

# Asymptotic Improvement of Resummations and Perturbative Predictions in Quantum Field Theory

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**Abstract** The improvement of resummation algorithms for divergent perturbative expansions in quantum field theory by asymptotic information about perturbative coefficients is investigated. Various asymptotically optimized resummation prescriptions are considered. The improvement of perturbative predictions beyond the reexpansion of rational approximants is discussed.

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# 1 INTRODUCTION

There is overwhelming evidence that perturbation series in quantum field theory and related disciplines diverge (see for example [1] and references therein). Consequently, resummation techniques, which allow to associate a finite value to a divergent series, are needed if divergent perturbation series are to be used for numerical purposes. The best known and most often applied resummation techniques are Padé approximants [2] and the Borel method [3, 4]. Recently, another summation method – the so-called delta transformation (Eq. (8.4-4) of [5]) – has gained some prominence as a summation method in various domains of physics [6–12]. There is evidence [7, 10] that the delta transformation is able to sum divergent series whose coefficients  $c_n$  grow essentially like  $n!$ ,  $(2n)!$  and even  $(3n)!$ . This is not achievable by employing Padé approximants [13].

In the case of both Padé approximants and the delta transformation, only the numerical values of a finite number of partial sums of the divergent input series are needed as input data. Within the framework of the Borel method, it is also necessary to know the large order asymptotics of the coefficients of the divergent power series. Consequently, the Borel method is slightly less general than the other two methods.

Rational approximations to (divergent) power series have an interesting feature: It is possible to extrapolate higher-order coefficients that were not used for the construction of the rational approximants. This feature, which was apparently first observed by Gilewicz [14], has so far been used quite extensively in the case of Padé approximants for divergent perturbation expansions from quantum field theory and related disciplines. We refer to the investigations by Elias, Steele and Chishtie *et al.* [15–21], by Samuel, Gardi, Karliner, Ellis *et al.* [22–41] and by Cvetič *et al.* [42–45]. An analogous approach also works in the case of the delta transformation [11, 12]. Recursive techniques for the Padé prediction of unknown series coefficients were developed recently [46].

It is the intention of this article to discuss – augmenting previous investigations [12] – how additional information on the large-order asymptotics of the perturbative coefficients can be incorporated effectively into resummation and prediction schemes. Our emphasis will be on the Borel-Padé method which was introduced by Graffi, Grecchi, and Simon [47]. This is a variant of the Borel method that uses Padé approximants for performing the analytic continuation of the Borel transformed series to a neighborhood of the positive real semiaxis. Further details on the Borel-Padé method as well as on other resummation methods can be found in Section 2.

We would like to mention here the investigations (based on the Borel method) by Fischer [48–50] and by Caprini and Fischer [51, 52] regarding the resummation of divergent perturbative expansions in quantum field theory. In [51], Caprini and Fischer use asymptotic information (location of the first IR and UV renormalon poles) for the construction of conformal mappings in the Borel plane. Here, we describe alternative ways in which the asymptotics of the coefficients of the perturbative expansion can be utilized to enhance the effectiveness of resummation procedures and perturbative predictions. These improvements, which provide an alternative to the methods presented in [51], are not necessarily restricted to the first IR and UV renormalons, but can take advantage of the location of all known poles in the Borel plane. Regarding asymptotic properties of perturbative coefficients in quantum field theory we also refer to [53, 54].

Cvetič and Yu in [45] consider the resummation of the *real* part of the (one-loop) QED effective action (or vacuum-to vacuum amplitude) in the presence of background electric and magnetic fields, of which the exact nonperturbative answer is known [56]. In the presence of an electric field, the QED effective action acquires an imaginary part proportional to the pair production amplitude for real electron-positron (or lepton-antilepton) pairs. The real part of the effective action does not constitute the full physical solution. The full, complex-valued answer requires an integration in the complex plane, for example along the special contour introduced in [57]. Only in the case of the pure magnetic field, which we consider in this article, the QED effective action is entirely real. For this particular example, the delta transformation employed in [12] or the Borel-Padé Cauchy principal value method used in [45] provide the full, i.e. entirely real, not complex physical solution. In view of the above, we again consider here only the resummation of the pure magnetic field. The electric field or the electric field combined with a magnetic field should be treated along the ideas introduced in [57].

We briefly discuss the relation of resummation and perturbative predictions: There is evidence (see for example [12, 57] and references therein) that the *resummation* of divergent series is ambiguous, especially if these series do not fulfill a Carleman condition (see [58] or Theorems XII.17 and XII.19 in [59]). However, the *prediction* of perturbative coefficients apparently does not suffer from such ambiguities. Perturbative predictions should be possible [12] even in those cases where the perturbation series is evaluated “on the

cut" in the complex plane. For the prediction of unknown perturbation series coefficients, the nonanalytic contributions, which are responsible for the ambiguities, are irrelevant. Only the analytic part of the function, which is represented by a divergent power series, matters. The terms of an otherwise more problematic divergent nonalternating series can be predicted just as well as the terms of an alternating series. That is to say, the resummation of a divergent series is not necessarily unique (see the different integration contours in [57]), but the perturbative coefficients, which can be extrapolated and predicted via rational approximants, are uniquely determined even though the complex integration along the different contours in [57] leads to different results for the nonperturbative, nonanalytic contributions.

This paper is organized as follows. In Section 2 we present a short account of various resummation methods for divergent series. The exploitation of additional available asymptotic information in resummation algorithms is discussed in Section 3. Asymptotically optimized perturbative predictions (i.e., predictions of higher-order unknown perturbative coefficients) are discussed in Sections 4 and 5 for various example cases. We conclude with a discussion of the results in Section 6.

## 2 A REVIEW OF RESUMMATION METHODS

In this Section, we provide a condensed description of the resummation methods under consideration in this article. Let

$$f(z) \sim \sum_{\nu=0}^{\infty} \gamma_{\nu} z^{\nu} \quad (1)$$

be a (formal) power series for some function  $f$ . Then, we define the  $(\kappa, \lambda)$ -generalized Borel integral transform according to

$$f(z) = \int_0^{\infty} t^{\lambda-1} \mathcal{B}^{(\kappa, \lambda)}(f; z t^{\kappa}) \exp(-t) dt \quad (2)$$

$$= z^{-\lambda/\kappa} \int_0^{\infty} s^{\lambda-1} \mathcal{B}^{(\kappa, \lambda)}(f; s^{\kappa}) \exp(-s/z^{1/k}) ds. \quad (3)$$

Here,

$$\mathcal{B}^{(\kappa, \lambda)}(f; z) = \sum_{\nu=0}^{\infty} \frac{\gamma_{\nu}}{\Gamma(\kappa\nu + \lambda)} z^{\nu} \quad (4)$$

is the  $(\kappa, \lambda)$ -generalized Borel transformed series of the power series (1) for  $f(z)$ . For  $\kappa = \lambda = 1$ , we recover the usual formulas for the Borel transformation:

$$f(z) = \int_0^{\infty} \mathcal{B}(f; zt) \exp(-t) dt \quad (5)$$

$$= \frac{1}{z} \int_0^{\infty} \mathcal{B}(f; s) \exp(-s/z) ds, \quad (6)$$

$$\mathcal{B}(f; z) = \mathcal{B}^{(1,1)}(f; z) = \sum_{\nu=0}^{\infty} \frac{\gamma_{\nu}}{\nu!} z^{\nu}. \quad (7)$$

There exists an extensive literature on the Borel method in general and on physical applications in special. Any attempt to provide something resembling a reasonably complete bibliography would clearly be beyond the scope of this article. Let us just mention that recent monographs on the Borel method and related topic were published by Shawyer and Watson [60] and Sternin and Shatalov [61].

Let us assume that the coefficients  $\gamma_n$  of the power series (1) possess the following large order asymptotics,

$$\gamma_n \sim A \Gamma(\kappa n + \lambda) B^n, \quad n \rightarrow \infty, \quad (8)$$

where  $A$ ,  $B$ ,  $\kappa$ , and  $\lambda$  are suitable constants. We say that a  $(\kappa', \lambda')$ -generalized Borel method for some power series (1) is *asymptotically optimized* if the parameters  $\kappa'$  and  $\lambda'$  agree with the parameters  $\kappa$  and  $\lambda$  in the large-order asymptotics (8) for the series coefficient  $\gamma_n$ . Thus, the leading (hyper)factorial growth of  $\gamma_n$  is exactly canceled out in this case. If we know in addition the parameter  $B$  in (8), then we can immediately deduce that the asymptotically optimized Borel transformed series (4) possesses a pole at  $z = 1/B$ .

