Formal Power Series and their Continued Fraction Expansion

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1 Introduction

1.1 Basics. The familiar continued fraction algorithm, normally applied to real numbers, can just as well be applied to formal Laurent series $\sum_{h=-m}^{\infty} g_h X^{-h}$ in a variable X^{-1}, with the 'polynomial portion' $\sum_{h=-m}^{0} g_h X^{-h}$ of the complete quotient taken to be its 'integer part'. Then the partial quotients are polynomials in X, and we learn that continued fraction expansions

$$[\, a_0(X), a_1(X), \ldots , a_h(X), \ldots]$$

with partial quotients polynomials of degree at least 1 in X and defined over some field apparently converge to formal Laurent series in X^{-1} over that field. It is an interesting exercise to prove that directly and to come to understand the sense in which the convergents provide best approximations to Laurent series.

Specifically, given a Laurent series $F(X)$ — unless the contrary is clearly indicated we will assume it not to be a rational function — define its sequence $(F_h)_{h\ge0}$ of *complete quotients* by setting $F_0 = F$, and $F_{h+1} = 1/(F_h - a_h(X))$. Here, the sequence $(a_h)_{h\ge0}$ of *partial quotients* of F is given by $a_h = \lfloor F_h \rfloor$ where $\lfloor\ \rfloor$ denotes the polynomial part of its argument. Plainly we have

$$F = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Only the partial quotients matter, so such a continued fraction expansion may be conveniently detailed by $[\, a_0 \, , \, a_1 \, , \, a_2 \, , \, a_3 \, , \ldots]$.

The truncations $[\, a_0 \, , \, a_1 \, , \, \ldots \, , \, a_h \,]$ are rational functions p_h/q_h. Here, the pairs of relatively prime polynomials $p_h(X)$, $q_h(X)$ are given by the matrix identities

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix}$$

and the remark that the empty matrix product is the identity matrix. This alleged correspondence, whereby these matrix products provide the sequences of *continuants* $(p_h)_{h\ge 0}$ and $(q_h)_{h\ge 0}$, and thus the *convergents* p_h/q_h for $h \ge 0$, may be confirmed by induction on the number of matrices on noticing the definition

$$[\, a_0\ , a_1\ , \dots\ , a_h\,] = a_0 + 1/[\, a_1\ , \dots\ , a_h\,], \qquad [\, a_0\,] = a_0\,.$$

It follows that the continuants q_h satisfy $\deg q_{h+1} = \deg a_{h+1} + \deg q_h$. It also clearly follows, from transposing the matrix correspondence, that

$$[\, a_h\ , a_{h-1}\ , \dots\ , a_1\,] = q_h/q_{h-1}, \quad \text{for} \quad h = 1, 2,\ \dots\ .$$

The matrix correspondence entails $p_h/q_h = p_{h-1}/q_{h-1} + (-1)^{h-1}/q_{h-1}q_h$ whence, by induction, $F = a_0 + \sum_{h=1}^{\infty}(-1)^{h-1}/q_{h-1}q_h$, and so

$$\deg(q_hF - p_h) = -\deg q_{h+1} < -\deg q_h\ ,$$

displaying the excellent quality of approximation to F provided by its convergents.

Proposition 1. *Let p, q be relatively prime polynomials. Then*

$$\deg(qF - p) < -\deg q$$

if, and only if, the rational function p/q is a convergent to F.

Proof. The 'if' part of the claim has already been noticed, so we may take h so that $\deg q_{h-1} \le \deg q < \deg q_h$, and note that supposing p/q is not a convergent entails that q is not a constant multiple of q_{h-1}. Because $p_hq_{h-1} - p_{h-1}q_h = \pm 1$, there are nonzero polynomials a and b such that

$$q = aq_{h-1} + bq_h$$
$$p = ap_{h-1} + bp_h\ ,$$

and so $qF - p = a(q_{h-1}F - p_{h-1}) + b(q_hF - p_h)$. Now suppose that the two terms on the right are of different degree, $\deg a - \deg q_h$ and $\deg b - \deg q_{h+1}$, respectively. In that case plainly $\deg(qF - p) > \deg(q_{h-1}F - p_{h-1}) > \deg(q_hF - p_h)$, confirming that the convergents provide the locally best approximations to F.

To verify the suggestion that the degrees of the two terms *are* different, notice that $\deg aq_{h-1} = \deg bq_h$, otherwise $\deg q < \deg q_h$ is not possible, so $\deg a - \deg q_h = \deg b - \deg q_{h-1} > \deg b - \deg q_{h+1}$. Moreover, $\deg a - \deg q_h = \deg(qF - p)$. So it remains to confirm that $\deg a - \deg q_h \ge -\deg q$. But that's plain because, of course, $\deg a$ must be at least as large as $\deg q_h - \deg q_{h-1}$. ∎

These arguments are noticeably clearer with a nonarchimedean absolute value, namely degree in X of a Laurent series in X^{-1}, than in the traditional archimedean case where one deals with the usual absolute value of real numbers.

1.2 Generalisations. There is of course an extensive literature touching on the topics of power series and continued fractions, going back to the very beginnings of modern mathematics. However the expansions involved are typically not the *simple* continued fractions we consider here but have the more general shape

$$F = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} =: [\, a_0 \,,\, b_1 : a_1 \,,\, b_2 : a_2 \,,\, b_3 : a_3 \,,\, \ldots \,] \,.$$

The abstract theory is not all that different from our 'basics' above, but now questions about the quality of convergence of the convergents are relevant and dominate. In brief, neither the series nor their continued fractions are *a priori* 'formal'. The bible of these matters is H. S. Wall [23]; there's a nice introduction in Henrici [11]. One might further study [12] and the very extensive literature on Padé approximation. And then there are wondrous identities à la Ramanujan; see for example the five volume series [5].

