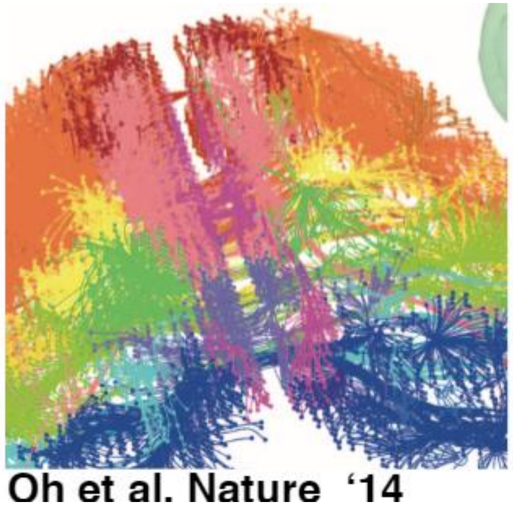
Single neuron scale

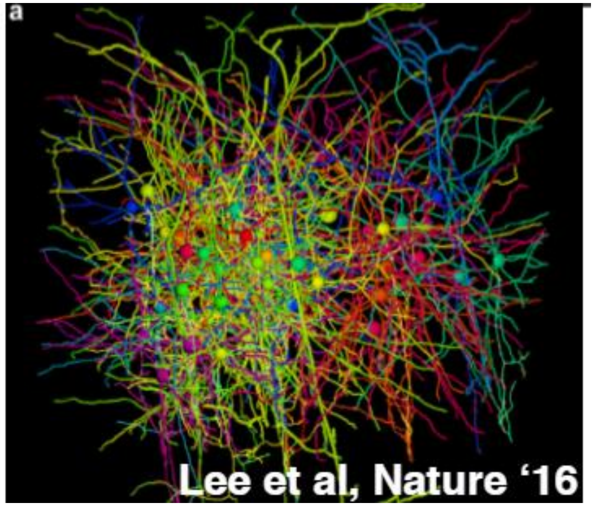


Whole brain scale

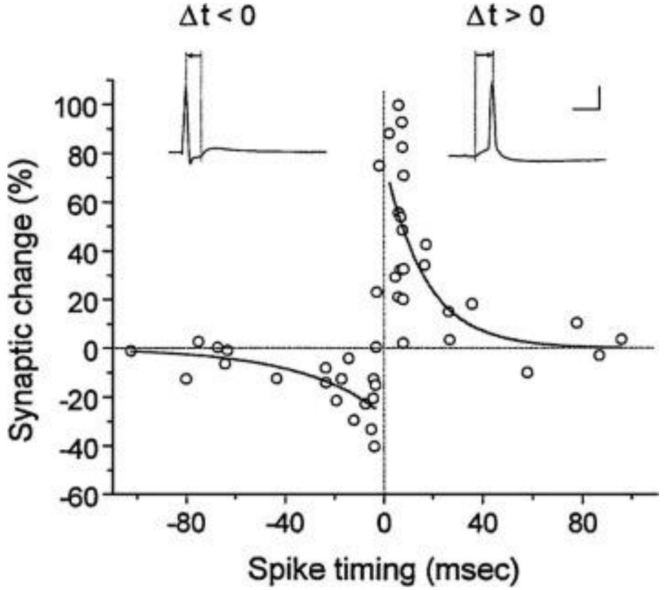


Connectivities on which the dynamics unfolds…



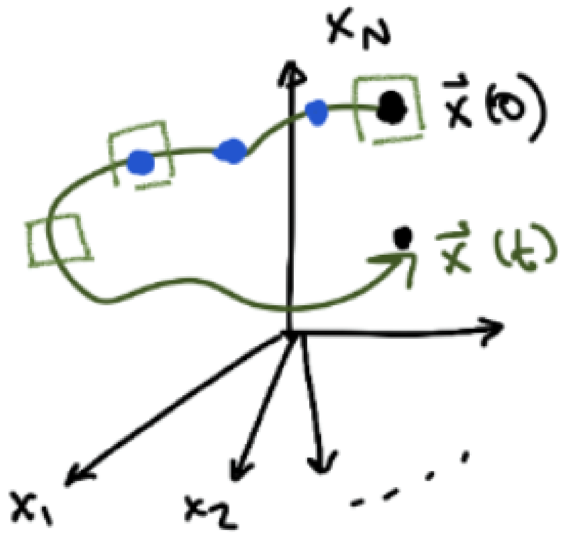


…the connectivity itself is dynamic!



Brain as a dynamical system

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \\ x_N(t) \end{pmatrix} = \bar{\boldsymbol{x}}$$



Discrete or continuous?

- **Time**
  $x(t), x(t+\ \Delta t) \ldots \Delta t \to 0?$
- **Space**

Dynamics = update rule:

$$x\ (t+\ \Delta t) = F\big(x(t)\big)$$

COMPUTATION

$$x\ (t+\ \Delta t) = x(t) + f\big(x(t)\big)\Delta t$$

$$\frac{x\ (t+\ \Delta t) - x(t)}{\Delta t} \approx \frac{dx}{dt} = f\big(x(t)\big)$$

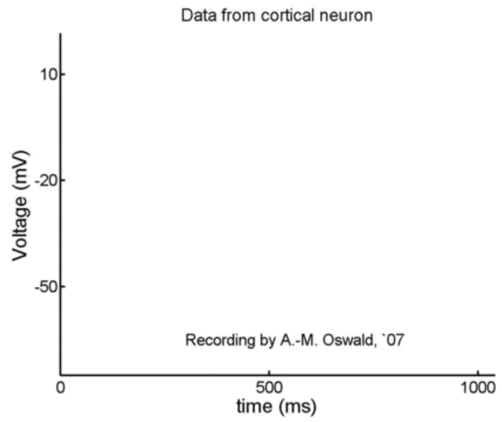
O. D. E.