The most difficult computational problem, which normally occurs in the context of a Borel summation process, is the construction of an analytic continuation for the Borel transformed series (4). If the coefficients  $\gamma_n$  of the power series (1) satisfy (8), then the  $(\kappa, \lambda)$ -generalized Borel transformed series (4) has a *nonzero* but *finite* radius of convergence. In order to be able to do the integration, we now need an analytic continuation which extends  $\mathcal{B}^{(\kappa, \lambda)}(f; z)$  from the interior of its circle of convergence to a neighborhood which contains the whole positive semiaxis. In this article, we emphasize the Borel-Padé method which was introduced by Graffi, Grecchi, and Simon [47] and which accomplishes the analytic continuation by converting the partial sums of the Borel transformed series (4) to Padé approximants. It may happen that the Padé approximants thus constructed exhibit poles along the positive real axis. In such a case, the function  $f$  is – strictly speaking – not Borel-Padé summable. However, by further generalizing the integrals (2) and (3) – either via a principal-value prescription [62, 63], or by employing conformal mappings [51, 52], or via the special integration contours used in [57] – it may nevertheless be possible to associate a finite value to the divergent power series (1) for  $f$ .

In the case of Padé approximants we use the notation and conventions of the monograph by Baker and Graves-Morris [2]. Thus, a Padé approximant  $[l/m]_f(z)$  to  $f(z)$  corresponds to the ratio of two polynomials  $P_l(z)$  and  $Q_m(z)$ , which are of degrees  $l$  and  $m$ , respectively, in  $z$ :

$$[l/m]_f(z) = \frac{P_l(z)}{Q_m(z)} = \frac{p_0 + p_1 z + \dots + p_l z^l}{1 + q_1 z + \dots + q_m z^m}. \quad (9)$$

The polynomials  $P_l(z)$  and  $Q_m(z)$  are constructed so that the Taylor expansion of the Padé approximation agrees with the original input series (1) up to terms of order  $l + m$  in  $z$ ,

$$f(z) - [l/m]_f(z) = O(z^{l+m+1}), \quad z \rightarrow 0. \quad (10)$$

This asymptotic error estimate leads to a system of linear equations by means of which the coefficients  $p_0, p_1, \dots, p_l$  and  $q_1, q_2, \dots, q_m$  in (9) can be computed. However, there are also several algorithms which permit a recursive computation of Padé approximants. A discussion of the merits and weaknesses of the various computational approaches can for instance be found in Section II.3 of the book by Cuyt and Wuytack [64].

An example of such a recursive algorithm is provided by Wynn's epsilon algorithm [65]:

$$\epsilon_{-1}^{(n)} = 0, \quad \epsilon_0^{(n)} = s_n, \quad n \in \mathbf{N}_0, \quad (11)$$

$$\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + 1/[\epsilon_k^{(n+1)} - \epsilon_k^{(n)}], \quad k, n \in \mathbf{N}_0. \quad (12)$$

Wynn [65] showed that if the input data  $s_n$  for the epsilon algorithm are the partial sums of the (formal) power series (1) for some function  $f(z)$  according to

$$s_n = f_n(z) = \sum_{\nu=0}^n \gamma_\nu z^\nu, \quad (13)$$

then the elements  $\epsilon_{2k}^{(n)}$  with *even* subscripts are Padé approximants to  $f$  according to

$$\epsilon_{2k}^{(n)} = [n + k/k]_f(z). \quad (14)$$

In contrast, the elements  $\epsilon_{2k+1}^{(n)}$  with *odd* subscripts are only auxiliary quantities which diverge if the whole process converges.

If one tries to sum a divergent power series or to accelerate the convergence of a slowly convergent series by converting its partial sums to Padé approximants, it is usually a good idea to use either *diagonal* Padé approximants, whose numerator and denominator polynomials have equal degrees, or – if this is not possible – to use Padé approximants with numerator and denominator polynomials whose degrees differ as little as possible. If we use the epsilon algorithm for the computation of the Padé approximants, then Eq. (14) implies that we then obtain the following staircase sequence in the Padé table (see Eq. (4.3-7) of [5]):

$$[0/0], [1/0], [1/1], \dots, [\nu/\nu], [\nu + 1/\nu], [\nu + 1/\nu + 1], \dots \quad (15)$$

This staircase sequence exploits the available information optimally if the partial sums  $f_m(z)$  with  $m \geq 0$  are computed successively and if after the computation of each new partial sum the element of the epsilon

table with the highest possible *even* transformation order is computed. With the help of the notation  $\llbracket x \rrbracket$  for the integral part of  $x$ , this staircase sequence can be written compactly as follows:

$$\epsilon_{2\llbracket n/2 \rrbracket}^{(n-2\llbracket n/2 \rrbracket)} = \left[ n - \llbracket n/2 \rrbracket / \llbracket n/2 \rrbracket \right]_f(z), \quad n = 0, 1, 2, \dots \quad (16)$$

The asymptotic error estimate (10) implies that all series coefficients, which are employed for the computation of the Padé approximant  $[l/m]_f(z)$ , are recovered by a Taylor expansion. Consequently, the higher order derivatives of the Padé approximant provide predictions for “unknown” series coefficients, i.e. to those series coefficients that were not used for computation of  $[l/m]_f(z)$ .

There is an enormous amount of literature on Padé approximants in general as well as on their application in theoretical physics. Let us just mention that the popularity of Padé approximants in theoretical physics can be traced back to a review by Baker [66], that the monograph by Baker and Graves-Morris [2] is the currently most complete source of information on Padé approximants, and that an account of the historical development of Padé approximants and related topics is given in a monograph by Brezinski [67].

The intense research on Padé approximants during the last decades of course also showed that Padé approximants suffer – like all other numerical techniques – from certain limitations and weaknesses. For example, Padé approximants are in principle limited to convergent or divergent power series, but cannot help in the case of many other types of slowly convergent or divergent sequences. Moreover, Padé approximants are either not useful or cannot be applied at all in the case of power series whose coefficients  $\gamma_n$  grow like  $(2n)!$  or even  $(3n)!$  [13]. Consequently, the intense research on Padé approximants also stimulated research on related techniques, the so-called sequence transformations.

Let us assume that  $\{s_n\}_{n=0}^\infty$  is a sequence, whose elements may for instance be the partial sums of an infinite series according to  $s_n = \sum_{k=0}^n a_k$ . A *sequence transformation* is a rule which maps a sequence  $\{s_n\}_{n=0}^\infty$  to a new sequence  $\{s'_n\}_{n=0}^\infty$  with hopefully better numerical properties. In this terminology, Padé approximants are just a special class of sequence transformations since they transform the partial sums of a (formal) power series to a doubly indexed sequence of rational approximants.

If  $\{s_n\}_{n=0}^\infty$  either converge to some limit  $s$  as  $n \rightarrow \infty$  or can be summed to the generalized limit  $s$  in the case of divergence, then a sequence element  $s_n$  can for all  $n \geq 0$  be partitioned into the (generalized) limit  $s$  and a remainder  $r_n$  according to

$$s_n = s + r_n. \quad (17)$$

Normally, a sequence transformation will not be able to determine the (generalized) limit  $s$  of  $\{s_n\}_{n=0}^\infty$  *exactly*. Thus, the elements of the transformed sequence  $\{s'_n\}_{n=0}^\infty$  can also be partitioned into the (generalized) limit  $s$  and a transformed remainder  $r'_n$  according to

$$s'_n = s + r'_n, \quad (18)$$

and the transformed remainders will in general be different from zero for all finite values of  $n$ .

In the literature on convergence acceleration it is said that a sequence transformation *accelerates convergence* if the transformed remainders  $\{r'_n\}_{n=0}^\infty$  vanish more rapidly than the original remainders  $\{r_n\}_{n=0}^\infty$  according to

$$\lim_{n \rightarrow \infty} \frac{r'_n}{r_n} = \lim_{n \rightarrow \infty} \frac{s'_n - s}{s_n - s} = 0, \quad (19)$$

and a divergent sequences, whose remainders  $r_n$  do not vanish as  $n \rightarrow \infty$ , is summed to its generalized limit  $s$  if the transformed remainders  $r'_n$  vanish as  $n \rightarrow \infty$ .

