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Isoperimetric Numbers of Graphs

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For $X \subseteq V(G)$, let ∂X denote the set of edges of the graph G having one end in X and the other end in $V(G) \backslash X$. The quantity $i(G) := \min\{|\partial X|/|X|\}$, where the minimum is taken over all non-empty subsets X of $V(G)$ with $|X| \leqslant |V(G)|/2$, is called the *isoperimetric number* of G. The basic properties of $i(G)$ are discussed. Some upper and lower bounds on $i(G)$ are derived, one in terms of $|V(G)|$ and $|E(G)|$ and two depending on the second smallest eigenvalue of the difference Laplacian matrix of G. The upper bound is a strong discrete version of the well-known Cheeger inequality bounding the first eigenvalue of a Riemannian manifold. The growth and the diameter of a graph G are related to $i(G)$. The isoperimetric number of Cartesian products of graphs is studied. Finally, regular graphs of fixed degree with large isoperimetric number are considered.

1. INTRODUCTION

Let G be a finite graph. If $X \subseteq V(G)$ is a set of vertices then ∂X denotes the set of edges of G having one end in X and the other end in $V(G) \backslash X$. The quantity

$$i(G) = \min_X \frac{|\partial X|}{|X|}, \tag{1.1}$$

where the minimum is taken over all non-empty subsets X of $V(G)$ satisfying $|X| \leqslant \frac{1}{2}|V(G)|$, is called the *isoperimetric number* of G. Clearly, $i(G)$ can be defined in a more symmetric form as

$$i(G) = \min \frac{|E(X, Y)|}{\min\{|X|, |Y|\}}, \tag{1.2}$$

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where the minimum runs over all partitions of $V(G) = X \cup Y$ into non-empty subsets X and Y, and $E(X, Y) = \partial X = \partial Y$ are the edges between X and Y.

The importance of $i(G)$ lies in various interesting interpretations of this number:

(a) From (1.2) it is evident that, in trying to determine $i(G)$, we have to find a small edge-cut $E(X, Y)$ separating as large a subset X (assume $|X| \leqslant |Y|$) as possible from the remaining larger part Y. So, it is evident that $i(G)$ can serve as a measure of *connectivity* of graphs. It seems that there might be possible applications in problems concerning connected networks and the ways to "destroy" them by removing a large portion of the network by cutting only a few edges.

(b) The problem of partitioning $V(G)$ into two equally sized subsets (to within one element) in such a way that the number of the edges in the cut is minimal, is known as the *bisection width* problem. It is important in VLSI design and some other practical applications (see [15] and the references therein). Clearly, it is related to the isoperimetric number.

(c) A large isoperimetric number indicates that G has a large growth rate. More precisely, if G has maximal vertex degree Δ and $B_k(v)$ is the set of vertices of G at distance at most k from v, then $|B_{k+1}(v)|/|B_k(v)| \geqslant i(G)/\Delta$ provided that $|B_k(v)| \leqslant |V(G)|/2$. So, $i(G)$ describes the expanding properties of G. A related quantity $i_V(G)$ (called the *magnifying constant* of G) is obtained by replacing, in (1.1), $|\partial X|$ by the number of vertices of $V(G)\backslash X$ which are adjacent to X. This quantity was studied by several authors, for example by Alon and Milman [1, 2] in relation to various appplications concerning the so-called expanders and superconcentrators. This study involves "magnifiers," graphs G with large $i_V(G)$. It is clear that graphs with large isoperimetric number are also good magnifiers since $i_V(G) \leqslant i(G) \leqslant \Delta i_V(G)$, where Δ stands for the maximal vertex degree.

(d) The quantity $i(G)$ is a discrete analogue of the (Cheeger) isoperimetric constant (see, e.g., [8]) measuring the minimal possible ratio between the length (area) of a subset X of M and the area (volume) of the smaller piece obtained by cutting M along X, where (usually) M is a Riemannian manifold. The reader is referred to [6, 7] where this analogy is described in more detail. In [6, 7] also the importance of $i(G)$ in the study of Riemann surfaces is outlined.

There are some related works on the isoperimetric numbers of infinite graphs, e.g., [3, 11, 13, 14, 17, 18]. For these the minimum in (1.1) is taken over all finite non-empty sets $X \subset V(G)$.

In our paper we explore basic properties of $i(G)$. The main result of Section 2 is that for a connected graph there is a partition $V(G) = X \cup Y$ such that the subgraphs induced on X and Y are connected and $i(G)=|E(X, Y)|/\min\{|X|, |Y|\}$. As a corollary we obtain in Section 3 a linear algorithm for calculating the isoperimetric numbers of trees. In general, the calculation of the isoperimetric number of graphs with multiple edges is NP-hard. In Section 2 a general upper bound on $i(G)$ is given (in terms of the number of vertices and the number of edges). Roughly speaking, $i(G)$ cannot exceed half of the average degree. The influence of $i(G)$ on the (distance) growth of the graph G is established. As a corollary, an upper bound on the diameter of a graph in terms of its isoperimetric number is derived.

In Section 4 we obtain lower and upper bounds on $i(G)$ in terms of the second smallest eigenvalue of the difference Laplacian matrix $D(G)$ of G. The upper bound is a discrete version of the well-known Cheeger inequality [9] bounding the first eigenvalue of a Riemannian manifold (cf. also [8]). Such bounds were discovered before in a similar setting (cf. [1, 2]). However, our version of the Cheeger inequality non-trivially improves the previously known results. The lower eigenvalue bound and a construction of Ramanujan graphs by Lubotzky *et al.* [16] give explicit examples of arbitrarily large k-regular graphs with isoperimetric numbers at least $\frac{1}{2}k-\sqrt{k-1}$, where k is such that $k \equiv 2 \pmod 4$ and $k-1$ is a prime. Asymptotically (in k) this gives explicit graphs with the largest possible isoperimetric number. See Section 6 for more details.