Hodgkin and Huxley model: the dynamics of spikes generation

$$C\frac{dV}{dt} = -G_L(V - V_L) - G_{Na}m^3h(V - E_{Na}) - G_K n^4(V - E_K) + I_e,$$

$$\begin{aligned}
\frac{dm}{dt} &= \alpha_m(V)(1-m) - \beta_m(V)m, \\
\frac{dh}{dt} &= \alpha_h(V)(1-h) - \beta_h(V)h, \\
\frac{dn}{dt} &= \alpha_n(V)(1-n) - \beta_n(V)n,
\end{aligned}$$

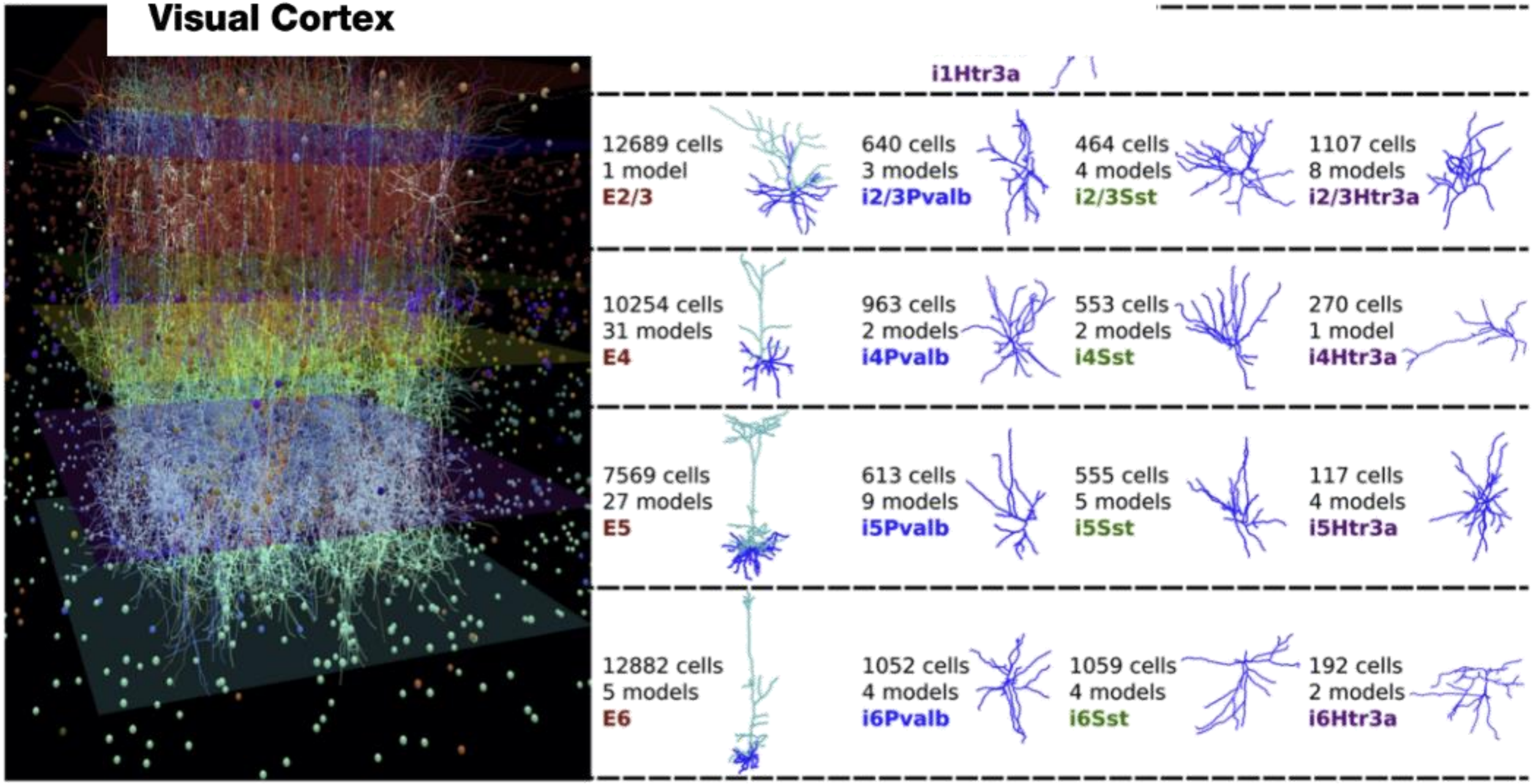
$$\begin{aligned}
\alpha_m(V) &= \frac{0.1(V+40)}{1-e^{-0.1(V+40)}}\ ; \quad \beta_m(V) = 4e^{-0.0556(V+65)}, \\
\alpha_h(V) &= 0.07e^{-0.05(V+65)}\ ; \quad \beta_h(V) = \frac{1}{1+e^{-0.1(V+35)}}, \\
\alpha_n(V) &= \frac{0.01(V+55)}{1-e^{-0.1(V+55)}}\ ; \quad \beta_n(V) = 0.125e^{-0.0125(V+65)}.
\end{aligned}$$



$$x\,(t + \Delta t) = x(t) + f\big(x(t)\big)\Delta t$$

$$\frac{dx}{dt} = f\big(x(t)\big)$$

Systematic Integration of Structural and Functional Data into Multi-scale Models of Mouse Primary Visual Cortex



