

Bergelson Notes 7/4

Jason Schuchardt

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1 Questions

Question: What is isomorphism, why are $2x \bmod 1$, the tent map, and the logistic map isomorphic?

Definition 1.1. If $f : X \rightarrow X$ then for all $x_0 \in X$, the *semiorbit* of x_0 is

$$\{f^n(x_0) : n \geq 0\} \text{ (note } f^0(x_0) = x_0\text{)}.$$

Why semiorbit? Because if f is invertible, then we can also talk about the *orbit*

$$\{f^n(x_0) : n \in \mathbb{Z}\}.$$

Sometimes we want to think about set valued functions, and talk about fixed points in this context. We won't do it now, but they're important in economics and game theory. However, you should feel that they are lurking in the background, since they show up naturally when we take preimages. See Figure 1

Question. Sensitive dependence on initial conditions should be for invertible functions? It's false for noninvertible functions. See Figure 1. We are talking about the condition

$$f^n(x) - f^n(y) > \epsilon$$

Exercise still remains for the map of

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

on \mathbb{T}^2 . Why is it invertible?

Suppose we write

$$\begin{pmatrix} 2 + \epsilon_1 & 1 + \epsilon_2 \\ 1 + \epsilon_3 & 1 + \epsilon_4 \end{pmatrix}$$

this is still a mapping mod 1.

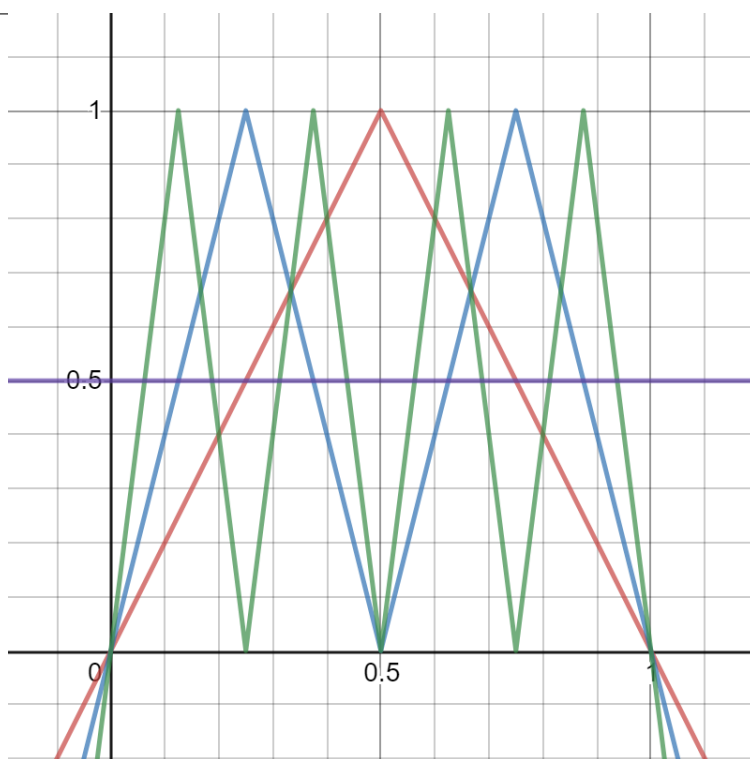
Seminar: Work more on \mathbb{T}^2 . Stress that $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, so

$$\mathbb{T} = \mathbb{R} / \mathbb{Z} = \{\text{the set of cosets, } \mathbb{Z} + t : t \in [0, 1]\}$$

Another example would be $X = \{0, 1\}^{\mathbb{Z}}$, $f : X \rightarrow X$ is shift, if x is a sequence, the shift σx is

$$(\sigma x)_n = x_{n+1}.$$

Question. Last time you mentioned topological entropy. Maybe we can talk about it.

Figure 1: Iterates of the tent map, and the line $y = 1/2$

Definition 1.2. Let (X, d) be a compact metric space, $T : X \rightarrow X$ is called *topologically mixing* if for all open nonempty $U, V \subseteq X$, there is some n such that

$$T^n U \cap V \neq \emptyset.$$

This is like in physics, we have laminar and turbulent flow. Periodic points are “laminar-like,” points with dense orbit are “turbulent-like.” Chaotic systems have both types together.

Question. Start with a function $f(x) = x$. Can you perturb by an arbitrarily small amount to make it chaotic?

