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NP-COMPLETENESS OF SOME PROBLEMS OF PARTITIONING AND COVERING IN GRAPHS

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Given a digraph $G=(X,U)$ such that $\forall x\in X$, $d^+(x)=d^-(x)=2$, we prove that the problem of determining whether U can be decomposed into two hamiltonian circuits is an NP-complete problem. From there, we deduce that it is NP-complete to determine the path-numbers of graphs and digraphs, even if these graphs have maximum degree four.

1. Introduction

A digraph G is a non-empty finite set X (the vertices), together with a finite family U of ordered pairs of distinct vertices (the arcs). A graph G is a non-empty finite set X (the vertices) together with a finite family E of unordered pairs of vertices (the edges).

An elementary path (resp. elementary chain) of a digraph (resp. graph) G is an alternating sequence of distinct vertices and arcs (resp. edges) $x_0u_1x_1u_2\cdots u_nx_n$, beginning and ending with vertices, in which u_i is an arc (resp. edge) from x_{i-1} to x_i. A circuit (resp. cycle) is an elementary path (resp. chain) with $x_0=x_n$. An anti-path is an alternating sequence of distinct vertices and arcs $x_0u_1x_1u_2\cdots u_nx_n$ beginning and ending with vertices, in which, for $1\le i\le n$, if u_i is an arc from x_{i-1} to x_i (resp. from x_i to x_{i-1}), then u_{i+1} is an arc from x_{i+1} to x_i (resp. from x_i to x_{i+1}). The terminology not defined here can be found in [2].

In this paper, we shall prove that several problems are NP-complete. The terminology and results of NP-completeness are given in [5].

We first consider the following problem: given a non-symmetric digraph $G=(X,U)$ such that $\forall x\in X$, $d^+(x)=d^-(x)=2$, the problem 2-DEC is to determine whether U can be partitioned by two hamiltonian circuits (such a partition will be called a decomposition of G).

Although several problems about the existence of hamiltonian circuits in digraphs had already been studied (see [5] and [10]), it is not the case for 2-DEC.

To prove that 2-DEC is NP-complete, we exhibit a polynomial reduction from the known NP-complete problem 3-SAT, which is defined as follows. A set of clauses $C=\{C_1,C_2,\dots,C_p\}$ in variables $u_1,u_2,\dots,u_s$ is given, each C_i consisting of three literals $x_{i,1},x_{i,2},x_{i,3}$ where a literal $x_{i,j}$ is either a variable u_k or its negation $\bar{u}_k$. The problem 3-SAT is to determine whether C is satisfiable, that is whether there is a truth assignment to the variables which simultaneously satisfies all the clauses of C, a clause being satisfied if one or more of its literals has value true.

Section 2 is devoted to the proof of the NP-completeness of 2-DEC. From Section 3 to Section 5, we shall prove the NP-completeness of several problems of partitioning or covering the edges (resp. the arcs) of a graph (resp. a digraph), which will be defined later, when studied. All these results will be deduced from the main theorem of Section 2.

2. NP-completeness of 2-DEC

First, we shall define several subgraphs which behave nicely with respect to Hamiltonian decomposition. Then, these graphs are used to effect a polynomial transformation from 3-SAT to 2-DEC.

2.1. Preliminary results

Let us consider the four digraphs $G_1=(X_1, U_1)$, $G_2=(X_2, U_2)$, $G_3=(X_3, U_3)$ and $G_4=(X_4, U_4)$ defined by Fig. 1.

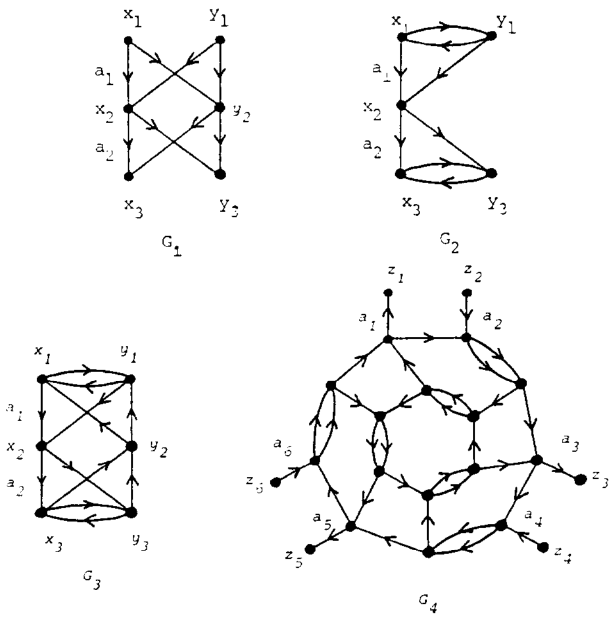

Fig. 1.

We now give some properties of these graphs which will be used farther to describe the polynomial transformation from 3-SAT to 2-DEC. The proofs of these lemmas being straightforward will be omitted.

Lemma 1. *Let $G=(X,U)$ be a decomposable digraph such that G_1 is the subdigraph of G induced by X_1. Then G_1 can be decomposed in two distinct ways: in the first one, the arcs a_1 and a_2 lie on the same circuit; in the second one, a_1 and a_2 belong to distinct circuits.*

Such a graph can be used as a 'switch' to represent the two possible values true and false of a variable. In the following, the first (resp. the second) decomposition will be called a T- (resp. a F-) decomposition.