Thus, a sequence transformation essentially tries to eliminate the remainders  $r_n$  from the sequence elements  $s_n$  as effectively as possible. Since, however, an in principle unlimited variety of different remainders can occur, it is necessary to make some assumptions – either explicitly or implicitly – which provide the basis for the construction of a sequence transformation. A detailed discussion of the construction of sequence transformations as well as many examples can be found in the book by Brezinski and Redivo Zaglia [68] or in [5].

Normally, the assumptions being made are incorporated into the transformation scheme via model sequences, whose remainders possess a particular simple structure and can be expressed by a finite number of terms:

$$\tilde{s}_n = \tilde{s} + \sum_{k=0}^{k-1} \tilde{c}_j \varphi_j(n). \quad (20)$$

Here, the  $\tilde{c}_j$  are unspecified coefficients, and the  $\varphi_j(n)$  are by assumption known functions of  $n$ .

The elements of this model sequence contain  $k + 1$  unknown, the (generalized) limit  $\tilde{s}$  and the  $k$  coefficients  $\tilde{c}_j$  with  $0 \leq j \leq k - 1$ . Since all unknowns occur *linearly*, it is possible to construct a sequence transformation  $\mathcal{T}$  – if necessary via Cramer’s rule – which is *exact* for the elements of this model sequence according to

$$\mathcal{T} = \mathcal{T}(\tilde{s}_n, \tilde{s}_{n+1}, \dots, \tilde{s}_{n+k}) = \tilde{s}, \quad (21)$$

if applied to the numerical values of  $k + 1$  sequence elements  $\tilde{s}_n, \tilde{s}_{n+1}, \dots, \tilde{s}_{n+k}$ .

Of course, simple model sequences of that kind normally do not occur in practical problems. However, their elements provide at least for sufficiently large values of  $k$  reasonably accurate approximations to the elements of the more realistic sequence

$$s_n = s + \sum_{j=0}^{\infty} c_j \varphi_j(n). \quad (22)$$

If we now apply this sequence transformation  $\mathcal{T}$  to the numerical values of  $k + 1$  elements of the sequence (22), then we have no reason to assume that  $\mathcal{T}$  might produce its exact (generalized) limit  $s$ . However, a more detailed mathematical analysis of the transformation process normally reveals that  $\mathcal{T}$  eliminates the first  $k$  terms  $c_j \varphi_j(n)$  with  $0 \leq j \leq k - 1$ . Thus, the transformed remainder  $r'_n$  starts with  $\varphi_k(n)$  instead of  $\varphi_0(n)$ , which for sufficiently large values of  $k$  normally constitutes a significant improvement.

Most sequence transformations can be constructed on the basis of model sequences of the type of Eq. (20). For example, Wynn [65] could show that his epsilon algorithm is exact for model sequences whose remainders can be expressed as a linear combination of exponential terms according to

$$\tilde{s}_n = \tilde{s} + \sum_{j=0}^{k-1} \tilde{c}_j \lambda_j^n. \quad (23)$$

Concerning the  $\lambda_j$  it is only assumed that they are different from zero and one and ordered according to magnitude, i.e.,  $\lambda_j \neq 0, 1$  and  $|\lambda_0| > |\lambda_1| > |\lambda_{k-1}| > 0$ . Thus, if the numerical values of  $2k + 1$  elements  $\tilde{s}_n, \tilde{s}_{n+1}, \dots, \tilde{s}_{n+2k}$  of this model sequence are available, then the epsilon algorithm is exact according to

$$\epsilon_{2k}^{(n)} = \tilde{s}. \quad (24)$$

Moreover, Wynn constructed in Theorems 16 and 17 of [69] asymptotic expansions ( $n \rightarrow \infty$ ) for the transformed remainders  $r'_n$  created by the application of  $\epsilon_{2k}^{(n)}$  to the elements of the sequence

$$s_n = s + \sum_{j=0}^{\infty} c_j \lambda_j^n, \quad (25)$$

which is an obvious generalization of the model sequence (23). He showed that the transformed remainders  $r'_n$  are proportional to  $\lambda_k^n$  which corresponds to an elimination of the first  $k$  exponential terms  $c_j \lambda_j^n$  on the right-hand side of (25). Since the  $\lambda_j$  are by assumption ordered in magnitude, this constitutes a significant achievement. Consequently, Wynn’s epsilon algorithm is *asymptotically optimal* for sequences of the type of (25). This means that no other sequence transformation, which also uses only the numerical values of the elements of the sequence (25) as input data, can produce a better asymptotic ( $n \rightarrow \infty$ ) truncation error.

Levin [70] introduced a class of sequence transformations which are exact for model sequences of the following type:

$$\tilde{s}_n = \tilde{s} + \omega_n z_n. \quad (26)$$

Here,  $\omega_n$  is an estimate for the truncation error  $\tilde{r}_n$ , and  $z_n$  is a correction term. Levin [70] assumed that  $z_n$  can be expressed as a truncated power series in  $1/(n + \zeta)$  where  $\zeta$  is a positive shift parameter. In Sections 7 – 9 of [5], several other sequence transformations were constructed which are also exact for the model sequence (26) but make different assumptions about the correction terms  $z_n$ . The remainder estimates  $\omega_n$  introduce additional degrees of freedom in the construction of the sequence transformation as compared to Padé approximants. One may draw an analogy between sequence transformations and Padé approximants on the one hand and the Gaussian integration and the Simpson rule on the other hand; the variable integration nodes and weight factors of the Gaussian integration yield additional degrees

of freedom which may be used in order to construct a potentially much more powerful algorithm for numerical integration.

In the following text, we will concentrate on sequence transformations which assume that  $z_n$  can be expressed as a truncated factorial series (Section 8 of [5]):

$$z_n = \sum_{j=0}^{k-1} \tilde{c}_j / (\zeta + n)_j. \quad (27)$$

Here,  $(n + \zeta)_j = \Gamma(n + \zeta + j) / \Gamma(n + \zeta)$  is a Pochhammer symbol, and  $\zeta$  is a positive shift parameter. The assumption (27) implies that the sequence transformations derived in this way are particularly well suited for sequences satisfying

$$s_n = s + \omega_n \sum_{j=0}^{\infty} c_j / (\zeta + n)_j. \quad (28)$$

It is *a priori* not obvious that the ratio  $[s_n - s] / \omega_n = r_n / \omega_n$  can be expressed as a factorial series. Nevertheless, this assumption leads to powerful sequence transformations which are apparently particularly well suited for the summation of factorially divergent series [6–8, 10–12]. Let  $s_n = \sum_{k=0}^n a_k$  be the partial sums of an infinite series. If the  $a_k$  strictly alternate in sign and decrease monotonously in magnitude, then the best simple estimate for the truncation error  $r_n = -\sum_{k=n+1}^{\infty} a_k$  is the first term  $a_{n+1}$  neglected in the partial sum  $s_n$ . Moreover, the first term neglected is also the best simple remainder estimate for many factorially divergent alternating series (see for example Theorem 13-2 of [5]). The mathematical structure of a factorially divergent series is expected for the perturbative expansions in quantum field theory. Further arguments supporting the general applicability of the delta transformation to the series of the mathematical structure as expected for quantum field theory will be discussed in [86].