In Section 5 we investigate the isoperimetric numbers of (Cartesian) products of graphs. In general $i(G\times H) \leqslant \min(i(G), i(H)\}$ with strict inequality being possible. In some cases we can guarantee equality and in general we can only say that the slack $\min\{i(G), i(H)\} - i(G\times H)$ cannot be too large.

Finally we mention that Boshier [5] proved that for a fixed genus g, the isoperimetric number of a cubic graph of genus g and with n vertices is at most $O(n^{-1/2})$ which seems to solve a problem which has grown up from a false conjecture of Buser [6].

2. Basic Results and Examples

At the beginning let us state some more or less trivial facts about the isoperimetric number of graphs:

(a) $i(G)=0$ if and only if G is disconnected.

(b) If G is k-edge-connected then $i(G) \geqslant 2k/|V(G)|$.

(c) If δ is the minimal degree of vertices in G then $i(G) \leqslant \delta$.

(d) If $e=uv$ is an edge of G and $|V(G)| \geqslant 4$ then $i(G) \leqslant [\deg(u)+\deg(v)-2]/2$.

(e) If Δ is the maximum vertex degree in G then $i(G) \leqslant (\Delta - 2) + 2/\lfloor |V(G)|/2 \rfloor$. If G has a cycle with a most half the vertices of G then $i(G) \leqslant \Delta - 2$. (If the minimum vertex degree in G is at least 3 this holds except if G is one of the graphs shown on Fig. 1.)

(f) If G has a bridge and δ is the minimum vertex degree in G then $i(G) \leqslant 1$ for $\delta = 1$ and $i(G) \leqslant 1/(\delta + 1)$ if $\delta \geqslant 2$.

For each of the inequalities stated above there are graphs for which these are tight.

It is relatively easy to determine the isoperimetric numbers of some nice graphs:

(a) For the complete graph K_n, $i(K_n) = \lceil n/2 \rceil$.

(b) The cycle C_n has $i(C_n) = 2/\lfloor n/2 \rfloor$.

(c) The path P_n on n vertices has $i(P_n) = 1/\lfloor n/2 \rfloor$.

(d) The complete bipartite graphs $K_{m,n}$ have their isoperimetric numbers equal to

$$i(K_{m,n}) = \begin{cases} mn/(m+n), & \text{if } m \text{ and } n \text{ are given} \\ (mn+1)/(m+n), & \text{if } m \text{ and } n \text{ are odd} \\ mn/(m+n-1), & \text{if } m+n \text{ is odd,} \end{cases}$$

which can be shortened to $i(K_{m,n}) = \lceil mn/2 \rceil / \lfloor (m+n)/2 \rfloor$.

(e) Petersen's graph has isoperimetric number equal to 1.

(f) The n-dimensional cube graph $Q_n = K_2^n$ has $i(Q_n) = 1$ (cf. Theorem 5.1).

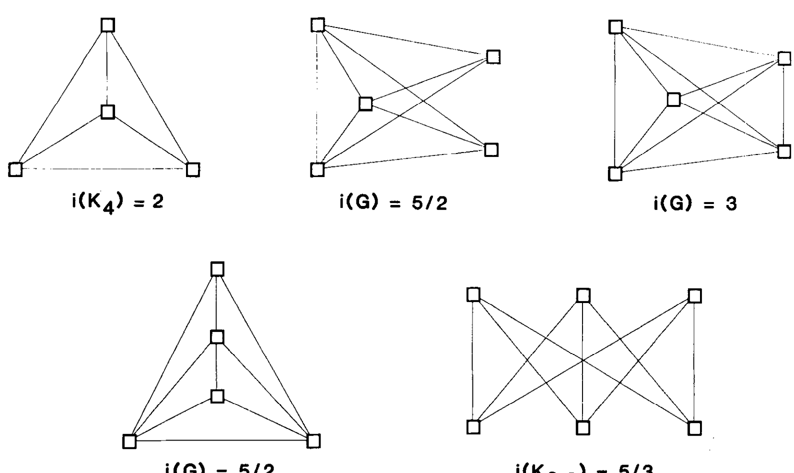

FIG. 1. Graphs with $i(G) > \Delta(G) - 2$.

If a set $X \subset V(G)$ with $|X| \leqslant \frac{1}{2}|V(G)|$ reaches the minimum $i(G) = |\partial X|/|X|$ we call it an *isoperimetric set.* For $U \subseteq V(G)$ denote by $G|U$ the subgraph of G induced on U (can be read as "G restricted to U").

PROPOSITION 2.1. *If G is a connected graph then it has an isoperimetric set X such that $G|X$ and $G|(V\backslash X)$ are connected subgraphs of G.*

Proof. Let X be an isoperimetric set of G, and let $X_1, X_2, ..., X_s$ be the partition of X determined by the connected components of $G|X$. We may assume that X_1 has the property that $|\partial X_1|/|X_1| \leqslant |\partial X_i|/|X_i|$ for $i = 1, 2, ..., s$. Since $|\partial X| = |\partial X_1| + \cdots + |\partial X_s|$ and $|X| = |X_1| + \cdots + |X_s|$ we get:

$$i(G) = \frac{|\partial X|}{|X|} = \frac{\sum |\partial X_i|}{\sum |X_i|} \geqslant \min \left\{ \frac{|\partial X_i|}{|X_i|} \right\} = \frac{|\partial X_1|}{|X_1|}.$$

It follows that X_1 is an isoperimetric set with $G|X_1$ being connected.

