Monstrous Menagerie with Vandehey 7/9

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$\omega_s(y)$.

$$\frac{\mu(E \cap C_s)}{\mu(C_s)} = \frac{\int_{E \cap T^k C_s} \omega_s(y)\, d\mu(y)}{\int_{T^k C_s} \omega_s(y)\, d\mu(y)}$$

If $T^k x = y$, with $x \in C_s$, then

$$\omega_s(y) = \frac{1}{|T^{k\prime} x|}$$

Remark. In general, $T(A \cap B) \neq TA \cap TB$, but it is in fact true

$$T^k(T^{-k} E \cap C_s) = E \cap T^k C_s.$$

1 Proving Ergodicity

Theorem 1.1. *Suppose we have a fibred system with the following*

1. $\mu(X) = 1$
2. *There exists a semi-algebra $\mathscr{A}$ that generates $\mathcal{A}$ such that each $A \in \mathscr{A}$ can be expressed as a disjoint union of countably many* full *cylinder sets. C_s is* full *if $T^{|s|} C_s = X$.*
3. *(Renyi's condition) There exists a uniform $M \geq 1$ such that for all admissible strings s, we have*

$$\frac{\sup_{y \in T^{|s|} C_s} \omega_s(y)}{\inf_{y \in T^{|s|} C_s} \omega_s(y)} \leq M.$$

Then T is ergodic.

This theorem goes by many names. It's even called the *folklore theorem.*

Usually you need something like this. You can break it a bit, but if you go too far, usually any hope of proving ergodicity is lost.

Example 1.1 (Base-b)**.** For base-b, this is easy. All cylinders are full, and Renyi's condition is satisfied because $\omega_s(y) = b^{-|s|}$.

Example 1.2 (β-expansions)**.** What about β-expansions? Here Renyi's condition is still easy, since we still have constant Jacobian, $\omega_s(y) = \beta^{-|s|}$. Condition 2 is not so easy. There will be a leftover bit corresponding to the last digit which won't be a full cylinder. The solution is to iterate the map enough times until you get a full cylinder inside the last digit cylinder. Then you can recurse on the remaining piece of the last cylinder. You will get countably many full cylinders that fill the bad cylinder.

Example 1.3 (RCF)**.** Full cylinders are easy again from drawing a graph. Renyi's condition is much less nice.

$\omega_s(y)$ can be related to the numerators and denominators of the convergents

$$p_{n-1}, p_n, q_{n-1}, q_n$$

in the continued fraction expansion of s.

$\omega_s(y)$ depends on these. Not sure exactly how, but it will work out.

This is ok for all the things we've talked about. However, if we go to 2-dimensions, e.g., the complex continued fractions expansion, we immediately lose these nice properties. (Like the full cylinder sets condition).

Proof. Suppose E is any invariant set of positive measure. We want to show that $\mu(E^C) = 0$. Let C_s be any full cylinder.

Then

$$\frac{\mu(E \cap C_s)}{\mu(C_s)} = \frac{\int_{E \cap T^k C_s} \omega_s(y)\, d\mu(y)}{\int_{T^k C_s} \omega_s(y)\, d\mu(y)}.$$

Since C_s is full, $T^{|s|} C_s = X$, so we have

$$\frac{\mu(E \cap C_s)}{\mu(C_s)} = \frac{\int_E \omega_s\, d\mu}{\int_X \omega_s\, d\mu} \geq \frac{\inf_{y \in X} \omega_s(y) \int_E\, d\mu}{\sup_{y \in X} \omega_s(y) \int_X\, d\mu} \geq \frac{\mu(E)}{M}$$

Let $A \in \mathscr{A}$. We assumed $A = \bigcup_{n \in \mathbb{N}} C_{s_n}$, with all of the C_{s_n} full. Therefore

$$\begin{aligned} \mu(E \cap A) &= \mu\left(E \cap \bigcup_{n \in \mathbb{N}} C_{s_n}\right) \\ &= \mu\left(\bigcup_{n \in \mathbb{N}} E \cap C_{s_n}\right) \\ &= \sum_{n \in \mathbb{N}} \mu(E \cap C_{s_n}) \\ &\geq \sum_{n \in \mathbb{N}} \frac{\mu(E)\mu(C_{s_n})}{M} \\ &= \frac{\mu(E)}{M} \sum_{n \in \mathbb{N}} \mu(C_{s_n}) \\ &= \frac{\mu(E)}{M} \mu\left(\bigcup_{n \in \mathbb{N}} C_{s_n}\right) \\ &= \frac{\mu(E)\mu(A)}{M}. \end{aligned}$$