If we now combine the assumption that  $z_n$  should be a truncated factorial series according to (27) with the remainder estimate

$$\omega_n = \Delta s_n = a_{n+1}, \quad (29)$$

which was introduced by Smith and Ford [71], we obtain the delta transformation which is defined by the following ratio of finite sums (Eq. (8.4-4) of [5]):

$$\begin{aligned} \delta_k^{(n)}(\zeta, s_n) &= \frac{\Delta^k [(\zeta + n)_{k-1} s_n / \Delta s_n]}{\Delta^k [(\zeta + n)_{k-1} / \Delta s_n]} \\ &= \frac{\sum_{j=0}^k (-1)^j \binom{k}{j} \frac{(\zeta + n + j)_{k-1}}{(\zeta + n + k)_{k-1}} \frac{s_{n+j}}{\Delta s_{n+j}}}{\sum_{j=0}^k (-1)^j \binom{k}{j} \frac{(\zeta + n + j)_{k-1}}{(\zeta + n + k)_{k-1}} \frac{1}{\Delta s_{n+j}}}. \end{aligned} \quad (30)$$

Here, the same notation as in [5] is used. Thus,  $\Delta$  stands for the difference operator defined by  $\Delta g(n) = g(n+1) - g(n)$ ,  $(a)_n = \Gamma(a+n) / \Gamma(a)$  is a Pochhammer symbol,  $k$  and  $n$  are nonnegative integers, and  $\zeta$  is a shift parameter which has to be positive to allow  $n = 0$  in Eq. (30). The most obvious choice, which is always used in this article, is  $\zeta = 1$ .

In Section 8.3 of [5], a simple recursive scheme is described which permits – depending upon the initial values – the recursive calculation of either the numerator or the denominator sum of  $\delta_k^{(n)}(\zeta, s_n)$ .

In the context of quantum field theory and related disciplines, the delta transformation (30) may be used for the summation of divergent perturbation expansions which are power series in some coupling constant. Thus, if we replace the input data  $s_n$  in (30) by the partial sums  $f_n(z) = \sum_{\nu=0}^n \gamma_{\nu} z^{\nu}$  of the (formal) power series (1) for  $f(z)$ , we obtain a rational expression, whose numerator and denominator polynomials are of degrees  $k+n$  and  $k$ , respectively, in  $z$ :

$$\delta_k^{(n)}(\zeta, f_n(z)) = \frac{\sum_{j=0}^k (-1)^j \binom{k}{j} \frac{(\zeta + n + j)_{k-1}}{(\zeta + n + k)_{k-1}} \frac{z^{k-j} f_{n+j}(z)}{\gamma_{n+j+1}}}{\sum_{j=0}^k (-1)^j \binom{k}{j} \frac{(\zeta + n + j)_{k-1}}{(\zeta + n + k)_{k-1}} \frac{z^{k-j}}{\gamma_{n+j+1}}}. \quad (31)$$

If the coefficients  $\gamma_n$  of the power series for  $f(z)$  are all different from zero, the rational function (31) satisfies the asymptotic error estimate (Eq. (4.29) of [10])

$$f(z) - \delta_k^{(n)}(\zeta, f_n(z)) = O(z^{k+n+2}), \quad z \rightarrow 0. \quad (32)$$

This estimate, which is formally very similar to the analogous estimate (10) for Padé approximants, implies that all terms of the formal power series, which are used for construction of the rational approximant  $\delta_k^{(n)}(\zeta, f_n(z))$ , are reproduced exactly by a Taylor expansion around  $z = 0$ . Moreover, the higher order derivatives provide just like in the Padé case predictions for those coefficients  $\gamma_{n+k+2}, \gamma_{n+k+3}, \dots$ , that were not used for the construction of the rational function.

As already discussed, the power of the delta transformation or of other Levin-type transformations results from the fact that an explicit estimate for the truncation error is incorporated into the transformation scheme. The truncation error estimate used by the delta transformation is the first term  $\gamma_{n+1}z^{n+1}$  neglected in the partial sum  $f_n(z) = \sum_{\nu=0}^n \gamma_\nu z^\nu$ . Consequently, for a proper application of the delta transformation *all* coefficients  $\gamma_n$  of the power series for  $f$  with  $n \geq 1$  have to be different from zero because otherwise the estimate for the truncation error makes no sense. This restriction also follows directly from the ratio (31), where undefined expressions occur if coefficients  $\gamma_n$  with  $n \geq 1$  are zero (cf. Eq. (11) in [45] which entails divisions by zero).

### 3 ASYMPTOTICALLY OPTIMIZED RESUMMATION

The problem of the resummation of divergent perturbative expansions in quantum field theory and related disciplines has been discussed in a number of recent publications, for example in [45, 51, 57, 62, 63, 74]. We investigate here asymptotically optimized resummation methods, i.e. methods which utilize information about large-order asymptotics of perturbative coefficients with the intention of enhancing the rate of convergence of the resummation algorithm.

We discuss here possible improvements of the Borel-Padé method on the basis of potentially available information about the large-order asymptotics of perturbative coefficients. We concentrate on the particular model example discussed recently by Dunne and Hall [55], by Cvetič and Yu in [45] and by ourselves in [12]. We discuss the QED effective action  $S_B$  in the presence of a constant background magnetic field. The exact nonperturbative result for  $S_B$  can be expressed as a proper-time integral:

$$S_B = -\frac{e^2 B^2}{8\pi^2} \int_0^\infty \frac{ds}{s^2} \left\{ \coth s - \frac{1}{s} - \frac{s}{3} \right\} \exp\left(-\frac{m_e^2}{eB} s\right). \quad (33)$$

Here,  $B$  is the magnetic field strength, and  $m_e$  and  $e$  are the mass and the charge of the electron, respectively (this result is given for example in Eq. (4-123) in [75]).

The integral representation (33) for  $S_B$  can be expressed as a strictly alternating perturbation series in the effective coupling  $g_B = e^2 B^2 / m_e^4$ :

$$S_B = -\frac{2e^2 B^2}{\pi^2} g_B \sum_{n=0}^{\infty} c_n g_B^n, \quad (34)$$

where

$$c_n = \frac{(-1)^{n+1} 4^n |\mathcal{B}_{2n+4}|}{(2n+4)(2n+3)(2n+2)}. \quad (35)$$

Here,  $\mathcal{B}_{2n+4}$  is a Bernoulli number. Thus, the perturbation expansion (34) has the remarkable feature that an unlimited number of series coefficients  $c_n$  are known analytically. Consequently, this series is particularly well suited as a model system for studying resummation methods.

Next, we utilize the fact that a Bernoulli number with even index can be expressed by a Riemann zeta function according to (see Section 23.2. on p. 807 of [76])

$$|\mathcal{B}_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n). \quad (36)$$



Inserting this into (34) and (35) yields

$$S_B = -\frac{e^2 B^2}{8\pi^2} g_B \sum_{n=0}^{\infty} (-1)^n (2n+1)! \frac{2\zeta(2n+4)}{\pi^{2n+4}} g_B^n \quad (37)$$

$$= -\frac{e^2 B^2}{8\pi^2} g_B S'_B, \quad (38)$$

where in the last line we define implicitly the scaled function  $S'_B$  which is also considered, e.g., in Table 2 of [12]. It is a direct consequence of the Dirichlet series (Section 23.2. of [76])

$$\zeta(s) = \sum_{m=0}^{\infty} (m+1)^{-s}, \quad (39)$$

that we have the inequality  $1 \leq \zeta(2n+4) \leq \zeta(4)$  for all nonnegative integers  $n$ . Consequently, the zeta function does not contribute to the factorial divergence of the perturbation series (34). Thus, the factorial  $(2n+1)!$  on the right-hand side of (37) implies that the perturbation series for  $S_B$  diverges for every coupling  $g_B \neq 0$ . Furthermore, it is clear from the representation (37) that an asymptotically optimized  $(\kappa, \lambda)$ -generalized Borel resummation scheme for  $S_B$  according to (2)–(4) requires the parameter setting  $\kappa = \lambda = 2$ .