Now let $Y_1, Y_2, ..., Y_r$ be the partition of $Y = V\backslash X$ determined by the components of $G|Y$. It will be assumed that X is an isoperimetric set with $G|X$ connected. It is clear that for $i = 1, ..., r$, $|Y_i| \leqslant \frac{1}{2}|V(G)|$ and $Y_i \neq \phi$ if $r \geqslant 2$ (which will be assumed henceforth). Therefore $|\partial Y_i|/|Y_i| \geqslant |\partial X|/|X| > |\partial Y_i|/|X|$, and consequently $|Y_i| < |X|$.

For $i = 1, 2, ..., r$ define $X_i := Y\backslash Y_i$. Since X is an isoperimetric set, $|X_i| \leqslant \frac{1}{2}|V(G)|$, and since $r \geqslant 2$, $X_i \neq \phi$. From

$$\begin{aligned} i(G) &\leqslant \frac{|\partial X_i|}{|X_i|} = \frac{|\partial X| - |\partial Y_i|}{|V| - |X| - |Y_i|} \\ &\leqslant \frac{|\partial X| - |\partial Y_i|}{|X| - |Y_i|} = \frac{i(G)\,|X| - (|\partial Y_i/|Y_i|) \cdot |Y_i|}{|X| - |Y_i|} \leqslant i(G) \end{aligned}$$

we conclude that the last inequality must be tight. But this is true only if $|\partial Y_i|/|Y_i| = i(G)$. So Y_i is an isoperimetric set and, clearly, $G|Y_i$ and $G|(V\backslash Y_i)$ are connected. ∎

In the study of $i(G)$ the related numbers

$$i_k(G) := \min \left\{ \frac{|\partial X|}{k} \,\middle|\, X \subseteq V(G), |X| = k \right\}$$

are also important. Clearly, $i(G) = \min\{i_k(G) \mid 1 \leqslant k \leqslant \lfloor \frac{1}{2}|V| \rfloor\}$. In general, the sequence $i_1(G)$, $i_2(G)$, ... may be quite irregular; for example, it may have many local extrema. On the other hand, it is reasonable to expect that it will be much better behaving if G is vertex-transitive, for example.

As we remarked at the beginning of this section, $i(G) \leqslant \Delta - 2$ with few exceptions. This result can be improved considerably as follows:

THEOREM 2.2. *Let G be a graph of order n with m edges. Then*

$$i_r(G) \leqslant \frac{2m(n-r)}{n(n-1)}, \qquad r = 1, 2, ..., n-1, \tag{2.1}$$

and consequently

$$i(G) \leqslant \begin{cases} m/(n-1), & \text{if } n \text{ is even} \\ m(n+1)/n(n-1), & \text{if } n \text{ is odd.} \end{cases} \tag{2.2}$$

Proof. Let $X_1, ..., X_{\binom{n}{r}}$ be all possible r-subsets of $V(G)$. The average of $|\partial X_i|$ is equal to

$$m \cdot 2 \cdot \frac{\binom{n-2}{r-1}}{\binom{n}{r}} = \frac{2m \cdot r \cdot (n-r)}{n \cdot (n-1)}$$

since each edge is in ∂X_i for exactly $2\binom{n-2}{r-1}$ different choices of i. Now (2.1) follows trivially and (2.2) is obtained by taking $r = \lfloor n/2 \rfloor$. ∎

Note that the upper bound of (2.2) is approximately equal to half of the average degree.

Let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$ be the degrees of vertices in G in the increasing order. If $e_r := (1/r)(d_1 + \cdots + d_r)$ then trivially $i_r(G) \leqslant e_r$. Thus:

$$i_r(G) \leqslant \min \left\{ e_r, \frac{2m(n-r)}{n(n-1)} \right\}. \tag{2.3}$$

The isoperimetric number imposes a lower bound on the growth of a graph. Let v be a vertex of G and denote, for $k = 0, 1, 2, ...$, by $B_k = B_k(v)$ the set of vertices of G at distance at most k from v. The *growth function* $f = f_v$ of G (with respect to v) is defined as $f_v(k) = |B_k(v)|$.

THEOREM 2.3. *Let G be a graph on n vertices with growth function $f = f_v$ ($v \in V(G)$) and maximal degree Δ. Then:*

(a) *If $f(k) \leqq n/2$ then*

$$\frac{f(k)}{f(k-1)} \geqq \frac{\Delta + i(G)}{\Delta - i(G)}.$$

(b) *If* $f(k-1) \geqq n/2$ *then*

$$\frac{n-f(k)}{n-f(k-1)} \leqq \frac{\Delta - i(G)}{\Delta + i(G)}.$$

Proof. (a) Since $\Delta(f(k)-f(k-1)) \geqq |\partial B_k| + |\partial B_{k-1}| \geqq i(G)(f(k)+f(k-1))$, the result follows easily.