Linear dynamical systems

$$\frac{d\vec{x}}{dt} = f(\vec{x})$$

$$f(\bar{x}) = 0$$

$$f(\vec{x})|_{x=\bar{x}} \approx 0 + A\vec{x} + \ \ldots$$

$$\frac{d\vec{x}}{dt} \approx A\vec{x}$$

Where you go is the sum of your parts!

$$\vec{x}\,(t) = a_1(t)\,\overrightarrow{v_1} + a_2(t)\,\overrightarrow{v_2} + \ldots$$

$$\frac{d\vec{x}}{dt} = A(a_1(t)\,\overrightarrow{v_1} + a_2(t)\,\overrightarrow{v_2} + \ldots) = A\,a_1(t)\,\overrightarrow{v_1} + A\,a_2(t)\,\overrightarrow{v_2} + \ \ldots$$

If $\{\overrightarrow{v_j}\}$ are eigenvectors of A, then $a_j(t) = e^{\lambda_j t}$

E.g., consider $x = \ a_1(t)\,\overrightarrow{v_1}$:

$$\frac{d\vec{x}}{dt} = \ A\,(a_1(t)\,\overrightarrow{v_1}) = a_1(t)\,\lambda_1\,\overrightarrow{v_1} = \frac{da_1(t)\,\overrightarrow{v_1}}{dt}$$

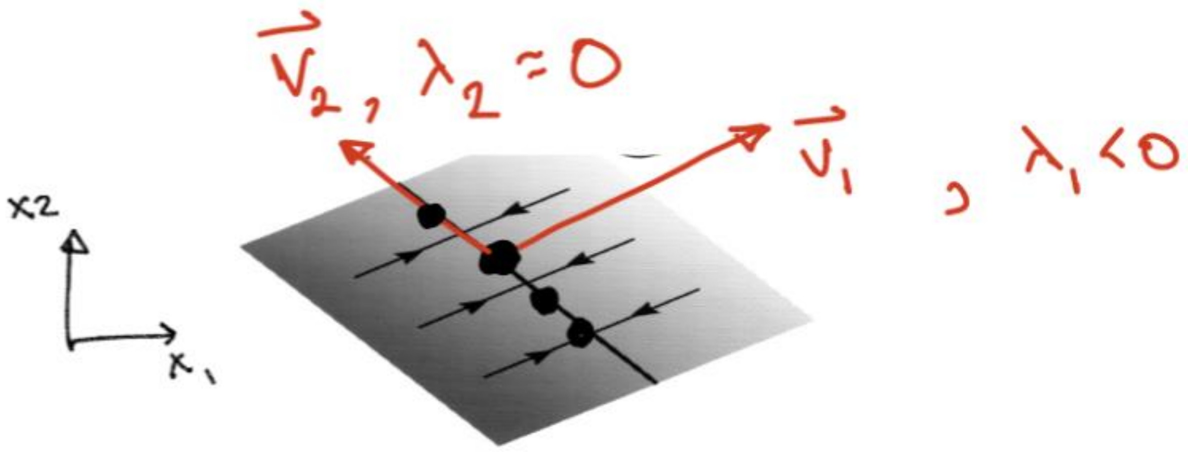
$$\frac{da_1(t)}{dt} = a_1(t)\,\lambda_1 \rightarrow a_1(t) = a_1(0)e^{\lambda_1 t}$$

Eigenvalues → timescales in neural networks

https://doi.org/10.1073/pnas.93.23.13339

How the brain keeps the eyes still

H. S. SEUNG



Eigenvalues → timescales in neural networks

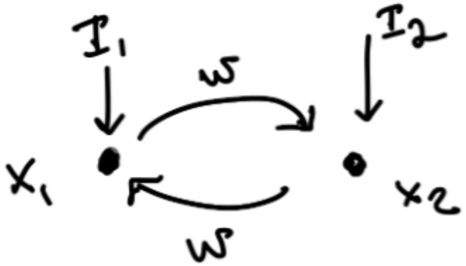
10.1037/0033-295x.108.3.550

The Time Course of Perceptual Choice:
The Leaky, Competing Accumulator Model

Marius Usher
Birkbeck College, University of London

James L. McClelland
Carnegie Mellon University and the Center for the Neural Basis of Cognition

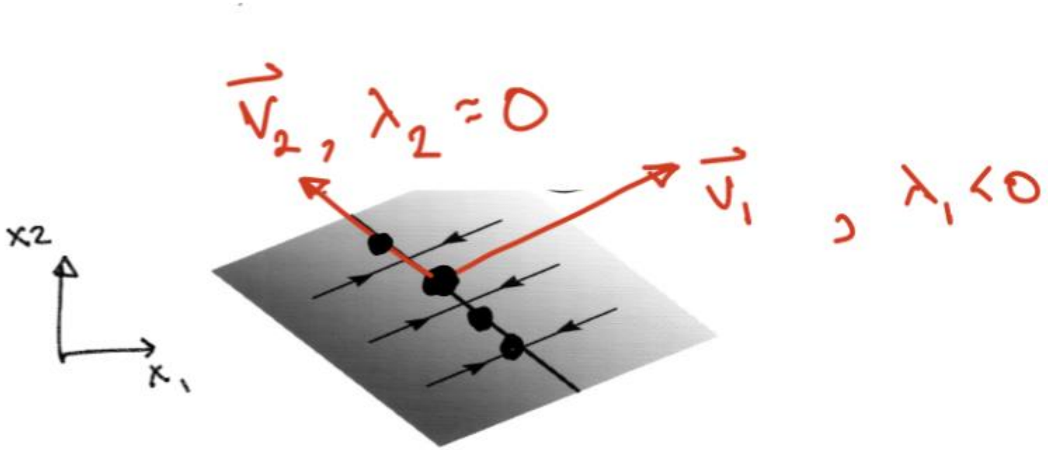
If w is large
https://doi.org/10.1146/annurev.neuro.29.051605.113038