2 Remarks and Allegations

2.1 A generic example. A brief computation by PARI GP reveals that

$$\begin{aligned}
&G(X) = \prod_{h\ge 0}(1 - X^{-2^h}) = \\
&[1 \,,\, -X{+}1 \,,\, -\tfrac12 X{-}1 \,,\, 2X^2{-}2X{+}4 \,,\, -\tfrac12 X \,,\, 2X^2{+}2X \,,\, \tfrac12 X{-}1 \,,\, X{+}\tfrac12 \,,\, \tfrac43 X{+}\tfrac{14}{9} \,, \\
&\quad -\tfrac{27}{8}X + \tfrac{9}{4} \,,\, -\tfrac{8}{81}X^2 - \tfrac{8}{81}X - \tfrac{16}{81} \,,\, \tfrac{81}{8}X \,,\, \tfrac{8}{81}X^2 - \tfrac{8}{81}X - \tfrac{16}{81} \,,\, \tfrac{81}{8}X - \tfrac{81}{4} \,, \\
&\tfrac{4}{243}X + \tfrac{10}{243} \,,\, -243X + \tfrac{729}{2} \,,\, -\tfrac{1}{486}X - \tfrac{1}{729} \,,\, -\tfrac{2187}{2}X - \tfrac{729}{8} \,,\, -\tfrac{32}{10935}X - \tfrac{56}{18225} \,, \\
&\tfrac{273375}{128}X - \tfrac{54675}{32} \,,\, -\tfrac{128}{4100625}X^2 - \tfrac{128}{4100625}X - \tfrac{256}{4100625} \,,\, -\tfrac{4100625}{2176}X + \tfrac{4100625}{2312} \,, \\
&\tfrac{39304}{4100625}X + \tfrac{26299}{2733750} \,,\, \tfrac{131220000}{83521}X + \tfrac{1960098750}{83521} \,,\, \tfrac{83521}{31492800000}X - \tfrac{83521}{1968300000} \,, \\
&\quad -\tfrac{472392000000}{83521}X^2 - \tfrac{472392000000}{83521}X \,,\, -\tfrac{83521}{31492800000}X - \tfrac{1085773}{62985600000} \,, \\
&\quad -\tfrac{8398080000}{1085773}X + \tfrac{802016640000}{14115049} \,,\, \tfrac{183495637}{184757760000}X + \tfrac{1256239361}{451630080000} \,, \\
&\tfrac{17884551168000}{102574061083}X - \tfrac{1681960743936000}{4410684626569} \,,\, -\tfrac{189659438942467}{5901901885440000}X + \tfrac{1292330595584717}{11066066035200000} \,, \\
&\quad -\tfrac{663963962112000000}{236505320361256349}X - \tfrac{535376274782976000000}{6858654290476434121} \,, \\
&\quad -\tfrac{19890097442381658950 9}{5228716201632000000 0}X - \tfrac{23792671733662749965749}{219606080468544000000 0} \,, \\
&\quad -\tfrac{11529319224598560000000}{2134207455567552005431 57}X + \tfrac{69395521428059904000000 0}{78965675855999424200968 09} \,, \\
&\quad -\tfrac{292173000667197869543581933}{2421157037165697600000000}X + \tfrac{16337998334606280867180297821}{403526172860949600000000000} \,, \\
&\quad -\tfrac{288232980614964000000000000}{2486392235677853869815882249 83}X - \tfrac{120407268844896818400000000000}{571870214205906390057652917460 9} \,, \\
&\quad -\tfrac{1011770378979680536255847469353 9}{664700547132468000000000000000}X - \tfrac{2342739085214054266441507954708313}{13144453319544554700000000000000 0} \,, \\
&\tfrac{673615975105911420000000000000 0}{183136425402298388189741858506129}X - \tfrac{182621794871498446530000000000000 0}{4523469707436770188286623905101386 3} \,,\, \ldots\ldots].
\end{aligned}$$

2.2 Thoughts and Remarks. The following are among the thoughts I mean to provoke by this example. **a**. PARI is a fine program indeed; the computation truly is brief. **b**. This *is* computational mathematics. It's nearly impossible to notice this kind of thing by hand; one thinks one must have blundered in the calculation. **c**. The example is no more than an example, and it seems quite special. But the general appearance of the expansion in fact is typical. **d**. It is striking that the complexity of the coefficients grows at a furious rate, yet the mindful eye sees pattern, of sorts. It will be worthwhile to hint at an explanation for that. **e**. Most of the partial quotients are of degree 1; the others have degree 2. It turns out that it is the partial quotients of degree 2 that should surprise. Partial quotients of formal Laurent series 'want' to have degree 1. A kind of sort of* 'quasi-repetition' in our particular example in fact 'perpetuates' an 'initial accident' which happens to yield a partial quotient of degree 2. **f**. One should give in to the temptation to wonder what happens to our example when it is considered to be defined over some field of characteristic $p \neq 0$. Of course, if p never occurs in a denominator of a partial quotient then the expansion has good reduction, and we can just reproduce it reduced modulo p. But what happens when the expansion has bad reduction at p?

By the way. It is not just that it's reasonable to reduce mod p. It's unreasonable not to. The example, let alone my claim that it is generic, shouts a reminder that formal power series want to be defined over a finite field, and not over $\mathbb{Q}$.

2.3 Two examples of reduction mod *p*. It's easy to begin to answer that last question by computing a few instances. For example, over $\mathbb{F}_3$ we find that

$$\prod_{h\geq 0}(1-X^{-2^h}) =$$
$$= [1\,,\ 2X+1\,,\ X+2\,,\ 2X^2+X+1\,,\ X\,,\ 2X^2+2X\,,\ 2X+2\,,\ X+2\,,\ X^2+2X+2\,,$$
$$2X^2+2X+1\,,\ X\,,\ X^2+2X+1\,,\ X+1\,,\ 2X^6+X^5+X^4+2X^3+X^2+2X+2\,,$$
$$X^2+X+2\,,\ 2X+2\,,\ X\,,\ 2X+1\,,\ 2X^4+X^3+X^2+2X+2\,,\ 2X+1\,,\ X+1\,,$$
$$2X^{12}+2X^{11}+X^8+X^7+X^4+X^3+2\,,\ X+2\,,\ 2X^2+2X+1\,,\ X^2+2X+2\,,\ X+1\,,$$
$$X^2+2X\,,\ 2X+1\,,\ X+2\,,\ \ldots\ldots].$$

Not too surprisingly, to the extent that the original expansion has good reduction the new expansion is its reduction; the first term with bad reduction 'collapses' to a term of higher degree. Beyond that term the expansion is not immediately recognisable in terms of the original.

