# Some Remarks on the Chromatic Index of a Graph

By

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**Introduction and Terminology.** In this paper we consider only finite graphs without loops; there may, however, be multiple edges (such graphs are in [12] called multigraphs, but we shall here just call them *graphs* or sometimes *general graphs*). A *simple graph* is a graph without multiple edges. A *multiple edge* is a set of edges joining the same pair of vertices; in particular a *double edge* is a multiple edge consisting of two edges.

Let $\Gamma$ be a graph. The set of vertices of $\Gamma$ is denoted by $\mathscr{V}(\Gamma)$ and the set of edges of $\Gamma$ by $\mathscr{E}(\Gamma)$. The *valency* of a vertex $x$ in $\Gamma$ is the number of edges incident with $x$ and is denoted by $v(x, \Gamma)$, and the maximal valency of any vertex in $\Gamma$ is denoted by $\sigma(\Gamma)$. A *regular graph* is a graph in which all vertices have the same valency. If $e \in \mathscr{E}(\Gamma)$ then $\Gamma - e$ is the graph obtained from $\Gamma$ by deleting the edge $e$. If $x \in \mathscr{V}(\Gamma)$ then $\Gamma - x$ is the graph obtained from $\Gamma$ by removing $x$ and deleting all edges incident with $x$.

For a real number $\alpha$, $\{\alpha\}$ denotes the least integer not less than $\alpha$ and $[\alpha]$ the greatest integer not greater than $\alpha$. For a set $S$, $|S|$ denotes the number of elements in $S$.

If the edges of a graph $\Gamma$ can be coloured with $k$ colours such that any two edges incident with the same vertex have different colours we say that there is a *k-colouring* of $\mathscr{E}(\Gamma)$. The *chromatic index* $q(\Gamma)$ of $\Gamma$ is the least number of colours required in a colouring of $\mathscr{E}(\Gamma)$ (this number is called the edge-chromatic number in [1] and [9], the line-chromatic number in [6], the chromatic class in [10] and the chromatic index in [2], [3]).

It was proved by Vizing ([10], [12], English version in [9], French in [3]) that if $s(\Gamma)$ denotes the maximal number of edges in a multiple edge of the graph $\Gamma$, then $\sigma(\Gamma) \leqq q(\Gamma) \leqq \sigma(\Gamma) + s(\Gamma)$. It follows that for a simple graph $q(\Gamma) = \sigma(\Gamma)$ or $\sigma(\Gamma) + 1$. Following [1] we shall say that a graph $\Gamma$ is of *class I* if $q(\Gamma) = \sigma(\Gamma)$ and of *class II* if $q(\Gamma) > \sigma(\Gamma)$. A graph of class II is said to be *critical* if any proper subgraph of it has smaller chromatic index than the graph itself (cf. Vizing [13]).

This paper is concerned with graphs of class II. In [8] L. S. Mel'nikov stated that the only then known examples of families of graphs of class II were the regular graphs with an odd number of vertices, and the graphs obtained from regular graphs with an even number of vertices by inserting a new vertex into an arbitrary edge. It is the object of the first section of this paper to present some simple constructions (which I do not think have been mentioned in the literature so far) of graphs of class II, thus extending somewhat the known families of such graphs. One of these constructions (Hajós' construction) is of particular interest in connection with the critical graphs, because it also provides a construction of new critical graphs from smaller ones. This is the subject of the second section, where also two elementary results known for critical simple graphs are generalized to graphs with multiple edges. The third and last section gives a characterization of critical graphs with a separating set of two edges.

**1. Graphs of class II.** A graph with maximal valency 2 is the vertex-disjoint sum of finite paths and/or circuits ([7], p. 17; a double edge defines a circuit). This remark leads to the following classification:

There are no graphs of class II with chromatic index 1 or 2. The graphs with chromatic index 1 are of course those whose connected components are complete 2-graphs (class I). The graphs with chromatic index 2 are those that are vertex-disjoint sums of finite paths and/or even circuits (class I). The graphs of class II with chromatic index 3 are those that are vertex-disjoint sums of finite paths and/or circuits and contain at least one odd circuit. We have here disregarded isolated vertices.

In the sequel we shall then be interested only in graphs with maximal valency at least 3 and graphs of class II with chromatic index at least 4.

We proceed to describe some constructions of graphs of class II partly by stating sufficient structural properties and partly by using smaller graphs of class II.