$$\frac{dx_1}{dt} = -kx_1 \ - wx_2 + I_1$$

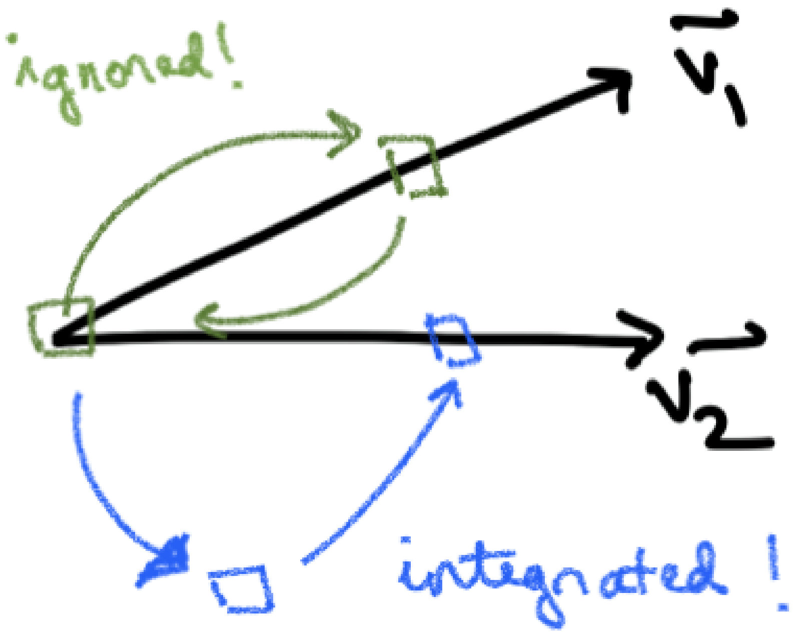
$$\frac{dx_2}{dt} = -kx_2 \ - wx_1 + I_2$$

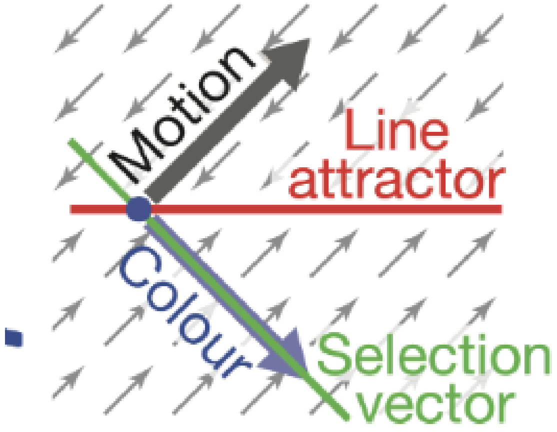
If $w = k$ →

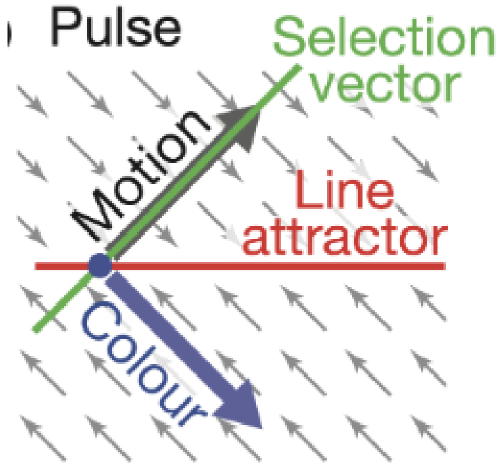


Non orthogonal eigenvectors

https://doi.org/10.1038/nature12742







Dynamics occurs that defy's the eigenvalues: **Non normal dynamics**

- Longer timescales
- Growth when λ's < 0

Where you go is the sum of your parts (2): response to time-dependent input

$$\frac{dx}{dt} = Ax + I(t)$$

$$\frac{dx_1}{dt} = Ax_1 + I_1(t) \qquad \frac{dx_2}{dt} = Ax_2 + I_2(t)$$

$$x(t) = x_1(t) + x_2(t)$$

$$\frac{dx}{dt} = Ax + I_1(t) + I_2(t)$$

$$= \sum_j a_j \delta_j(t)$$

All I need to know to characterize response is to know the response to a unitary input (**impulse response**)

$$x(t) = \int d\tau h_A(t-\tau) I(\tau)$$

Where you go is the sum of your parts (2): response to time-dependent input

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All I need to know to characterize response is to know the response to a unitary input (**impulse response**)

