Monstrous Menagerie with Vandehey 7/2

Jason Schuchardt

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1 Probability and Invariant measures.

Definition 1.1. μ is a *probability measure* if $\mu(X) = 1$.

If $\mu(X)$ is finite, we can always renormalize to get a probability measure:

$$\mu^*(A) = \frac{\mu(A)}{\mu(X)}.$$

Sometimes $\mu(X)$ is infinite. These are basically the only two possibilities (other than the zero measure, which is boring).

Definition 1.2. Given $(X, \mathcal{A}, T, \mu)$ (space, σ-algebra, transformation, measure) we say μ is *T-invariant,* if for all $A \in \mathcal{A}$,

$$\mu(T^{-1}A) = \mu(A),$$

where $T^{-1}(A) = \{x \in X : Tx \in A\}$.

Two questions here: Why is it important? Because almost everything we want to do requires invariance.

If μ is not invariant, define $\mu_k(A) = \mu(T^{-k}A)$, and then we can take a sort of limit of μ_k to get an invariant measure.

The other question is: Why is it $T^{-1}A$, why not just TA? Because T^{-1} preserves all of the information. I can start with two points x and y and apply T and get a single point. For example with base b expansions, two points which differ in their first digit end up at the same point after applying T.

On the other hand, for T^{-1} we know where we came from, we can just apply T to any point in $T^{-1}A$. $TT^{-1}x = x$, but $T^{-1}Tx = ?$.

1.1 Proving invariance

Theorem 1.1. *Suppose $\mathscr{A}$ is a semi-algebra that generates a σ- algebra $\mathcal{A}$. If $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathscr{A}$, then μ is T-invariant.*

Proof. Notetaker's proof sketch:

Define $\mu'(A) = \mu(T^{-1}A)$. μ' is also a measure on $(X, \mathcal{A})$, so by the uniqueness of the Caratheodory Extension Theorem, and the fact that the given information is that $\mu'(A) = \mu(A)$ for $A \in \mathscr{A}$, we conclude that $\mu = \mu'$ as desired. (Uniqueness requires σ-finiteness) ∎

Example 1.1. For base b, λ is T-invariant.

$$T^{-1}x = \left\{ \frac{0}{b} + \frac{x}{b}, \frac{1}{b} + \frac{x}{b}, \dots, \frac{b-1}{b} + \frac{x}{b} \right\}$$

$$\lambda(T^{-1}[x,y]) = \lambda\left(\bigcup_{i=0}^{b-1}\left[\frac{i+x}{b}, \frac{i+y}{b}\right]\right) = \sum_{i=0}^{b-1}\lambda\left(\left[\frac{i+x}{b}, \frac{i+y}{b}\right]\right) = \sum_{i=0}^{b-1}\frac{y-x}{b} = y-x$$

Since the semi-algebra of intervals generates the Lebesgue σ-algebra, λ is T-invariant.
Exercise 1. Prove that λ is not invariant for regular CF expansions.
Definition 1.3. An *algebra* is a collection $\mathscr{A}$ of subsets of X satisfying

1. $\varnothing \in \mathscr{A}$,
2. if $A, B \in \mathscr{A}$, then $A \cap B \in \mathscr{A}$,
3. if $A \in \mathscr{A}$, then $X \setminus A \in \mathscr{A}$.

To get an algebra from a semi-algebra, take all finite disjoint unions of subsets and add them to the semi-algebra. (This is also the smallest algebra containing the semi-algebra).
Definition 1.4. A monotone class is a collection $\mathscr{C}$ of subsets of X satisfying

1. If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ are all in $\mathscr{C}$, then
$$\bigcup_i E_i \in \mathscr{C}$$
2. If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ are all in $\mathscr{C}$, then
$$\bigcap_i F_i \in \mathscr{C}$$

Once again, if you start with any collection that you'd like, there is a smallest monotone class containing that collection.
Lemma 1.1. *Let $\mathscr{A}$ be any algebra of X. Then the monotone class generated by $\mathscr{A}$ is the same as the σ-algebra generated by $\mathscr{A}$, $\sigma(\mathscr{A})$.*

Proof. Omitted, see example below. You want to prove that the monotone class is a σ-algebra and vice-versa. ∎

Example 1.2. If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ are in the σ-algebra, then

$$\bigcup_i E_i$$

is in the σ-algebra too.