**Lemma 1.** *Let $\Gamma_1$ and $\Gamma_2$ be two graphs both with chromatic index at most $k$. Let $\Gamma$ be the graph obtained by identifying a vertex $x_1$ of $\Gamma_1$ and a vertex $x_2$ of $\Gamma_2$. If $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) \leqq k$, then $\Gamma$ has chromatic index at most $k$.*

Proof. There is a $k$-colouring of $\mathscr{E}(\Gamma_1)$ such that the $v(x_1, \Gamma_1)$ edges incident with $x_1$ all have different colours, and there is a $k$-colouring of $\mathscr{E}(\Gamma_2)$ such that the $v(x_2, \Gamma_2)$ edges incident with $x_2$ all have different colours. Since $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) \leqq k$ the same $k$ colours may be used in both $k$-colourings in such a way that the $v(x_1, \Gamma_1) + v(x_2, \Gamma_2)$ edges incident with $x_1$ and $x_2$ all have different colours. This yields a $k$-colouring of $\mathscr{E}(\Gamma)$, so $q(\Gamma) \leqq k$ and the lemma follows.

**Theorem 1.** *Let $\Gamma_1$ be a graph with chromatic index $k$ and of class II. Let $\Gamma_2$ be a graph either with chromatic index $k$ and of class II or with chromatic index less than $k$. Let $x_1 \in \mathscr{V}(\Gamma_1)$ and $x_2 \in \mathscr{V}(\Gamma_2)$ be such that $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) \leqq k - 1$. Then the graph obtained from $\Gamma_1$ and $\Gamma_2$ by identifying $x_1$ and $x_2$ has chromatic index $k$ and is of class II.*

Proof. $\Gamma_1$ and $\Gamma_2$ are both graphs with chromatic index at most $k$. Hence by Lemma 1 $q(\Gamma) \leqq k$. $\Gamma$ contains a graph, namely $\Gamma_1$, with chromatic index $k$, so $q(\Gamma) = k$. $\sigma(\Gamma) = \max(\sigma(\Gamma_1), \sigma(\Gamma_2), v(x_1, \Gamma_1) + v(x_2, \Gamma_2)) \leqq k - 1$, hence $\Gamma$ is of class II. This proves Theorem 1.

A set of edges of a graph is called *independent* if no two of them are incident with the same vertex.

**Lemma 2** ([3], p. 242). *Let $\Gamma$ be a graph and let $t$ denote the maximal number of independent edges of $\Gamma$. Then* $q(\Gamma) \geqq \max\left(\sigma(\Gamma), \left\{\frac{|\mathscr{E}(\Gamma)|}{t}\right\}\right)$.

Proof. Let $q(\Gamma) = k$. Any $k$-colouring of $\mathscr{E}(\Gamma)$ partitions $\mathscr{E}(\Gamma)$ into $k$ sets of independent edges, each set containing at most $t$ edges. Hence $|\mathscr{E}(\Gamma)| \leqq k \cdot t$ and consequently $q(\Gamma) = k \geqq \left\{\frac{|\mathscr{E}(\Gamma)|}{t}\right\}\cdot$ The lemma follows.

**Theorem 2.** *Let $d$ be a natural number ($\geqq 3$) and let $\Gamma$ be a graph with an odd number of vertices such that each vertex has valency $d$ with the possible exception that there are $v$ vertices $x_1, \dots, x_v$, $1 \leqq v \leqq d - 1$, each having valency at most $d - 1$, such that $\sum_{i=1}^{v} v(x_i, \Gamma) \geqq (v - 1)\,d + 1$. Then we have*

a) $q(\Gamma) \geqq d + 1$.

b) *The deletion of not more than $m$ edges, where*

$$m = \max\left(0, \left[\tfrac{1}{2}\left(\sum_{i=1}^{v} v(x_i, \Gamma) - (v - 1)\,d - 1\right)\right]\right),$$

*does not lower the chromatic index* $\left(m \geqq 1 \textit{ for } \sum_{i=1}^{v} v(x_i, \Gamma) \geqq (v - 1)\,d + 3\right)$.

c) *If in addition $\Gamma$ is a simple graph then $q(\Gamma) = d + 1$.*

Proof. Let the number of vertices of $\Gamma$ be $2r + 1$. If all vertices have valency $d$ we put $v = 0$ and $\sum_{i=1}^{v} v(x_i, \Gamma) = 0$.

