Monstrous Menagerie with Vandehey 7/9

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1 Applications of the Ergodic Theorem

Last time we talked about the pointwise ergodic theorem, now we'll see some applications of it.

Let's suppose (X, T, μ) is an ergodic system with probability measure. Consider $f = 1_A$. Since

$$\int f \, d\mu = \mu(A),$$

therefore

$$\int |f| \, d\mu = \int f \, d\mu = \mu(A) < \infty.$$

Thus $f \in L^1(X, \mu)$ for any $A \in \mathcal{A}$.

We can therefore apply the pointwise ergodic theorem. Namely for almost all $x \in X$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(Tx) = \int f \, d\mu = \mu(A).$$

Thus for an ergodic system, almost all points spend $\mu(A)$ fraction of the time on average in A.

Can consider $A = C_s$ for fibred systems. In words, for almost all x, the limiting frequency that s appears in the expansion of x should be the measure of the corresponding cylinder set, C_s.

Example 1.1. For example, with base-b expansions,

$$\lambda(C_{[a_1,\ldots,a_k]}) = b^{-k}.$$

Returning to general fibred systems, for each s, let B_s denote the set of x where this *doesn't* happen. Note that $\mu(B_s) = 0$, so

$$\mu\left(\bigcup_s B_s\right) = 0,$$

since there are only countably many finite strings.

Definition 1.1. A point $x \in X$ is *T-normal* if for every admissible string s we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} 1_{C_s}(T^i x) = \mu(C_s).$$

Theorem 1.1 (Borel Normal Number Theorem). *Almost all $x \in X$ are T-normal.*

Almost all is doing a lot of work here. A system for which every point is normal, is called *uniquely ergodic*. For example, an irrational rotation of the circle is uniquely ergodic. However, this often fails. In particular, if there are any periodic points, this will almost always fail to be true.

Proposition 1.1. *Almost all (Lebesgue) $x \in [0,1)$ are* absolutely normal*, i.e., normal in every base $b \geq 2$ simultaneously.*

This proposition follows immediately from the Borel Normal Number Theorem, since there are only countably many bases b.

Side note: There is also the notion of a number looking random. Our condition says that the expectation of a base string appearing is what it should be. What about the variance? If the variances are what they should be, then the number "looks random." It is also true that almost every number looks random, but ergodicity doesn't cover it, so we won't go into it.

Proposition 1.2. *If x is base-b normal, then it is base b^r normal for any rational r such that b^r is an integer.*

This should not be surprising. For example, for bases 2 and 4, we obtain base 2 from expanding the digits in base 4, and obtain base 4 from grouping the digits in base 2.

Proposition 1.3. *Suppose $b \neq c^r$ for any rational r. Then the set of base-b but not base-c normal numbers is uncountable.*

In fact, this set has positive Hausdorff dimension.

This was first proved for base-2, base-3, by proving that almost all of the points in the middle thirds cantor set are base-2 normal.

What if we throw in CF expansions?

Proposition 1.4 (Due to Vandehey)**.** Assuming GRH*, there exists $x \in [0,1)$ that is RCF-normal, but not base- b normal for any $b \geq 2$.*

So far they haven't been able to find a proof that doesn't rely on the Riemann hypothesis.

If we ask for a particular number, π, or e, the question of whether it is normal in a particular base or CF expansion is either obvious (like e in RCF), or entirely intractable with our current methods.

It is hard to construct normal numbers. Alan Turing worked on it.

Theorem 1.2 (Champernowne, 1922)**.**

$$0.123456789101112131415161718192021222324\ldots$$

is base-10 normal.

Champernowne published as an undergrad, then moved on to economics.

Theorem 1.3 (Copeland-Erodős)**.**

$$0.23571113171923\ldots$$

is base-10 normal.

These theorems generalize in the sense that we can apply these constructions by writing the integers or primes in order in other bases to obtain normal numbers in other bases.

Definition 1.2.

$$\omega(n) = \#\text{ of distinct prime factors of } n$$

For example, $\omega(12) = 2$.

Let $\omega_y(n)$ be the string of digits of the last y portion of $\omega(n)$. For example, if $\omega(N) = 5290$, then $\omega_{1/2}(N) = 90$. Then we have the theorem

Theorem 1.4 (Vandehey)**.**

$$0.\omega_y(1)\omega_y(2)\omega_y(3)\omega_y(4)\ldots$$

is normal if and only if $0 < y \leq 1/2$.

