Monstrous Menagerie with Vandehey 7/2

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1 Probability and Invariant measures.

Definition 1.1. μ is a *probability measure* if $\mu(X) = 1$.

If $\mu(X)$ is finite, we can always renormalize to get a probability measure:

$$\mu^*(A) = \frac{\mu(A)}{\mu(X)}.$$

Sometimes $\mu(X)$ is infinite. These are basically the only two possibilities (other than the zero measure, which is boring).

Definition 1.2. Given $(X, \mathcal{A}, T, \mu)$ (space, σ-algebra, transformation, measure) we say μ is T*-invariant,* if for all $A \in \mathcal{A}$,

$$\mu(T^{-1}A) = \mu(A),$$

where $T^{-1}(A) = \{x \in X : Tx \in A\}$.

Two questions here: Why is it important? Because almost everything we want to do requires invariance.

If μ is not invariant, define $\mu_k(A) = \mu(T^{-k}A)$, and then we can take a sort of limit of μ_k to get an invariant measure.

The other question is: Why is it $T^{-1}A$, why not just TA? Because T^{-1} preserves all of the information. I can start with two points x and y and apply T and get a single point. For example with base b expansions, two points which differ in their first digit end up at the same point after applying T.

On the other hand, for T^{-1} we know where we came from, we can just apply T to any point in $T^{-1}A$. $TT^{-1}x = x$, but $T^{-1}Tx = ?$.

1.1 Proving invariance

Theorem 1.1. *Suppose $\mathscr{A}$ is a semi-algebra that generates a σ- algebra $\mathcal{A}$. If $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathscr{A}$, then μ is T-invariant.*

Proof. Define $\mu'(A) = \mu(T^{-1}A)$. μ' is also a measure on $(X, \mathcal{A})$, so by the uniqueness of the Caratheodory Extension Theorem, and the fact that the given information is that $\mu'(A) = \mu(A)$ for $A \in \mathscr{A}$, we conclude that $\mu = \mu'$ as desired. ■

Example 1.1. For base b, λ is T-invariant.

$$T^{-1}x = \left\{\frac{0}{b} + \frac{x}{b}, \frac{1}{b} + \frac{x}{b}, \ldots, \frac{b-1}{b} + \frac{x}{b}\right\}$$

$$\lambda(T^{-1}[x,y]) = \lambda\left(\bigcup_{i=0}^{b-1}\left[\frac{i+x}{b},\frac{i+y}{b}\right]\right) = \sum_{i=0}^{b-1}\lambda\left(\left[\frac{i+x}{b},\frac{i+y}{b}\right]\right) = \sum_{i=0}^{b-1}\frac{y-x}{b} = y-x$$

Since the semi-algebra of intervals generates the Lebesgue σ-algebra, λ is T-invariant.

Prove that λ is not invariant for regular CF expansions.