$$2\,|\mathscr{E}(\Gamma)| = (2r - (v - 1))\,d + \sum_{i=1}^{v} v(x_i, \Gamma) \geqq 2rd + 1$$

by assumption. Then by Lemma 2

$$q(\Gamma) \geqq \left\{\frac{|\mathscr{E}(\Gamma)|}{t}\right\} \geqq \left\{\frac{|\mathscr{E}(\Gamma)|}{r}\right\} \geqq \left\{d + \frac{1}{2r}\right\} = d + 1\,.$$

This proves a). c) follows then from the Theorem of Vizing for simple graphs. To prove b) we observe that if $\Gamma^*$ is a graph obtained from $\Gamma$ by deleting $m$ edges, then

$$2\,|\mathscr{E}(\Gamma^*)| = 2rd - (v - 1)\,d + \sum v(x_i, \Gamma) - 2m \geqq 2rd + 1\,,$$

and as above we see that $q(\Gamma^*) \geqq d + 1$. This proves Theorem 2.

Remark 1. There cannot exist more than $d - 1$ vertices of valency at most $d - 1$ with the property that their valency-sum is at least $(v - 1)\,d + 1$, because we have $v(d - 1) \geqq \sum v(x_i, \Gamma) \geqq (v - 1)\,d + 1$, implying $v \leqq d - 1$.

Remark 2. There are certain restrictions as to what can occur in Theorem 2. If $d$ is even and there are vertices of valency less than $d$ then there cannot be an odd number of these vertices with odd valency. If $d$ is odd and there are $v$ vertices of valency less than $d$, then if $v$ is even there cannot be an even number of these $v$

vertices with odd valency and if $\nu$ is odd there cannot be an odd number of these $\nu$ vertices with odd valency. In particular the case $\nu = 1$, $v(x_1, \Gamma) = 1$ cannot occur.

Fig. 1 shows two examples with $d = 4$, $\nu = 2$, where there are two vertices of valency 3:

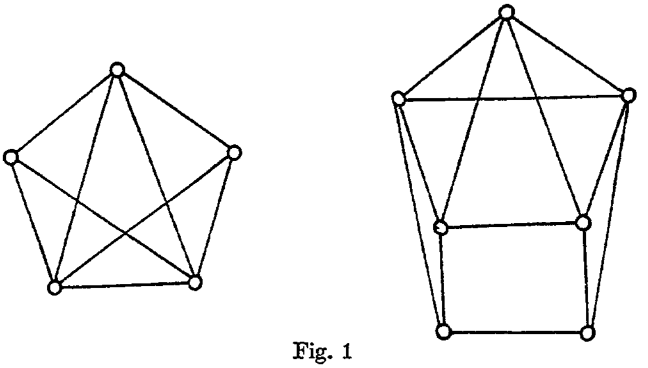


Fig. 1

Remark 3. There exist graphs $\Gamma$ for which $q(\Gamma) > \sigma(\Gamma) + 1$ (they must of course have multiple edges). The graph of Fig. 2 is an example with maximal valency 6 and chromatic index 8; it is regular and thus fits the description of Theorem 2 showing that a) cannot be strengthened to equality.

b) is not best possible as the graph of Fig. 3 shows; here $d = 3$, $\nu = 1$, $m = 0$, but the edge $e$ can be deleted without lowering the chromatic index, which is 4.

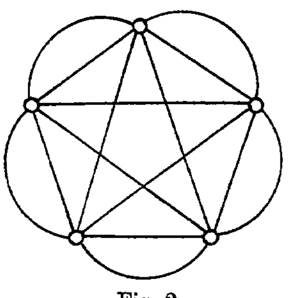


Fig. 2

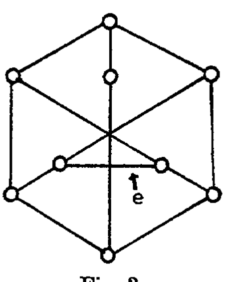



Fig. 3

Theorem 2 generalizes

**Corollary 1.** *Let $\Gamma$ be a graph with an odd number of vertices where each vertex has valency $d$ except one, which has valency $d' \leqq d$. Let $\Gamma^*$ be any graph obtained from $\Gamma$ by deleting not more than $\frac{1}{2}d' - 1$ edges; then $q(\Gamma^*) \geqq d + 1$, with equality for simple graphs.*

This is proved for simple graphs by Beineke and Wilson in [1].

