1 Computing CF Expansions

Consider $\sqrt{2}$.

We know

$$\sqrt{2} = 1 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots$$

so

$$\sqrt{2} - 1 = \frac{a_1 + \theta_1}{,}$$

and

$$\frac{1}{\sqrt{2}-1} = a_1 + \theta_1,$$

so $a_1 = \left\lfloor \frac{1}{\sqrt{2}-1} \right\rfloor$.

Then

$$\sqrt{2} = 1 + \frac{1}{a_1+}\frac{1}{a_2+\theta_2},$$

and

$$\frac{1}{\sqrt{2}-1} = a_1 + \frac{1}{a_2+\theta_2},$$

so

$$\frac{1}{\frac{1}{\sqrt{2}-1} - a_1} = a_2 + \theta_2.$$

Thus

$$a_2 = \left\lfloor \frac{1}{\frac{1}{\sqrt{2}-1} - a_1} \right\rfloor$$

1.1 Comparing to base-10

Consider $x \in [0,1)$. a_1 corresponds to d where $x \in [\frac{d}{10}, \frac{d+1}{10})$. a_2 corresponds to d where $x \in [\frac{a_1}{10} + \frac{d}{100}, \frac{a_1}{10} + \frac{d+1}{100})$

Consider instead $x_1 = 10x - a_1$, then $a_2 = d$, where $x_1 \in [\frac{d}{10}, \frac{d+1}{10})$, $x_2 = 10x - a_2 = \{10x\}$. $a_3 = d$ where $x_2 \in [\frac{d}{10}, \frac{d+1}{10})$ and so on.

2 Dynamical system (fibred system)

Definition 2.1. A *dynamical system* consists of

1) A space X, e.g., $[0,1)$,

2) A set of digits $\mathscr{D}$, finite or countably infinite,

3) A partition $\mathscr{X}$ of X into
$$\{X_d : d \in \mathscr{D}\},$$

4) And a transformation, $T : X \to X$, which is injective on each X_d,

5) A σ-algebra $\mathcal{F}$ and a measure μ.

Given $x \in X$, let $a_1(x) = d$, where $x \in X_d$. Let $a_2(x) = a_1(Tx)$. In general, let $a_n(x) = a_1(T^{n-1}x)$.

Definition 2.2. The sequence $(a_1(x), a_2(x), a_3(x), \ldots, a_n(x), \ldots)$ is the expansion of x in this system.

Example 2.1. For base-b, we have $X = [0,1)$, $\mathscr{D} = \{0,1,\ldots,b-1\}$, $X_d = [\frac{d}{b}, \frac{d+1}{b})$, and $Tx = \{bx\}$.

Picture: Graph the transformation for base 2, base 3.

Example 2.2. β-expansion is not great. We'll use *greedy* β-expansion, which tries to maximize leftward digits. With greedy β-expansion, we have $X = [0,1)$, $\mathscr{D} = \{0,1,\ldots,\lceil\beta\rceil - 1\}$,

$$X_d = \begin{cases} \left[\dfrac{d}{\beta}, \dfrac{d+1}{\beta}\right) & d < \lceil\beta\rceil - 1 \\ \left[\dfrac{d}{\beta}, 1\right) & d = \lceil\beta\rceil - 1, \end{cases}$$

and $Tx = \{\beta x\}$

Picture: Graph the transformation.

Note $\{b\{bx\}\} = \{b^2x\}$, when $b \in \mathbb{Z}$, but not when $b = \beta$ is not an integer.

Does this do what we want it to do?

We have

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}.$$

Then

$$\{\beta x\} = \left\{a_1 + \sum_{i=1}^{\infty} \frac{a_{i+1}}{\beta}\right\}.$$

It's not clear that the fractional part just deletes the a_1, but if we're using greedy expansions, this should be true.

Example 2.3. For regular CF, $X = [0,1)$, $\mathscr{D} = \mathbb{N} \cup \{\infty\}$,

$$X_d = \begin{cases} \left(\dfrac{1}{d+1}, \dfrac{1}{d}\right] & d \in \mathbb{N} \\ \{0\} & d = \infty \end{cases},$$

and $Tx = \left\{\frac{1}{x}\right\}$.

Checking that this works, we have

$$T\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\cdots = \left\{a_1 + \frac{1}{a_2+}\frac{1}{a_3+}\cdots\right\} = \frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\cdots$$

What's with the ∞? $T(1/5) = \{5\} = 0$.

Picture: Plot Tx.

Even CF picture: Reflects $1/x$ back and forth between $y = 0$ and $y = 1$.

Odd CF: Flips the intervals that the even one does not.

Example 2.4. For Lüroth series, we have $X = (0,1]$, $\mathscr{D} = \mathbb{N}_{\geq 2}$, $X_d = \left(\frac{1}{d}, \frac{1}{d-1}\right]$, $Tx = \{d(d-1)x\}$, for $x \in X_d$.

Picture: Lines of increasing slope.

This is why Lüroth series have the same periodic points as base-b expansions. We keep multiplying by an integer, and subtracting an integer, so we stay rational, and maybe shrink the denominator.

The injectivity of T on X_d is necessary for the uniqueness of representations, but not necessarily sufficient.

3 Convergence