For intersections, observe that we have this helpful equality

$$\bigcap_i F_i = X \setminus \bigcup_i (X \setminus F_i).$$

Proof of theorem. Let

$$\mathscr{C} = \{A \in \mathcal{A} : \mu(T^{-1}A) = \mu(A)\}.$$

We know by assumption, $\mathscr{A} \subseteq \mathscr{C}$, and by construction $\mathscr{C} \subseteq \mathcal{A}$. Moreover, if $\mu(T^{-1}A_i) = \mu(A_i)$, for some collection of disjoint A_is, then it must hold for their union as well.

$$\mu\left(T^{-1}\bigcup_i B_i\right) = \mu\left(\bigcup_i T^{-1}B_i\right) = \sum_i \mu(T^{-1}B_i) = \sum_i \mu(B_i) = \mu\left(\bigcup_i B_i\right).$$

Thus $\mathscr{C}$ contains the algebra generated by $\mathscr{A}$. Next we'd like to apply the lemma, by showing that $\mathscr{C}$ is a monotone class.

Let $E_1 \subseteq E_2 \subseteq \cdots$ be in $\mathscr{C}$. Let $E = \bigcup_i E_i$.

$$\begin{aligned}
\mu(T^{-1}E) &= \mu\left(T^{-1}\bigcup_i E_i\right) \\
&= \lim_{i\to\infty} \mu(T^{-1}E_i) \\
&\text{(maybe requires } \mu \text{ finite/}\sigma\text{-finite, certainly true when } \mu(X) = 1) \\
&= \lim_{i\to\infty} \mu(E_i) \\
&= \mu\left(\bigcup_i E_i\right) \\
&= \mu(E).
\end{aligned}$$

Similarly, we get the other property, so $\mathscr{C}$ is a monotone class.

Recall $\mathscr{C} \subseteq \mathcal{A}$. By lemma, $\mathcal{A}$ is the smallest monotone class containing the algebra, but $\mathscr{C}$ is a monotone class containing the algebra. Thus $\mathscr{C} = \mathcal{A}$.

Recalling that $\mathscr{C}$ is defined to be the collection of all invariant sets, we have invariance of the whole σ-algebra, so we are done. ■

Example 1.3. Other invariant measures:

Let

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

be the Dirac measure.

For base-2,

$$\frac{1}{2}(\delta_{1/3} + \delta_{2/3})$$

is an invariant probability measure.

Story time: There was a mathematician, perhaps Serre, who would shout out in lectures, “Your notation sucks!”

The graduate students appreciated this, so they decided to make him a “Your notation sucks!” shirt, but they needed to presernt it correctly.

So they designed a lecture with the worst possible notation, the conjugate of capital xi, Ξ divided by itself:

$$\frac{\bar{\Xi}}{\bar{\Xi}}$$

He said nothing. So they had to present the shirt after the lecture. (End of story time)

$\delta_{1/3}$ ($\delta_{2/3}$) only cares about 1/3 (2/3).

$$T^{-1}\frac{1}{3} = \left\{\frac{1}{6}, \frac{2}{3}\right\},$$

which still has measure 1/2.

In general, we can construct invariant measures from any periodic point.

2 Why is invariance actually useful? - Poincaré Recurrence

Theorem 2.1 (Poincaré Recurrence)**.** *Let (X, T, μ) be a dynamical system, with μ a T-invariant probability measure.*

Then for any set A with $\mu(A) > 0$, almost all points in A return to A infinitely often (as we iterate T).

It's possible that when we apply T to a point in A, it might leave A and never return, but this happens almost never. Almost every point will land in A infinitely often.

Application: Consider base-b expansion. For every cylinder set C_s, C_s has positive measure. So for almost all points x that start with s, we see s infinitely often in x.

This is maybe not surprising in base-b.

Is it true for β-expansions? Continued fractions? Lüroth series? Answer: yes, with the appropriate invariant measure.