For the sake of completeness let us explicitly state two corollaries (of both Theorem 2 and Corollary 1) that are well-known for simple graphs:

**Corollary 2.** *Let $\Gamma$ be a regular graph of (even) valency $d$ and with an odd number of vertices. Then $q(\Gamma) \geqq d + 1$, with equality for simple graphs.*

**Corollary 3.** *Let $\Gamma$ be a regular graph of valency $d$ and with an even number of vertices. Let $\Gamma^*$ be any graph obtained from $\Gamma$ by inserting a new vertex into an arbitrary edge of $\Gamma$. Then $q(\Gamma^*) \geqq d + 1$, with equality for simple graphs.* (This is an extreme case of Theorem 2 with $\nu = 1$ and $v(x_1, \Gamma) = 2$.)

Corollary 2 and Corollary 3 were proved for simple graphs by Vizing in [12].

**Hajós' Construction.** Let $\Gamma_1$ and $\Gamma_2$ be two graphs. Then the graph $\Gamma$ is said to be obtained from $\Gamma_1$ and $\Gamma_2$ using *Hajós' construction* if $\Gamma$ is the resulting graph when

1) a vertex $x_1$ of $\Gamma_1$ is identified with a vertex $x_2$ of $\Gamma_2$,
2) an edge $e_1$ of $\Gamma_1$ incident with $x_1$ is deleted,
3) an edge $e_2$ of $\Gamma_2$ incident with $x_2$ is deleted,
4) the two end-vertices other than $x_1$ and $x_2$ of the two deleted edges are joined by an edge $e$.

The reason for the name "Hajós' construction" is that the construction is identical with one introduced by G. Hajós ([5]) in connection with vertex-colourings of graphs (note: identical, not dual!).

We shall introduce some further notation. The end-vertex of $e_1$ different from $x_1$ is denoted by $z_1$, and the end-vertex of $e_2$ different from $x_2$ is denoted by $z_2$. The subgraph of $\Gamma$ consisting of $\Gamma_1 - e_1$ together with the vertex $z_2$ joined by the edge $e$ to $z_1$ is denoted by $\Gamma_1'$. Analogously $\Gamma_2'$ is the subgraph of $\Gamma$ consisting of $\Gamma_2 - e_2$ together with the vertex $z_1$ joined by the edge $e$ to $z_2$. Finally let $x$ denote the vertex obtained by identifying $x_1$ and $x_2$.

**Lemma 3.** *Let $\Gamma_1$ and $\Gamma_2$ be two graphs both with chromatic index at least $k$. Then any graph $\Gamma$ obtained from $\Gamma_1$ and $\Gamma_2$ using Hajós' construction has chromatic index at least $k$. If $x_1 \in \mathscr{V}(\Gamma_1)$ and $x_2 \in \mathscr{V}(\Gamma_2)$ are the two identified vertices, then*

$$\sigma(\Gamma) \leqq \max(\sigma(\Gamma_1), \sigma(\Gamma_2), v(x_1, \Gamma_1) + v(x_2, \Gamma_2) - 2).$$

Proof. In any $(k-1)$-colouring of $\mathscr{E}(\Gamma_1')$ there is an edge of $\Gamma_1 - e_1$ incident with $x$ which has the same colour as $e$, because otherwise $e_1$ could be given the colour of $e$ resulting in a $(k-1)$-colouring of $\Gamma_1$; also in any $(k-1)$-colouring of $\mathscr{E}(\Gamma_2')$ there is an edge of $\Gamma_2 - e_2$ incident with $x$ which has the same colour as $e$. A $(k-1)$-colouring of $\mathscr{E}(\Gamma)$ induces a $(k-1)$-colouring of $\mathscr{E}(\Gamma_1')$ and a $(k-1)$-colouring of $\mathscr{E}(\Gamma_2')$, hence in every $(k-1)$-colouring of $\mathscr{E}(\Gamma)$ there is an edge of $\Gamma_1 - e_1$ incident with $x$ having the colour of $e$ and also an edge of $\Gamma_2 - e_2$ incident with $x$ having the colour of $e$, so there are two edges having the same colour incident with one vertex. Hence $\mathscr{E}(\Gamma)$ cannot be coloured with $k-1$ colours and consequently $q(\Gamma) \geqq k$. The vertex $x$ has valency $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) - 2$ in $\Gamma$, hence $\sigma(\Gamma)$ is clearly at most $\max(\sigma(\Gamma_1), \sigma(\Gamma_2), v(x_1, \Gamma_1) + v(x_2, \Gamma_2) - 2)$.

This proves Lemma 3.