Of course,

$$\prod_{h\geq 0}(1+X^{-2^h}) = 1/(1-X^{-1}) = X/(X-1)\,,$$

* For those not Anglophone: The phrase 'kind of sort of', though rarely produced in print, is thought often. It's, well, kind of sort of a little more vague than just 'kind of', or 'sort of', alone.

and so we should not be shocked to find that over $\mathbb{F}_2$, for example

$$(1-X^{-1})(1-X^{-2})(1-X^{-4})(1-X^{-8})(1-X^{-16})(1-X^{-32})(1-X^{-64}) \\ = [\,1\,,\,X+1\,,\,X^{126}+X^{125}+\cdots+X+1\,].$$

In this case, the collapse to high degree is exceptionally vivid.

2.4 An atypical example. On the other hand, consider

$$\begin{aligned} &X^{-1}+X^{-2}+X^{-3}+X^{-5}+\cdots+X^{-F_h}+\dots \\ &= [0\,,\,X-1\,,\,X^2+2X+2\,,\,X^3-X^2+2X-1\,,\,-X^3+X-1\,,\,-X\,,\,-X^4+X\,,\,-X^2, \\ &-X^7+X^2\,,\,-X-1\,,\,X^2-X+1\,,\,X^{11}-X^3\,,\,-X^3-X\,,\,-X\,,\,X\,,\,X^{18}-X^5, \\ &\quad -X\,,\,X^3+1\,,\,X\,,\,-X\,,\,-X-1\,,\,-X+1\,,\,-X^{29}+X^8\,,\,X-1\,,\,\dots]\,. \end{aligned}$$

Here the sequence of exponents (F_h) of the series is defined by the recurrence relation $F_{h+2}=F_{h+1}+F_h$ and the initial values $F_2=1$, $F_3=2$.

The following thoughts and remarks will surely have sprung to the reader's mind. **a**. This example is likely to have first been noticed by persons excessively interested in Fibonacci numbers. **b**. The continued fraction expansion appears to have good reduction everywhere; that entails that on replacing X by any integer of absolute value at least 2 we obtain a numerical expansion defective only to the extent that it may include nonpositive integer partial quotients.

Indeed, Jeff Shallit had long known that

$$\begin{aligned} &2^{-1}+2^{-2}+2^{-3}+2^{-5}+\cdots+2^{-F_h}+\cdots \\ &= [0\,,\,1\,,\,10\,,\,6\,,\,1\,,\,6\,,\,2\,,\,14\,,\,4\,,\,124\,,\,2\,,\,1\,,\,2\,,\,2039\,,\,1\,,\,9\,,\,1\,,\,1\,,\,1\,,\,262111\,,\,2\,, \\ &\quad 8\,,\,1\,,\,1\,,\,1\,,\,3\,,\,1\,,\,536870655\,,\,4\,,\,16\,,\,3\,,\,1\,,\,3\,,\,7\,,\,1\,,\,140737488347135\,,\,\dots]\,. \end{aligned}$$

It seemed difficult to explain the apparent patterns of the numerical expansion; the more *rigid* formal power series case appeared relatively accessible [20]. **c**. One expects, with considerable confidence, that the example is representative of the nature of the continued fraction expansion of a very wide class of power series. After all, it's well known that for mathematical purposes the Fibonacci numbers have no property not generalised by way of units of real quadratic number fields or, according to the case, by higher order recurrence sequences, say those 'generated' by Pisot numbers. **d**. On the other hand, this example is fairly startling in that the sequential truncations of the series do not themselves provide convergents. It shares that property with the product with which we began. What if the exponent 2^h in that product were replaced with powers of larger integers?

3 Various Hilfsätze and Related Principles

3.1 Negating the negative. The following very simple lemmata provide most of the results we will need. We will here occasionally write $\overline{}$ to denote $-\ $. We draw attention to the following.

Lemma 1. $-\beta = [\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,0\,,\,\beta\,]$.

Proof. According to taste, study either of the two columns of computation below.

$$\begin{aligned} -\beta &= 0+\overline{\beta} & \text{or } -\beta &= [\,0+\overline{\beta}\,] = [\,0\,,\,-1/\beta\,] \\ -1/\beta &= \overline{1}+(\beta-1)/\beta & &= [\,0\,,\,\overline{1}+(\beta-1)/\beta\,] = [\,0\,,\,\overline{1}\,,\,\beta/(\beta-1)\,] \\ \beta/(\beta-1) &= 1+1/(\beta-1) & &= [\,0\,,\,\overline{1}\,,\,1+1/(\beta-1)\,] = [\,0\,,\,\overline{1}\,,\,1\,,\,\beta-1\,] \\ \beta-1 &= \overline{1}+\beta & &= [\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}+\beta\,] = [\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,1/\beta\,] \\ 1/\beta &= 0+1/\beta & &= [\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,0+1/\beta\,] = [\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,0\,,\,\beta\,] \\ \beta &= \beta \end{aligned}$$

∎

One needs to recall, say by the matrix correspondence, that $[\,a\,,\,0\,,\,b\,] = [\,a+b\,]$. Since, of course, $-[\,b\,,\,\gamma\,] = [\,-b\,,\,-\gamma\,]$, we have, for example,

$$\begin{aligned} -\pi &= -[\,3\,,\,7\,,\,15\,,\,1\,,\,292\,,\,1\,,\,\ldots\,] = [\,-3\,,\,-[7\,,\,15\,,\,1\,,\,292\,,\,1\,,\,\ldots\,]\,] \\ &= [\,-3\,,\,0\,,\,\overline{1}\,,\,1\,,\,\overline{1}\,,\,0\,,\,7\,,\,15\,,\,1\,,\,292\,,\,\ldots\,] = [\,-4\,,\,1\,,\,6\,,\,15\,,\,1\,,\,292\,,\,\ldots\,]. \end{aligned}$$

Corollary 1. *Alternatively,* $-\beta = [\,0\,,\,1\,,\,\overline{1}\,,\,1\,,\,0\,,\,\beta\,]$.

