TYPE THEORY HW 2

All problems are in the simply typed lambda calculus.

Exercise 1. Define addition on the natural numbers in a different way from that done in class.

Exercise 2. Define an exponential function (that takes two natural numbers m, n to m^n) on the natural numbers.

Exercise 3. Consider the type $\mathbb{B}$ defined in the last homework. Show that this acts like $\mathbb{N}/2$; that is:

(1) Define a function $\texttt{mod2} : \mathbb{N} \implies \mathbb{B}$ that sends every even number to 0 and every odd number to 1

(2) Check for a few specific natural numbers n that $\texttt{mod2}(\texttt{mult}(2, n)) = 0$. Think about what would be needed in the type theory to prove that

$$\texttt{mod2}(\texttt{mult}(2, n)) = 0.$$

for all $n : \mathbb{N}$.

(3) Define functions $f : \mathbb{N} \times \mathbb{B} \implies \mathbb{N}$ and $g : \mathbb{N} \implies \mathbb{N} \times \mathbb{B}$ that are metatheoretically inverse to each other (that is, for given any specific $(n, b) : \mathbb{N} \times \mathbb{B}$, you could show that $gf(n, b) = (n, b)$ and given any specific $n : \mathbb{N}$, you could show that $fg(n) = n$).

Exercise 4.

(1) Consider the following rule that is part of the definition of $\mathbb{N}$ (in the empty context).

$$\frac{\vdash z : T \qquad x : T \vdash t(x) : T}{n : \mathbb{N} \vdash j_{z,t}(n) : T}$$

Turn this rule into a function in the type theory.

(2) Generalize this to take into account any context.

$$\frac{\Gamma \vdash z : T \qquad \Gamma, x : T \vdash t(x) : T}{\Gamma, n : \mathbb{N} \vdash j_{z,t}(n) : T}$$

Exercise 5. In set-based mathematics, a *pointed magma structure* on a set M consists of a point $e \in M$ and a binary operation that takes $m, n \in M$ to some $m \cdot n \in M$. The operation is not required to be associative and the point is not required to be a unit. Note that every group and every monoid has an underlying magma.

(1) Using the types that we have already defined in the simply typed lambda calculus, for any type T, construct the type of magma structures on T, emulating the set-based definition above. Call this type $\texttt{Magma}(T)$.

(2) Construct an interesting magma structure on $\mathbb{N}$.

(3) Construct an interesting magma structure on $T \implies T$ for any type T.

(4) Construct, for any types S and T, a magma structure on $S \times T$ from a magma structure on S and a magma structure T.