**Theorem 3.** *Let $\Gamma_1$ and $\Gamma_2$ be two graphs both of class II and with chromatic index $k$. Let $\Gamma$ be a graph obtained from $\Gamma_1$ and $\Gamma_2$ using Hajós' construction where $x_1 \in \mathscr{V}(\Gamma_1)$ and $x_2 \in \mathscr{V}(\Gamma_2)$ are identified. If $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) \leqq k + 1$, then $\Gamma$ is of class II with chromatic index $k$.*

Proof. By Lemma 3 $q(\Gamma) \geqq k$. There is a $k$-colouring of $\mathscr{E}(\Gamma_1)$ such that the $v(x_1, \Gamma_1)$ edges incident with $x_1$ all have different colours, and there is a $k$-colouring of $\mathscr{E}(\Gamma_2)$ such that the $v(x_2, \Gamma_2)$ edges incident with $x_2$ all have different colours. Since $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) - 1 \leqq k$ the same $k$ colours may be used in both $k$-colourings in such a way that the $v(x_1, \Gamma_1) + v(x_2, \Gamma_2)$ edges incident with $x_1$ and $x_2$ all have different colours except that $e_2$ has the same colour as $e_1$. By giving $e$ the common colour of $e_1$ and $e_2$ we thus obtain a $k$-colouring of $\mathscr{E}(\Gamma)$, so we have $q(\Gamma) = k$. As $\sigma(\Gamma_1) \leqq k - 1$, $\sigma(\Gamma_2) \leqq k - 1$ and $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) - 2 \leqq k - 1$ it follows that $\sigma(\Gamma) \leqq k - 1$, hence $\Gamma$ is of class II. This completes the proof of Theorem 3.

**2. Critical graphs.** Practically all investigations of graphs that are critical with respect to edge-colourings have hitherto concentrated on simple graphs (see [1], [11], and [12]). Here, however, we consider general graphs.

It follows from the remarks in the beginning of the first section that there are no critical graphs with chromatic index 1 or 2 because a critical graph by definition is of class II. It also follows that the critical graphs with chromatic index 3 are the odd circuits.

It remains to characterize the critical graphs with chromatic index greater than 3. Only a rather modest amount of (published) research has been carried out concerning this, but the problem appears to be about as difficult as the well-known and much investigated corresponding problem for vertex-colourings. The Theorem of Tait ([7], p. 202) shows that the four colour problem is equivalent to the statement that any planar regular graph of valency 3 without bridges has chromatic index 3; a characterization of critical graphs with chromatic index 4 would thus be very helpful if not settle the four colour problem. This indicates how difficult to obtain such a characterization can be expected to be.

In this paper we shall, however, be content with stating a few very simple structural properties of critical graphs and a construction-procedure. In another paper we shall present a more detailed discussion of critical graphs with chromatic index 4.

A cut-vertex of a connected graph is a vertex the removal of which makes the graph disconnected. We have:

**Theorem 4.** *A critical graph is connected and has no cut-vertex.*

Proof. It is obvious that a critical graph must be connected. Assume now that there exists a critical graph $\Gamma$ with $q(\Gamma) = k$ such that $\Gamma$ has a cut-vertex $x$. Divide the connected components of $\Gamma - x$ into two disjoint, non-empty classes $\mathscr{K}_1$ and $\mathscr{K}_2$ and let for $i = 1$ and $2$ $\Gamma_i$ be the subgraph of $\Gamma$ consisting of $\mathscr{K}_i$ together with $x$ and all edges from $x$ to $\mathscr{K}_i$. Then $q(\Gamma_i) \leqq k - 1$ for $i = 1$ and $2$ since $\Gamma$ is critical, and $v(x, \Gamma_1) + v(x, \Gamma_2) = v(x, \Gamma) \leqq k - 1$ since $\Gamma$ is of class II, hence by Lemma 1 $q(\Gamma) \leqq k - 1$ contrary to hypothesis. Theorem 4 follows.

Vizing stated Theorem 4 for simple graphs in [12] (identical proof).