We now discuss the construction of the asymptotically optimized Borel transform explicitly. We start from the scaled series,

$$S'_B(g_B) = \sum_{n=0}^{\infty} (16 c_n) g_B^n = \sum_{n=0}^{\infty} (-1)^n (2n+1)! \frac{2\zeta(2n+4)}{\pi^{2n+4}} g_B^n. \quad (40)$$

The  $(2, 2)$ -generalized Borel transformed series of  $S'_B$  is given by

$$\mathcal{B}^{(2,2)}(S'_B; z) = \sum_{n=0}^{\infty} \frac{16 c_n}{(2n+1)!} g_B^n = \sum_{n=0}^{\infty} (-1)^n \frac{2\zeta(2n+4)}{\pi^{2n+4}} z^n. \quad (41)$$

This series can be brought into a form which clearly shows the singularity structure of the Borel transformed series. For that purpose, we replace the Riemann zeta function by its Dirichlet series according to (39) and interchange the order of the two infinite nested summations:

$$\mathcal{B}^{(2,2)}(S'_B; z) = \frac{2}{\pi^4} \sum_{m=0}^{\infty} (m+1)^{-4} \sum_{n=0}^{\infty} \left\{ -z / [\pi(m+1)]^2 \right\}^n. \quad (42)$$

Thus,  $\mathcal{B}^{(2,2)}(S'_B; z)$  is essentially a superposition of geometric series with arguments  $z / [\pi(m+1)]^2$ . If we now use  $\sum_{k=0}^{\infty} (-x)^k = 1/(1+x)$ , we obtain:

$$\mathcal{B}^{(2,2)}(S'_B; z) = \frac{2}{\pi^4} \sum_{m=0}^{\infty} \frac{(m+1)^{-4}}{1 + z / [\pi(m+1)]^2}. \quad (43)$$

This representation shows that the poles of the Borel transformed series  $\mathcal{B}^{(2,2)}(S'_B; z)$  are located along the negative real axis according to

$$z = -n^2 \pi^2, \quad (44)$$

where  $n$  is a nonzero positive integer. Moreover, we obtain the following representations for the QED effective action  $S_B$  as a  $(2, 2)$ -generalized Borel integral according to Eqs. (2) and (3):

$$S_B = -\frac{e^2 B^2}{8\pi^2} g_B \int_0^{\infty} t \mathcal{B}^{(2,2)}(S'_B; g_B t^2) \exp(-t) dt \quad (45)$$

$$= -\frac{e^2 B^2}{8\pi^2} \int_0^{\infty} s \mathcal{B}^{(2,2)}(S'_B; s^2) \exp(-s/g_B^{1/2}) ds. \quad (46)$$

Cvetič and Yu in [45] use a  $(2, 2)$ -generalized Borel transformed series for the QED effective action, constructed according to Eq. (45). As explained in Section 1, this transformation is asymptotically optimized in the sense that the leading factorial growth of the perturbative coefficients in Eq. (37) is divided

out. It could appear from the Eqs. (4) and (8) in [45] that a (1,1)-generalized Borel transform is used where the  $n$ th perturbative coefficient is divided by a factor of  $n! = \Gamma(n+1)$ . This is, however, not the case. Note that in [45], the perturbative expansions are written in a very peculiar parameter, which is

$$\tilde{b} = e B/m_e^2. \quad (47)$$

In normal QED terminology, this would correspond to an expansion in powers of  $\sqrt{\alpha}$ . By consequence, all even-order perturbative coefficients vanish in the analysis presented in Ref. [45]. In this context, one may note that the expression for the delta transformation according to Eq. (11) in [45] is actually undefined since it involves divisions by zero. The expansion parameter used here is  $g_B = e^2 B^2/m_e^4 = \tilde{b}^2$ ; this parameter is also used in [12] and in [55].

Due to the special, mathematically compact form of the perturbative coefficients in Eq. (35), the asymptotically optimized Borel transform (42) is simply a superposition of geometric series. Such a simple mathematical structure cannot be expected to be of general importance concerning series occurring in quantum field theory. Note that this series can be brought in a form which clearly shows that Wynn's epsilon algorithm, which computes Padé approximants according to (14), is optimal, as discussed in Section 2. For that purpose, we rewrite the  $n$ th partial sum of the series (43) as follows:

$$\sum_{\nu=0}^n (-1)^\nu \frac{2\zeta(2\nu+4)}{\pi^{2\nu+4}} z^\nu = \mathcal{B}^{(2,2)}(S'_B; x) - \sum_{\nu=n+1}^{\infty} (-1)^\nu \frac{2\zeta(2\nu+4)}{\pi^{2\nu+4}} z^\nu. \quad (48)$$

On substituting the Dirichlet series (39) into the infinite series on the right-hand side and interchanging the order of summations, we obtain

$$\sum_{\nu=0}^n (-1)^\nu \frac{2\zeta(2\nu+4)}{\pi^{2\nu+4}} z^\nu = \mathcal{B}^{(2,2)}(S'_B; z) + \frac{2(-1)^n}{\pi^4} \sum_{m=0}^{\infty} \frac{z(m+1)^{-6}}{\pi^2 + z/(m+1)^2} \left( \frac{z}{[\pi(m+1)]^2} \right)^n. \quad (49)$$

Thus, the partial sum of the Borel transformed series (43) possesses the following general structure:

$$s_n = s + (-1)^n \sum_{j=0}^{\infty} c_j \lambda_j^n. \quad (50)$$

This sequence is obviously a special case of the sequence (25), for which Wynn's epsilon algorithm is – as discussed in Section 2 – asymptotically optimal.

The conclusions drawn by Cvetič and Yu in [45] appear to be restricted, at least in part, to the particular model example studied in their paper. In this context it should be emphasized that the superiority of the delta transformation over Padé approximants if applied *directly* to factorially divergent series, cannot be assumed to persist after the Borel transformation by which the leading factorial divergence is divided out. I.e., the delta transformation is more powerful than the Padé technique for factorially divergent series, but this finding by no means allows us to conclude that, or in fact has any connection to the assumption that the combined Borel-delta technique should be numerically superior to the Borel-Padé method. This consideration is relevant for the interpretation of the conclusions drawn by Cvetič and Yu with regard to the variant of the Borel method proposed in [45], which uses the delta transformation for the analytic continuation of the Borel transformed series and which is called the Borel-Weniger method by the authors of [45].

We return now to the discussion of further improvements of the asymptotically optimized Borel-Padé method. The leading asymptotics do not only permit to modify (optimize) the Borel transform accordingly, but indeed it is the leading large-order asymptotics which determine the location of the poles in the Borel plane. In view of Eq. (43), the singularities of the function  $\mathcal{B}^{(2,2)}(S'_B; z)$  defined in (41) are at  $z = -n^2 \pi^2$ , i.e. along the negative real axis. This is where one would expect them to lie in (distant) analogy to the renormalon theory [77].

Using the information on the location of the poles, it is possible to construct further improved Padé approximants. To this end, we utilize the known location of the poles in order to construct improved Padé approximants to the function  $\mathcal{B}^{(2,2)}(S'_B; z)$ . Normally, the Borel integral (43) would be evaluated with the upper- or lower-diagonal Padé approximants to  $\mathcal{B}^{(2,2)}(S'_B; z)$  in the integrand. We use upper-diagonal Padé approximants here, as they can be computed by Wynn's epsilon algorithm according to (14). We denote by  $\mathcal{P}_n(z)$  the upper-diagonal Padé approximant,

$$\mathcal{P}_n(z) = [\llbracket (n+1)/2 \rrbracket / \llbracket n/2 \rrbracket]_{\mathcal{B}^{(2,2)}(S'_B)}(z). \quad (51)$$

Table 1: Resummation of the divergent series  $S'_B$  given in Eq. (40) with  $g_B = 10$ . The numerical data is presented normalized to a number in the interval  $(0, 1)$  via multiplication by a factor of 100.

$n$	$\mathcal{T}'S'_{B,n}$ asymptotically optimized, see Eq. (52)	$\mathcal{T}'''S'_{B,n}$ improved transforms def. in Eq. (59)
0	-2.222 222 222 222 222 222	-2.222 222 222 222 222 222
1	-0.779 860 343 938 511 943	-0.846 447 993 882 134 544
2	-0.832 545 950 383 972 556	-0.807 698 096 764 310 129
3	-0.804 166 791 460 607 115	-0.805 669 649 913 560 215
4	-0.806 776 251 699 410 322	-0.805 634 754 493 393 579
5	-0.805 531 742 010 943 471	-0.805 634 061 009 148 890
6	-0.805 700 473 754 628 870	-0.805 633 961 318 348 380
7	-0.805 626 062 286 745 674	-0.805 633 975 558 025 628
8	-0.805 638 540 560 183 781	-0.805 633 975 330 131 982
9	-0.805 633 322 312 161 338	-0.805 633 975 322 121 382
10	-0.805 634 321 402 303 670	-0.805 633 975 321 669 649
11	-0.805 633 919 056 749 287	-0.805 633 975 321 697 521
12	-0.805 634 003 326 479 025	-0.805 633 975 321 696 356
13	-0.805 633 970 320 418 441	-0.805 633 975 321 696 294
14	-0.805 633 977 694 044 469	-0.805 633 975 321 696 292
exact	-0.805 633 975 321 696 292	-0.805 633 975 321 696 292