(b) Here we estimate, as in (a), the number $n-f(k-1)$ and $n-f(k)$ instead of $f(k)$ and $f(k-1)$. ▮

COROLLARY 2.4. *Let G be a graph of order n with maximal vertex degree* Δ. *Then*

$$\operatorname{diam}(G) \leqq 2 \left\lceil \frac{\log(n/2)}{\log((\Delta + i(G))/(\Delta - i(G)))} \right\rceil.$$

Proof. Let $f^{-}(t)$ be the smallest k such that $f(k) \geqq t$. Clearly, $\operatorname{diam}(G) \leqq 2 \cdot f^{-}(n/2+1)$. By Theorem 2.3, $f(k) > ((\Delta + i(G))/(\Delta - i(G)))^k$ if $f(k) \leqq n/2$ and $k \geqq 1$. Consequently, $k \geqq \log(n/2)/\log((\Delta + i(G))/(\Delta - i(G)))$ implies $f(k) > n/2$, and $f^{-}(n/2+1) \leqq \lceil \log(n/2)/\log((\Delta + i(G))/(\Delta - i(G))) \rceil$. ▮

3. How Difficult Is the Calculation of $i(G)$?

From the algorithmic (complexity) point of view the calculation of $i(G)$ seems to be a difficult problem. Here we outline a simple recursive algorithm to determine $i(G)$. The proof (using Proposition 2.1) is left to the reader.

Algorithm 1

For $X, Y \subset V(G)$, $X \cap Y = \varnothing$, define recursively:

$$i(X, Y) = \begin{cases} \infty, & \text{if } |X| > \frac{1}{2}|V(G)|; \\ |\partial X|/|X|, & \text{if } |X| = \lfloor |V(G)|/2 \rfloor \text{ or } X \cup Y = V(G); \\ i(X \cup X', Y), & \text{if } X' \text{ and } Y \text{ are separated by } X; \\ i(X, Y \cup Y'), & \text{if } X \text{ and } Y' \text{ are separated by } Y; \\ \min\{i(X \cup \{v\}, Y), i(X, Y \cup \{v\})\}, & \\ & \text{for any } v \in V(G) \backslash (X \cup Y). \end{cases}$$

Then $i(G) = i(\phi, \phi)$.

The choice of $v \in V(G) \backslash (X \cup Y)$ in the last possibility in the definition of

$i(X, Y)$ is arbitrary. Any rule for its choice will be good. We may add an additional rule:

$$i(X, Y) = \infty \qquad \text{if} \quad X \text{ separates } Y.$$

On the other hand, we may omit the two "separation rules" in Algorithm 1.

To determine $i(G)$ effectively (i.e., in polynomial time) it sufficies to have a polynomial algorithm for solving the following decision problem (referred to as RESTRICTED CUTSET) for every t:

INSTANCE: Graph G and integer k with $k \leqslant |V(G)|/2$.

QUESTION: Is there a subset $X \subset V(G)$ with $|X| = k$ and $|\partial X| \leqslant kt$.

Equivalently, we might try to solve the MINIMAL RESTRICTED CUTSET problem asking for the minimum of $|\partial X|$ where $|X| = k$, $X \subset V(G)$. This is equivalent to calculate $i_k(G)$. However, these problems are NP-complete in general. Note that we cannot conclude from this that computing $i(G)$ is also NP-hard. On the other hand, if these problems turn out to be in P for some restricted class of graphs, then also the isoperimetric number of these graphs can be determined effectively.

THEOREM 3.1. *For graphs with multiple edges the computation of the isoperimetric number is NP-hard.*

Proof. Assume we have a polynomial time algorithm for $i(G)$. Given a graph G on n vertices and m edges, we can determine $i_{\lfloor n/2 \rfloor}(G)$ in polynomial time as follows. Let H be the graph obtained from G by adding, between each pair of vertices, m parallel edges. Let X be an isoperimetric set of H found by our algorithm. We claim that $|X| = \lfloor n/2 \rfloor$. This is easy to see since $|\partial_H X|/|X| = |\partial_G X|/|X| + m(n - |X|)$ and $m > |\partial_G X|/|X|$. Consequently, X is an optimal bipartition determining $i_{\lfloor n/2 \rfloor}(G)$, the calculation of which is known to be NP-hard [12]. ∎

In view of Proposition 2.1 the isoperimetric number of a tree T is equal to $1/k$ for some integer k satisfying $1 \leqslant k \leqslant |V(T)|/2$, since every set F of edges whose removal from T gives two connected components must consist of one edge only.

PROPOSITION 3.2. *Let T be a tree with n vertices. Then:*

(a) *$i(T) \leqslant \frac{1}{2}$ unless $T = K_{1,n-1}$ in which case $i(T) = 1$.*

(b) *$i(T) = \frac{1}{2}$ if and only if $|V(T)| \geqslant 4$ and T has a vertex v such that $T - v$ consists of $p \geqslant 1$ copies of the graph K_2 and $n - 2p - 1 \geqslant 0$ isolated vertices. (See Fig. 2.)*

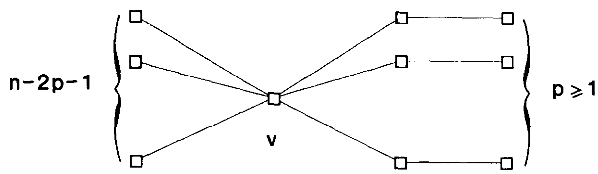

FIGURE 2

The proof of this proposition is an easy task, so we continue with additional properties.

In view of the remark from the beginning of this section, to compute $i(T)$ it suffices to find an edge of T such that its removal yields two trees of similar size (as close as possible). This describes an easy algorithm of quadratic complexity. But we can do even better.

Orient the edges of a tree T in such a way that for each vertex v at most one edge has v as its endpoint. At the same time we may define a linear relation $<$ on $V(T)$ in such a way that $v < u$ implies there is no directed path from v to u. This can be done in linear time by a breadth-first search on T.