Using Lemma 1 we readily remove negative partial quotients from expansions. Thus $[\,a\,,\,\overline{b}\,,\,c\,,\,\delta\,] = [\,a\,,\,0\overline{1}1\overline{1}0\,,\,b\,,\,-c\,,\,-\delta\,] = [\,a-1\,,\,1\,,\,b-1\,,\,-c\,,\,-\delta\,]$, and that's $[\,a-1\,,\,1\,,\,b-1\,,\,0\overline{1}1\overline{1}0\,,\,c\,,\,\delta\,] = [\,a-1\,,\,1\,,\,b-2\,,\,1\,,\,c-1\,,\,\delta\,]$. If $b=1$ one proceeds differently.

It seems best to work from first principles, applying the Lemma repeatedly, rather than trying to apply consequent formulas.

3.2 Removing and Creating Partial Quotients. For continued fractions of real numbers the 'admissible' partial quotients are the positive integers. That makes it useful to have techniques for removing inadmissible partial quotients, specifically 0 and negative integers; it's rather more difficult to neatly remove more complicated quantities. For continued fraction expansions of formal power series, however, the corresponding admissibility criterion is that the partial quotients be polynomials of degree at least 1. It is now constant partial quotients that are inadmissible but which can be dealt with fairly readily.

We had best first remark that $x[\,a\,,\,b\,,\,c\,,\,\delta\,] = [\,xa\,,\,x^{-1}b\,,\,xc\,,\,x^{-1}\delta\,]$, a fact that is obvious but that is somehow not terribly widely known.

Lemma 2. $[\,a\,,\,x\,,\,\gamma\,] = [\,a+x^{-1}\,,\,-x^2\gamma-x\,]$.

Proof. Set $F = [\,a\,,\,x\,,\,\gamma\,]$, so $xF = [\,xa\,,\,1\,,\,x\gamma\,] = [\,xa\,,\,1\,,\,0\overline{1}1\overline{1}0\,,\,-x\gamma\,]$. Then $xF = [\,xa+1\,,\,-x\gamma-1\,]$ yields F as claimed. ∎

Corollary 2. *Conversely,* $[\,a+x\,,\,\gamma\,] = [\,a\,,\,x^{-1}\,,\,-x^2\gamma-x\,]$.

We see that 'moving x' propagates through the tail of the expansion as alternate multiplication and division by x^2. I suggest — this is 'philosophy', not mathematics — that the explosive increase in complexity of the rational coefficients of the partial quotients in the continued fraction expansion of a 'typical' formal power series is the consequence of a sequence of 'movings' of rational quantities. I will illustrate this explicitly for the example function G.

3.3 Paperfolding. The matrix correspondence readily yields the following extraordinarily useful result; I learned it from Mendès France [14]. As above, we have set $[\, a_0 \,,\, a_1 \,,\, \ldots \,,\, a_h \,] = p_h/q_h$ for $h = 0,\, 1,\, \ldots$. For convenience, we think of the string of symbols $a_1 \cdots a_h$ as the 'word' w_h. Naturally, given that, by $\overleftarrow{w_h}$ we then mean the word $a_h \cdots a_1$, and by $-\overleftarrow{w_h}$ the word $\overline{a}_h \cdots \overline{a}_1$.

Proposition 2 (Folding Lemma). *We have*

$$p_h/q_h + (-1)^h/xq_h^2 = [\, a_0 \,,\, w_h \,,\, x - q_{h-1}/q_h \,] = [\, a_0 \,,\, w_h \,,\, x \,,\, -\overleftarrow{w_h} \,]\,.$$

Proof. Here $\longleftrightarrow$ denotes the 'correspondence' between 2 by 2 matrices and continued fractions. We have

$$[\, a_0 \,,\, w_h \,,\, x - q_{h-1}/q_h \,] \longleftrightarrow \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix} \begin{pmatrix} x - q_{h-1}/q_h & 1 \\ 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} xp_h - (p_h q_{h-1} - p_{h-1} q_h)/q_h & p_h \\ xq_h & q_h \end{pmatrix} \longleftrightarrow p_h/q_h - (-1)^{h+1}/xq_h^2\,,$$

as alleged. Moreover, $[\, x - q_{h-1}/q_h \,] = [\, x \,,\, -\overleftarrow{w_h} \,]$. ∎

Why, 'folding'? Iteration of the perturbed symmetry $w \longrightarrow w, x, -\overleftarrow{w}$ yields a pattern of signs corresponding to the pattern of creases in a sheet of paper repeatedly folded in half; see [8].

For example, the continued fraction expansion of the sum

$$F = X \sum_{h \geq 0} X^{-2^h} = 1 + X^{-1} + X^{-3} + X^{-7} + X^{-15} + X^{-31} + X^{-63} + X^{-127} + \cdots$$

is given sequentially by $1 + X^{-1} = [\, 1 \,,\, X \,]$, $1 + X^{-1} + X^{-3} = [\, 1 \,,\, X \,,\, \overline{X} \,,\, \overline{X} \,]$, $1 + X^{-1} + X^{-3} + X^{-7} = [\, 1 \,,\, X \,,\, \overline{X} \,,\, \overline{X} \,,\, \overline{X} \,,\, X \,,\, X \,,\, \overline{X} \,]$, $\ldots$, where the addition of each term is done by a 'fold' with $x = -X$; see [19].

There is a different way of producing that folded sequence, but I'll use the more conventional symbols 0 and 1 in place of X and $\overline{X}$. We'll just 'fill the spaces' with '$1 \cdot 0 \cdot$ ' repeatedly ... ; having begun with '$0 \cdot 1 \cdot$ '. Here the $\cdot$ s denote a space about to be filled.

$$\begin{array}{l}
0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0{\cdot}1{\cdot}0 \ldots \\
011{\cdot}001{\cdot}011{\cdot}001{\cdot}011{\cdot}001{\cdot}011{\cdot}001{\cdot}011{\cdot}001{\cdot}011{\cdot}001{\cdot}0 \ldots \\
0111001{\cdot}0110001{\cdot}0111001{\cdot}0110001{\cdot}0111001{\cdot}0110001{\cdot}0 \ldots \\
011100110110001{\cdot}011100100110001{\cdot}011100110110001{\cdot}0 \ldots \\
0111001101100011011100100110001{\cdot}01110011011000100 \ldots \\
011100110110001101110010011000110111001101100 0100 \ldots
\end{array}$$

This remark actually seems useful in understanding some continued fraction expansions of formal power series. For me, it motivated the following result.