Lemma 2. *Suppose G_2 is the subdigraph induced by X_2 of a digraph G. In a decomposition of G, G_2 can be decomposed in two distinct ways: in the first one, denoted T, a_1 and a_2 lie on the same circuit; in the second one, denoted F, a_1 and a_2 belong to different circuits.*

Lemma 3. *G_3 can be decomposed in two distinct ways: one called a T-decomposition (where a_1 and a_2 lie on the same circuit), the other a F-decomposition (a_1 and a_2 belonging to distinct circuits).*

Lemma 4. *Suppose G_4 is the subdigraph induced by X_4 of a digraph G which admits a decomposition. Then:*
(i) *The arcs a_1 and a_2 (resp. a_3 and a_4, (resp. a_5 and a_6)) lie on the same circuit.*
(ii) *All the arcs a_i, $1\le i\le 6$, cannot belong to the same circuit.*
(iii) *We can have either a_2 and a_4 or a_2 and a_6 or a_4 and a_6 on the same circuit.*

We now define for any couple $(p_1,p_2)\in\mathbb{N}^2$ a digraph G_{p_1,p_2} by:
- $G_{1,0}\cong G_{0,1}\cong G_3$.
- If $p_1+p_2\ge 2$, G_{p_1,p_2} is obtained from two digraphs isomorphic with G_2 and denoted H_1 and $H_{p_1+p_2}$, and (p_1+p_2-2) digraphs isomorphic with G_1 and called H_i, $2\le i\le p_1+p_2-1$. We shall denote by $x_1^i, x_2^i, x_3^i, y_1^i, y_2^i, y_3^i$ the vertices of H_i, according to the notations of Fig. 1. To obtain G_{p_1,p_2}, we must add arcs between the digraphs H_i and H_{i+1}, $1\le i\le p_1+p_2-1$, and we shall distinguish between two cases:

Case 1. If we have
(a) $i=1$, $p_1\ge 2$ or $p_2=0$,
(b) $2\le i\le p_1+p_2-1$ and $p_1=i$,
(c) $i+1=p_1+p_2$, $p_1<i$ or $p_2=0$,
we add the following arcs, called of type 1:

$$[x_3^i, x_1^{i+1}],\quad [y_3^i, y_1^{i+1}],\quad [x_3^{i+1}, y_1^i],\quad [y_3^{i+1}, x_1^i].$$

Case 2. For
(a) $i=1$, $p_1=1$,
(b) $2\le i\le p_1+p_2-1$, $p_1\ge i+1$ or $p_1<i$,
(c) $i+1=p_1+p_2$, $p_1=i$,

we add the arcs of type 2:

$$[x_3^i, x_1^{i+1}],\quad [y_3^i, y_1^{i+1}],\quad [x_3^{i-1}, x_1^i],\quad [y_3^{i+1}, y_1^i]$$

Example. $p_1=3$, $p_2=2$. See Fig. 2.

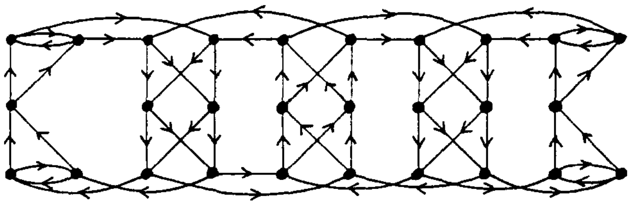

Fig. 2.

Lemma 5. *For every couple* $(p_1, p_2)\in\mathbb{N}^2$, G_{p_1,p_2} *is decomposable. Furthermore, there exist only two distinct decompositions of* G_{p_1,p_2}: *in the first one (resp. the second one), all the* H_i, $1\le i\le p_1$, *are* T *(resp.* F*) -decomposed and the* H_j, $p_1+1\le j\le p_1+p_2$, *are* F *(resp.* T*) -decomposed.*

Proof. Suppose $p_1+p_2\ge 2$. We shall prove that the choice of a truth assignment to the decomposition of H_1 implies a truth assignment to the decomposition of the other H_i. More precisely, we shall prove that if two disjoint paths C and $\bar{C}$ beginning and ending in H_i and going through all the vertices of $H_1, H_2, \ldots, H_i$ have been built, we can extend them to H_{i+1}. Furthermore, if H_i is X-decomposed (where $X\in\{T,F\}$), then H_{i+1} is $\bar{X}$-decomposed only when $i=p_1$; in the other cases, H_{i+1} is X-decomposed.

Among the 6 possible cases, we only describe one of them, leaving the other cases for the reader.

