Visualization Homework Assignment 1

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1 Tasks

Question 1.1

Curvilinear Grid around a Cylinder

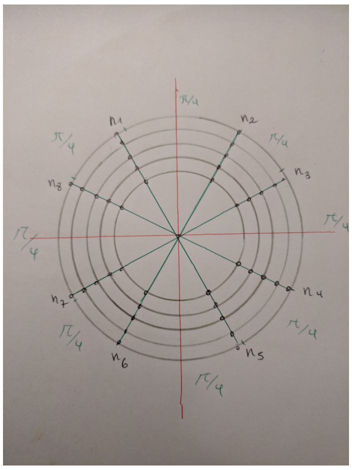

Figure 1: Top view of the curvilinear grid

Figure 1 shows the top view from the Z-axis. We know that $n_z = 8$ and that the angle varies from $0°$ degree to $360° = 2\pi$.

$$\phi = \frac{2\pi}{n_z} = \frac{2\pi}{8} = \frac{\pi}{4} = 45°$$

Furthermore we know that the radius r is equal to 5 and that the grid's thickness is 5, and $n_r = 5$, the distance between each cylinder grid is equal to $\frac{thickness}{n_r} = \frac{5}{5} = 1$. The radius variation can be then described as $R = 5 + j$ where $1 \leq j \leq 5$.

From the insights mentioned above, the circular grid on XY-axis can be general described as :

$$\begin{cases} X = R\cos\frac{\pi}{4}i \\ Y = R\sin\frac{\pi}{4}i \end{cases}$$

$\begin{cases} X = (5+j)\cos\frac{\pi}{4}i \\ Y = (5+j)\sin\frac{\pi}{4}i \end{cases}$
where $1 \leq j \leq 5$ and $1 \leq i \leq 8$
Taking into account the Z-axis, the general description of the grid is :
$\begin{cases} X = (5+j)\cos\frac{\pi}{4}i \\ Y = (5+j)\sin\frac{\pi}{4}i \\ Z = k \end{cases}$
where $1 \leq j \leq 5$, $1 \leq i \leq 8$ and $1 \leq k \leq 20$

Question 1.2

Order Independence of Bi-linear Interpolation

Using image[1] (2) :

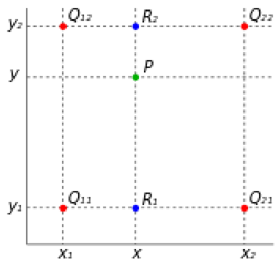

Figure 2: Bi-linear interpolation

For deriving the bilinear interpolation we need to apply the linear interpolation three times either two times in the x-axis and one time in the y-axis or two times in the y-axis first and one time in the x-axis

- Two times in the x-axis and one time in the y-axis

$$f(x, y_1) \approx \frac{x_2 - x}{x_2 - x_1} f(Q_{11}) + \frac{x - x_1}{x_2 - x_1} f(Q_{21})$$

$$f(x, y_2) \approx \frac{x_2 - x}{x_2 - x_1} f(Q_{12}) + \frac{x - x_1}{x_2 - x_1} f(Q_{22})$$

$$f(x, y) \approx \frac{y_2 - y}{y_2 - y_1} f(x, y_1) + \frac{y - y_1}{y_2 - y_1} f(x, y_2)$$

$$f(x, y) \approx \frac{y_2 - y}{y_2 - y_1} \left[\frac{x_2 - x}{x_2 - x_1} f(Q_{11}) + \frac{x - x_1}{x_2 - x_1} f(Q_{21}) \right] + \frac{y - y_1}{y_2 - y_1} \left[\frac{x_2 - x}{x_2 - x_1} f(Q_{12}) + \frac{x - x_1}{x_2 - x_1} f(Q_{22}) \right]$$

[1] Image from ”en.wikipedia.org/wiki/Bilinear$_{i}nterpolation$”

$$f(x,y) \approx \tau \Big[(x_2-x)(y_2-y)f(Q_{11})+(x-x_1)(y_2-y)f(Q_{21})+(x_2-x)(y-y_1)f(Q_{12})+(x-x_1)(y-y_1)f(Q_{22})\Big] \tag{1}$$

where

$$\tau = \frac{1}{(x_2-x_1)(y_2-y_1)}$$

- Two times in the y-axis and one time in the x-axis

$$f(x_1,y) \approx \frac{y_2-y}{y_2-y_1}f(Q_{11}) + \frac{y-y_1}{y_2-y_1}f(Q_{12})$$

$$f(x_2,y) \approx \frac{y_2-y}{y_2-y_1}f(Q_{21}) + \frac{y-y_1}{y_2-y_1}f(Q_{22})$$