**Corollary 4.** *A critical graph contains no vertex of valency* 1 *and no vertex only incident with a multiple edge.*

**Theorem 5.** *Let $\Gamma$ be a critical graph with chromatic index $k$ and let $x$ and $y$ be two vertices of $\Gamma$ joined by $s$ edges, $s \geqq 1$. Then $v(x, \Gamma) + v(y, \Gamma) \geqq k + s$.*

Proof. Assume on the contrary that $v(x, \Gamma) + v(y, \Gamma) < k + s$. Let $e$ be an edge of $\Gamma$ joining $x$ and $y$. $\mathscr{E}(\Gamma - e)$ can be coloured with $k-1$ colours since $\Gamma$ is critical. The number of edges of $\Gamma$ apart from $e$ that are incident with either $x$ or $y$ or both is $(v(x, \Gamma) - s) + (v(y, \Gamma) - s) + (s - 1) = v(x, \Gamma) + v(y, \Gamma) - s - 1 < k + s - s - 1 = k - 1$ by assumption. But then in any $(k-1)$-colouring of $\mathscr{E}(\Gamma - e)$ the edges adjacent to $e$ are coloured with at most $k-2$ colours, hence $e$ can be given a $(k-1)$'th colour thus extending the $(k-1)$-colouring to $\mathscr{E}(\Gamma)$. This contradiction proves Theorem 5.

**Corollary 5** (Vizing [12]). *In a critical simple graph with chromatic index $k$ any two vertices having a valency-sum less than $k+1$ are not joined by an edge.*

**Theorem 6.** *Let $\Gamma_1$ and $\Gamma_2$ be two graphs both critical with chromatic index $k$ and let $x_1 \in \mathscr{V}(\Gamma_1)$ and $x_2 \in \mathscr{V}(\Gamma_2)$ such that $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) \leqq k + 1$. Then the graph $\Gamma$ obtained from $\Gamma_1$ and $\Gamma_2$ by Hajós' construction in which $x_1$ and $x_2$ are identified is critical with chromatic index $k$.*

Proof. Let the notation be as before for Hajós' construction. By Theorem 3 $\Gamma$ is of class II with chromatic index $k$. It is then sufficient to prove that deleting any one edge of $\Gamma$ lowers the chromatic index of $\Gamma$. Let $e'$ be an edge of $\Gamma_1 - e_1$. $\mathscr{E}(\Gamma_1 - e')$ can be coloured with $k-1$ colours; let the colour of $e_1$ in such a $(k-1)$-colouring be denoted by $c_1$. This gives rise to a $(k-1)$-colouring of $\mathscr{E}(\Gamma_1' - e')$ where $e$ has the colour $c_1$ and the edges of $\Gamma_1 - e_1 - e'$ incident with $x$ have at most the $v(x_1, \Gamma_1) - 1$ different colours $c_2, \ldots, c_{v(x_1, \Gamma_1)}$, each of these colours being different from $c_1$. $\mathscr{E}(\Gamma_2')$ can be coloured with $k-1$ colours such that $e$ has the colour $c_1$; then as shown in the proof of Lemma 3 there is an edge of $\Gamma_2 - e_2$ incident with $x$ having the colour $c_1$, and since $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) - 2 \leqq k - 1$ the notation may be chosen so that the other edges of $\Gamma_2 - e_2$ incident with $x_2$ are coloured with the $v(x_2, \Gamma_2) - 2$ different colours $c_{v(x_1,\Gamma_1)+1}, \ldots, c_{v(x_1,\Gamma_1)+v(x_2,\Gamma_2)-2}$, all these colours being different from $c_1$. The $(k-1)$-colouring of $\mathscr{E}(\Gamma_1' - e')$ and the $(k-1)$-colouring of $\mathscr{E}(\Gamma_2')$ may then be combined to furnish a $(k-1)$-colouring of $\mathscr{E}(\Gamma - e')$. Analogously it is proved that when $e' \in \mathscr{E}(\Gamma_2 - e_2)$ then $\mathscr{E}(\Gamma - e')$ can be coloured with $k-1$ colours. Finally it follows from Lemma 1 that $\mathscr{E}(\Gamma - e)$ is $(k-1)$-colourable since $\Gamma - e$ is obtained by identifying a vertex of $\Gamma_1 - e_1$ and a vertex of $\Gamma_2 - e_2$ and $(v(x_1, \Gamma_1) - 1) + (v(x_2, \Gamma_2) - 1) \leqq k - 1$. Hence whatever edge is deleted, the edges of the resulting graph can be coloured with $k-1$ colours. But then $\Gamma$ is critical with chromatic index $k$ and Theorem 6 has been proved.