Suppose we have $2\le i\le p_1+p_2-1$, $p_1=i$, H_i T-decomposed and arcs of type 1 between H_i and H_{i+1}. Call C the path containing $[x_1^i, x_2^i]$. From the hypothesis, C contains $[x_1^i, x_2^i, x_3^i]$ and $[y_1^i, y_2^i, y_3^i]$. As the vertices x_3^{i-1} and y_1^{i-1} are already on C, we must put the arcs $[y_3^i, y_1^{i+1}]$ and $[y_3^{i-1}, y_1^i]$ in C. Then, from Lemma 1, we must put, for instance, $[y_1^{i+1}, y_2^{i+1}, x_3^{i+1}]$ and $[x_1^{i+1}, x_2^{i+1}, y_3^{i+1}]$ in C. Of course, we put the arcs of H_{i+1} and those connecting H_i and H_{i+1} and not mentioned above in $\bar{C}$. This shows that C and $\bar{C}$ can be extended to H_{i+1} and H_{i+1} is then F-decomposed.

Finally, H_{i+1} is $\bar{X}$-decomposed when H_i is X-decomposed only when $i=p_1$; this proves the existence of only two distinct decompositions of G_{p_1,p_2}, as claimed in Lemma 5. □

With three digraphs isomorphic with G_1, denoted G_5^1, G_5^2 and G_5^3, we build a digraph G_5 as in Fig. 3.

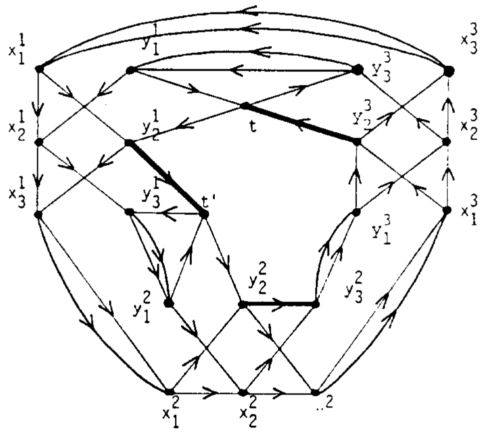

Fig. 3.

Lemma 6. *G_5 can be decomposed in four distinct ways. These decompositions can be summarized by TTF, TFT, FTT and FFF (where TTF, for example, means that G_5^1 is T-decomposed, G_5^2 T-decomposed and G_5^3 F-decomposed).*

Proof. Suppose G_5 is decomposed by the hamiltonian circuits C and $\bar{C}$. Let us remark that if G_5^j $(1 \le j \le 3)$ is F-decomposed by $(C, \bar{C})$, then C enters G_5^j through the vertex x_1^j (resp. y_1^j) and leaves it through the vertex y_3^j (resp. x_3^j).

So, in a decomposition of G_5, one can have only an odd number of G_5^j F-decomposed (otherwise a circuit $[x_1^1, x_2^1, x_3^1, \dots, x_3^3, x_1^1]$ would be created). As any G_5^j can be F- or T-decomposed, we obtain the four decompositions announced.

Suppose G_5 decomposed by C and $\bar{C}$, and call C the circuit which contains $[x_1^1, x_2^1]$. Put $u_1 = [y_2^1, t']$, $u_3 = [y_2^2, y_3^2]$, $u_3 = [y_2^3, t]$. Then, it is easy to see that
in the TTF-decomposition, we have $u_1 \in C, u_2 \in \bar{C}, u_3 \in \bar{C}$,
in the TFT-decomposition, we have $u_1 \in C, u_2 \in C, u_3 \in \bar{C}$,
in the FTT-decomposition, we have $u_1 \in \bar{C}, u_2 \in C, u_3 \in \bar{C}$,
in the FFF-decomposition, we have $u_1 \in \bar{C}, u_2 \in \bar{C}, u_3 \in \bar{C}$. □

We can now build a digraph G_6 as in Fig. 4: G_6 is obtained from G_4 and G_5 by joining them with the arcs u_1, u_2 and u_3.

Lemma 7. *G_6 is decomposable in three distinct ways, corresponding to TTF, TFT and FTT-decompositions.*

Proof. G_6 is obtained from G_4 and G_5 by joining them with arcs u_1, u_2 and u_3. But u_1, u_2 and u_3 lie on the same circuit only on the FFF-decomposition. So, by Lemma 4, this cannot happen in G_6 and so, we have the announced result. □

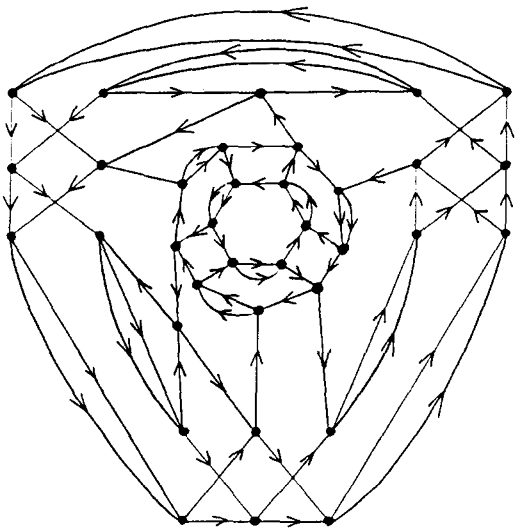

Fig. 4.