In the upper-diagonal case, we evaluate the transforms  $\mathcal{T}S'_{B,n}$  where

$$\mathcal{T}S'_{B,n} = \int_0^\infty t \mathcal{P}_n(g_B t^2) \exp(-t) dt, \quad (52)$$

and observe numerical convergence of the transform at large transformation order  $n$ . When the location of the poles is known, we may improve the convergence of the transforms by the following replacement,

$$\mathcal{P}_n(z) \rightarrow \mathcal{P}'_n(z) = \frac{\mathcal{Q}_n(z)}{\prod_{i=1}^{\llbracket n/2 \rrbracket} (1 + z/(n^2 \pi^2))}, \quad (53)$$

where  $\mathcal{Q}_n(z)$  is the  $[\llbracket (n+1)/2 \rrbracket/0]$ -Padé approximant to the function  $\mathcal{R}_n(z)$ ,

$$\mathcal{Q}_n(z) = [\llbracket (n+1)/2 \rrbracket/0]_{\mathcal{R}_n}(z), \quad (54)$$

and  $\mathcal{R}_n(z)$  is given by

$$\mathcal{R}_n(z) = \prod_{i=1}^{\llbracket n/2 \rrbracket} (1 + z/(n^2 \pi^2)) \mathcal{B}^{(2,2)}(S'_B; z). \quad (55)$$

The asymptotic enhancement is possible only if additional asymptotic information is available on the perturbative coefficients. Such information may be available (renormalon poles), but this is not necessarily

provided. In this context it should be noted that there is currently no general proof of the assumption that the renormalon poles are the only relevant poles in the Borel plane [79], but the factorial divergence of the perturbative coefficients is a commonly accepted assumption [1, 77, 80–85].

The pole-structure improved transforms are obtained from (52) by the replacement (53),

$$\mathcal{T}'S'_{B,n} = \int_0^\infty t \mathcal{P}'_n(g_B t^2) \exp(-t) dt. \quad (56)$$

Similar improvement of the convergence of transforms can also be expected in those cases where the final evaluation of the Borel integral proceeds in the complex plane along the integration contours introduced in [57].

Further improvement of the rate of convergence is possible by taking the transforms  $\mathcal{T}'S'_{B,n}$  as input data to the epsilon algorithm (11) in order to accelerate the convergence of the sequence of the pole-structure improved transforms  $\{\mathcal{T}'S'_{B,n}\}_{n=0}^\infty$ . The application of the epsilon algorithm defined in Eq. (11) to the pole-structure improved transforms results in a sequence of upper-diagonal Padé approximants which we denote by

$$\mathcal{T}''S'_{B,n} = \epsilon_{2\llbracket n/2 \rrbracket}^{(n-2\llbracket n/2 \rrbracket)}. \quad (57)$$

As input data for the epsilon algorithm, we use

$$s_n = \mathcal{T}'S'_{B,n}. \quad (58)$$

In a second epsilon transformation we may in turn employ the  $\mathcal{T}''S'_{B,n}$  as input data for a further application of the epsilon algorithm,

$$\mathcal{T}'''S'_{B,n} = \epsilon_{2\llbracket n/2 \rrbracket}^{(n-2\llbracket n/2 \rrbracket)}, \quad (59)$$

where we use  $s_n = \mathcal{T}''S'_{B,n}$  as input data. This results in a sequence of pole-structure and doubly epsilon-improved transforms  $\mathcal{T}'''S'_{B,n}$ . The application of the epsilon algorithm further enhances the rate of convergence of the pole-structure improved transforms.

In Table 1 we present numerical data for the Borel transforms calculated according to Eq. (52) (in passing we note that these correspond to the method proposed in [45]) and the transforms (59). The first 15 transforms calculated according to Eq. (52) exhibit convergence to 8 significant digits, whereas the pole-structure and epsilon improved transforms coincide with the exact result to within 18 significant digits.

The delta transformation is a general-purpose transformation which has been proven to be applicable to a *wide variety* of alternating factorially divergent series [5–8, 10–12]. In the context of the delta transformation, additional asymptotic information could be used in order to modify the remainder estimates  $\omega_n$  defined in Eq. (29) (see also Eqs. (7.3-8), (8.2-7), (8.4-1) and (8.4-4) of [5]). Also, we note a rescaling of the perturbative coefficients as a potential source for further improvements [7]. Work along these lines is currently in progress and will be presented elsewhere [86].

## 4 ASYMPTOTICALLY OPTIMIZED PREDICTIONS

We refer here to the predictions of unknown higher-order perturbative coefficients as perturbative predictions or perturbative extrapolations. As outlined in Section 2 and [28], these extrapolations are obtained by reexpanding certain rational approximants in powers of the coupling parameter. The next higher-order term obtained after the reexpansion can then be interpreted as a prediction for that perturbative coefficient. The rational approximants discussed in Section 2 fulfill accuracy-through-order relations, i.e., upon reexpansion in the coupling parameter, all the perturbative terms used for the construction of the rational approximant are reproduced [see Eq. (10) for Padé approximants and Eq. (32) for the delta transformation]. The coefficients of the Borel transformed series are related to those of the input series by Eq. (4). Therefore, we can either predict the perturbative coefficients of the original series or the coefficients of the Borel transformed series.

We consider here the asymptotic improvement of three different prediction methods: (i) asymptotically optimized predictions based on the combination of Borel and Padé techniques, (ii) predictions based on the delta transformation, and (iii) the direct application of Padé approximants to the perturbation series. We consider here the following improvements beyond reexpansion of the rational approximant,

1. A-posteriori corrections. These are further corrections to the perturbative predictions obtained by estimating not only the coefficient, but also the probable error in making that estimate. Similar methods in the context of Padé approximants have been introduced, e.g., in [18].
2. Fixing poles. In the context of the asymptotically improved Borel-Padé method, it is possible to improve predictions if the leading and subleading large-order asymptotics of the perturbative predictions are known. These asymptotics determine the location of the poles in the Borel plane, which can be put in by hand (see also Eqs. (53)–(55) in Section 3 and, in part, [92]).
3. Renormalization group. The renormalization group can be used to enhance perturbative predictions for certain classes of diagrams; this has been used e.g. in [90] for the anomalous magnetic moment of the muon.

It is natural to assume that combinations of these techniques should be investigated where appropriate.

The basic idea of a-posteriori corrections is as follows. The errors made in the “prediction” of lower-order coefficients are available by the time we come to higher order, so they may be utilized for an estimate of the error which is to be expected in a prediction of the next higher-order coefficient. We denote the predictions which are obtained by extrapolating the coefficients and the “prediction errors” (in contrast to the coefficients alone) by the term *a-posteriori improved predictions* because the further correction due to the extrapolated error is applied after the reexpansion of the rational approximant which yields the “first-order” prediction. The a-posteriori improvement of predictions is useful in both lower and higher orders of perturbation theory. In higher orders, the transient, pre-asymptotic behavior of the perturbative coefficients has died away, and the extrapolations of the coefficients as well as the a-posteriori corrections become more accurate. A particular merit of the a-posteriori corrections is the fact that they can be applied to *any* of the prediction algorithms proposed above, in order to achieve an improvement of the prediction beyond the reexpansion of the rational approximant used.