$$f(x,y) \approx \frac{x_2-x}{x_2-x_1}f(x_1,y) + \frac{x-x_1}{x_2-x_1}f(x_2,y)$$

$$f(x,y) \approx \frac{x_2-x}{x_2-x_1}\left[\frac{y_2-y}{y_2-y_1}f(Q_{11})+\frac{y-y_1}{y_2-y_1}f(Q_{12})\right]+\frac{x-x_1}{x_2-x_1}\left[\frac{y_2-y}{y_2-y_1}f(Q_{21})+\frac{y-y_1}{y_2-y_1}f(Q_{22})\right]$$

$$f(x,y) \approx \tau \Big[(x_2-x)(y_2-y)f(Q_{11})+(x-x_1)(y_2-y)f(Q_{21})+(x_2-x)(y-y_1)f(Q_{12})+(x-x_1)(y-y_1)f(Q_{22})\Big] \tag{2}$$

where

$$\tau = \frac{1}{(x_2-x_1)(y_2-y_1)}$$

We see that equations (1) and (2) is similarly the same therefore the result of the bi-linear interpolation is independent of the axis by with the linear interpolation is first applied to.

Question 1.3

Bilinear Interpolation in the Unit Square

The general relation for bi-linear interpolation in 2D is :

$$f(x,y) = \begin{bmatrix} \frac{b-x}{b-a} & \frac{x-a}{b-a} \end{bmatrix} \begin{bmatrix} z_{a,c} & z_{a,d} \\ z_{b,c} & z_{b,d} \end{bmatrix} \begin{bmatrix} \frac{d-y}{d-c} \\ \frac{y-c}{d-c} \end{bmatrix}$$

where $z_{i,j}$ is the value function at a specific point of (x,y).
(a) In our example the function value is :

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

The weights on the X-side is :

$$\begin{bmatrix} \frac{1-\frac{1}{2}}{1-0} & \frac{\frac{1}{2}-0}{1-0} \end{bmatrix}$$

The weights on the Y-side is :

$$\begin{bmatrix} \frac{1-\frac{1}{2}}{1-0} \\ \frac{\frac{1}{2}-0}{1-0} \end{bmatrix}$$

then :

$$f(\frac{1}{2},\frac{1}{2}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$f(\frac{1}{2},\frac{1}{2}) = \frac{5}{2}$$

(b) Now we need to calculate the gradient of the f(x,y) interpolation function.

$$\frac{\partial f(x,y)}{\partial x} = \begin{bmatrix} \frac{-1}{b-a} & \frac{1}{b-a} \end{bmatrix} \begin{bmatrix} z_{a,c} & z_{a,d} \\ z_{b,c} & z_{b,d} \end{bmatrix} \begin{bmatrix} \frac{d-y}{d-c} \\ \frac{y-c}{d-c} \end{bmatrix}$$

$$\frac{\partial f(x,y)}{\partial y} = \begin{bmatrix} \frac{b-x}{b-a} & \frac{x-a}{b-a} \end{bmatrix} \begin{bmatrix} z_{a,c} & z_{a,d} \\ z_{b,c} & z_{b,d} \end{bmatrix} \begin{bmatrix} \frac{-1}{d-c} \\ \frac{1}{d-c} \end{bmatrix}$$

$$\nabla f(x,y) = \frac{\partial f(x,y)}{\partial x} i + \frac{\partial f(x,y)}{\partial y} j$$

(c)

$$\nabla f(\frac{1}{2},\frac{1}{2}) = \frac{\partial f(\frac{1}{2},\frac{1}{2})}{\partial x} i + \frac{\partial f(\frac{1}{2},\frac{1}{2})}{\partial y} j$$

$$\frac{\partial f(\frac{1}{2},\frac{1}{2})}{\partial x} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\frac{\partial f(\frac{1}{2},\frac{1}{2})}{\partial y} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix} \begin{bmatrix}3 & 2\\ 2 & 3\end{bmatrix} \begin{bmatrix}-1\\ 1\end{bmatrix}$$

