Tutorial 5: Part 2

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8.1 T test words in background noise $\underline{x} = (x_1, \dots, x_T)$

$x_t = 1$ correct

$x_t = 0$ incorrect

$z = \sum_{t=1}^{T} x_t \quad P(x_t = 1) = \omega$

iid.

8.1a $\hat{\omega}_{ML}$?

$$P(Z = z) = \binom{T}{z} \omega^z (1-\omega)^{T-z}$$

$$\max_{\omega} \log\left(\binom{T}{z} \omega^z (1-\omega)^{T-z}\right)$$

$$\max_{\omega} z \log(\omega) + (T-z)\log(1-\omega)$$

$$\Rightarrow \frac{z}{\omega} - \frac{(T-z)}{1-\omega} = 0$$

$$z(1-\omega) - (T-z)\omega = 0$$

$$z - T\omega = 0 \qquad \hat{\omega}_{ML} := \frac{z}{T}$$

$$\boxed{\hat{\omega}_{ML} = \frac{\sum_{t=1}^{T} x_t}{T}}$$

- $\dfrac{d\log(\omega)}{d\omega} = \dfrac{1}{\omega}$
- $\dfrac{\partial^2 \log(\omega)}{\partial \omega^2} = -\dfrac{1}{\omega^2}$ ✓
- $\dfrac{d\log(1-\omega)}{d\omega} = \dfrac{-1}{1-\omega}$
- $\dfrac{d^2\log(1-\omega)}{\partial \omega^2} = \dfrac{-(-1)(-1)}{(1-\omega)^2}$ ✓

... a single one of them.

8.1.b $f_W(w)$ consider a single one of them.

$$f_W(w) \propto \sqrt{I(w)}$$

$$I(w) = -E_X\left[\frac{\partial^2 \ln f_{X|W}(x|w)}{\partial^2 w}\right]$$

$P(X=1|w) = w$

$P(X=0|w) = 1-w.$

$$I(w) = -P(X=1|w)\frac{\partial^2 \ln(w)}{\partial w^2} - P(X=0|w)\frac{\partial^2 \ln(1-w)}{\partial w^2} =$$

$$= +w\frac{1}{w^2} + (1-w)\cdot\frac{1}{(1-w)^2} = \frac{1}{w} + \frac{1}{1-w} = \frac{1-w+w}{w(1-w)} = \frac{1}{w(1-w)}$$

$$I(w) = \frac{1}{\sqrt{w(1-w)}}$$

$$\Rightarrow f_W(w) \propto \frac{1}{\sqrt[4]{(w)(1-w)}}$$

8.3 $\underline{X} = (X_1, \dots, X_T)$

$X_0 = 0$ $\qquad W_t \sim N(0,1)$ iid

$X_t = aX_{t-1} + cW_t \quad t \geq 1 \qquad A_0 \sim N(\mu_0, \sigma_0^2)$

$x_t = aX_{t-1} + cW_t \quad t \geq 1$ $\qquad$ $A_0 \sim N(\mu_0, \sigma_0^2)$

$$f_{A|\underline{X}}(a|\underline{x}) \propto f(\underline{x}|a) f(a)$$

$$f(\underline{x}_1^T|a) = f(x^T|\underline{x}_1^{T-1}; a)\, f(\underline{x}_1^{T-1}|a) = f(x^T|x^{T-1}; a)\, f(x_1^{T-1}|a)$$

$$= \prod_{i=2}^{T} f(x^i|x^{i-1}, a) \cdot f_W(x_1)$$

It is clear that $f_W(x_1) f(a)$ is proportional to a Gaussian in a.

If multiplying an arbitrary Gaussian in a by $f(x^i|x^{i-1}, a)$ remains Gaussian, by induction $f(\underline{x}_1^T|a) f(a)$ is Gaussian $\forall T$.

$$f(x^i|x^{i-1}, a) \propto e^{-\frac{(x^i - ax^{i-1})^2}{2c^2}}$$

$$\prod_{j=2}^{i-1} f(x^j|x^{j-1}, a) f(x_1) f(a) \propto e^{-\frac{(a-\mu)^2}{2\sigma^2}}$$

$$e^{-(a-\mu)^2/2\sigma^2} \quad e^{-\frac{(x^i - ax^{i-1})^2}{2c}} \quad \propto$$

$$e^{-\frac{\left(\boxed{x^i/x^{i-1}}-a\right)^2}{2\boxed{c^2/(x^{i-1})^2}}}\, e^{-(a-\mu)^2/2\sigma^2}$$

$$\propto e^{-\left(\frac{a^2\sigma^2 - 2x^i/x^{i-1}\,a\sigma^2 + a^2 c^2/(x^{i-1})^2 - 2\mu\frac{c^2}{(x^{i-1})^2}a}{2(\sigma^2)\,c^2/(x^{i-1})^2}\right)}$$

divide by this num and denom.