Proposition 3 (Ripple Lemma). $[\,z\,,\,a\,,\,b\,,\,c\,,\,d\,,\,e\,,\,f\,,\,g\,,\,h\,,\,i\,,\,j\,,\ldots] =$

$$[\,z-1\,,\,1\,,\,\overline{a}\,,\,\overline{1}\,,\,b\,,\,1\,,\,\overline{c}\,,\,\overline{1}\,,\,\;d\,,\,1\,,\,\overline{e}\,,\,\overline{1}\,,\,f\,,\,1\,,\,\overline{g}\,,\,\overline{1}\,,\,h\,,\,1\,,\,\overline{i}\,,\,\overline{1}\,,\,j\,,\,1\,,\ldots].$$

Proof. Appropriately apply Lemma 1, equivalently the Corollary to Lemma 2 with $x = \pm 1$, again, and again, and . . . . ∎

The series F is given by the functional equation $1+X^{-1}F(X^2) = F(X)$. But it's easy to see that the folded continued fraction $\mathcal{F}(X)$ claimed for F above has the property that $X\mathcal{F}(X) - X$ is just a rippled version of $\mathcal{F}(X^2)$, providing a new proof that F has the continued fraction expansion $\mathcal{F}(X)$. This new viewpoint [17] readily allowed a noticeable simplification and generalisation of the work of [1] detailing various more delicate properties of the expansion.

4 Some Details

We display a computation illustrating how there is an explosion in complexity of the rational coefficients of the partial quotients of formal power series, and mention the effect of reduction mod p on such continued fraction expansions.

4.1 A Painful Computation. Suppose we have discovered, either laboriously by hand, or aided by the miracle of PARI, that

$$G(X;1) = (1-X^{-1})(1-X^{-2}) = [\,1\,,\,-X+1\,,\,-\tfrac{1}{2}X-\tfrac{3}{4}\,,\,8X-4\,].$$

We bravely set out to compute, by hand, the continued fraction expansion of $G(X;2) = (1-X^{-1})G(X^2;1)$. Replacing X by X^2 is easy, but then we'll want to divide by X, and multiply by $X-1$.

To that end we first ready the expansion for being divided by X by repeated applications of Lemma 2; thus generalised 'rippling'. We see that

$$\begin{aligned}[\,1\,,\,-X^2+1\,,\,-\tfrac{1}{2}X^2-\tfrac{3}{4}\,,\,8X^2-4\,] &= [\,0\,,\,1\,,\,X^2-2,\,\tfrac{1}{2}X^2+\tfrac{3}{4}\,,\,-8X^2+4\,]\\ &= [\,0\,,\,1\,,\,X^2\,,\,-\tfrac{1}{2}\,,\,-2X^2-1\,,\,2X^2-1\,]\\ &= [\,0\,,\,1\,,\,X^2\,,\,-\tfrac{1}{2}\,,\,-2X^2\,,\,-1\,,\,-2X^2+2\,],\end{aligned}$$

so we'll divide $[\,0\,,\,1\,,\,X^2\,,\,-\tfrac{1}{2}\,,\,-2X^2\,,\,-1\,,\,-2X^2\,,\,\tfrac{1}{2}\,]$ by X. We obtain

$$\begin{aligned}[\,0\,,\,X\,,\,X\,,\,-\tfrac{1}{2}X\,,\,-2X\,,\,-X\,,\,-2X\,,\,\tfrac{1}{2}X\,] &= [\,0\,,\,X-1\,,\,1\,,\,-X-1\,,\,\tfrac{1}{2}X\,,\ldots]\\ &= [\,0\,,\,X-1\,,\,1\,,\,-X+1\,,\,-\tfrac{1}{2}\,,\,\underline{-2X+2}\,,\,-\tfrac{1}{2}X\,,\,-4X\,,\,-\tfrac{1}{2}X\,,\,2X\,],\end{aligned}$$

where we've started to ripple the expansion to ready it for multiplication by $X-1$. The exciting feature is the underlined term $-2X+2$. It is 'accidentally' ready — without our having had to ripple it into submission. It's that unlikely to be repeated accident that causes the expansion of G to have partial quotients of degree 2. Next

$$[\,0\,,\,X-1\,,\,1\,,\,-X+1\,,\,-\tfrac12\,,\,-2X+2\,,\,-\tfrac12X\,,\,-4X+4\,,\,-\tfrac14\,,\,8X+4\,,\,-\tfrac18X\,]$$
$$=[\,\dots\,,\,-\tfrac14\,,\,8X-8\,,\,\tfrac1{12}\,,\,18X-12\,]=[\,\dots\,,\,-\tfrac14\,,\,8X-8\,,\,\tfrac1{12}\,,\,18X-18\,,\,\tfrac16\,].$$

On multiplying by $X-1$, as we now can easily do, we see that $G(X;2)$ is

$$[\,0\,,\,1\,,\,X-1\,,\,-1\,,\,-\tfrac12(X-1)\,,\,-2\,,\,-\tfrac12X(X-1)\,,\,-4\,,$$
$$-\tfrac14(X-1)\,,\,8\,,\,\tfrac1{12}(X-1)\,,\,18\,,\,\tfrac16(X-1)\,].$$

Finally, we tidy that up, again risking an increase in complexity of the rational coefficients. When the dust has settled we obtain

$$G(X;2)=[1\,,\,-X+1\,,\,-\tfrac12X-1\,,\,2X^2-2X+3\,,\,-X+\tfrac12\,,\,-\tfrac43X-\tfrac{14}9\,,\,\tfrac{27}8X-\tfrac94\,].$$

I don't want to claim that the method used here is a sensible way of pursuing the computation; it's far more convenient to type a few lines on one's computer. But I do suggest that it enables us to see both how complexity of the coefficients propagates, and that it requires an unlikely accident — so unlikely as to be near impossible other than at the beginning of the expansion, when the coefficients still are orderly — to newly create a partial quotient of degree other than 1. It is the functional equation satisfied by $G(X)$ that generates the 'kind of sort of quasi-periodicity' I vaguely spoke of above.