Consider now the digraph H_1' described by Fig. 5. Let us denote H_1^1 (resp. H_1^2) the digraph obtained from H_1' by deleting the vertex t and the arcs $[y_1^1, y_2^2]$ and $[x_1^2, y_2^1]$ and adding the arcs $[y_1^1, y_2^1]$ and $[y_1^1, x_2^1]$ (resp. $[x_1^2, x_2^2]$ and $[x_1^2, y_2^2]$). Then, the subdigraph of H_1^j, $j=1$ or 2, induced by $\{x_i^j, y_i^j \mid 1 \le i \le 3\}$ is isomorphic with G_1. We shall say that H_1^j is X-decomposed, $X=T$ or F, if the subdigraph of H_1^j isomorphic with G_1 is X-decomposed, as in Lemma 1. Then, we have:

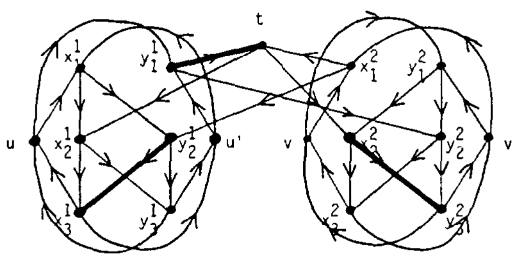

Fig. 5.

Lemma 8. (1) *H_1' is decomposable in four distinct ways.*

(2) *Any decomposition of H_1^i, $i=1,2$, gives rise to a decomposition of H_1'.*

Proof. Suppose, for example, that H_1^1 is T-decomposed by

$$C' = [x_1^1, x_2^1, x_3^1, u, y_1^1, y_2^1, y_3^1, u', x_1^1], \qquad \bar{C}' = [x_1^1, y_2^1, x_3^1, u', y_1^1, x_2^1, y_3^1, u, x_1^1]$$

and that H_1^2 is F-decomposed by

$$C'' = [x_1^2, x_2^2, y_3^2, v', y_1^2, y_2^2, x_3^2, v, x_1^2], \qquad \bar{C}'' = [x_1^2, y_2^2, y_3^2, v, y_1^2, x_2^2, x_3^2, v', x_1^2].$$

These decompositions give rise to the following partition for H_1':

$$C=[x_1^1, x_2^1, x_3^1, u, y_1^1, t, x_2^2, y_3^2, v', y_1^2, y_2^2, x_3^2, v, x_1^2, y_2^1, y_3^1, u', x_1^1],$$

$$\bar{C}=[x_1^1, y_2^1, x_3^1, u', y_1^1, y_2^2, y_3^2, v, y_1^2, x_2^2, x_3^2, v', x_1^2, t, x_2^1, y_3^1, u, x_1^1].$$

So, from any decomposition of H_1^1 and any decomposition of H_1^2, we obtain a decomposition of H_1'. By Lemma 1, there are two distinct ways for each H_1^i and this gives four possibilities for H_1'.

Conversely, we have to prove that there do not exist other decompositions than those described above, the vertices u and u' (resp. v and v') being considered as non-distinguishable. Suppose, for example, that we try to construct an hamiltonian circuit beginning with $[x_1^1, x_2^1, x_3^1, u, y_1^1, y_2^2]$; in order to pass through all the vertices of H_1^2, we must leave H_1^2 by the vertex x_1^2; if we choose the arc $[x_1^2, y_2^1]$, the obtained circuit will not pass through t; if we choose $[x_1^2, t]$, we cannot go further; so, in each case, this solution cannot lead to a hamiltonian circuit.

If we call $u_1=[y_1^1, t]$, $u_2=[y_2^1, x_3^1]$, $u_3=[x_2^2, y_3^2]$, it is easy to see that
in the TT-decomposition, $u_1 \in C$, $u_2 \in \bar{C}$ and $u_3 \in \bar{C}$,
in the TF-decomposition, $u_1 \in C$, $u_2 \in \bar{C}$ and $u_3 \in C$,
in the FT-decomposition, $u_1 \in C$, $u_2 \in C$ and $u_3 \in \bar{C}$,
in the FF-decomposition, $u_1 \in C$, $u_2 \in C$ and $u_3 \in C$. □

Now, let us consider the digraph H_2' described by Fig. 6.

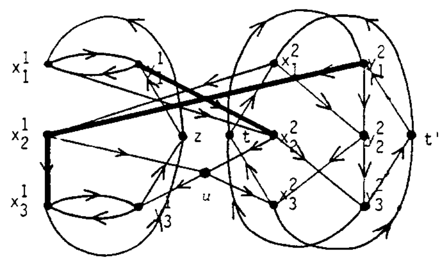

Fig. 6.

Let us denote by H_2^1 (resp. H_2^2) the subdigraph of H_2' obtained by deleting the vertex u and the arcs $[y_1^1, x_2^2]$, $[x_1^1, x_2^2]$, $[x_1^2, x_2^1]$ and $[y_1^2, x_2^1]$, $[x_1^1, x_2^1]$, $[y_1^1, x_2^1]$ and $[x_2^1, y_3^1]$ (resp. $[x_1^2, x_2^2]$, $[y_1^2, x_2^2]$ and $[x_2^2, x_3^2]$). Then, the subdigraph of H_2^1 (resp. H_2^2) induced by the $\{x_i^1, y_i^1 \mid 1 \le i \le 3\}$ (resp. $\{x_i^2, y_i^2 \mid 1 \le i \le 3\}$) is isomorphic with G_2 (resp. G_1). So we apply the same convention as for H_1'.