For each oriented edge (u, v) of T let $S(u, v)$ denote the order (number of vertices) of the component of $T - uv$ containing vertex u. The values of $S(u, v)$ can be calculated by the use of the formula,

$$S(u, v) = 1 + \sum_{\substack{w \sim v \\ w < v}} S(v, w),$$

and applying it first for "smaller" vertices v (w.r.t. $<$) and their unique neighbours u such that uv is oriented from u to v. At the same time we may determine the number k as the maximum of $\min\{S(u, v), |V(T)| - S(u, v)\}$ taken over all oriented edges (u, v) of T. Then, clearly, $i(T) = 1/k$.

PROPOSITION 3.3. *The algorithm described above determines, in linear time, the isoperimetric number of an arbitrary tree.*

We believe that the isoperimetric number of planar graphs can be obtained in polynomial time, too.

4. SPECTRAL BOUNDS

Let $A = A(G)$ be the adjacency matrix of a graph G, and let $D = \operatorname{diag}(\deg(v)) - A$ be the corresponding *difference Laplacian* matrix. Several authors [1, 2, 3, 11, 17] have observed a relation between the isoperimetric number and the spectrum of A or D (the spectrum is the set of eigenvalues together with their multiplicities), mostly inspired by the results about the spectrum of the Laplacian on Riemannian manifolds (cf. [8, 9]).

It can be shown easily (see, for example, [2]) that for the second smallest eigenvalue λ_1 of D and for an arbitrary $X \subseteq V(G)$ the following relation holds:

$$\lambda_1 \leqslant |\partial X| \left(\frac{1}{|X|} + \frac{1}{|V \backslash X|} \right). \tag{4.1}$$

Consequently:

THEOREM 4.1. *Let G be a graph on n vertices and let λ_1 be the second smallest eigenvalue of its difference Laplacian matrix D. Then for every k $(1 \leqslant k \leqslant n-1)$,*

$$i_k(G) \geqslant \frac{(n-k)\lambda_1}{n} \tag{4.2}$$

and, consequently, $i(G) \geqslant \lambda_1/2$.

There are graphs for which this inequality is tight. For example, K_n has $\lambda_1 = n$ and $i(K_n) = \lceil n/2 \rceil$, and the graph Q_n of the n-dimensional cube has $\lambda_1 = 2$ and $i(Q_n) = 1$.

In the setting of Riemannian manifolds, the smallest (positive) eigenvalue λ_1 of the Laplacian provides also an upper bound for the isoperimetric number. This is known as Cheeger inequality [9] (see also [8]). Its discrete versions for infinite and finite graphs were obtained by several authors [1, 3, 11, 17]. Our next result is a strong version of a Cheeger-like inequality.

THEOREM 4.2. *Let G be a graph with maximal vertex degree Δ and let λ_1 be the second smallest eigenvalue of its difference Laplacian matrix. If G is not equal to any of K_1, K_2, or K_3 then*

$$i(G) \leqslant \sqrt{\lambda_1(2\Delta - \lambda_1)}. \tag{4.3}$$

Before proving Theorem 4.2 we need some lemmas concerning λ_1.

LEMMA 4.3. *Let G be a graph with adjacency matrix A and difference Laplacian matrix D. If Δ and δ are the maximum and the minimum vertex degrees of G, respectively, then the second largest eigenvalue μ_1 of A and the second smallest eigenvalue λ_1 of D are related by*

$$\delta - \lambda_1 \leqslant \mu_1 \leqslant \Delta - \lambda_1. \tag{4.4}$$

Proof. μ_1 is the second largest eigenvalue of A, and $\delta - \lambda_1$ is the second largest eigenvalue of $\delta I - D = A - (\mathrm{diag}(\deg(v)) - \delta I)$, which differs from A only on the diagonal, where the non-negative values $\deg(v) - \delta$ are subtracted. Consequently, $\delta - \lambda_1 \leqslant \mu_1$. In a similar way also the other inequality is obtained. ∎

LEMMA 4.4. *If G is not a complete graph then, with the notations of Lemma* 4.3,

$$\mu_1 \geqslant 0 \qquad \textit{and} \qquad \lambda_1 \leqslant \Delta. \tag{4.5}$$

Proof. G contains the path P_3 as an induced subgraph. By the interlacing theorem for the eigenvalues of a graph and its vertex-deleted subgraphs (see, e.g., [10]), it follows that $\mu_1(G) \geqslant \mu_1(P_3) = 0$. The second inequality of (4.5) follows from this by Lemma 4.3. ∎

Proof of Theorem 4.2. If G is disconnected then $i(G) = 0$ and $\lambda_1 = 0$, so (4.3) trivially holds. It is also easy to prove (4.3) for the complete graphs K_n, $n \geqslant 4$. We also assume that $\lambda_1 \leqslant$ minimal vertex degree since otherwise (4.3) follows directly by Lemma 4.4 and the fact that $i(G) \leqslant$ minimal degree.

We shall assume henceforth that $G \neq K_n$ is connected.