Then

$$\frac{\partial f(\frac{1}{2},\frac{1}{2})}{\partial x} = 0$$

$$\frac{\partial f(\frac{1}{2},\frac{1}{2})}{\partial y} = 0$$

The gradient is :

$$\nabla f(\frac{1}{2},\frac{1}{2}) = 0i + 0j$$

Question 1.4

Bi-linear Interpolation

The Bi-linear Interpolation follow is as follow :

$$f(x,y) = \begin{bmatrix}\frac{b-x}{b-a} & \frac{x-a}{b-a}\end{bmatrix} \begin{bmatrix}z_{a,c} & z_{a,d}\\ z_{b,c} & z_{b,d}\end{bmatrix} \begin{bmatrix}\frac{d-y}{d-c}\\ \frac{y-c}{d-c}\end{bmatrix}$$

in our case

$$f(x,y) = \begin{bmatrix}\frac{b-x}{b-a} & \frac{x-a}{b-a}\end{bmatrix} \begin{bmatrix}f_{0,0} & f_{0,1}\\ f_{2,0} & f_{2,1}\end{bmatrix} \begin{bmatrix}\frac{d-y}{d-c}\\ \frac{y-c}{d-c}\end{bmatrix}$$

$$f(x,y) = \begin{bmatrix}\frac{b-x}{b-a} & \frac{x-a}{b-a}\end{bmatrix} \begin{bmatrix}f_{0,0} & f_{0,1}\\ f_{2,0} & f_{2,1}\end{bmatrix} \begin{bmatrix}\frac{d-y}{d-c}\\ \frac{y-c}{d-c}\end{bmatrix}$$

$$f(1,\frac{1}{2}) = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix} \begin{bmatrix}1 & 5\\ 3 & 7\end{bmatrix} \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

$$f(1,\frac{1}{2}) = 4$$

Question 1.5

Interpolation

We have 3 points with their corresponding function values :

- $f(0,0) = 1$
- $f(1,0) = 2$
- $f(0,1) = 3$

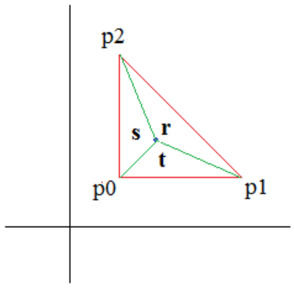

Figure 3: Triangle

(a) We can divide the area in three different parts, where we can define the barycentric value of the middle point of the triangle figure 3 as follows :

$$f(x,y) = r * f(0,0) + s * f(1,0) + t * f(0,1)$$

where $r + s + t = 1$ and $0 <= (r,s,t) <= 1$

The point P, is unknown (x,y). the value of r,s,t can be calculated by the determinants as follow :

$$t = \frac{area(f_{0,0}, f_{1,0}, f_{x,y})}{area(f_{0,0}, f_{1,0}, f_{0,1})} = \frac{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ x & y & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = y$$

$$r = \frac{area(f_{0,0}, f_{1,0}, f_{x,y})}{area(f_{0,0}, f_{1,0}, f_{0,1})} = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ x & y & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = 1 - y - x$$

$$s = \frac{area(f_{0,0}, f_{1,0}, f_{x,y})}{area(f_{0,0}, f_{1,0}, f_{0,1})} = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ x & y & 1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = x$$

We have :

$r + s + t = (y) + (1 - y - x) + (x) = 1$

then the barycentric linear interpolation becomes :

$$f(x,y) = (1 - y - x) * f(0,0) + (x) * f(1,0) + y * f(0,1)$$

(b) The Shepard interpolation in our case (p = 2) is defined by :

$$f(x, y) = \sum_{k=1}^{n} \frac{\omega_k(x, y)}{\sum_{j=1}^{n} \omega_j(x, y)} f_k$$

where

$$\omega_k(x, y) = \frac{1}{\sqrt{(x - x_k)^2 + (y - y_k)^2}}$$

the function f(x,y) can be described as :

$$f(x, y) = \frac{\omega_{0,0}(x, y)}{\zeta} f_{0,0} + \frac{\omega_{1,0}(x, y)}{\zeta} f_{1,0} + \frac{\omega_{0,1}(x, y)}{\zeta} f_{0,1}$$

where

$$\omega_{0,0}(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

$$\omega_{1,0}(x, y) = \frac{1}{\sqrt{(x - 1)^2 + y^2}}$$

$$\omega_{0,1}(x, y) = \frac{1}{\sqrt{x^2 + (y - 1)^2}}$$

and

$$\zeta = \frac{1}{\sqrt{x^2 + y^2}} + \frac{1}{\sqrt{(x - 1)^2 + y^2}} + \frac{1}{\sqrt{x^2 + (y - 1)^2}}$$

(c) The interpolation for the barycentric interpolation is :

- $f(1/2, 0) = 1.5$
- $f(0, 1/2) = 2.0$
- $f(1/2, 1/2) = 2.5$
- $f(2/3, 2/3) = 3.0$

The interpolation for the shepard interpolation is :

- $Shepard(1/2, 0) = 1.774$
- $Shepard(0, 1/2) = 2.0$
- $Shepard(1/2, 1/2) = 2.0$
- $Shepard(2/3, 2/3) = 2.075$

References