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CONNECTIVITY AND EDGE-DISJOINT SPANNING TREES *

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1. Introduction

The purpose of this note is to discuss some simple, but useful consequences of a theorem due independently to Tutte and Nash-Williams [1,2]. The most important consequence is that any undirected graph with edge connectivity 2k must contain at least k mutually edge-disjoint spanning trees. This result by itself is an interesting property of connectivity, and has applications in multi-casting in computer networks, in generalizations of the k-paths problem, in the subgraph homeomorphism problem, and in wire-routing problems. The theorem of Tutte/Nash-Williams also has applications in the study of graph vulnerability, allowing a more refined measure of vulnerability than that based on simple edge connectivity. Using algorithms for matroid partition, a close approximation of the measure can be computed in polynomial time.

Definition 1. Let $G = (N, E)$ be a connected undirected (multi-)graph. For S a subset of edges, let G/S be the graph resulting from deleting the edges in S from G, and let $q(G/S)$ be the number of connected components in G/S.

Definition 2. The *edge connectivity* of a connected undirected graph G is the minimum number of edges of any set S such that $q(G/S) \geqslant 2$.

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Definition 3. Two spanning trees of G are called *mutually edge-disjoint* if they have no edges in common. A set of spanning trees is mutually edge-disjoint if every pair of trees is. Let M denote the size of the largest set of mutually edge-disjoint spanning trees in G.

2. The maximum number of edge-disjoint spanning trees

Theorem (Tutte, Nash-Williams [1,2]). *A necessary and sufficient condition for* G *to have* k *mutually edge-disjoint spanning trees is that*

$$k[q(G/S) - 1] \leqslant |S|$$

for all subsets S *of* E. *Hence*

$$M = \min_{S \subseteq E} \lfloor |S|/[q(G/S) - 1] \rfloor .$$

Corollary 1. *If* G *has edge connectivity* 2k, *then* $M \geqslant k$, *and this is the best possible bound.*

Proof. Let

$$\min_{S \subseteq E} |S|/[q(G/S) - 1] = \frac{x}{y},$$

where $|S| = x$ and $y = q(G/S) - 1$. Then G/S consists of $y + 1$ connected components $C(1), \ldots, C(y + 1)$. Let H be the multi-graph resulting from contracting each C(i) in G to a single node. Each node in H must have degree $\geqslant 2k$, since G has edge connectivity 2k. Hence H has at least $2k(y+1)/2$ edges, and $|S| \geqslant k(y+1)$. Therefore

$$\frac{x}{y} \geqslant \frac{k(y+1)}{y} > k \quad \text{and} \quad M = \left\lfloor \frac{x}{y} \right\rfloor \geqslant k.$$

To see that this is the best possible bound, note that the complete graph with n nodes has connectivity $n-1$, and contains only enough edges for $\frac{1}{2}(n-1)$ spanning trees. □

One of the most important interpretations of connectivity is that in a graph with edge connectivity k there are at least k mutually edge-disjoint paths between any two vertices in G. This is Menger's theorem [3]. It is not immediate, however, what this pairwise structure implies for the structure of the entire graph. The above corollary relates the connectivity of a graph to the global structure of the graph, and hence is in itself an important statement about connectivity.

It is interesting to note that a similar, but stronger, observation was made for the case of directed graphs [4]. There, connectivity is defined as the minimum number of edges to remove so that between some pair of nodes i, j, there is no directed path from i to j. In that case, if the directed graph has connectivity k, then there are k edge-disjoint spanning branchings, where the root of each branching may be chosen arbitrarily. It would be interesting to get a proof of Corollary 1 from this directed graph theorem.

Corollary 1 has a direct application in a generalization of the k-paths problem. The k-paths problem is the following: Given a graph G and 2k distinct vertices $s(1),\ldots,s(k)$ and $t(1),\ldots,t(k)$, do there exist k mutually edge-disjoint paths P(i) for $i = 1,\ldots,k$ such that P(i) connects s(i) and t(i) for each i? The k-paths problem is a specialization of the multi-commodity flow problem [5] and has applications in wire routing and circuit design. The k-paths problem can be generalized as follows: Given G and k arbitrary subsets of nodes N(i) for $i = 1,\ldots,k$, do there exist k mutually edge-disjoint sets of edges E(i) for $i = 1,\ldots,k$ such that E(i) connects the nodes in N(i) for all i? The k-paths problem is the case when $|E(i)| = 2$ for all i. The generalized path problem arises in multi-casting in computer networks, i.e., when there are several subsets of nodes in the network, and for each subset, there are broadcast messages that must reach every node in the subset, but need not be sent to nodes outside the subset. Since broadcast traffic can seriously degrade the performance of the system, it is desirable to have totally edge-disjoint sets of edges allocated for the communication paths of each of the subsets.