4.2 On the other hand ... . Consider $H_4(X)=\prod_{h\ge0}(1+X^{-4^h})$. The surprise this example provides is not just that its partial quotients have high degree. The coefficients of the partial quotients all are integers!

Just as in the clumsy approach just tried, we obtain sequentially that

$$H_4(X;0)=[\,1\,,\,X^4\,]\,;\qquad H_4(X;1)=\frac{X-1}{X}[\,1\,,\,X^4\,]=$$
$$=\frac{X-1}{X}[\,0\,,\,1\,,\,-X^4\,,\,\overline{1}\,]=(X-1)[\,0\,,\,X\,,\,-X^3\,,\,-X\,].$$

Next, we ripple again, now to permit the multiplication by $X-1$. We obtain

$$(X-1)[\,0\,,\,X-1\,,\,1\,,\,X^3-1\,,\,X\,]$$
$$=[\,0\,,\,1\,,\,X-1\,,\,(X^3-1)/(X-1)\,,\,X(X-1)\,]$$
$$=[\,1\,,\,-X\,,\,-(X^3-1)/(X-1)\,,\,-X(X-1)\,].$$

The reader caring to pursue this process will find that each iteration adds just two partial quotients, and that all have integer coefficients. Indeed, Mendès France and I remark in [15] that for $k\ge3$ the truncations of $H_k(X)$ are readily seen to be convergents of H_k and, when k is even, not 2, they are every second convergent of H_k. However, for $k\ge3$ odd, we show in [2] with the aid of Allouche that the expansion is 'normal'* — up to the partial quotients given by the truncations of the product, and their 'quasi-repetition' occasioned by the functional equation. In the case $k=3$ the partial quotients whose existence is given by the truncations of the product too have degree 1. However, the ingenious proof in [2] that indeed all the partial quotients of H_3 have degree 1 — which relies on multiplying by using the 'Raney automata' [21] — was nugatory. It is remarked by Cantor [7] that the degree of the partial quotients of H_3 is an obvious consequence of the fact that $H_3(X)$ is $(1+X^{-1})^{-1/2}$ when reduced mod 3 and that *its* partial quotients all have degree 1.

4.3 Beal's Principle. Cantor's observation follows from a general principle. We'll need to mind our ps and qs, since we'll want to use p to denote a prime; so our convergents will here be x/y. Given a series F, we denote its sequence of partial quotients by (a_h), and of its complete quotients by (F_h). My remarks are inspired by a question put to me by Guillaume Grisel (Caen) at Eger, 1996.

The principle underlying Grisel's question was that it seemed likely that every reduction $\overline{F}$, mod p, of F has no more partial quotients than does F itself. Notice that F must have reduction at p for this to make sense at all and that, naturally, if we mention the number of partial quotients then we must apparently be alluding to the number of partial quotients of rational functions; thus of truncations of the series F.

However, I've now realised that the idea is to understand the first principles genesis of the sequence of polynomials (y_h) yielding the convergents to F. Recall that, by Proposition 1, those are the polynomials of least degree not exceeding d_h, say, respectively, so that the Laurent series y_hF has no terms of degree -1, -2, …, and $-d_h$. There is no loss of generality (but there is a significant change in definition of the y_h) in our determining that the y_h have been renormalised so that each has integer coefficients not sharing a common factor. Now consider this story in characteristic p. It can be told in the same words, other than that it's not relevant to fuss about normalisation of the $\overline{y_h}$ and that we mark all quantities with an overline $\overline{}$.

Theorem 1. *The distinct reductions $\overline{y_h}$ of the y_h yield all the convergents of $\overline{F}$.*

Proof. Certainly, each $\overline{y_h}$ yields a convergent to $\overline{F}$, because

$$\deg(y_hF-x_h)<-\deg y_h \text{ implies that } \deg(\overline{y_hF-x_h})<-\deg y_h\le-\deg\overline{y_h}\,.$$

However, some of the $\overline{y_h}$ may coincide. Denote representatives of the *distinct* $\overline{y_h}$ by $\overline{y_{h(0)}}$, $\overline{y_{h(1)}}$, … , $\overline{y_{h(j)}}$, … , where each $h(j)$ is maximal; that is $\overline{y_{h(j)}}=\overline{y_{h(j)-1}}=\cdots=\overline{y_{h(j-1)+1}}$. Then

$$\deg(y_{h(j)}F-x_{h(j)})=-\deg y_{h(j)+1} \text{ entails } \deg(\overline{y_{h(j)}F-x_{h(j)}})\le-\deg y_{h(j)+1}\,.$$

* That is, all the partial quotients are of degree 1. But I also intend to invoke the notion that the coefficients of those polynomials are 'typical' and explode in complexity.

The last inequality informs us that the corresponding next partial quotient of $\overline{F}$, let's call it $\overline{b_{j+1}}$, has degree at least $\deg y_{h(j)+1} - \deg \overline{y_{h(j)}}$. But

$$\sum_{j=0}^{n}\bigl(\deg y_{h(j)+1} - \deg \overline{y_{h(j-1)+1}}\,\bigr) \geq \sum_{j=0}^{n}\bigl(\deg y_{h(j)+1} - \deg y_{h(j-1)+1}\bigr) = \deg y_{h(n)+1}\,,$$

where we recall $\overline{y_{h(j)}} = \overline{y_{h(j-1)+1}}$, and that by the formalism $\overline{y_{h(-1)+1}} = 1$, so that $y_{h(-1)+1}$ is a constant, and thus is of degree zero.