**3. Critical graphs that are separated by two edges.** A set of edges $\mathscr{H}$ of a connected graph $\Gamma$ is said to *separate* $\Gamma$ if the graph that remains after all the edges of $\mathscr{H}$ have been deleted is disconnected. Two edges incident with the same vertex does not separate a critical graph except when they are the two edges incident with a vertex of valency 2; this follows from Theorem 4. We may thus restrict ourselves to consider separation by two independent edges. The following theorem holds:

**Theorem 7.** *A critical graph with chromatic index $k$ is separated by two independent edges if and only if it can be obtained from two smaller critical graphs both also with* *chromatic index $k$ using Hajós' construction where a vertex of valency 2 in one of the graphs is identified with a vertex in the other.*

(Cf. Theorem 1 in [4] which is about edge-critical $k$-chromatic graphs; the "line-graph" ([6], p. 71) of a critical graph with chromatic index $k$ is vertex-critical $k$-chromatic.)

Proof of Theorem 7. Let $\Gamma_1$ and $\Gamma_2$ be two critical graphs both with chromatic index $k$, and let $x_1 \in \mathscr{V}(\Gamma_1)$, $x_2 \in \mathscr{V}(\Gamma_2)$ such that $v(x_1, \Gamma_1) = 2$. Hajós' construction can now be applied on $\Gamma_1$ and $\Gamma_2$ with identification of $x_1$ and $x_2$, so as to obtain a graph which is critical with chromatic index $k$ by Theorem 6 because $v(x_1, \Gamma_1) + v(x_2, \Gamma_2) = v(x_2, \Gamma_2) + 2 \leqq k + 1$. It is clearly separated by two independent edges.

Let conversely $\Gamma$ be a critical graph with chromatic index $k$ separated by two independent edges, $e_1$ and $e_2$, say. $\Gamma - e_1 - e_2$ has two connected components $C_1$ and $C_2$ because there are no cut-vertices in $\Gamma$; let $v_1$ and $v_2$ be the end-vertices of $e_1$ and $e_2$ respectively in $C_1$, and let $z_1$ and $z_2$ be the end-vertices of $e_1$ and $e_2$ respectively in $C_2$. Let $\Gamma'$ denote the subgraph of $\Gamma$ consisting of $C_1$ together with $z_1$ and $z_2$ joined by the edges $e_1$ and $e_2$ respectively to $C_1$, and let $\Gamma''$ denote the subgraph of $\Gamma$ consisting of $C_2$ together with $v_1$ and $v_2$ joined by the edges $e_1$ and $e_2$ respectively to $C_2$. Both $\mathscr{E}(\Gamma')$ and $\mathscr{E}(\Gamma'')$ are $(k-1)$-colourable because $\Gamma$ is critical and the notation may be chosen so that in any $(k-1)$-colouring of $\mathscr{E}(\Gamma')$ $e_1$ and $e_2$ have the same colour while in any $(k-1)$-colouring of $\mathscr{E}(\Gamma'')$ $e_1$ and $e_2$ have different colours, because otherwise a $(k-1)$-colouring of $\mathscr{E}(\Gamma)$ could be obtained.

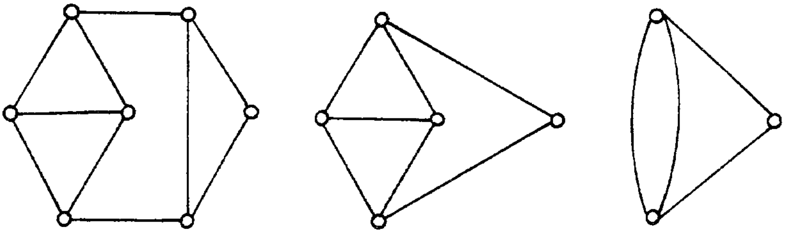
Let now $\Gamma_1$ be the graph consisting of $C_1$ and a new vertex $z$ joined by an edge $f_1$ to $v_1$ and by an edge $f_2$ to $v_2$ (i.e. $\Gamma_1$ is obtained from $\Gamma'$ by identifying $z_1$ and $z_2$), and let $\Gamma_2$ be the graph consisting of $C_2$ together with a new edge $e^*$ joining $z_1$ and $z_2$ (they may of course be joined already in which case we get a multiple edge). Neither $\mathscr{E}(\Gamma_1)$ nor $\mathscr{E}(\Gamma_2)$ are $(k-1)$-colourable, they have in fact both chromatic index $k$; this follows from the above statement about any $(k-1)$-colouring of $\mathscr{E}(\Gamma')$ and of $\mathscr{E}(\Gamma'')$. Furthermore $\Gamma$ is obtained from $\Gamma_1$ and $\Gamma_2$ using Hajós' construction where $z$ is identified with e.g. $z_1$.