By an arbitrarily small amount, we mean that for any ϵ , we can add a function g with

$$\sup_{x \in [0,1]} |g(x)| < \epsilon.$$

Exercise 1. Given $X = [0, 1]$, equipped with Lebesgue measure, μ , a transformation $T : [0, 1] \rightarrow [0, 1]$ is called *mixing* if for any measurable $A, B \subseteq [0, 1]$, $\mu(A), \mu(B) > 0$, then

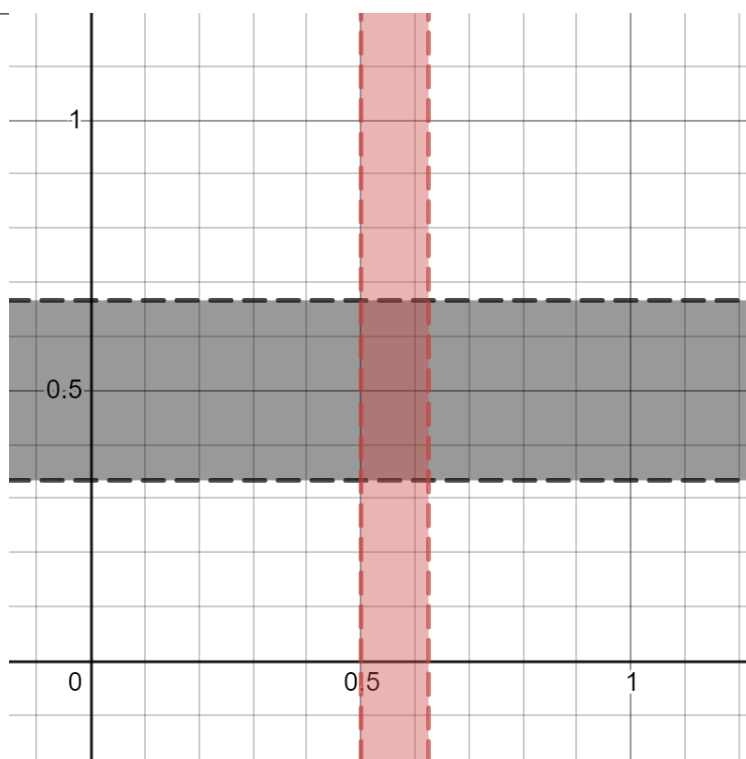
$$\mu(A \cap T^{-n}B) \rightarrow \mu(A)\mu(B)$$

as $n \rightarrow \infty$.

Assume that for all measurable $A \subseteq [0, 1]$, $\mu(T^{-1}A) = \mu(A)$. I.e., that T is *measure preserving*.

Why is this a good definition?

$$\mu(A \cap B) = \mu(A)\mu(B)$$

Figure 2: Two independent sets in $[0, 1] \times [0, 1]$.

is the definition of the independence of sets A and B . Consider the sets in Figure 2, they are independent, and this is a typical picture of what independence looks like in practice. The sets somehow come from different coordinates.

Exercise 2. (i) Find a sequence, (A_n) , of pairwise independent sets in $[0, 1]$. (ii) Find a sequence $(A_n) \subseteq [0, 1]$ of *mutually independent sets*. This means that for any k , and for all $A_{n_1}, A_{n_2}, \dots, A_{n_k}$, with the indices distinct,

$$\mu(A_{n_1} \cap A_{n_2} \cap \dots \cap A_{n_k}) = \mu(A_{n_1})\mu(A_{n_2}) \cdots \mu(A_{n_k}).$$

What is an example of three mutually independent sets? Consider the space $[0, 1]^3$. Take the sets $A \times [0, 1]^2$, $[0, 1] \times B \times [0, 1]$, $[0, 1]^2 \times C$, where A, B, C are positive measure subsets of $[0, 1]$.

Another way to think about mixing in terms of independence is that sets tend to become independent as we iterate T^{-1} .

Question. Why did we take T^{-1} in the definition of mixing?

Example 1.1. This example should show why we want T^{-1} .

Consider the tent map, $2x \bmod 1$, the map

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \rightarrow \mathbb{T}^2,$$

and the logistic map.

Is it clear that the first three preserve Lebesgue measure?

The first two have slope two, so the image has twice the area, but the preimage of an interval consists of two intervals, each with half the area.

Exercise 3. In order to establish that T is measure-preserving (for Lebesgue measure), it is enough to check that it holds for intervals.

The reason is that it suffices to check that the measure-preserving property holds for a generating set for the σ -algebra.

Thus preimage preserves measure, but the direct image shrinks measure, so it's not as good for us.

In general, mathematics prefers working with preimages. This is true in topology as well. We define f to be continuous to be that $f^{-1}(U)$ is open for all open U in the codomain.

There is a famous example, that we can't tell in this class, because it involves mixing cocktails. Instead the story will be about milkshakes.

You take milk and chocolate syrup, and start shaking it. Shaking is a measure preserving transformation, since liquids are incompressible. If the quality of the shaking is good, then it will be mixing, and you will get a good cocktail... , milkshake.

Note: σ -additive and countably additive will have the same meaning for us.

Seminar: Please give the general definition of measure, as a σ -additive function.