Let $y = (y_v \mid v \in V)$ be an eigenvector of λ_1 such that the set $W := \{v \in V \mid y_v > 0\}$ has cardinality $|W| \leqslant |V|/2$. Put

$$g_v := \begin{cases} y_v, & \text{if } v \in W \\ 0, & \text{otherwise} \end{cases}$$

and denote by EW the set of edges $e \in E(G)$ which have both ends in W. Then

$$\begin{aligned}
\lambda_1 \sum_{v \in W} y_v^2 &= \sum_{v \in W} \left(\deg(v)\, y_v - \sum_{u \sim v} y_u\right) y_v \\
&= \sum_{v \in W} \sum_{u \sim v} (y_v - y_u)\, y_v \\
&= \sum_{vu \in EW} [(y_v - y_u)\, y_v + (y_u - y_v)\, y_u] + \sum_{vu \in \partial W} (y_v - y_u)\, y_v \\
&= \sum_{vu \in E(G)} (g_v - g_u)^2 - \sum_{vu \in \partial W} y_u y_v.
\end{aligned} \tag{4.6}$$

Similarly we get

$$(2\Delta - \lambda_1) \sum_{v \in W} y_v^2 \geqslant \sum_{E(G)} (g_v + g_u)^2 + \sum_{\partial W} y_u y_v. \tag{4.7}$$

Combining (4.6) and (4.7) and writing $\alpha := \sum_{\partial W} y_u y_v$ we see that

$$\lambda_1(2\Delta - \lambda_1)\left(\sum_{v\in W} y_v^2\right)^2 \geqslant \sum_{E(G)} (g_v + g_u)^2 \sum_{E(G)} (g_v - g_u)^2 - \alpha\left(4\sum_{EW} y_u y_v + \alpha\right). \tag{4.8}$$

In applying (4.8) we shall need two more details,

$$\alpha \leqslant 0 \tag{4.9}$$

which is clear since for any $vu \in \partial W$, $y_u y_v \leqslant 0$. The second inequality is

$$\begin{aligned} 4\sum_{EW} y_u y_v + \alpha &= 2\sum_{EW} y_u y_v + \sum_{v\in W} y_v \sum_{u\sim v} y_u \\ &= 2\sum_{EW} y_u y_v + \sum_{v\in W} (\deg(v) - \lambda_1)\, y_v^2 \geqslant 0. \end{aligned} \tag{4.10}$$

The last conclusion on non-negativity follows from the fact that $y_u y_v > 0$ if $vu \in EW$ and from our initial assumption $\lambda_1 \leqslant$ minimal vertex degree.

If we introduce

$$\square := \sum_{vu\in E(G)} |g_v^2 - g_u^2|$$

then by (4.8), (4.9), (4.10), and the Cauchy–Schwartz inequality,

$$\begin{aligned} \square^2 = \left(\sum_{E(G)} |g_v^2 - g_u^2|\right)^2 &\leqslant \sum_{E(G)} (g_v + g_u)^2 \sum_{E(G)} (g_v - g_u)^2 \\ &\leqslant \lambda_1(2\Delta - \lambda_1)\left(\sum_{v\in W} g_v^2\right)^2. \end{aligned} \tag{4.11}$$

On the other hand, a lower bound on $\square$ can be obtained in a similar way as in the setting of Riemannian manifolds (cf. [7] or [6]). Let $0 = t_0 < t_1 < \cdots < t_m$ be all the different values of g_v ($v \in V$). For $k = 0, 1, ..., m$ define $V_k := \{v \in V \mid g_v \geqslant t_k\}$. Note that for $k \geqslant 1$, $|V_k| \leqslant |W| \leqslant \frac{1}{2}|V|$. Then

$$\begin{aligned} \square &= \sum_{k=1}^{m} \sum_{\substack{vu\in E \\ g_u < g_v = t_k}} (g_v^2 - g_u^2) = \sum_{k=1}^{m} \sum_{vu\in\partial V_k} (t_k^2 - t_{k-1}^2) \\ &= \sum_{k=1}^{m} |\partial V_k|\,(t_k^2 - t_{k-1}^2) \geqslant i(G) \sum_{k=1}^{m} |V_k|\,(t_k^2 - t_{k-1}^2) \\ &= i(G) \sum_{k=0}^{m} t_k^2(|V_k| - |V_{k+1}|) = i(G) \sum_{v\in V} g_v^2 = i(G) \sum_{v\in W} g_v^2. \end{aligned} \tag{4.12}$$

In the equality in the first line above we have used the fact that $g_v^2 - g_u^2 = (t_k^2 - t_{k-1}^2) + (t_{k-1}^2 - t_{k-2}^2) + \cdots + (t_{q+1}^2 - t_q^2)$ with $g_u = t_q$, and so this term contributes $t_j^2 - t_{j-1}^2$ for every j with $vu \in \partial V_j$. In coming from the second to the third line we used $t_0 = 0$ and $|V_{m+1}| = 0$.

The proof is finished since (4.11) and (4.12) clearly imply (4.3). ∎

If we have available, instead of the Laplacian eigenvalues, the eigenvalues of the adjacency matrix of a graph, we might, instead of Theorem 4.2, use the following weaker version:

COROLLARY 4.5. *If μ_1 is the second largest eigenvalue of the adjacency matrix of a graph G, and Δ is its maximal vertex degree, then*

$$i(G) \leqslant \sqrt{\Delta^2 - \mu_1^2}. \tag{4.13}$$

Proof. By Lemma 4.3, $\lambda_1 = \Delta - \mu_1 - a$ for some $a \geqslant 0$. Thus

$$\begin{aligned} \lambda_1(2\Delta - \lambda_1) &= (\Delta - (\mu_1 + a))(\Delta + (\mu_1 + a)) \\ &= \Delta^2 - (\mu_1 + a)^2 \leqslant \Delta^2 - \mu_1^2. \end{aligned}$$

So Theorem 4.2 implies (4.13). ∎

5. PRODUCTS

If G and H are graphs then their *Cartesian product* is the graph $G \times H$ with vertex set $V(G) \times V(H)$ and vertices (v, u), (v', u') adjacent iff v is adjacent to v' in G and $u = u'$, or $v = v'$ and u adjacent to u' in H. It is well known (cf. [10]) that the (adjacency matrix) eigenvalues of $G \times H$ are equal to sums of pairs of eigenvalues of graphs G and H. It can be shown that the same is true for the Laplacian spectrum. In particular, $\lambda_1(G \times H) = \min\{\lambda_1(G), \lambda_1(H)\}$. Theorem 4.1 implies that, if $\lambda_1(G)$ and $\lambda_1(H)$ are large than $G \times H$ will have large isoperimetric number (this fact is used in [2]).

