Metric Space Review with Luke and Evan

Jason Schuchardt

June 24, 2019

1 Metric Spaces

Just as rings are abstractions of the properties of numbers, metric spaces are the abstraction of the concept of distance.

Definition 1.1. A *metric space* is a set X of elements, called *points*, equipped with a distance function, $d : X \times X \to \mathbb{R}$ such that

1 $d(x, x) = 0$.

2 If $x \neq y$, $d(x, y) > 0$.

3 $d(x, y) = d(y, x)$.

4 $d(x, z) \leq d(x, y) + d(y, z)$.

Example 1.1. Examples include the line, $\mathbb{R}$, the plane, $\mathbb{R}^2$, euclidean space, $\mathbb{R}^3$, and so on. Notice that we're ignoring a lot of structure on $\mathbb{R}^n$.

Example 1.2. Another example is a subspace.

We can measure the distance between points in a subspace using the metric of the ambient space.

Example 1.3. If $X = \{\text{subway stations}\}$, we can let $d(x, y) =$ length of the shortest path from x to y in the subway network

Example 1.4. The continuous functions on the interval,

$$X = C[0, 1] := \{\text{cts fns } f | f : [0, 1] \to \mathbb{R}\}.$$

$$d(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$$

Similarly, we have

$$X = C^1[0, 1] := \{\text{continuously differentiable fns } f | f : [0, 1] \to \mathbb{R}\},$$

and we can give it the distance

$$d_{C^1}(f, g) = \max_x |f(x) - g(x)| + \max_x |f'(x) - g'(x)|.$$

Question: Can we use the $C[0, 1]$ metric for continuously differentiable functions? Answer: Yes, but notice that they can be very different.

Consider $g(x) = \sin(100x)$ and $f(x) = 0$ the distance $d_C(f, g) \leq 1$, whereas $d_{C^1}(f, g) \approx 100$.

2 Properties of metric spaces

Definition 2.1. The (open) *ball* with center x and radius $r > 0$ is the set

$$B_d(x,r) = \{y \in X | d(x,y) < r\}$$

Example 2.1. A ball in $\mathbb{R}^2$ is a disc, and a ball in $\mathbb{R}^3$ is the interior of a sphere of radius r.

Definition 2.2. A subset U of X is *open* if for all $x \in U$, there exists $r > 0$ such that

$$B_d(x,r) \subseteq U$$

Example 2.2. Open intervals and open balls are open subsets.

Proposition 2.1. *The open ball, $B_d(x,r)$ is open.*

Proof. Let $y \in B_d(x,r)$. By definition $d(x,y) < r$. Let $R = r - d(x,y)$.

Then we claim that $B_d(y,R) \subseteq B_d(x,r)$. To see this, let $z \in B_d(y,R)$. Thus $d(y,z) < R$, so

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + R = d(x,y) + r - d(x,y) = r,$$

and therefore $z \in B_x(r)$.

Hence $B_d(y,R) \subseteq B_d(x,r)$ as desired. ∎

This is why we also call balls open balls.

Exercise: An arbitrary union of open sets is open, and a finite intersection of open sets is open.

Note that

$$\bigcap_{r>0} B_d(x,r) = \{x\},$$

and single point sets are not open in $\mathbb{R}^n$ for example. so arbitrary intersections of open sets are not generally open. They may be in weird metrics, like the subway metric.

Definition 2.3. A subset F of X is *closed* if its complement is open.

Warning! Despite the terminology, it is possible for sets to be neither open nor closed or even both open and closed.

Definition 2.4. If A is a subset of X, $x \in X$ is a limit point of A if for all $\epsilon > 0$, $B_d(x,\epsilon) \cap A$ contains a point other than x.

In other words, every ball around x contains another point of A. Intuitively, there is no space between x and $A \setminus \{x\}$.

Example 2.3. The limit points of $(0,1)$ are $[0,1]$, and the set

$$\left\{\frac{1}{n} : n \in \mathbb{N}_{>0}\right\} = \{1, 1/2, 1/3, \ldots, 1/n\}$$

has only the limit point 0.

Definition 2.5. In contrast to a limit point, an *isolated point* of a subset A is a point $a \in A$ which is not a limit point of A.

Definition 2.6. A *Cauchy sequence* in a metric space (X,d) is a sequence $\{x_n\}_{n\in\mathbb{N}} = x_0, x_1, x_2, \ldots$ in X such that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$,

$$d(x_n, x_m) < \epsilon.$$

Example 2.4. $3, 3.1, 3.14, \ldots$ $\epsilon = 10^{-k}$, $N = k+2$.

Not a Cauchy sequence: $a_k = \sum_{j \in \mathbb{N}_{>0}} \frac{1}{j}$.

Exercise: Cauchy sequences are bounded.
Definition 2.7. A sequence $\{x_n\}_{n \in \mathbb{N}} = x_0, x_1, \ldots$ in a metric space (X, d) is *convergent*, and we say it *converges* to a *limit* y if for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n > N$,

$$d(x_n, y) < \epsilon.$$

Exercise: Convergent sequences are Cauchy sequences.

An example with function spaces.

Question: Is $f_n = x^n \in C[0,1]$ Cauchy?
Definition 2.8. A metric space (X, d) is *complete* if every Cauchy sequence converges.
Example 2.5. $\mathbb{Q}$ is not complete, but $\mathbb{R}$ is. $(0,1)$ is not complete, but $[0,1]$ is.

Question: Is $C[0,1]$ complete?

3 Maps of metric spaces

If (X, d_X) and (Y, d_Y) are metric spaces, we don't want to consider all maps from X to Y, but ones with special properties.
Definition 3.1. $f : X \to Y$ is *continuous* if for all convergent sequences $x_0, x_1, \ldots \to x$, the image sequence

$$f(x_0), f(x_1), \ldots \to f(x).$$

Intuitively a function is continuous it has no gaps.
Definition 3.2. f is a *homeomorphism* if f^{-1} exists both f and f^{-1} are continous.

Note: If X and Y are *homeomorphic* (i.e., there is a homeomorphism from one to the other) then X and Y have the same topology (the open sets are the same), but they may have different metrics.
Definition 3.3. We say f is an isometry if f^{-1} exists, and f preserves distances in the sense that for all $x_1, x_2 \in X$,

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$

Definition 3.4. We say f is a contraction if there exists a constant β, with $0 < \beta < 1$ such that f shrinks distances in the sense that for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) < \beta d_X(x_1, x_2).$$

Note that contractions and isometries are continuous.
Theorem 3.1. *A contraction $f : X \to X$ on a complete metric space (X, d) has a unique fixed point. I.e. there is a unique point $x \in X$ such that $f(x) = x$.*