However, it's plain by induction on a remark in the introduction, that

$$\sum_{j=0}^{n}\deg \overline{b_{j+1}} = \deg \overline{y_{h(n)+1}} \leq \deg y_{h(n)+1}\,.$$

It follows that the 'polite' inequalities above (where we wrote '$\leq$' because we could not be certain that we were allowed to write '$=$') all are equalities, that is, $\deg \overline{y_{h(j-1)+1}} = \deg y_{h(j-1)+1}$ and $\deg y_{h(j)+1} - \deg y_{h(j)} = \deg b_{j+1}$, and the $y_{h(j)}$ must account for all the convergents of $\overline{F}$ as claimed. ∎

This yields a verification of Beal's principle in the best sense, because we show that the convergents of $\overline{F}$ arise from a subset of those of F, so that it always makes sense to claim in that sense that the number of convergents of $\overline{F}$ cannot exceed that of F.

It is then a triviality that if $\deg \overline{q_h} = h$ for all h necessarily the same, that is $\deg q_h = h$ for all h, is true for the original function. But it is easily confirmed that $\deg \overline{q_h} = h$ for all h for $(1+X^{-1})^{-1/2}$ over $\mathbb{F}_3$, so of course also $\deg q_h = h$ is true for the product $\prod_{h\geq 0}(1+X^{-3^h})$, as Cantor pointed out.

5 In Thrall to Fibonacci

We remark that to our surprise, and horror, continued fraction expansion of formal power series appears to adhere to the cult of Fibonacci.

5.1 Specialisable Continued Fraction Expansions. Suppose $(g_h)_{h\geq 0}$ is a sequence of positive integers satisfying $g_{h+1} \geq 2g_h$. Then the Folding Lemma, together with Lemma 2 whenever $g_{h+1} = 2g_h$, readily shows that every series $\sum_{h\geq 0} \pm X^{-g_h}$ has a continued fraction expansion with partial quotients polynomials with integer coefficients. Since such expansions are precisely the expansions that continue to make sense when X is replaced by an integer at least 2, we call them *specialisable*.

When the exponents g_h increase less rapidly more *ad hoc* tricks become necessary. Shallit and I noticed [20], mostly experimentally but with proofs for several simpler cases, that certain series $\sum X^{-T_h}$ are specialisable, where the recurrence sequence (T_h) satisfies $T_{h+n} = T_{h+n-1} + T_{h+n-2} + \cdots + T_h$ — the dreaded Fibonacci, Tribonacci [*sic*], and more generally, forgive me, Polynacci⋆ sequences.

⋆ Surely 'n-acci' is no better?

Since it seemed absurd that continued fractions be in thrall to Fibonacci, I was keen to discover a larger class of examples of which those instances were part.

5.2 A Shocking Surprise. It seems one should study the continued fractions of the sequence of sums

$$\bigl(X^{G_{m+1}}(X^{-G_{m+1}}+X^{-G_{m+2}}+\cdots+X^{-G_{m+h}})\bigr)_{h\geq 0}\,.$$

Having, somehow, obtained the expansion for h, one changes $m \longrightarrow m{+}1$, divides by $X^{G_{m+2}-G_{m+1}}$, and finally one adds 1. The ripple lemma makes it feasible to do this 'by hand' and to see what 'really' happens in moving $h \longrightarrow h{+}1$. Whatever, one finds that the folded sequence obtained is first perturbed after $n+1$ steps by the behaviour of the 'critical' exponent $G_{m+n+2}-2G_{m+n+1}+G_{m+1}$. Call this quantity $G_{m,n}$. Specifically, if for some n, $G_{m,n-1}>0$ but $G_{m,n}<0$ then the expansion is not specialisable; and if both are positive, then n is not critical. So the interesting case is $G_{m,n-1}>0$ and $G_{m,n}=0$.

In that case, and only that case, the expansion *is* specialisable. But, *horribile dictu*, the condition $G_{m,n}=0$ for all m says that (G_h) is some constant translate of the Polynacci sequence of order n. Contrary to decency and common sense, it does seem that these cases really are special when it comes to specialisable continued fraction expansion.

The perturbation caused by the vanishing of $G_{m,n}$ spreads through the expansion by the inductive step. One also notices that specialisability is lost if one makes arbitrary changes to the signs of the terms. Finally, one obtains cases such as the examples of [20], see §2.4 above, by presuming $G_0=G_1=\cdots=G_{n-2}=1$, $G_{n-1}=1$ and taking $m=n-1$. There's still work to do then to show that the expansion remains specialisable on division by X. That division further perturbs the pattern in the expansion, explaining why the work of [20] was so complicated.

6 Concluding Remarks

6.1 Normality. That, for formal power series over an infinite field, all partial quotients are almost always of degree one, is just the observation that remainders $\sum_{h\geq 1}a_hX^{-h}$ have a reciprocal with polynomial part of degree greater than one, and thus give rise to a partial quotient of degree greater than one, if and only if $a_1=0$. Moreover, if $a_1\neq 0$, the partial quotient is $a_1^{-1}X-a_2a_1^{-1}$ and the next remainder is $(a_2^2-a_1a_3)a_1^{-3}X^{-1}+$ terms of lower degree in X.

This same viewpoint shows that the reduction mod p of a formal power series almost always has partial quotients of degree greater than one, since now the nonvanishing of the coefficient of X^{-1} of all remainders is as unlikely as the nonappearance of the digit 0 in the base p expansion of a random real number.

These two remarks combine to explain my claim that one should expect a formal power series with integer coefficients to have partial quotients of degree one, that the continued fraction expansion will have bad reduction at all primes, and that — noting the shape of the coefficient of X^{-1} of the 'next' remainder — the coefficients of the partial quotients will quickly explode in complexity.

6.2 Announcements. Other remarks also are announcements with hint of their eventual proof; one might call them conjectures with some justification. Thus, I certainly give no proof that the partial quotients of $G(X)$, see §2.1, are all of degree at most 2; notwithstanding the strong hints of §4.1. Here, I'm not sure that a proof warrants the effort. On the other hand, §5 both reports some results proved in [20] and the announcement that I now know how to prove the conjectures of that paper; evidently with details to appear elsewhere.