It is easily seen that

$$i(G \times H) \leqslant \min\{i(G), i(H)\} \tag{5.1}$$

for arbitrary graphs G and H. Also, there are graphs G and H for which, in (5.1), strict inequality holds. For example, let G be obtained from $K_a \cup K_b$ by adding a bridge between the two components, and let H be obtained in a similar way from K_c and K_d. Then $G \times H$ is as shown on Fig. 3.

Clearly, $i(G) = 1/a$ (if $a \leqslant b$) and $i(H) = 1/d$ (if $d \leqslant c$). By taking the

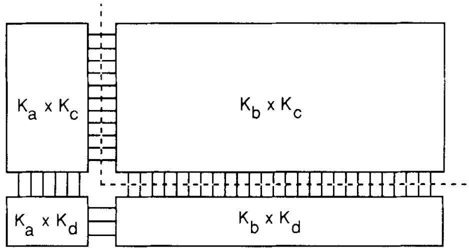

FIGURE 3

cutset denoted on Fig. 3 we see that $i(G \times H) \leqslant (b+c)/(ac+ad+bd)$, assuming $bc \geqslant \frac{1}{2}|V(G \times H)|$. It is easily seen that there are values of a, b, c, d for which $i(G \times H) < \min\{i(G), i(H)\}$.

THEOREM 5.1. *If G is a graph having an even number of vertices then for every $n \geqslant 1$,*

$$i(G \times K_{2n}) = \min\{i(G), n\}.$$

Proof. Let $V(K_{2n}) = \{1, 2, ..., 2n\}$, $|V(G)| = 2m$, and $X \subseteq V(G \times K_{2n})$ with at most half of the vertices of $G \times K_{2n}$. For $i = 1, ..., 2n$ let $X_i := X \cap (V(G) \times \{i\})$. We assume that for $i = 1, 2, ..., t$, $|X_i| = m - e_i \leqslant m$ (so $e_i \geqslant 0$) and for $i = t+1, ..., 2n$, $|X_i| = m + d_i \geqslant m$ (so $d_i \geqslant 0$). If $D = \sum d_i$ and $E = \sum e_i$ then it is easy to see that $D \leqslant E$. This implies

$$\begin{aligned}\sum_{i=1}^{t} \sum_{j=t+1}^{2n} |X_i + X_j| &\geqslant \sum_{i=1}^{t} \sum_{j=t+1}^{2n} (e_i + d_j) \\ &= (2n-t)E + tD \geqslant 2nD, \qquad (5.2)\end{aligned}$$

where $X_i + X_j$ denotes the symmetric difference of sets X_i and X_j.

In what follows we shall use the letter α to denote $\min\{i(G), n\}$.

The edges in ∂X fall into two classes: the edges in copies of G in $G \times K_{2n}$ and the edges in $\{v\} \times K_{2n}$ ($v \in V(G)$). The number of edges of the first type in $G \times i$ is $|\partial X_i| \geqslant i(G)\,|X_i|$ (if $i \leqslant t$) and $|\partial X_i| \geqslant i(G)(2m - |X_i|)$ if $i > t$. The number of edges of the second type between the ith and jth copy of G is equal to $|X_i + X_j|$. Consequently,

$$\begin{aligned}|\partial X| \geqslant i(G) \sum_{i=1}^{t} |X_i| + i(G) \sum_{i=t+1}^{2n} (2m - |X_i|) \\ + \sum_{i=1}^{2n} \sum_{j=i}^{2n} |X_i + X_j|\end{aligned}$$

$$\geqslant \alpha\,|X| + \alpha \sum_{i=t+1}^{2n} (2m - 2\,|X_i|)$$

$$+ \sum_{i=1}^{t} \sum_{j=t+1}^{2n} |X_i + X_j|$$

$$\geqslant \alpha\,|X| - 2\alpha D + 2nD \geqslant \alpha\,|X|.$$

This completes the proof. ∎

For the product of paths $P_n \times P_m$ equality holds in (5.1), without restriction on parity of n and m.

At the end we note that the eigenvalue argument shows that the slack $\min\{i(G), i(H)\} - i(G \times H)$ cannot be too large.

6. Graphs with Large Isoperimetric Number

P. Buser noticed [6, 7] the close relationship between cubic graphs and Riemannian surfaces. In the study of these, the cubic graphs with high isoperimetric number play an important role. The first success was Buser's paper [7] in which the author realized that for any $n > 0$ there exists a cubic graph G of order $\geqslant n$ with $i(G) \geqslant 1/128$. If we define

$$F(n, k) := \max\{i(G) \mid G \text{ is } k\text{-regular on } n \text{ vertices}\}$$

and

$$f(k) := \limsup_{n \to \infty} F(n, k)$$

then Buser's result means just that $f(3) \geqslant 1/128$. We are able to prove the following results on $f(k)$.