Then we have the following result:

Lemma 9. (1) *H_2' is decomposable in four distinct ways.*

(2) *Any decomposition of H_2^i, $i=1,2$, gives rise to a decomposition of H_2'.*

Proof. It is similar to the one of Lemma 8. □

Furthermore, we denote, in H_2':

$$u_1 = [x_2^1, y_3^1], \qquad u_2 = [x_1^2, x_2^1] \quad \text{and} \quad u_3 = [x_2^2, y_3^2].$$

Finally, we can define the digraphs H_1 and H_2 which will be used in the transformation $\mathcal{T}$.

H_1 (resp. H_2) is obtained from H_1' (resp. H_2') by deleting the arcs u_1, u_2 and u_3 and connecting the ends of these arcs with the vertices $z_1, \ldots, z_6$ of a digraph isomorphic with G_4, and by splitting the vertices u, u', v and v' (resp. z, t and t') to obtain pendent arcs (see Fig. 7 for H_1). Then, we have:

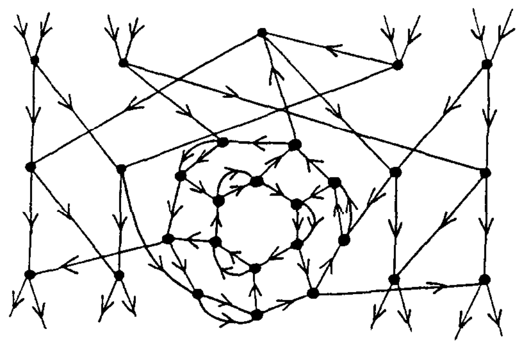

Fig. 7.

Lemma 10. *H_1 (resp. H_2) is decomposable in three distinct ways corresponding to TT, TF and FT-decompositions of the subdigraphs H_1^1 and H_1^2 (resp. H_2^1 and H_2^2).*

Proof. We only have to apply Lemma 4 and the remarks made in Lemmas 8 and 9 about the arcs u_1, u_2 and u_3. □

2.2. *The transformation $\mathcal{T}$*

Let $C = \{C_1, C_2, \ldots, C_p\}$ be a set of clauses in variables $u_1, u_2, \ldots, u_s$ such that each clause C_i consists of three literals $x_{i,1}, x_{i,2}, x_{i,3}$.

We now describe a transformation $\mathcal{T}$ which associates with C a digraph $G = (X, U)$ such that $\forall x \in X$, $d^+(x) = d^-(x) = 2$.

(i) With each clause C_i, we associate a digraph isomorphic with G_6, denoted $G_6(i)$.

(ii) With each variable u_j, we associate a digraph isomorphic with G_{p_1,p_2}, where p_1 (resp. p_2) is the number of clauses C_i which contain u_j (resp. $\bar{u}_j$), and denoted $G_{p_1,p_2}(j)$.

(iii) We define now how to connect the digraphs associated with the clauses C_i and the variables u_j in (i) and (ii):

Suppose u_k (resp. $\bar{u}_k$) is a literal of C_i. Then we connect a subdigraph isomorphic with G_1 (and not yet used) of $G_6(i)$ with one of the p_1 first subdigraphs (resp. one of the p_2 last subdigraphs) not yet used of $G_{p_1,p_2}(k)$, building a digraph isomorphic with H_1 or H_2 according as the subdigraph of $G_{p_1,p_2}(k)$ is isomorphic with G_1 or G_2.

So, we have defined a polynomial transformation $\mathcal{T}$ which associates with a set C of clauses a digraph $G=(X,U)$.

2.3. The main result

We can now prove:

Theorem 1. *C is satisfiable iff G has a decomposition.*

Proof. First, suppose G has a decomposition $\{C,\bar{C}\}$. For each k, and by Lemma 5, this decomposition gives a truth assignment to $G_{p_1,p_2}(k)$ (the truth assignment of one of the p_1 first subdigraphs isomorphic with G_1). Let us consider a clause C_i. By Lemma 7, one of the subdigraphs G_5^j of $G_6(i)$ is F-decomposed. In the transformation $\mathcal{T}$, G_5^j is connected to a subdigraph H_h of $G_{p_1,p_2}(k)$. Then, by Lemma 9 or 10, H_h is T-decomposed. So, if we take for truth assignment of u_k the truth assignment of the decomposition of $G_{p_1,p_2}(k)$ at least one literal of C_i has value true, for any i, which shows that C is satisfiable.

Conversely, suppose C is satisfiable, and consider, for any k, a corresponding truth assignment for the literal u_k. Let $C_i=\{x_{i,1},x_{i,2},x_{i,3}\}$ be a given clause; we choose a variable u_s such that $x_{i,j}$ is true. Then, we F-decompose the subdigraph G_5^h of $G_6(i)$ connected to $G_{p_1,p_2}(s)$ and we T-decompose the two other subdigraphs G_5^l of $G_6(i)$. In last, for any k, we decompose the digraph $G_{p_1,p_2}(k)$ according to the truth assignment of the variable u_k.