Clearly if a graph has k mutually edge-disjoint spanning trees, then the answer to the generalized k-paths problem is always "yes", no matter what the k sets N(i) are. For the k-paths problem, the best result today [6] states that if a graph is $2k-3$ edge connected, then the answer to the k-paths problem is always "yes". Corollary 1 states that for a little more connectivity (2k instead of $2k-3$) the answer to the generalized k-paths problem is always "yes".

Corollary 1 also has a useful interpretation in the edge version of the Subgraph Homeomorphism problem [7]. The problem is: Given a graph G with N nodes, and a pattern graph P with $n < N$ nodes and k edges, is there a mapping of the nodes of P to distinct nodes of G such that edges of P are mapped to mutually edge-disjoint paths in G? The corollary then says that the answer is "yes" unless every induced subgraph of G with at least n nodes has connectivity less than 2k. In particular, if G has connectivity 2k, then the answer is "yes". Note that in the directed case, if G has directed connectivity k, and P is a directed graph, then the answer is "yes". Note also that a guaranteed "yes" to the k-paths problem implies a guaranteed "yes" to subgraph homeomorphism only when P consists of k edges that share no endpoints.

3. Graph vulnerability

The area of graph vulnerability concerns the question of how much communication in a network is disrupted by the deletion of edges from the graph. The most fundamental measure of graph vulnerability of a connected graph is the edge connectivity of the graph. More refined measures of vulnerability include the number of edges that must be removed so that at least r disconnected components are created, or the number of edges needed to be removed to disconnect each pair of nodes in a given set of r nodes. More refined measures of vulnerability, such as the two above, are typically hard to compute, and therefore edge connectivity has remained the practical measure of vulnerability. The theorem of Tutte/Nash-Williams can be used to define another useful measure of graph vulnerability. We can interpret $q(G/S) - 1$ as the number of additional components that are created by removing the edge set S from G. Then the set S that minimizes $|S|/[q(G/S) - 1]$ is the set whose removal from G maximizes the number of additional components created, per edge removed. Let T denote this maximum ratio. It seems reasonable that T can be used as a measure of vulnerability; the larger the ratio, the more vulnerable is the graph to large amounts of disconnection for few edge deletions. This seems most useful in comparing two graphs which have the same connectivity. We can further use T to bound from below the above two measures of vulnerability: If $r - 1$ new components result by removing edges, then at least $(r - 1)/T$ edges must be deleted. Such bounds could be useful in branch and bound schemes to compute the measure exactly. A restatement of Corollary 1 leads to the following observation about T.

Corollary 1′. *If graph* G *has connectivity* k, *then* $T \geqslant 1/k$, *and*

$$\frac{1}{M+1} \leqslant T \leqslant \frac{1}{M} \leqslant \frac{2}{k}.$$

The bound $T \leqslant 2/k$ is again a statement relating the structure of the whole graph to its connectivity. Clearly at least k edges must be removed in order to create the first new component, but it is conceivable that some very 'strategic' first cut(s) could be found so that each successive component requires substantially fewer deleted edges. Corollary 1′ allows some improvement over k edges per each new component, but less than $\frac{1}{2}k$ edges per new component is impossible.

It is not clear whether T can be computed in polynomial time, however, M can be computed in polynomial time by matroid partitioning algorithms [5], and so T can be very closely approximated using Corollary 1′. The matroid partition algorithm finds a partition of the edges of G into the fewest number of mutually edge-disjoint forests, and has the property that it lexicographically maximizes (over all partitions of G into forests) the number of edges in the forests found. Hence if there are M mutually edge-disjoint spanning trees, then the matroid partition algorithm will find them all.

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