Theorem 6.1. *If p is a prime congruent to* 1 *modulo* 4 *then*

$$f(p+1) \geqslant \tfrac{1}{2}(p+1) - \sqrt{p}. \tag{6.1}$$

Proof. If q is a prime, $q \equiv 1 \pmod 4$, and $q \neq p$ then there exists a $(p+1)$-regular graph $G^{p,q}$ with $n = q(q^2-1)$ vertices if the Legendre symbol $(p/q) = -1$, and with $n = q(q^2-1)/2$ vertices if $(p/q) = 1$, which has the second largest adjacency matrix eigenvalue at most $2\sqrt{p}$. This was shown by Lubotzky *et al.* [16]. So $\lambda_1(G^{p,q}) \geqslant (p+1) - 2\sqrt{p}$ and by Theorem 4.1 we have that $i(G^{p,q}) \geqslant (p+1)/2 - \sqrt{p}$. ∎

Adding edges to a graph cannot decrease its isoperimetric number. The Ramanujan graphs used in the proof of Theorem 6.1 have an even number of vertices, hence we may add to these graphs $t-1$ disjoint 1-factors and thus obtain arbitrarily large $(p+t)$-regular graphs with the isoperimetric number at least $(p+1)/2-\sqrt{p}$. Together with known results on the distribution of primes in arithmetic progressions this implies that

$$f(k) \geqslant \frac{k}{2} + O(k^{1-\varepsilon}) \tag{6.2}$$

for some $\varepsilon > 0$. On the other hand, (2.7) implies

$$f(k) \leqslant \frac{k}{2} \tag{6.3}$$

which shows that (6.2) is asymptotically the best possible result.

The best possible eigenvalue bound on $f(k)$ would be

$$f(k) \geqslant \frac{k}{2} - (k-1)^{1/2} \tag{6.4}$$

since large k-regular graphs have $\lambda_1 \leqslant k - 2(k-1)^{1/2} + o(1)$ (cf. [16]). On the other hand, it was conjectured by N. Alon that almost every k-regular graph has $\lambda_1 \geqslant k - 2(k-1)^{1/2} - o(1)$ which would then imply that (6.4) is actually true. We have several reasons to believe that, for every k, there is a slightly better bound than (6.4):

Conjecture 6.2. There are constants l and u such that for each $k \geqslant 3$,

$$f(k) = \frac{k}{2} - c_k(k-1)^{1/2}, \tag{6.5}$$

where $\frac{1}{2} < l < c_k < u < 1$.

Any sequence of graphs with $i(G)$ tending to (6.5), assuming the conjecture is true, could be called *super expanding* since its members are much better expanders than one could conclude on the eigenvalue basis.

We can also use the known bounds on λ_1 of random k-regular graphs to estimate $f(k)$.

THEOREM 6.3. $f(2k) \geqslant k - O(k^{1/2})$.

Proof. It was shown by J. Friedman [19] that the expected value for the second eigenvalue λ_1 of the difference Laplacian of $2k$-regular graphs is at least $2k - O(k^{1/2})$. The theorem now follows trivially by Theorem 4.1. ∎

At then end of this section we include a result the proof of which was kindly submitted to the author by Noga Alon.

THEOREM 6.4. *As* $k \to \infty$, $f(k) > k/2 - (1 + o(1))\sqrt{k}$.

Proof. Given k, let p_1 be the largest prime congruent to 1 modulo 4, which is smaller than $k-1$. Let p_2 be the largest prime congruent to 1 modulo 4, which is smaller than $k-1-p_1$, etc. One can find, by the known results about the distribution of primes in arithmetic progressions, some constant number m (independent of k) such that

$$p_i \leqslant k - p_1 = O(k^{1-\varepsilon}), i = 1, ..., m \qquad \text{and} \qquad k \geqslant \sum_{i=1}^{m} (p_i + 1) \geqslant k - k^{0,1}.$$

Now, by the Ramanujan graphs [16] we can find infinitely many values of n (order of graphs) for which there are $(p_i + 1)$-regular ($i = 1, ..., m$) graphs H_i on n vertices each, and with $i(H_i) \geqslant (p_i + 1)/2 - \sqrt{p_i}$. If n is large enough we may construct a graph G on n vertices which is an edge disjoint union of all these graphs H_i. This follows from a result of Sauer and Spencer [20, 21]. Trivially, the isoperimetric number of an edge disjoint union of graphs is at least as large as the sum of the corresponding isoperimetric numbers. Now an easy computation gives the desired estimate of the theorem. ∎

It is worth noting that Sauer and Spencer's proof is constructive, and hence the above proof really produces, constructively, graphs with large isoperimetric number.

7. ISOPERIMETRIC NUMBER AND GENUS

Boshier [5] proved the following result on isoperimetric numbers of graphs of given genus:

PROPOSITION 7.1. *Let G be a graph of order n, with maximal vertex degree Δ and with the genus equal to g. If $n > 18(g+2)^2$ then*

$$i(G) \leqslant \frac{3(g+2)\,\Delta}{(n/2)^{1/2} - 3(g+2)}.$$

Proposition 7.1 shows that the isoperimetric numbers of graphs of a fixed genus and with bounded degrees tend to 0 as the order of the graphs increases. This is the result which should be considered as a substitute for the false conjecture of Buser [6].

On the other hand, there are arbitrarily large planar graphs with high $i(G)$. For example, the double pyramid, $D_n = C_n * 2K_1$ has $i(D_n) = 2 + O(1/n)$ for all n. In a sense this is the worst case that can happen since the average degree of graphs of any fixed genus is at most $6 + O(1/n)$, and hence their isoperimetric number is at most $3 + O(1/n)$, by Theorem 2.2.

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Note added in proof. Some recent results can be used to improve our statements in Section 6. Bollobas [*Eur. J. Comb.* **9** (1988), 241–244] estimated the isoperimetric number of random regular graphs. On the other hand constructions of Ramanujan graphs with arbitrary degrees are known, cf. [*Notices AMS* **34** (1989), 5–22].

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