Coinitial semantics for redecoration of triangular matrices

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Simple inductive types—W-types—are characterized categorically as initial algebras of a polynomial functor. Dually, *co*inductive types are characterized as terminal *co*algebras of polynomial functors. In the case of coinductive types, the meta-theoretic notion of equality is not adequate: thus, the notion of *bisimulation* was introduced by Aczel [1].

The characterization of inductive types as initial algebras has been extended to *heterogeneous*—also called *nested*—inductive data types, e.g., the type of λ-terms, in various formulations [3, 4]. The main goal of those works is to characterize not only the data type via a universal property, but rather the data type *equipped with a well-behaved substitution operation*.

In the present work we study a specific *co*inductive *heterogeneous* data type—the type family Tri of infinite triangular matrices—and its *redecoration* operation: the codata type is parametrized by a fixed set E for entries not on the diagonal, and indexed by another, *variable*, set A for entries on the diagonal. The respective types of its specifying destructors top and rest are given in Figure 1, together with the destructors for the coinductively defined bisimilarity relation on it. Equipped with the redecoration operation, the type Tri is shown by Matthes and Picard [5] to constitute what they call a "weak constructive comonad".

In this work, we first identify those weak constructive comonads as an instance of the more general notion of *relative comonad*. Indeed, a weak constructive comonad is precisely a comonad relative to the functor $\mathsf{eq} : \mathsf{Set} \to \mathsf{Setoid}$ from the category of sets to that of setoids that is left adjoint to the forgetful functor.

Afterwards, we characterize the codata type Tri, equipped with the cosubstitution operation of redecoration, as a terminal object of some category. For this, we dualize the approach by Hirschowitz and Maggesi [4], who characterize the heterogeneous inductive type of lambda terms—equipped with a suitable substitution operation—as an initial object in a category of algebras for the

$$\frac{t : \mathsf{Tri}(A)}{\mathsf{top}_A(t) : A} \qquad \frac{t : \mathsf{Tri}(A)}{\mathsf{rest}_A(t) : \mathsf{Tri}(E \times A)}$$

$$\frac{t \sim t'}{\mathsf{top}(t) = \mathsf{top}(t')} \qquad \frac{t \sim t'}{\mathsf{rest}(t) \sim \mathsf{rest}(t')}$$

Fig. 1. Destructors and bisimilarity for the coinductive family of setoids Tri

signature of lambda terms. In their work, the crucial notions are the notion of monad and, more importantly, *module over a monad*. It turns out that more work than a simple dualization is necessary for two reasons:

- the lambda calculus can be seen as a monad on sets and thus, in particular, as an endofunctor. The codata type Tri, however, associates to any *set* of potential diagonal elements a *setoid* of triangular matrices. We thus need a notion of comonad whose underlying functor is not necessarily endo: the already mentioned *relative* comonads;
- the category-theoretic analysis of the destructor rest is more complicated than that of the heterogeneous constructor of abstraction of the lambda calculus.

Finding a suitable categorical notion to capture the destructor rest and, more importantly, its interplay with the comonadic redecoration operation on Tri, constitutes the main contribution of the present work. These rather technical details shall not be explained in this extended abstract.

Once we have found such a categorical notion, we can use it to give a definition of a "coalgebra" for the signature of infinite triangular matrices, together with a suitable notion of *morphism* of such coalgebras. We thus obtain a category of coalgebras for that signature. Any object of this category comes with a comonad relative to the aforementioned functor $\mathsf{eq} : \mathsf{Set} \to \mathsf{Setoid}$ and a suitable comodule over this comonad, modeling in some sense the destructor rest. Our main result then states that this category has a terminal object built from the codata type Tri and its destructor rest, which are seen as a relative comonad and a comodule over that relative comonad, respectively. This universal property of coinitiality characterizes not only the codata type of infinite triangular matrices but also the **bisimilarity** relation on it as well as the **redecoration** operation.

All our definitions, examples, and lemmas have been implemented in the proof assistant Coq. The Coq source files and HTML documentation are available on the project web page [2]. While parts of our work seem to be specific to the particular codata type Tri, we believe that our work proves the suitability of the notion of relative (co)monads and (co)modules thereover for a categorical characterization of coinductive data types.

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[2] Benedikt Ahrens and Régis Spadotti. *Coinitial semantics for redecoration of triangular matrices*. http://benediktahrens.github.io/coinductives/. arXiv:1401.1053.

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[4] André Hirschowitz and Marco Maggesi. "Modules over monads and initial semantics". In: *Inf. Comput.* 208.5 (2010), pp. 545–564.

[5] Ralph Matthes and Celia Picard. "Verification of redecoration for infinite triangular matrices using coinduction". In: *TYPES*. Ed. by Nils Anders Danielsson and Bengt Nordström. Vol. 19. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2011, pp. 55–69. DOI: 10.4230/LIPIcs.TYPES.2011.55.