Then, by Lemmas 5, 7, 9 and 10 the decompositions described just above give rise to a decomposition of G. □

Remark. It is possible to work only with simple digraphs (i.e. digraphs without parallel arcs) in the previous theorem; we have only to substitute the digraph $G=(X,U)$ with $X=\{x,y,z,z'\}$, $U=\{[x,z],[x,z'],[z,z'],[z',z],[z,y],[z',y]\}$ for two parallel arcs $[x,y]$.

3. NP-completeness of 2-ANTIDEC

Let $G=(X,U)$ be a digraph such that $\forall x\in X$, $d^+(x)=d^-(x)=2$. We shall prove that determining whether G can be partitioned by two hamiltonian anti-circuits, problem called 2-ANTIDEC, is NP-complete.

For this, we shall exhibit a polynomial reduction $\mathcal{T}'$ from 2-DEC, which uses the following digraph $G_7=(X_7,U_7)$ with

$$X_7=\{x,t,t',y\},$$
$$U_7=\{[t,x],[t,t'],[t',t],[t',x],[y,t],[y,t']\}.$$

Lemma 11. *There is only one minimal anti-path partition of G_7.*

Proof. The partition is $\{[x, t, t', y], [x, t', t, y]\}$. $\square$

Thus, we define $\mathscr{T}'$ by:

(i) We associate a graph $G_7(i)$, isomorphic with G_7, with any vertex i. We call x_i, t_i, t_i', y_i the vertices of $G_7(i)$.

(ii) We associate an arc $[y_i, x_j]$ with an arc $[i, j]$.

With $\mathscr{T}'$, we obtain a digraph $G' = (X', U')$ and we have:

Theorem 2. *G has a decomposition iff G′ has a partition by two hamiltonian anti-circuits.*

Proof. First, suppose G is decomposed by $C_1 = [1, i_2, i_3, \dots, i_n, 1]$ and $C_2 = [1, j_2, \dots, j_n, 1]$. Then

$$C_1' = [y_1, x_{i_2}, t_{i_2}, t_{i_2}', y_{i_2}, x_{i_3}, \dots, y_{i_n}, x_1, t_1, t_1', y_n],$$

$$C_2' = [y_1, x_{j_2}, t_{j_2}', t_{j_2}, x_{j_3}, \dots, y_{j_n}, x_1, t_1', t_1, y_1]$$

are two hamiltonian anti-circuits which partition U'.

Conversely, suppose U' is partitioned by two anti-circuits C_1' and C_2'. Suppose C_i', $i = 1$ or 2, contains the arc $[y_j, x_k]$. Then, by Lemma 11, C_i' must contain an arc $[x_t, t_k]$ or an arc $[x_k, t_k']$. So, by replacing a subanti-path $[x_k, t_k', t_k, y_k]$ (or $[x_k, t_x, t_k', y_k]$) by the vertex k, the two anti-circuits C_1' and C_2' lead to hamiltonian disjoint circuits C_1 and C_2 or G, which gives the result. $\square$

We also deduce from 2-DEC the NP-completeness of several problems. These problems are concerned with some invariants we now define. Let $\mathscr{P}$ be a family of elementary paths (resp. elementary chains) of a digraph (resp. graph) G. If each arc (resp. edge) of G lies on at least one element of $\mathscr{P}$, then $\mathscr{P}$ is a path-covering (resp. chain-covering) of G; if each arc (resp. edge) lies on exactly one element of $\mathscr{P}$, then $\mathscr{P}$ is a path-partition (resp. chain-partition) of G. Similarly, we define an anti-path partition of a digraph G. One defines the following invariants:

$r'(G)$ is the minimum cardinality of a path-covering of a digraph G,

$\varrho'(G)$ is the minimum cardinality of a chain-covering of a graph G,

$p'(G)$ is the minimum cardinality of a path-partition of a digraph G,

$\pi'(G)$ is the minimum cardinality of a chain-partition of a graph G,

$p''(G)$ is the minimum cardinality of an anti-path-partition of a digraph G. These invariants have been introduced by Harary [6] for the graphs, Chaty and Chein [3] for digraphs and studied by many authors, in particular [7, 9, 11, 1, 3, 4, 8].

4. NP-completeness of 2-PAR an 2-REC

We shall call 2-PAR (resp. 2-REC) the following problem: given a graph $G=(X,E)$ with maximum degree 4, does it satisfy $\pi'(G)$ (resp. $\varrho'(G)=2$)?

We first prove that 2-PAR is NP-complete by exhibiting a polynomial transformation $\mathscr{T}_1$ from 2-DEC. For this transformation, we need two particular graphs $G_8=(X_8,E_8)$ and $G_9=(X_9,E_9)$ described in Fig. 8.

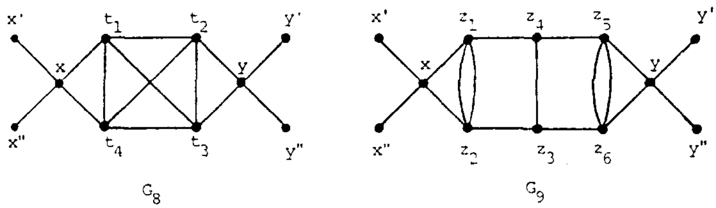

Fig. 8.