So let $\delta = \frac{\mu(E)}{M}$. Then Knopp's lemma applies with this δ, so $\mu(E^C) = 0$. ■

2 Asymptotic notation

If we write $f(x) = O(g(x))$, then we mean that there exists $C > 0$ such that

$$|f(x)| \leq C|g(x)|$$

(for large enough x).

$f(x) = o(g(x))$ will mean

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

$f(x) \asymp g(x)$ will mean that both $f = O(g)$ and $g = O(f)$. Finally, if we write $f(x) \sim g(x)$, we mean $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

We might also use it to write something like $x^2 + 3x + 2 = x^2 + O(x)$, to hide lower order terms.

3 Copeland-Erdős method

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. Consider the number

$$0.f(1)f(2)f(3)f(4)f(5)f(6)\cdots,$$

where $f(1)f(2)$ means write $f(1)$ and $f(2)$ in base-b and concatenate them.

The idea here is that: If most $f(n)$ look like normal numbers on a small finite scale, the whole thing should also look normal.

Definition 3.1. Let a be an integer, $a \geq 1$. Let s be a string. Let

$$\nu_s(a) = \#\text{ of times } s \text{ appears in } a.$$

Let

$$\nu(a) = \#\text{ of digits in } a$$

We say a is (ϵ, k) normal if for all s with $|s| = k$, we have

$$\left| \frac{\nu_s(a)}{\nu(a)} - \frac{1}{b^k} \right| \leq \epsilon$$

So if most $f(n)$ are (ϵ, k)-normal, then the frequency with which any s of length k appears in our construction should be

$$\frac{1}{b^k} + O(\epsilon).$$

We have to be careful. We need to worry about strings that start in $f(n)$ and ends in $f(n+1)$. These should be small if most $f(n)$ are large. We also need to worry if some $f(n)$ is explosively large. We say $S \subseteq \mathbb{N}$ is *meager* if

$$\#\{n \in S : n \leq m\} \leq m^{1-\delta}$$

for some fixed δ (and all large m).

We say $S \subseteq \mathbb{N}$ has asymptotic density 0 if

$$\#\{n \in S : n \leq m\} = o(m).$$

Note. The set of primes has asymptotic density 0, but is *not* meager.

Definition 3.2. We say $f : \mathbb{N} \to \mathbb{N}$ is *almost bijective* if the preimage of any meager set has asymptotic density 0.

Theorem 3.1. *Suppose $f : \mathbb{N} \to \mathbb{N}$ is almost bijective. If, in addition,*

$$m = o\left(\sum_{n=1}^{m} \nu(f(n))\right),$$

and

$$m \cdot \max_{1 \leq n < m} \nu(f(n)) = O\left(\sum_{n=1}^{m} \nu(f(n))\right),$$

then

$$0.f(1)f(2)f(3)f(4)\cdots$$

is base-b normal.

The almost bijective criterion guarantees that most things are (ϵ, k) normal. The first asymptotic criterion makes sure that the $f(i)$ grow larger and larger. The other asymptotic criterion makes sure that individual numbers aren't too large.

$f = \mathrm{id}$ works, but so does the Euler φ function, or indeed $f = \varphi^n$.

Lemma 3.1. *Let $c \in (0, 1)$. Then as $n \to \infty$,*

$$\binom{n}{cn} = \frac{n!}{(cn)!((1-c)n)!} = \frac{1}{\sqrt{2\pi c(1-c)n}}\left[c^{-c}(1-c)^{-(1-c)}\right]^n (1 + O(1/n))$$

Proof. Stirling's formula. (This is the one thing to remember from this class)

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n)).$$

Everything here is not too hard, except for the constant of 2π.

Consider

$$\log(n!) = \sum_{i=1}^{n} \log i \approx \int_1^n \log x \, dx = x \log x - x\big|_{x=1}^{n} = n \log n - n + 1 = \log\left(\frac{n}{e}\right)^n + 1$$

■