6.3 Power Series over Finite Fields. There is a well studied analogy between number fields and function fields in positive characteristic leading, for example, to a theory of diophantine approximation of power series in finite characteristic as in the work of de Mathan at Bordeaux. Iteration of references from the recent paper [13] will readily lead the reader into that literature. By the way, Theorem C sketched on p.224 of that paper is a trivial application of the Folding Lemma, Proposition 2 above. I might add that it was the work of Baum and Sweet [3] that first interested me in the present questions. Beal's Principle informs on these matters; description of that will be the subject of future work.

6.4 Power Series with Periodic Continued Fraction Expansion. It is of enormous interest to find infinite classes of positive integers D for which the continued fraction expansion of $\sqrt{D}$ has a 'long' period, in principle of length $O(\sqrt{D}\log\log D)$, but in practice of length $O\big((\log D)^k\big)$ since no better is known than some cases with small k. One approach, as exemplified by [9], leads to a study of families $\sqrt{f(n)}$, with f a given polynomial taking integer values at the integers, and for integers n. Here a theorem of Schinzel [22] shows that the period has uniformly bounded length, thus independent of n, essentially when the power series $\sqrt{f(X)}$ has a periodic expansion with good reduction at all primes, perhaps other than 2. More precisely, polynomials providing solutions to the Pell equation must be in $\frac{1}{2}\mathbb{Z}[X]$. These issues connect closely with recent work of Bombieri and Paula Cohen [6] showing that the coefficients of simultaneous Padé approximants to algebraic functions are large — this is essentially the explosive growth of the coefficients of partial quotients of which we make much above — in effect unless a 'generalisation' of Schinzel's conditions holds. For the hyperelliptic case of this phenomenon, which goes back to Abel, see [5].

References

1. J.-P. Allouche, Anna Lubiw, M. Mendès France, A. J. van der Poorten and J. O. Shallit, 'Convergents of folded continued fractions', *Acta Arith.*, **77** (1996), 77–96.
2. J.–P. Allouche, M. Mendès France and A. J. van der Poorten, 'An infinite product with bounded partial quotients', *Acta Arith.* **59** (1991), 171–182.
3. L. Baum and M. Sweet, 'Continued fractions of algebraic power series in characteristic 2', *Ann. of Math.* **103** (1976), 593–610.
4. Bruce C. Berndt, *Ramanujan's notebooks*, Springer-Verlag. Part I, with a foreword by S. Chandrasekhar, New York-Berlin, 1985; Part II, New York-Berlin, 1989; Part III, New York, 1991; Part IV, New York, 1994; Part V, New York, 1998.

5. T. G. Berry, 'On periodicity of continued fractions in hyperelliptic function fields', *Arch. Math.* **55** (1990), 259–266.
6. Enrico Bombieri and Paula B. Cohen, 'Siegel's Lemma, Padé Approximations and Jacobians' (with an appendix by Umberto Zannier), to appear in the De Giorgi volume, Annali Scuola Normale Superiore, Pisa
7. David G. Cantor, 'On the continued fractions of quadratic surds', *Acta Arith.* **68** (1994), 295–305.
8. Michel Dekking, Michel Mendès France and Alf van der Poorten, 'FOLDS!', *The Mathematical Intelligencer* **4** (1982), 130-138; II: 'Symmetry disturbed', *ibid.* 173-181; III: 'More morphisms', *ibid.* 190-195.
9. E. Dubois et R. Paysant-Le Roux, 'Sur la longueur du développement en fraction continue de $\sqrt{f(n)}$', *Astérisque* 198–200 (1991), 107–119
10. David Goss, David R. Hayes and Michael I. Rosen eds., *The arithmetic of function fields*, Proceedings of the workshop held at The Ohio State University, Columbus, Ohio, June 17–26, 1991. Ohio State University Mathematical Research Institute Publications, **2**. Walter de Gruyter & Co., Berlin, 1992. viii+482 pp.
11. Peter Henrici, *Applied and computational complex analysis*, Vol. 2, *Special functions—integral transforms—asymptotics—continued fractions*. Reprint of the 1977 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1991. x+662 pp.
12. William B. Jones and Wolfgang J. Thron, *Continued fractions. Analytic theory and applications*, with a foreword by Felix E. Browder and an introduction by Peter Henrici. Encyclopedia of Mathematics and its Applications, **11**. Addison-Wesley Publishing Co., Reading, Mass., 1980. xxix+428 pp.
13. Alain Lasjaunias, 'Diophantine approximation and continued fractions expansions of algebraic power series in positive characteristic', *J. Number Theory* **65** (1997), 206–225.
14. Michel Mendès France, 'Sur les fractions continues limitées', *Acta Arith.* **23** (1973), 207–215.
15. M. Mendès France and A. J. van der Poorten, 'From geometry to Euler identities', *Theoretical Computing Science* **65** (1989), 213–220.
16. M. Mendès France and A. J. van der Poorten, 'Some explicit continued fraction expansions', *Mathematika* **38** (1991), 1–9.
17. Michel Mendès France, Alf van der Poorten and Jeffrey Shallit, 'On lacunary formal power series and their continued fraction expansion', to appear in the Proceedings of the Number Theory Conference in honour of Andrzej Schinzel on the occasion of his 60th birthday, K. Győry, H. Iwaniec and J. Urbanowicz eds, 6pp.
18. A. J. van der Poorten, 'An introduction to continued fractions', in J. H. Loxton and A. J. van der Poorten eds., *Diophantine Analysis* (Cambridge University Press, 1986), 99–138.
19. A. J. van der Poorten and J. Shallit, 'Folded continued fractions', *J. Number Theory* **40** (1992), 237–250 .
20. A. J. van der Poorten and J. Shallit, 'A specialised continued fraction', *Canad. J. Math.* **45** (1993), 1067-1079.
21. G. N. Raney, 'On continued fractions and finite automata', *Math. Ann.* **206** (1973), 265–283.
22. A. Schinzel, 'On some problems of the arithmetical theory of continued fractions', *Acta Arith.* **7** (1962), 287–298.
23. H. S. Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand Company, Inc., New York, N. Y., 1948; xiii+433 pp.