Lemma 12. (1) $\pi'(G_8)=2$.

(2) $\pi'(G_9)=3$ *and no minimal chain-partition can contain neither* $[x',x,x'']$ *nor* $[y',y,y'']$.

Proof. We just prove the second part of the lemma. First,

$$\mu_1=[x',x,z_1,z_2,z_3,z_4,z_5,z_6,y,y'],$$

$$\mu_2=[x'',x,z_2,z_1,z_4], \qquad \mu_3=[y'',y,z_5,z_6,z_3]$$

is a chain-partition of G_9.

Now, consider a chain-partition of G_9 including the chain $\mu_1'=[x',x,x'']$. If xz_1 and xz_2 lie on a same chain μ_2', two other chains are needed to partition the parallel edges z_1z_2, so the path-partition cannot be minimal. If xz_1 belongs to a chain μ_2' and xz_2 to a chain μ_3', four vertices have odd degree in $G_9-\{x\}$, so the chain-partition cannot be minimal. □

Now, we can define $\mathscr{T}_1$. Let $G=(X,U)$ a digraph such that $\forall x\in X$, $d^+(x)=d^-(x)=2$. We can suppose $X=\{1,2,\dots,n\}$.

(i) Let $i\geq 3\in X$. We substitute to i a graph $G_8(i)$ isomorphic with $G_8-\{x',x'',y',y''\}$ with vertices x_i,y_i,t_j^i, $1\leq j\leq 4$. We substitute to 1 (resp. 2) a graph $G_9(1)$ (resp. $G_9(2)$) isomorphic with $G_9-\{x',x'',y',y'\}$ with vertices x_1,y_1,z_j^1, $1\leq j\leq 6$ (resp. x_2,y_2,z_j^2, $1\leq j\leq 6$).

(ii) We replace the arc $[i,j]$ by an edge x_iy_j. With $\mathscr{T}_1$, which is obviously polynomial, we build a graph $G'=(X',E')$ with exactly four vertices of degree three $(z_3^1,z_4^1,z_3^2,z_4^2)$, the others being of degree four.

Theorem 3. *G has a decomposition iff* $\pi'(G')=2$.

Proof. Suppose G is decomposed by the two hamiltonian circuits $C_1 = [1, i_2, i_3, \dots, i_n, 1]$ and $C_2 = [2, j_2, \dots, j_n, 2]$. With C_1, we define the chain

$$\mu_1 = [z_4^1, z_1^1, z_2^1, x_1, y_{i_2}, \xrightarrow{i_2}, x_{i_2}, y_{i_3}, \dots, y_1, z_5^1, z_6^1, z_3^1]$$

and with C_2, the chain

$$\mu_2 = [z_4^2, z_1^2, z_2^2, x_2, y_{j_2}, \overset{j_2}{\rightsquigarrow}, x_{j_2}, y_{j_3}, \dots, y_2, z_5^2, z_6^2, z_3^2]$$

where

$$\xrightarrow{i} = [y_i, t_4^i, t_3^i, t_1^i, t_2^i, x_i] \quad \text{and} \quad \overset{i}{\rightsquigarrow} = [t_i, t_3^i, t_2^i, t_4^i, t_1^i, x_i].$$

Then, $\{\mu_1, \mu_2\}$ is a chain-partition of G': if the edge e comes from an arc u of G, e is on μ_i iff u were on C_i. If the edge e belongs to a G_i, e is either on μ_i or in μ_2, because $\{\xrightarrow{i}, \overset{i}{\rightsquigarrow}\}$ is a chain-partition of G_i.

Conversely, suppose $\pi'(G') = 2$ and let μ_1 and μ_2 the elements of a chain-partition of G'. By Lemma 12, a chain μ_i joins a cubic vertex of $G_9(1)$ (for example) to the other cubic vertex of $G_9(1)$. By deleting all the subchains of μ_i of the kind $\xrightarrow{i}$ or $\overset{i}{\rightsquigarrow}$, we obtain a circuit C_i in G; it is easy to see that C_1 and C_2 decompose G. $\square$

Remark. If we want to work only with simple graphs (i.e. graphs without parallel edges), we have only to substitute the graph $G_7 - \{x', x'', y', y''\}$ for two parallel edges xy.

Let us remark that the graph G' obtained with $\mathcal{T}_1$ satisfies the following property:

$$\text{(H)} \qquad \begin{cases} X' = X_1 \cup X_2 & \text{with } X_1 \cap X_2 = \emptyset, \qquad |X_1| = 4 \quad \text{and} \\ \forall x \in X_1, \quad d(x) = 3, & \forall x \in X_2, \quad d(x) = 4. \end{cases}$$

Call $\mathcal{G}$ the set of graphs satisfying (H). The NP-completeness of 2-REC will be an immediate consequence of the following property:

Proposition 4. *Let $G \in \mathcal{G}$. $\pi'(G) = 2$ iff $\varrho'(G) = 2$.*

Proof. Obvious. $\square$

5. NP-completeness of 2-$\overrightarrow{\text{PAR}}$, 2-$\overrightarrow{\text{ANTIPAR}}$ and 2-$\overrightarrow{\text{REC}}$

Let $G = (X, U)$ be a digraph such that $\forall x \in X$, $d^+(x) \le 2$ and $d^-(x) \le 2$. For such a digraph, we shall call 2-$\overrightarrow{\text{PAR}}$ (resp. 2-$\overrightarrow{\text{ANTIPAR}}$ (resp. 2-$\overrightarrow{\text{REC}}$)) the following problem: does it satisfy $p'(G) = 2$ (resp. $p''(G) = 2$, (resp. $r'(G) = 2$))?