The reason for the $y \leq 1/2$ is that $\omega(N) \approx \log\log N$, and $\log\log$ changes its digits so slowly that something will dominate, preventing normality.

The first 9 appears at digit $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$, since that is the first number for which $\omega(n) = 9$.

The goal for this week is to show you that Champernowne's theorem is true. Before we get there, we'll show that some things are ergodic.

2 How to prove that a system is ergodic?

Lemma 2.1 (Knopp's Lemma)**.** *Suppose (X, μ) is a probability space. Suppose that $\mathscr{A}$ is a semi-algebra that generates $\mathcal{A}$. Let $E \in \mathcal{A}$. If there exists $\delta > 0$ such that for all $A \in \mathscr{A}$,*

$$\mu(E \cap A) \geq \delta\mu(A)$$

then $\mu(E) = 1$.

Similarly, if there exists $\eta < 1$ such that for all $A \in \mathscr{A}$,

$$\mu(E \cap A) \leq \eta\mu(A),$$

then $\mu(E) = 0$.

We will use one unproved fact. Fact: For any $E \in \mathcal{A}$, there exists B which is a finite disjoint union of sets in $\mathscr{A}$ such that $\mu(E\Delta B)$ is arbitrarily small.

Proof. We prove the first half. The second follows similarly. Let $\epsilon > 0$, so there exist $A_1, A_2, \ldots, A_k \in \mathscr{A}$ such that

$$\mu\left(E^C \Delta \bigcup_{i=1}^{k} A_i\right) < \epsilon.$$

Then

$$\begin{aligned}
0 &= \mu(E \cap E^C) \\
&\geq \mu\left(E \cap \bigcup_{i=1}^{k} A_i\right) - \mu\left(E^C \Delta \bigcup_{i=1}^{k} A_i\right) \\
&> \mu\left(E \cap \bigcup_{i=1}^{k} A_i\right) - \epsilon \\
&= \mu\left(\bigcup_{i=1}^{k} (E \cap A_i)\right) - \epsilon \\
&= \sum_{i=1}^{k} \mu(E \cap A_i) - \epsilon \\
&\geq \sum_{i=1}^{k} \delta\mu(A_i) - \epsilon
\end{aligned}$$

$$
\begin{aligned}
&= \delta\mu\left(\bigcup_{i=1}^{k} A_i\right) - \epsilon \\
&\geq \delta\left(\mu(E^C) - \mu\left(E^C \Delta \bigcup_{i=1}^{k} A_i\right)\right) - \epsilon \\
&\geq \delta\mu(E^C) - (1+\delta)\epsilon
\end{aligned}
$$

So $\mu(E^C) \leq \frac{(1+\delta)\epsilon}{\delta}$. Since ϵ was arbitrary, $\mu(E^C) = 0$. ■

This lemma makes it much easier to check ergodicity, since now we only have to check how things intersect the cylinder sets.

Example 2.1 (Base-b)**.** Suppose E is T-invariant. Let C_s be any cylinder set. Consider

$$\lambda(E \cap C_s) = \lambda(T^{-|s|}E \cap C_s),$$

by invariance of E. So

$$\lambda(E \cap C_s) = \frac{\lambda(E)}{b^{|s|}} = \lambda(E)\lambda(C_s),$$

using the fact that we know how T^s acts on C_s

Let $\delta = \lambda(E)$, and apply Knopp's lemma to see that $\lambda(E) = 0$ or 1.

This is complicated to generalize, since we relied on the fact that we knew the explicit definition of T.

2.1 More general setting

Suppose E is a set of positive measure and that E is T-invariant. Suppose $C_s = C_{[a_1, a_2, \ldots, a_k]}$.

Then we consider

$$\frac{\mu(E \cap C_s)}{\mu(C_s)},$$

and we want to show that this is bounded below by some positive δ. Now we have

$$\frac{\mu(E \cap C_s)}{\mu(C_s)} = \frac{\mu(T^{-k}E \cap C_s)}{\mu(C_s)} = \frac{\int_{T^{-k}E \cap C_s} d\mu}{\int_{C_s} d\mu}.$$

Then we have

$$= \frac{\int_{T^k(T^{-k}E \cap C_s)} \omega_s(y)\, d\mu}{\int_{T^k C_s} \omega_s(y)\, d\mu},$$

where $\omega_s(y)$ is the Jacobian of T^k.

$$= \frac{\int_{E \cap T^k C_s} \omega_s(y)\, d\mu}{\int_{T^k C_s} \omega_s(y)\, d\mu}.$$

Note that in base-b, $\omega_s(y) = b^{-k}$.