We will be investigating in the sequel the Borel transform of the QED effective action defined in (43). In order to investigate the extrapolation of coefficients of the Borel transform, we define auxiliary quantities (coefficients)  $\hat{c}_{2n+1}$  by the relation

$$\hat{c}_{2n+1} = -\frac{16 c_n}{(2n+1)!}, \quad (60)$$

where the  $c_n$  are defined in Eq. (35). We additionally set  $\hat{c}_{2n} = 0$  for even-order coefficients. In terms of the coefficients  $c_j(p)$  which are defined in Eq. (5) in [45], the  $\hat{c}_{2n+1}$  are given by  $\hat{c}_{2n+1} = (-1)^n c_{2n+1}(0)$ . The coefficient  $\hat{c}_{2n+1}$ , written in terms of the Bernoulli numbers, reads [see also Eq. (35)]

$$\hat{c}_{2n+1} = (-1)^n \frac{2^{2n+4} |\mathcal{B}_{2n+4}|}{(2n+4)!}. \quad (61)$$

We concentrate here on the coefficient  $\hat{c}_{13}$  defined in Eq. (60) and we discuss how the prediction for the coefficient  $\hat{c}_{13}$  can be improved on the basis of a-posteriori corrections and other asymptotic improvements. We define *correction factors*  $\xi_n$  by

$$\hat{c}_n = \xi_n \bar{c}_n \quad (62)$$

where  $\hat{c}_n$  is the exact  $n$ th order coefficient and  $\bar{c}_n$  is the estimate obtained by reexpanding the rational approximant which is used for the prediction. In the case of Borel-Padé approximants, these would be Padé approximants applied to the Borel transform of the QED effective action (43). Specifically, for the prediction of the  $n$ th perturbative coefficient, these are the approximants  $[(n-1)/2]/[n/2]$  for the lower-diagonal and  $[n/2]/[(n-1)/2]$  for the upper-diagonal case. For the prediction of  $\hat{c}_{13}$ , the previous errors made in the “prediction” of  $\hat{c}_7$ ,  $\hat{c}_9$  and  $\hat{c}_{11}$  can be analyzed. Note that the exact values of  $\hat{c}_7$ ,  $\hat{c}_9$  and  $\hat{c}_{11}$  must be assumed as available, exploitable information by the time we try to predict  $\hat{c}_{13}$ .

An estimate for  $\xi_{13}$  can be obtained for example by fitting the natural logarithms of the quantities  $\xi_7 - 1$ ,  $\xi_9 - 1$  and  $\xi_{11} - 1$  with a linear model in order to obtain an estimate for  $\xi_{13} - 1$ . This *linear* fit of the logarithms of the  $\xi_i$  is based on the empirical observation that relative errors of the predictions decrease exponentially in higher order, a phenomenon which has been observed in a number of applications, including variants of anharmonic oscillators. In the context of Padé approximants, a similar error dependence has been conjectured (see [18]). The leading coefficient of the decay of the relative errors may

Table 2: Prediction of perturbative coefficients  $\hat{c}_{13}$  defined in Eq. (60).

method	result	relative error
Asympt. opt. [5/6]-Borel-Padé (see [45])	$2.221\,459\,447\,36 \times 10^{-8}$	$6 \times 10^{-7}$
Asympt. opt. [6/5]-Borel-Padé (for comparison)	$2.221\,454\,724\,11 \times 10^{-8}$	$3 \times 10^{-6}$
Asympt. opt. [6/5]-Borel-Padé with a-posteriori correction (this work)	$2.221\,460\,221\,71 \times 10^{-8}$	$3 \times 10^{-7}$
Asympt. opt. [6/5]-Borel-Padé with one pole (this work)	$2.221\,460\,950\,35 \times 10^{-8}$	$3 \times 10^{-8}$
Asympt. opt. [6/5]-Borel-Padé with three poles (this work)	$2.221\,460\,901\,25 \times 10^{-8}$	$1 \times 10^{-8}$
Asympt. opt. [6/5]-Borel-Padé with five poles (this work)	$2.221\,460\,880\,27 \times 10^{-8}$	$5 \times 10^{-10}$
Asympt. opt. [6/5]-Borel-Padé with five poles + a-posteriori (this work)	$2.221\,460\,878\,35 \times 10^{-8}$	$3 \times 10^{-10}$
$\hat{c}_{13}$ : exact value	$2.221\,460\,878\,99 \times 10^{-8}$	

depend on the problem considered and on the extrapolation scheme used, but the exponential improvement of perturbative predictions in higher order appears to be a rather general feature. Details on this point will be presented elsewhere [86].

Using a linear least-squares fit of the respective logarithms  $\ln(\xi_7 - 1)$ ,  $\ln(\xi_9 - 1)$  and  $\ln(\xi_{11} - 1)$ , we obtain an estimate of  $\xi_{13} - 1 = 2.47 \times 10^{-6}$  in the case of upper-diagonal Borel-Padé approximants. This leads to the data presented in Table 2. Note that even the crude linear model for the  $\ln(\xi_i - 1)$  used here already doubles the accuracy of the prediction of the coefficient  $\hat{c}_{13}$  as compared to the plain Borel-Padé prediction used, for example, in [45]. Note also that an averaging of upper- and lower-diagonal Borel-Padé approximants does not improve the situation in favor of the plain Borel-Padé extrapolation. Further improvements of the *a-posteriori* corrected predictions is possible with more elaborate extrapolation schemes [86].

We would like to mention that knowledge of the leading asymptotics of the perturbative coefficients is required for the construction of an asymptotically optimized Borel-Padé transformation; this information is not available for many of the phenomenologically interesting series currently investigated [15–21]. It is helpful to note that the delta transformation is (like Padé) a rather general-purpose method for the prediction of perturbative coefficients, and that knowledge of large-order asymptotics is not required for its construction or application in a particular case. This property is helpful especially in cases of practical interest where little is rigorously known about the large-order asymptotics of the perturbative coefficients, and where only a limited number of perturbative coefficients are available. A number of practically interesting examples of delta-based predictions were discussed in [12], and it was observed that the delta transformation yields more accurate predictions than the Padé technique in many cases.

We now turn to a discussion of topologically new effects in higher orders of perturbation theory and improvements of predictions based on the renormalization group. Topologically new effects have caused problems for perturbative predictions in the past. We refer to the quartic Casimirs in the QCD beta

function [36, 40, 89] and to light-by-light scattering graphs in the tenth order anomalous magnetic moment of the muon [90]. Analogous considerations might hold for the perturbation series investigated in [92]. The topologically new effects cannot be taken into account by straight extrapolations, nor by renormalization-group improved extrapolations, which lead to resummation of certain classes of diagrams. For the anomalous magnetic moment of the muon discussed in [90], the contribution of the topologically new light-by-light scattering diagrams originally analyzed in [91] could not be reproduced by renormalization-group techniques introduced in [90]. By reexpansion of the delta approximant to the perturbation series for the muon anomaly, an estimate of  $a_\mu^{(10)} = 711$  has been obtained [87]. If an a-posteriori correction based on a combination of the delta transformation and Padé approximants is added to this prediction, then the estimate for the 10th order coefficient changes to  $a_\mu^{(10)} \sim 970$  [86]. This improved estimate is in excellent agreement with the analytically obtained approximate result of 930(170) from [91], comprising the topologically new effects which are present in five-loop order.

Now it is of course not permissible to conclude that topologically new effects can be taken into account in the general case by considering a-posteriori corrections, and/or that a-posteriori corrections necessarily give an accurate estimate for the size of these problematic, topologically new effects. On the other hand, the reverse proposition, which is that perturbative predictions are scientifically unsound because of their general inability to include topologically new effects, does not appear to be generally valid, either.

## 5 THE ANOMALOUS DIMENSION

We consider the divergent perturbation series for the  $\gamma$  function (anomalous dimension) of the Yukawa coupling as studied in [78]. Specifically, we consider the resummation of the perturbation series for the anomalous dimension of a fermion field with a Yukawa interaction  $g \bar{\psi} \sigma \psi$  at  $d_c = 4$ , which is given in Eq. (17) in [78]. This calculation comprises an evaluation of the contribution of all nested self-energy diagrams to the anomalous dimension  $\gamma$  function of the Yukawa theory up to the 30-loop level (an analogous analysis is performed in [78] for the  $(4 - \epsilon)$ -dimensional  $\phi^4$  theory). We restrict the discussion here to the Yukawa case. With the convention

$$a = \frac{g^2}{4\pi^2}, \quad (63)$$

the result for the anomalous dimension  $\gamma$  function as considered in [78] reads,

$$\tilde{\gamma}_{\text{hopf}}(a) \sim \sum_{n=1}^{\infty} (-1)^n \frac{\tilde{G}_n}{2^{2n-1}} a^n. \quad (64)$$

The perturbative coefficients  $\tilde{G}_n$  are listed in Table 3. The coefficients grow factorially in absolute magnitude; in [78] little change is observed in the quantities

$$\tilde{S}_n = \frac{\tilde{G}_n}{2^{n-1} \Gamma(n + 1/2)} \quad (65)$$

for large  $n$ . The evaluation [78] confirms in a concrete, 30-loop calculation the assumption originally put forth by Dyson [88] that the convergence radius of the quantum field theoretic perturbative expansion is zero. For a large number of quantum field theoretic observables like the anomalous magnetic moment of the muon (see Section 4) only a few perturbative terms are known. Although rapid growth of the perturbative coefficients is observed even in relatively low order (see the large number of examples discussed in [28]), one may argue that the factorial growth of the coefficients has not been demonstrated in concrete calculations, and that it is unclear if severe cancellations between different classes of diagrams occur in higher order.