It remains only to prove that $\Gamma_1$ and $\Gamma_2$ both are critical.

Let $e$ be an edge of $\Gamma_1$; if $e$ is one of $f_1$, $f_2$ it is clear from the above that $\mathscr{E}(\Gamma_1 - e)$ is $(k-1)$-colourable. If $e$ is not one of these edges then $e$ is an edge of $C_1$. $\mathscr{E}(\Gamma - e)$ can then be coloured with $k-1$ colours such that $e_1$ and $e_2$ have different colours because any $(k-1)$-colouring of $\mathscr{E}(\Gamma - e)$ induces a $(k-1)$-colouring of $\mathscr{E}(\Gamma'')$ and in any such $e_1$ and $e_2$ have different colours. This $(k-1)$-colouring of $\mathscr{E}(\Gamma - e)$ furnishes a $(k-1)$-colouring of $\mathscr{E}(\Gamma_1 - e)$ if $f_1$ is given the colour of $e_1$ and $f_2$ the colour of $e_2$. We have now proved that $\Gamma_1$ is critical.

Let now $e$ be an edge of $\Gamma_2$; if $e$ is $e^*$ it is clear that $\mathscr{E}(\Gamma_2 - e)$ is $(k-1)$-colourable. If $e$ is not $e^*$ then $e$ is an edge of $C_2$. $\mathscr{E}(\Gamma - e)$ can then be coloured with $k-1$ colours such that $e_1$ and $e_2$ have the same colour because any $(k-1)$-colouring of $\mathscr{E}(\Gamma - e)$ induces a $(k-1)$-colouring of $\mathscr{E}(\Gamma')$ and in any such $e_1$ and $e_2$ have the same colour. This $(k-1)$-colouring of $\mathscr{E}(\Gamma - e)$ furnishes a $(k-1)$-colouring of $\mathscr{E}(\Gamma_2 - e)$ if $e^*$ is given the common colour of $e_1$ and $e_2$. We have now proved that also $\Gamma_2$ is critical. This completes the proof of Theorem 7.

We close this paper by giving an illustration of Theorem 7. The graph in Fig. 4(a) is a critical graph with chromatic index 4 obtained from the graphs in Fig. 4(b) and (c) (likewise critical with chromatic index 4) using Hajós' construction. Moreover, this example shows that a critical *simple* graph can be obtained from critical graphs with multiple edges using this construction. This shows that in connection with edge-colourings it is worthwhile to study general graphs even if one is primarily interested in simple graphs.



(a) (b) (c)

Fig. 4

## References


[1] L. W. BEINEKE and R. J. WILSON, On the edge-chromatic number of a graph. To appear in Discrete Mathematics.

[2] C. BERGE, Théorie des graphes et ses applications. Paris 1958.

[3] C. BERGE, Graphes et hypergraphes. Paris 1970.

[4] G. A. DIRAC, On the structure of 5- and 6-chromatic abstract graphs. J. reine angew. Math. **214/215**, 43—52 (1964).

[5] G. HAJÓS, Über eine Konstruktion nicht $n$-färbbarer Graphen. Wiss. Z. Martin Luther Univ., Halle-Wittenberg **A 10**, 116—117 (1961).

[6] F. HARARY, Graph Theory. Reading 1969.

[7] D. KONIG, Theorie der endlichen und unendlichen Graphen. Leipzig 1936; reprinted New York 1950.

[8] L. S. MEL'NIKOV, Chromatic class and the location of a graph on a closed surface (Russian). Mat. Zametki **7**, 671—681 (1970).
English translation in Math. Notes **7**, 405—411 (1970).

[9] O. ORE, The Four-Color Problem. New York-London 1967.

[10] V. G. VIZING, On an estimate of the chromatic class of a $p$-graph (Russian). Diskret. Analiz **3**, 25—30 (1964).

[11] V. G. VIZING, Critical graphs with a given chromatic class (Russian). Diskret. Analiz **5**, 9—17 (1965).

[12] V. G. VIZING, The chromatic class of a multigraph (Russian). Kibernetika (Kiev) **1965**, No. 3, 29—39.
English translation in Cybernetics **1965**, No. 3, 32—41.

[13] V. G. VIZING, Some unsolved problems in graph theory (Russian). Uspehi Mat. Nauk **23**, 117—134 (1968).
English translation in Russian Math. Surveys **23**, 125—142 (1968).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