Consider topologically mixing transformation T , we can consider

$$U \cap T^{-n}V \stackrel{?}{=} T^{-n}(T^n U \cap V) \stackrel{?}{\neq} \emptyset,$$

and the claim is that it doesn't matter whether we have n or $-n$ in the definition for topologically mixing, since we don't want to measure the size.

Exercise 4. If you have a matrix A acting on \mathbb{R}^n , then A is measure preserving (for the Lebesgue measure) if and only if $|\det(A)| = 1$.

Can we generalize this? If we have a general differentiable map, how can we determine if it is measure preserving? Sketch: At each point, we can take the best linear approximation to the function, the Jacobian matrix. The determinant of the Jacobian should measure how much the function changes Lebesgue measure locally.

2 Isomorphism

Definition 2.1. Let X_1, X_2 be spaces. If they are topological spaces, then

$$X_1 \cong X_2$$

if X_1 and X_2 are homeomorphic.

Now assume (for convenience) that X_1, X_2 are compact metric spaces, and $f_i : X_i \rightarrow X_i$ are continuous maps. Then the systems, (X_1, f_1) and (X_2, f_2) are isomorphic as topological dynamical systems if there exists a homeomorphism $\varphi : X_1 \rightarrow X_2$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ \downarrow f_1 & & \downarrow f_2 \\ X_1 & \xrightarrow{\varphi} & X_2 \end{array}$$

commutes.

Example 2.1. We can easily give an example of things that are not isomorphic. The easiest way is to select nonhomeomorphic spaces, like $[0, 1]$ and $\{0, 1\}^{\mathbb{N}}$.

Another reasonable question is: Given two rotations of the circle, $r_\alpha : x \mapsto x + \alpha$, $r_\beta : x \mapsto x + \beta$. When are they isomorphic? Suppose that α and β are irrational. Otherwise, if α, β are rational, it is easy, to see rotation by 180 and 90 degrees are different.

Exercise 5. (i) Given two irrational rotations of the circle, r_α, r_β . When are they isomorphic? They are not in general. (ii) Similarly, if we take matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 5 & 1 \\ 19 & 4 \end{pmatrix}$$

when are they isomorphic as maps $\mathbb{T}^2 \rightarrow \mathbb{T}^2$.

The answer might depend on a computable invariant of the system.

We have four examples:

1. $2x \bmod 1$,
2. the tent map,
3. $S^1 \rightarrow S^2, z \mapsto z^2$,
4. $\{0, 1\}^{\mathbb{N}} \xrightarrow{\sigma} \{0, 1\}^{\mathbb{N}}$, the shifting map. $(\sigma x)_n = x_{n+1}$.

They should all be isomorphic. However, they are not isomorphic in our topological sense. The first map is not even continuous. S^1 and $[0, 1]$ are different topological spaces, since removing one point from $(0, 1)$ disconnects the space, but removing any point from S^1 still leaves a connected space. Finally the last topological space is homeomorphic to the Cantor set, and therefore by considering connectedness, it cannot be the same as S^1 or $[0, 1]$.

The correct notion of isomorphism for our purposes will be measure theoretic, and should have the same general shape.

We now want $\varphi : X_1 \rightarrow X_2$ to be an isomorphism of measure spaces. How would we define this?

Definition 2.2. $\varphi : (X_1, \mu_1) \rightarrow (X_2, \mu_2)$ is an isomorphism of measure spaces if it is a measure preserving almost everywhere bijection. I.e., φ is defined on a subset of X_1 of full measure and has values on some set of full measure, and we have $\varphi^* \mu_2 = \mu_1$, meaning that for all measurable subsets B of X_2 ,

$$\mu_2(B) = \mu_1(\varphi^{-1}(B)).$$

More precisely again, (X_i, μ_i) are measure spaces, $\mu_i(X_i) = 1$. Then X_1 is isomorphic to X_2 if there exist sets $\widetilde{X}_i \subseteq X_i$, with $\mu_i(\widetilde{X}_i) = 1$, and there is $\varphi : \widetilde{X}_1 \rightarrow \widetilde{X}_2$ such that for all $A \subseteq X_2$ (or \widetilde{X}_2), such that

$$\mu_1(\varphi^{-1}(A)) = \mu_2(A).$$

The next step to rigorizing this is to allow φ to represent a class of maps defined up to measure zero.

Exercise 6. With this definition, $[0, 1]$ and $\{0, 1\}^{\mathbb{N}}$ are isomorphic as measure spaces, indeed as dynamical systems.

Exercise 7. $[0, 1]^2$ and $[0, 1]$ are also measure theoretically isomorphic. We suddenly lose a notion of dimension when we move to measure theoretic dynamics.

This is good for dynamics, since we no longer worry about peculiarities of spaces, but instead the peculiarities of dynamics.

