1 What is logic? Alternatively: Could you reason with an alien?

Disclaimer: I am not a logician.

The main question I want to address in this part of my talk is the following:
Question 1. What is logic? What is a logical argument?

Before I talk about that though, I'd like to take a couple polls:

1. Does every statement have a truth value?

2. Is truth independent of the existence of human beings?

3. Who thinks that if you could communicate with a sentient alien life form that you would be able to reaon with it?

Returning to logic. In its most basic form, logic is what allows us to make deductions. For example, if I know that it is raining outside, I should be able to logically conclude that if I walk outside then I will get wet. This sort of logic is produced by our brains without much conscious thought as to why we are making the deductions we are making. To many of us, particularly as mathematicians, logic is intuitive and obvious.

However, humans are not rational beings (for some/most definitions of rationality). We cannot just rely on our brains to naturally only produce true statements. Our brains are prey to all sorts of natural cognitive biases, like selection bias, negativity bias, loss aversion, and the framing effect.

Quite aside from the imperfections in our brains' reasoning abilities, we also have the question of what we mean when we say a statement is *true*.

For a classical example, consider the *liar paradox*. We have the statement

"This statement is false."

Is the statement true or false? I'm sure many of you have seen this before, but if the statement is assumed to be true, then it is false. If the statement is assumed to be false, then the statement is true. In either case, we arrive at a contradiction.

The only reasonable (according to what logic?) conclusion is that we cannot assign a coherent truth value to the statement "This statement is false."

However, notice that the very notion of truth we are using here is one that has been constructed by our brains for us. It is intuitive and imprecisely defined.

2 Formal Logic

What then is the solution to these philosophical problems that I have raised? It is formal logic.

Formal logic resolves the problem of being meaningful by sidestepping the question entirely. Formal logic is a syntactical game. Questions of semantics can then be studied independently of the underlying logical rules.

2.1 Semantics

As an example of what I mean by this, consider this rule from propositional logic, known as *modus ponens*:

$$\frac{P \qquad P \implies Q}{Q}$$

In the traditional semantics for propositional logic, we think of P and Q as being variables representing *propositions*, statements which have well defined truth values. Then this rule says that if we know P is true, and the conditional $P \implies Q$ is true, then it must be the case that Q is also true.

However, there's no particular reason we have to interpret this rule in this way. Type theory gives us another interpretation. One example of how type theory does this, is that we can interpret P, Q, and $P \implies Q$ as sets, with P and Q being arbitrary sets, and

$$P \implies Q = \mathrm{Hom}_{\mathbf{Set}}(P, Q).$$

In this context, the meaning of this rule is that if we have an element p of P, and a function $f : P \to Q$, then we have an element $f(p) \in Q$.

The study of assigning meaning to a language, or a logical theory is referred to as semantics.

3 Syntax

Many formal logics, including first order logic, propositional logic, and the various type theories can be discussed with a common framework.

In this framework, we have an *alphabet*, a *language* on this alphabet describing the grammatically valid formulae, and a collection of *judgements* we can potentially make about the valid formulae, and finally we have *inference rules* that allow us to produce new judgements from old judgements.

Definition 3.1. Now I can define propositional logic. Let

$$L = \{(,), \iff, \implies, \neg, \vee, \wedge, T, F\}$$

be the set of logical symbols and parentheses. Then for any set V of symbols disjoint from L that we call *propositional variables*, we have *propositional logic* with variables V:

1. The alphabet is $V \cup L$.

2. I won't describe the language defining the valid formulae, except to say it is the obvious one, given that each of the logical symbols other than the parentheses is an operator that takes the number of arguments described in Table 1.

3. Classical propositional logic has a single judgement, TRUE that asserts that a given formula is true. For a formula φ, we write φ TRUE, or more usually, just the formula by itself, φ, to express this judgement (since there is only a single judgement, so there is no possibility of confusion).

   Thus for example, if P is a propositional variable, we want to assert the judgement that P is true, we would write simply P. If we want to assert P is false, in classical logic, we can instead write $\neg P$, and assert that $\neg P$ is true.

4. Finally, propositional logic has a decently sized list of inference rules that allow us to produce new judgements from old ones.j

$\iff$	2
$\implies$	2
$\neg$	1
$\vee$	2
$\wedge$	2
T	0
F	0

Table 1: Arguments of the logical symbols

Negation introduction	$\dfrac{p \implies q \qquad p \implies \neg q}{\neg p}$
Negation elimination	$\dfrac{\neg p}{p \implies r}$
Double negation elimination	$\dfrac{\neg\neg p}{p}$
Conjunction introduction	$\dfrac{p \qquad q}{p \wedge q}$
Conjunction elimination	$\dfrac{p \wedge q}{p} \quad \dfrac{p \wedge q}{q}$
Disjunction introduction	$\dfrac{p}{p \vee q} \quad \dfrac{q}{p \vee q}$
Disjunction elimination	$\dfrac{p \vee q \qquad p \implies r \qquad q \implies r}{r}$
Biconditional introduction	$\dfrac{p \implies q \qquad q \implies p}{p \iff q}$
Biconditional elimination	$\dfrac{p \iff q}{p \implies q} \quad \dfrac{p \iff q}{q \implies p}$
Modus ponens (conditional elimination)	$\dfrac{p \qquad p \implies q}{q}$
Conditional proof (conditional introduction)	$\dfrac{p \vdash q}{p \implies q}$

Table 2: Inference rules (see Wikipedia, Propositional Calculus, natural deduction)

Notice that the last condition is different. A symbol I didn't mention for propositional logic appears here. It is the symbol $\vdash$. In reality, propositional logic allows for contexts as well (although how this is handled seems to differ from source to source).

The only additional rule we need to handle contexts mostly correctly (says the nonlogician, noticing Wikipedia being confusing) is

$$\overline{\Gamma, p \vdash p}$$

Exercise 1. Prove the following logical arguments

1. (Modus Tollens)
$$(p \implies q) \wedge \neg q \vdash \neg p$$

2. (Disjunctive Syllogism)
$$(p \vee q) \wedge \neg p \vdash q$$

3. (Composition)
$$(p \implies q) \wedge (q \implies r) \vdash (p \implies r)$$

4. (De Morgan's Theorem)
$$\neg(p \wedge q) \vdash \neg p \vee \neg q$$

5. (Law of excluded middle)
$$\vdash p \vee \neg p$$

6. (Law of non-contradiction)
$$\vdash \neg(p \wedge \neg p)$$

4   Propositional Logic and Type Theory

Which of the propositional logical inference rules either are part of the simply typed lambda calculus's rules, and which can be proven? Can they all be proven?