We shall exhibit polynomial transformations $\mathcal{T}_2$ and $\mathcal{T}_3$ from 2-DEC to 2-$\overrightarrow{\text{PAR}}$

and $\overrightarrow{\text{2-ANTIPAR}}$. For these transformations, we need the following two digraphs $G_{10}=(X_{10}, U_{10})$ and $G_{11}=(X_{11}, U_{11})$, see Fig. 9.

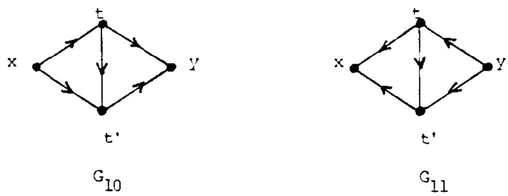

Fig. 9.

Lemma 13. (1) *$p'(G_{10})=3$ and there is only one minimal path-partition of G_{10} containing a hamiltonian path from x to y.*

(2) *$p''(G_{11})=3$ and there is only minimal anti-path-partition of G_{11} which contains an hamiltonian antipath from x to y.*

Proof. It is straightforward. $\square$

We can now define the transformations $\mathscr{T}_2$ and $\mathscr{T}_3$.

Let $G=(X, U)$ be a digraph such that $\forall x \in X$, $d^+(x)=d^-(x)=2$ (we assume $X=\{1,2,\dots,n\}$).

(i) Consider any two vertices of G, say 1 and 2. We substitute a digraph $G_{10}(1)$ (resp. $G_{10}(2)$) isomorphic with G_{10} for the vertex 1 (resp. 2). Let us denote by x_i, t_i, t_i', y_i, $i=1$ or 2, the vertices of $G_{10}(i)$.

For $i=1$ or 2, we substitute an arc $[j, x_i]$ (resp. $[y_i, k]$) for an arc $[j, i]$ (resp. $[i, k]$).

This transformation $\mathscr{T}_2$ gives rise to a digraph $G'=(X', U')$.

(ii) We associate a digraph $G_{11}(i)$, isomorphic with G_{11}, with the vertex i, for $i=$ or 2, and a digraph $G_7(j)$, isomorphic with G_7, for any vertex j, with $j \geq 3$. We denote by x_i, t_i, t_i', y_i the vertices for $G_{11}(i)$ (resp. $G_7(i)$).

We associate an arc $[y_i, x_j]$ with an arc $[i, j]$ of G.

We obtain so a digraph $G''=(X'', U'')$ with this transformation $\mathscr{T}_3$.

We then have:

Theorem 5. (1) *G has a decomposition iff $p'(G')=2$.*

(2) *G has a decomposition iff $p''(G'')=2$.*

Proof. (1) Suppose G is decomposed by

$$C_1=[1=i_1, i_2, \dots, i_n, 1] \quad \text{and} \quad C_2=[2=j_1, j_2, \dots, j_n, 2].$$

The paths

$$\mu_1 = [t_1', y_1, i_2, \dots, x_2, t_2, t_2', y_2, \dots, i_n, x_1, t_1],$$

$$\mu_2 = [t_1', y_2, j_2, \dots, x_1, t_1, t_1', y_1, \dots, j_n, x_2, t_2]$$

partition the arcs of G'.

Conversely, suppose $p'(G')=2$ and let $\{\mu_1, \mu_2\}$ be a path-partition of G'. Each μ_i partitions the arcs of $G_{11}(1)$ and $G_{11}(2)$, so by Lemma 13, μ_1 and μ_2 are of the kind described in the first part of the proof. So each path gives rise to a hamiltonian circuit in G, disjoint of the other.

(2) The demonstration is similar to the previous one. □

Finally, we can prove that $\overrightarrow{2\text{-REC}}$ is an NP-complete problem.

Let us remark that the digraph G' obtained with the transformation $\mathcal{T}_2$ satisfies the property:

$$(\mathrm{H}') \qquad \begin{cases} X = X_1 \cup X_2 \cup X_3 & \text{with } X_i \cap X_j = \emptyset \quad \text{if } i \neq j, \\ |X_2| = |X_3| = 2, & \forall x \in X_1, \quad d^+(x) = d^-(x) = 2, \\ \forall x \in X_2, \quad d^+(x) = 2, \quad d^-(x) = 1, & \forall x \in X_3, \quad d^+(x) = 1, \quad d^-(x) = 2. \end{cases}$$

Let us denote $\mathcal{G}'$ the set of digraphs satisfying (H′). Then, the NP-completeness of $\overrightarrow{2\text{-REC}}$ is an easy consequence of Theorem 5 and of the following property:

Proposition 6. *Let $G \in \mathcal{G}'$. Then $p'(G)=2$ iff $r'(G)=2$.*

Proof. It is straightforward. □

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