$$x(t) = \int d\tau h_A(t-\tau) I(\tau) \qquad x(\omega) = h(\omega) I(\omega)$$

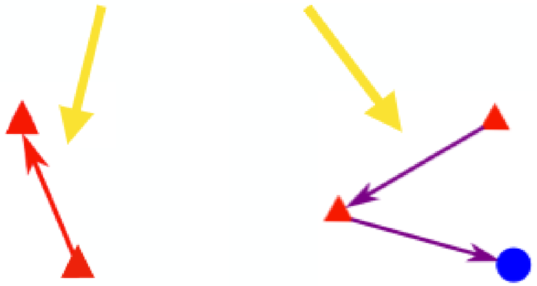
Networks of linear systems

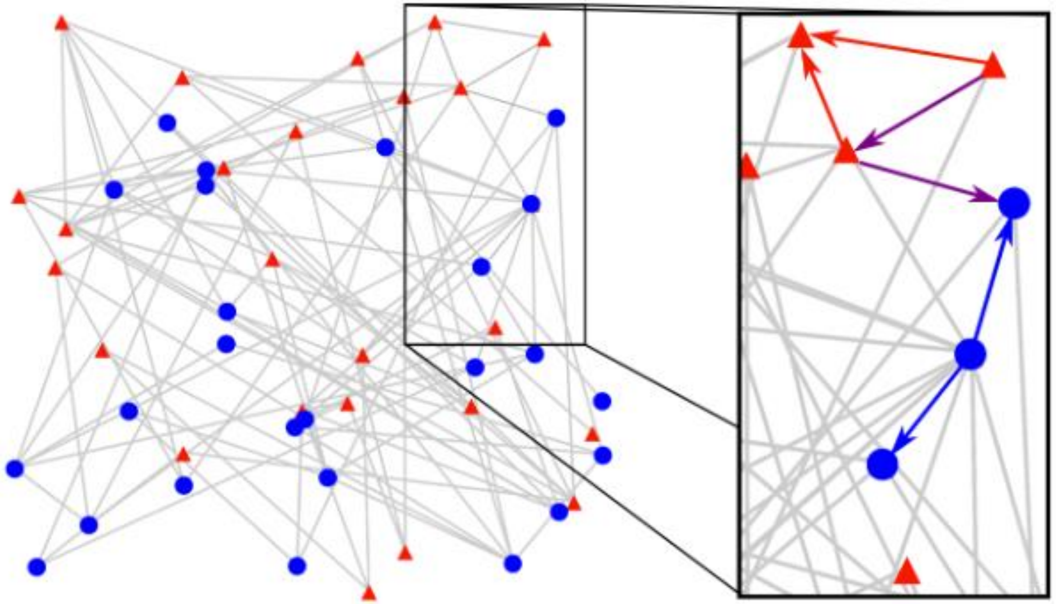
$$x_i(t) = \int d\tau h_A(t-\tau)\left[I(\tau) + \sum_{ij} W_{ij} x_j\,(t)\right]$$

$$\vec{x}(\omega) = h(\omega)[\vec{I}(\omega) + W\vec{x}(\omega)]$$

$$\vec{x}(\omega) = [Id\ - h(\omega)W]^{-1} h(\omega)\vec{I}(\omega)$$

$$\vec{x}(\omega) = [Id + hW + h^2W^2 + \cdots]h(\omega)\vec{I}(\omega)$$

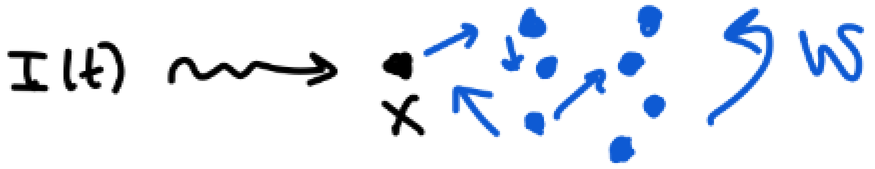




$$dynamics = f(W) = f(motifs)$$

Stochastic linear dynamical systems

$$x(t) = \int d\tau h_A(t-\tau) I(\tau)$$



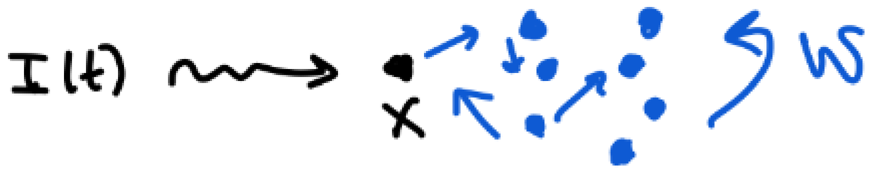
If I_t (and $x(0)$) are jointly gaussian then x is gaussian

Explicit formula for covariance!

$$\vec{x}(\omega) = [Id \; - h(\omega)W]^{-1} h(\omega)\vec{I}(\omega)$$

$$Cov\left(\vec{x_i}(\omega)\vec{x_j}(\omega)\right) \sim [Id \; - hW]^{-1} h Cov(I) h [Id \; - hW]^{-T} = f(W)$$