Is $L^2[0, 1]$ a space of functions? It's a space of equivalence classes of functions. This is different. We can't evaluate any element of $L^2[0, 1]$ at any point, since we can always change the function on any set of measure 0.

Exercise 8. Can you give me an example of a Hilbert space that is a functional space, which is made not of classes of equivalence functions, but of actual functions. The answer is yes.

Seminar: Touch on Hilbert and Banach spaces.

3 Banach Fixed Point Applications

Examples of applications of the Banach fixed point theorem.

3.1 Finding roots

Let $F : [a, b] \rightarrow \mathbb{R}$, and assume that F is continuous, $F(a) < 0$, $F(b) > 0$. We know that a root exists, but how can we find it? Assume also that $F'(x)$ exists, and

$$0 < c_1 \leq F'(x) \leq c_2.$$

Let

$$f(x) = x - \lambda F(x) \quad (\lambda \neq 0). \quad (*)$$

We are looking for x such that $f(x) = x$, in which case $F(x) = 0$. Now differentiating $(*)$, we get

$$f'(x) = 1 - \lambda F'(x),$$

so

$$1 - \lambda c_2 \leq f'(x) \leq 1 - \lambda c_1.$$

Now choose λ so that $|f'(x)| \leq c < 1$ for all $x \in [a, b]$.

Then Banach fixed point theorem gives us a fixed point, which is a zero of our original function.

The derivative condition follows from the mean value theorem We want

$$|f(x) - f(y)| < c|x - y|,$$

this follows from $f(x) - f(y) = f'(\theta_{xy})(x - y)$ for some $\theta_{xy} \in (x, y)$ together with the uniform bound on the derivative.

3.2 Solving linear equations

We have a transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $Tx = Ax + b$, with $x, b \in \mathbb{R}^n$, $A \in M_{n \times n}(\mathbb{R})$. We want solutions to $Tx = x$.

What does it mean for our matrix to be a contraction? It depends on the metric on \mathbb{R}^n . We will go through three important cases.

We want to find conditions under which T is a contraction. The three cases to consider are d_1 , d_2 , d_∞ .

d_∞ : The condition is

$$\sum_{j=1}^n a_{ij} \leq c < 1$$

for all $i = 1, \dots, n$.

d_1 : The condition is

$$\sum_{i=1}^n a_{ij} \leq c < 1$$

for all $j = 1, \dots, n$.

d_2 : The condition is

$$\sum_{i,j} a_{ij}^2 \leq c < 1.$$

In \mathbb{R}^n we can introduce norm $\|x\|_2$ by

$$\sqrt{\sum_i x_i^2} = d_2(x, 0).$$

Then $d_2(x, y) = \|x, y\|$.

Question: Is the general condition that we take p -norm of rows, q -norm of these p -norms.

Exercise 9. Give the answer for d_p , for general $p \in [1, \infty]$.

Question: For Euclidean norm, is there a way to express it in terms of traces and eigenvalues?

This is still n^2 operations, so do we have sufficient conditions that are easier to check?

Exercise 10. If $|a_{ij}| < \frac{1}{n}$ for all i, j , then all three conditions are satisfied.

If $a_{ij} > \frac{1}{n}$ for all i, j , then none of the three conditions are satisfied.

3.3 Solving differential equations

What is this thing:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0?$$

It is an ordinary differential equation, since we only have one variable. How do we know a solution exists?

The general result wasn't obtained for a while. Special cases were known by some people. The first one was Picard. He established solution in a neighborhood of (x_0, y_0) .

Exercise 11. Look up these people: Picard, Brouwer, Sperner, Lebesgue, Peano, Lipschitz.

If you want a general solution, the best theorem is by Peano. Peano is famous for Peano axioms, a space filling curve, and his theorem on the existence of global solutions to ODEs.

Convenient assumption is that

$$f(x, y_1) - f(x, y_2) \leq M|y_1 - y_2|$$

for all (x, y_i) in some domain G around x_0, y_0 . This condition is called a Lipschitz condition on y .

Theorem 3.1 (Picard). *There exists $d > 0$ such that on the interval*

$$|x - x_0| \leq d$$

our equation has a unique solution $y = \varphi(x)$.

Proof sketch. 1. Move to an equivalent integral equation,

$$\varphi(x) = y_0 + \int_{x_0}^x f(t, \varphi(t)) dt$$

2. Now we need to choose t . Since f is continuous in G , and G is compact there exists K such that

$$|f(x, y)| \leq K$$

for all $x, y \in G$. Then choose d so that (i) if $|x - x_0| < \delta$, then $|y - y_0| < Kd$, and (ii) $Md < 1$.

3. Then Md will be the constant for the Banach fixed point theorem.

4. We are moving to a subspace to make everything work.

■

Question: Is the convex body exercise still true for star shaped sets?

Answer: Yes.

Note taker's comment: I think you'll want star shaped about every point in an open neighborhood.