$\triangleq \alpha = \sigma^2 + c^2/(x^{i-1})^2$

$\beta \triangleq \frac{x^i}{\alpha x^{i-1}}\sigma^2 + \frac{\mu c^2}{(x^{i-1})^2\alpha}$

$$e^{-\left(a^2 - \frac{2x_i}{\alpha x^{i-1}}a\sigma^2 - \frac{2\mu c^2}{(x^{i-1})^2\alpha}a\right)}$$

$$\propto e^{\quad} \qquad e^{-\beta^2}$$

$$\propto e^{-(a^2-\beta)^2}$$

and therefore it is proportional to a Gaussian in $\underline{a}$.

We can conclude then that

$f_{A|X}$ is Gaussian and is the scaled version of the product of a gaussian $\sim \mathcal{N}(\mu_0, \sigma_0^2)$

a gaussian $\sim \mathcal{N}(\mu_0, \ldots)$

and $t = 2, \ldots, T$ s.t.

$$\sim \mathcal{N}\left(x^t / x^{t-1}, \frac{c^2}{(x^{t-1})^2}\right)$$

Un-normalized Gaussian resulting of the product of N Gaussians:

$$\sigma^2 = \left(\sum_{i=1}^{N} \sigma_i^{-2}\right)^{-1}$$

$$\mu = \sigma^2 \sum_{i=1}^{N} \sigma_i^{-2} \mu_i$$

$$\sigma_T^2 = \frac{1}{\frac{1}{\sigma_0^2} + \sum_{t=1}^{T-1} \frac{(x_t)^2}{c^2}} = \boxed{\frac{c^2}{\frac{c^2}{\sigma_0^2} + \sum_{t=1}^{T-1} (x_t)^2}}$$

$$\mu_T = \frac{c^2}{\frac{c^2}{\sigma_0^2} + \sum_{t=1}^{T-1} (x_t)^2} \left(\frac{\mu_0}{\sigma_0^2} + \sum_{t=1}^{T} \frac{(x^{t-1})^2}{c^2} \frac{x^t}{x^{t-1}} \right) = \boxed{\frac{\left(c^2 \mu_0 / \sigma_0^2 + \sum_{t=1}^{T} x^{t-1} x^t \right)}{c^2/\sigma_0^2 + \sum_{t=1}^{T-1} (x_t)^2}}$$

8.3.b μ_T and variance σ_T^2

as a recursive update

as a recursive update

$$\sigma_{T-1}^2 = \frac{c^2}{\frac{c^2}{\sigma_0^2} + \sum_{t=1}^{T-2} (x_t)^2} \qquad \sigma_T^2 = \frac{c^2}{c^2/\sigma_0^2 + \sum_{t=1}^{T-1} (x_t)^2}$$

$$\sigma_{T-1}^{-2} = \frac{c^2/\sigma_0^2 + \sum_{t=1}^{T-2} (x_t)^2}{c^2} \qquad \sigma_T^{-2} = \frac{c^2/\sigma_0^2 + \sum_{t=1}^{T-1} (x_t)^2}{c^2}$$

$$\boxed{\sigma_T^{-2} = \sigma_{T-1}^{-2} + \frac{(x_{T-1})^2}{c^2}}$$

$$\mu_T = \sigma_T^2 \left(\mu_0/\sigma_0^2 + \sum_{t=1}^{T} x^{t-1} x^t \right)$$

$$\mu_{T-1} = \sigma_{T-1}^2 \left(\mu_0/\sigma_0^2 + \sum_{t=1}^{T-1} x^{t-1} x^t \right)$$

$$\boxed{\mu_T = \frac{\sigma_T^2}{\sigma_{T-1}^2} \mu_T + \sigma_T^2 X^{T-1} X^T}$$

8.3.C. Mean and variance of next future sample X_{T+1} given $(x_1, \dots, x_T)$ including uncertainty effect A.

given $(x_1, \dots, x_T)$ including uncertainty ...

$f_{A|X}$ is fully characterized.

$$f_{X_{T+1}|X_T, A} = \int \underbrace{f_{X^{t+1}/X_T,A}(x|..)}_{\text{gaussian}} \underbrace{f_{A|\underline{X}_1^T}(a|\underline{X}_1^T)}_{\text{gaussian}}\, da$$

with mean aX_T — with mean μ_T

variance c^2 — variance σ_T^2

$$\propto \int e^{-\frac{(x-ax_T)^2}{2c^2}} e^{-\frac{(a-\mu_T)^2}{2\sigma_T^2}}\, da$$

$$\propto \int e^{-\frac{(x/x_T - a)^2}{2c^2/x_T^2}} e^{-\frac{(a-\mu_T)^2}{2\sigma_T^2}}\, da \qquad \propto e^{-\left(\frac{(x/x_T - \mu_T)^2}{2(c^2/x_T^2+\sigma_T^2)}\right)}$$

$f(\tau - a)\, g(a)$

0+μ_T

sum of variances.

Gaussian convolution

$$e^{-\left(\frac{(x - x_T\mu_T)^2}{2(c^2+x_T^2\sigma_T^2)}\right)}$$

e

mean $X_T \mu T$

Variance $c^2 + X_T^2 \sigma_T^2$