Stochastic linear dynamical systems



$\boldsymbol{Cov = f(W)}$**, then:**

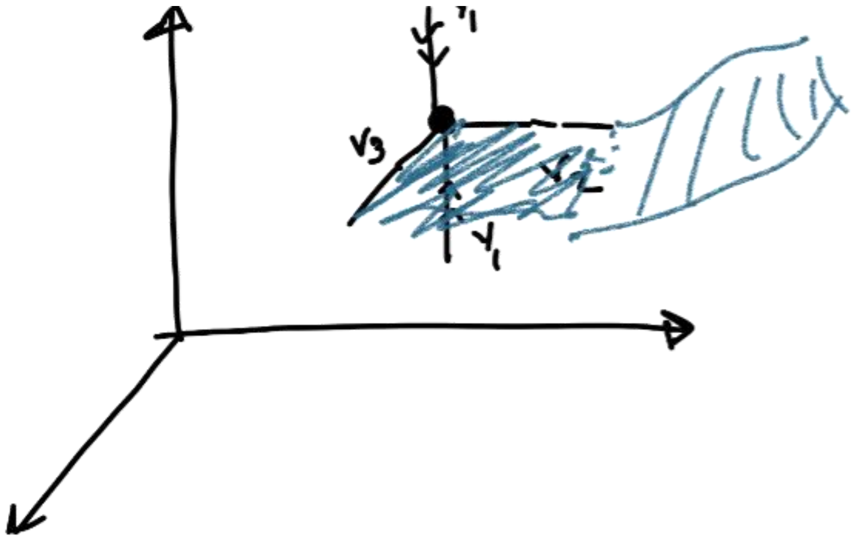
- $Cov = f(motifs)$
- $Dimension = f(W) = f(motifs)$
- $W \rightarrow W + \Delta W$

LEARNING

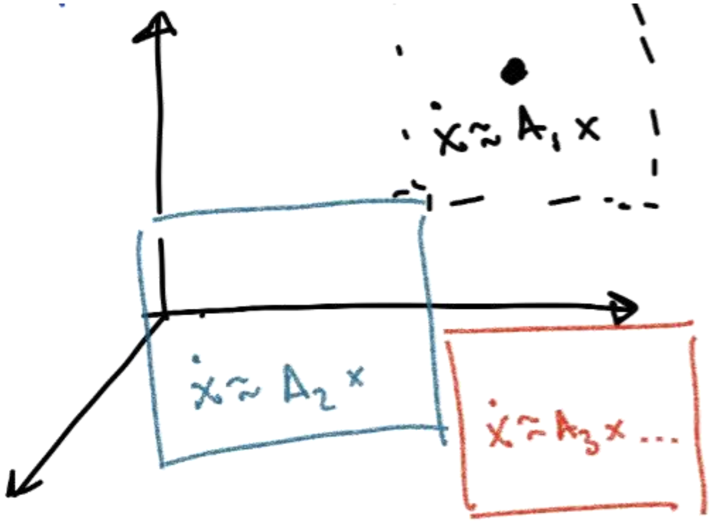
Say $\Delta W = f(Cov)$ (as in STDP),
then $\Delta W = f(Cov) = f(g(W))$

Closed form dynamical system also for the connectivity!

Beyond linear dynamical systems



Stable manifold (low-D non linear dynamics)



Piecewise linear dynamical systems

Tutorial 1: deterministic 1 and 2D linear systems

- Simulate a 1D deterministic linear system $$\frac{dx}{dt} = ax$$

Analytical solution:

$$x(t) = x(0)e^{at}$$

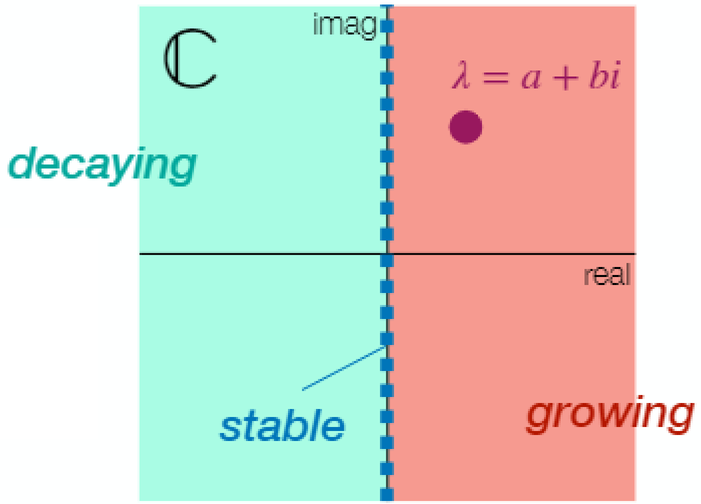
- If $a < 0$: exponential decay
- If $a = 0$: nothing changes
- If $a > 0$: exponential growth

Tutorial 1: deterministic 1 and 2D linear systems

$$\frac{dx}{dt} = \lambda x$$

If λ is a complex number: $x(t) = x(0)e^{\lambda t} = x(0)e^{(a+ib)t} = x(0)e^{at}[\cos(bt) + i\sin(bt)\ ]$

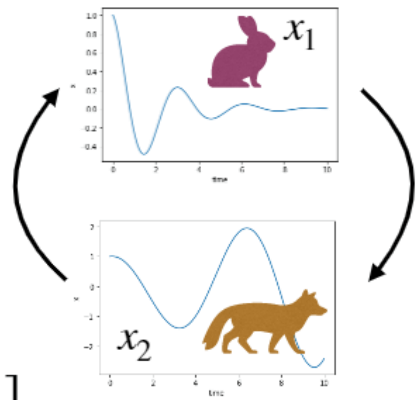
$b \sim$ frequency of oscillation



Tutorial 1: deterministic 1 and 2D linear systems

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2$$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

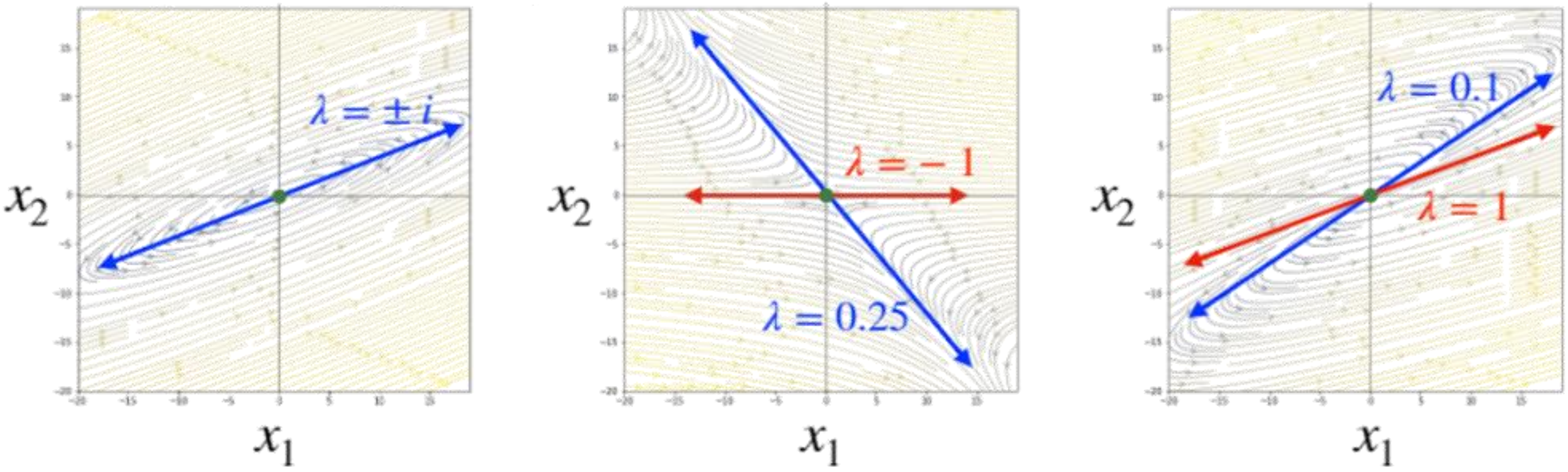


https://www.youtube.com/watch?v=PFDu9oVAE-g

$$\dot{\vec{x}} = A\,\vec{x}$$

$$\boldsymbol{x}(t) = \boldsymbol{v}_1\boldsymbol{x}_0\boldsymbol{v}_1^T e^{\lambda_1 t} + \boldsymbol{v}_2\boldsymbol{x}_0\boldsymbol{v}_2^T e^{\lambda_2 t}$$

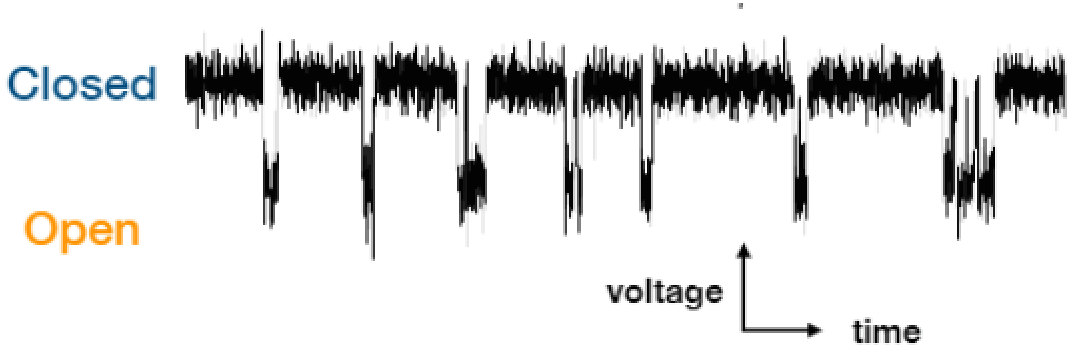
Behavior of the system depends on eigenvalues and eigevectors of A

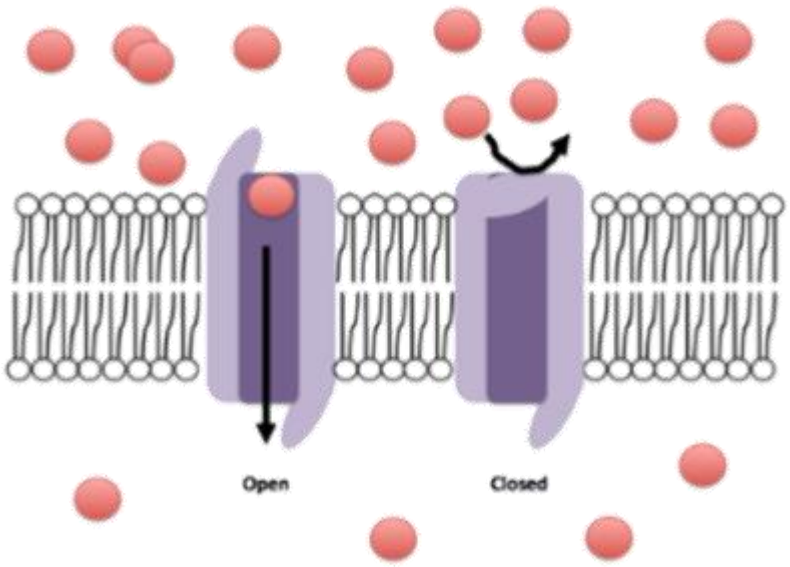


Tutorial 2: Markovian dynamical system

A system is markovian if the **present** state determines the probability of transitions to the next state

Opening and closing of ion channels as random events





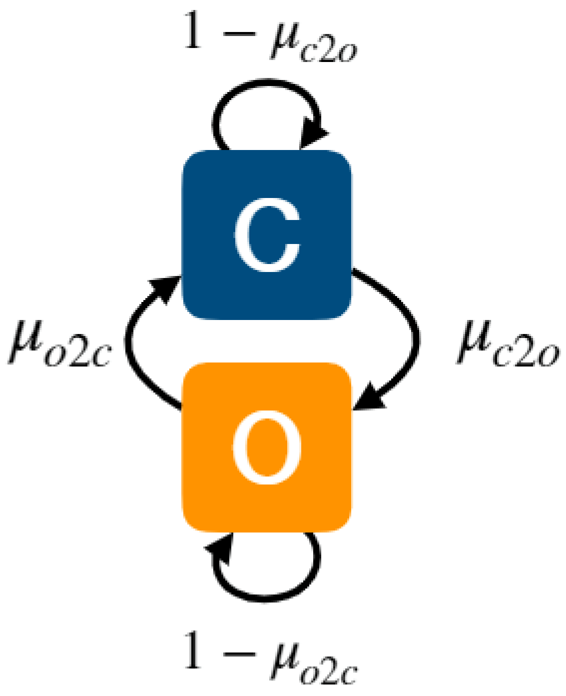
A telegraph process

Transition from open to close: P(o2c)

Transition from close to open: P(c2o)

Tutorial 2: Markovian dynamical system

Instead of keeping track of single channels and perform many simulations, now keep track of the probability state



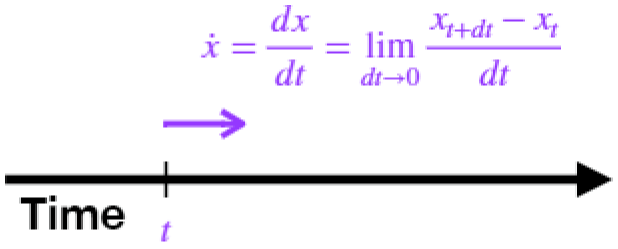
State vector at time k: probability of being in state O or C

$$\begin{bmatrix} C \\ O \end{bmatrix}_{k+1} = \mathbf{A} \begin{bmatrix} C \\ O \end{bmatrix}_k = \begin{bmatrix} 1-\mu_{c2o} & \mu_{o2c} \\ \mu_{c2o} & 1-\mu_{o2c} \end{bmatrix} \begin{bmatrix} C \\ O \end{bmatrix}_k$$

Tutorial 2: Markovian dynamical system

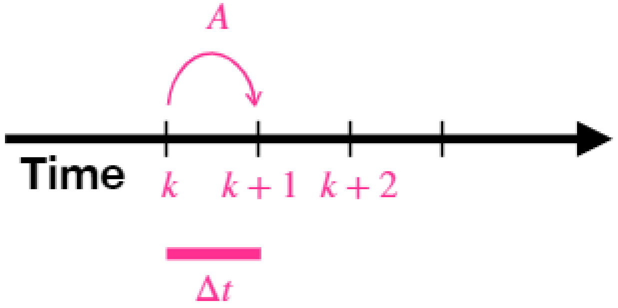
Continuous Time Formulation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$



Discrete Time Formulation

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$



Stable solution, when things don't change

$$\mathbf{A} = 0$$

$$\mathbf{A} = 1$$

Tutorial 2: Markovian dynamical system

Continuous Time Formulation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

Discrete Time Formulation

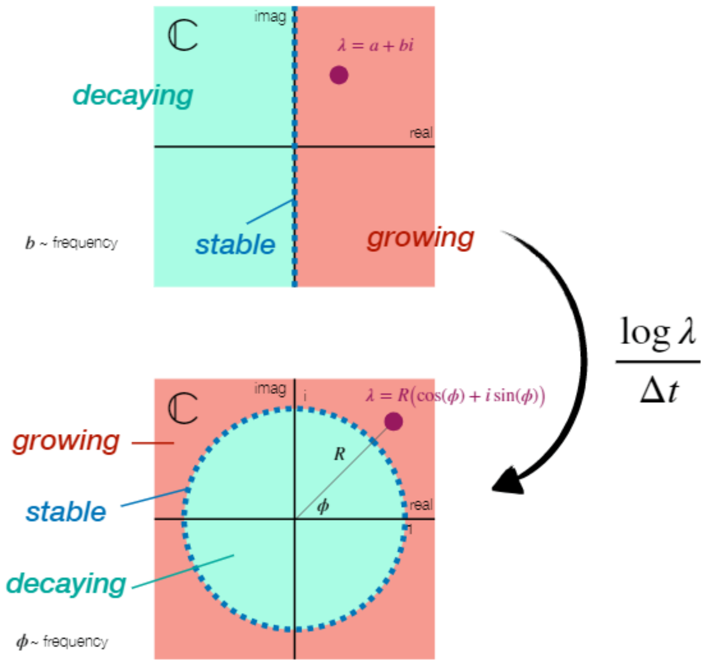
$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$$

Stable solution, when things don't change

$$\mathbf{A} = 0$$

$$\mathbf{A} = 1$$

Eigenvalue Spectrum of A



- Identify the stable eigenvalue of **A** and its corresponding eigenvector
- Compare the equilibrium distribution solutions with the eigenvalues and eigenvectors of **A** matrix