The 30-loop calculation by Broadhurst and Kreimer may indicate in one particular example at least, that the factorial divergence is likely to persist, and that possible cancellations due to the renormalization or between different sets of diagrams do not contradict the concept of ultimate factorial divergence of the perturbative coefficients. From the observation made in [78] that the  $S_n$  defined in Eq. (65) change little at large  $n$ , one may tentatively infer the leading factorial divergence of the perturbative coefficients,

$$\tilde{G}_n \sim 2^{n-1} \Gamma(n + 1/2) \quad n \rightarrow \infty. \quad (66)$$

This asymptotic behavior, of course, leads to a vanishing radius of convergence of the perturbative expansion (64).

Table 3: Perturbative coefficients  $\tilde{G}_n$  for the anomalous dimension  $\gamma$  function of the Yukawa coupling [see Eq. (64)].

$n$	$\tilde{G}_n$
1	1
2	1
3	4
4	27
5	248
6	2 830
7	38 232
8	593 859
9	10 401 712
10	202 601 898
11	4 342 263 000
12	101 551 822 350
13	2 573 779 506 192
14	70 282 204 726 396
15	2 057 490 936 366 320
16	64 291 032 462 761 955
17	2 136 017 303 903 513 184
18	75 197 869 250 518 812 754
19	2 796 475 872 605 709 079 512
20	109 549 714 522 464 120 960 474
21	4 509 302 910 783 496 963 256 400
22	194 584 224 274 515 194 731 540 740
23	8 784 041 120 771 057 847 338 352 720
24	414 032 133 398 397 494 698 579 333 710
25	20 340 342 746 544 244 143 487 152 873 888
26	1 039 819 967 521 866 936 447 997 028 508 900
27	55 230 362 672 853 506 023 203 822 058 592 752
28	3 043 750 896 574 866 226 650 924 152 479 935 036
29	173 814 476 864 493 583 374 050 720 641 310 171 808
30	10 272 611 586 206 353 744 425 870 217 572 111 879 288

As observed by Broadhurst and Kreimer, the perturbation series (64) can be resummed with the help of an asymptotically improved Borel-Padé technique. From Eq. (22) in [78], it is clear that a  $(1, 1/2)$ -generalized Borel-Padé transformation is used by Broadhurst and Kreimer [for the definition of generalized Borel-Padé transformations see Eqs. (2)–(4) in this article]. This is not completely obvious because the transformation as used by Broadhurst and Kreimer has been modified additionally such as to normalize the first coefficient of the Borel transform (not of the input series) to unity, and the transformation is additionally rewritten such as to reflect the vanishing coefficient of zeroth order in  $a$  in Eq. (64). From the leading asymptotics in Eq. (66), the singularity of the  $(1, 1/2)$ -generalized Borel transform closest to the origin can be inferred. This singularity was explicitly “put in by hand” by Broadhurst and Kreimer (see Eq. (22) in [78]).

It is also possible to resum the alternating divergent series (64) by a delta transformation, even at large coupling. At a large Yukawa coupling of  $g = 30$ , we obtain a relative accuracy of 6 significant figures in the resummed results with a plain, unmodified delta transformation.

We add here a remark on the relation of the asymptotically optimized Borel-Padé based predictions to those obtained using the delta transformation. We consider the relative accuracy of perturbative predictions for the coefficient  $\tilde{G}_{30}$  of the perturbation series defined in Eq. (64) using various methods. The coefficient  $\tilde{G}_{30}$  is known (see Table 3), therefore we merely check the accuracy to which this coefficient can be reproduced by considering the first 29 perturbative coefficients of the series (64). With an asymp-



totically optimized  $(1, 1/2)$ -generalized Borel-Padé technique, the coefficient  $\tilde{G}_{30}$  can be reproduced with a relative accuracy of  $5 \times 10^{-16}$ . The delta transformation, without any modifications, leads to a prediction with a relative error of  $3 \times 10^{-16}$ ; this result is more accurate than the prediction provided by the asymptotically optimized Borel-Padé transformation. In accordance with the results of Section 4, the Borel-Padé transformation can be significantly enhanced by including the pole closest to the origin (see Eq. (66) above and Eq. (22) in [78]). When this pole is included, a prediction is obtained with a relative error of  $4 \times 10^{-17}$ . We do not consider a-posteriori improvements to either of these predictions, here.

## 6 CONCLUSION

We have considered the resummation of divergent perturbation series and the prediction of unknown higher-order perturbative coefficients (perturbative predictions or perturbative extrapolations). We have mentioned and discussed the following resummation prescriptions,

- the direct application of Padé approximants to a divergent series,
- the direct application of the nonlinear (delta) sequence transformation,
- asymptotically improved variants of the Borel-Padé technique.

The direct application of Padé approximants is less efficient in the resummation of divergent perturbation series than both the delta transformation and the combined Borel and Padé techniques. The combined, asymptotically improved Borel and Padé techniques, and the delta transformation are complementary. On the one hand, it can hardly be overemphasized that the asymptotically improved Borel-Padé technique is less general than the delta transformation because it depends on the availability of information on the leading large-order asymptotics of the coefficients. By contrast, there is considerable evidence that the plain, unmodified delta transformation can sum factorially divergent alternating series which diverge as strongly as  $(3n)!$  [7, 10]. This is beyond the power of directly applied Padé approximants and also beyond the power of the  $(1, 1)$ -generalized Borel-Padé transformation (“usual”) Borel-Padé transformation defined in Eq. (4).

If additional information is available on the input series, then the asymptotically optimized Borel-Padé technique is rather attractive. As discussed in Section 3, it is possible to enhance the rate of convergence simply by utilizing the location of known poles in the Padé approximants to the Borel transform of the input series. These improvements are not restricted to the first UV and IR renormalon poles, but, as shown in Eqs. (53)–(55) and exemplified by the numerical results in Table 1, can take advantage of an in principle unlimited number of poles in the Borel plane. Other techniques for possible improvements of the Borel-Padé algorithm have been described in [92]. Note that it is also possible to generalize the Borel-Padé technique to those cases where there are poles along the positive real axis in which case the Borel integral in Eq. (2) is actually undefined [57]. Using the special integration contours in [57], it is even possible to derive nonperturbative imaginary parts from real, not complex, perturbative coefficients. Concerning nonperturbative effects in quantum field theory we also refer to the recent investigation [93].

As discussed in Section 3, it is possible to accelerate the convergence of the resulting Borel-Padé transforms by subsequent application of Wynn’s epsilon algorithm (Borel-Padé-Wynn technique). These techniques lead to an improved rate of convergence. Note that the use of explicit information of the location of the poles in the Borel plane and the subsequent improvement of the convergence of the transforms by Wynn’s epsilon algorithm should also lead to accelerated convergence in the case of the complex integrations discussed in [57].

Because all the resummation prescriptions discussed above fulfill accuracy-through-order relations, they can be used to predict perturbative coefficients (we refer to this procedure as perturbative predictions or perturbative extrapolations). The straightforward predictions are obtained by reexpansion of the rational approximant in powers of the coupling parameter. That is to say, we consider here perturbative predictions based on

- the reexpansion of Padé approximants directly applied to the perturbative (input) series,
- the reexpansion of nonlinear (delta) sequence transformations directly applied to the perturbative (input) series,

- and the reexpansion of Padé approximants applied to the asymptotically improved Borel transform of the input series.

As it has been demonstrated in [12] and [45], the predictions based on the delta transformation and on the combined Borel and Padé techniques yield better results for the perturbative coefficients of the QED effective action than the Padé approximants alone. In [12] we also presented a number of more realistic and practically interesting examples in which the delta transformation leads to better predictions than the Padé approximants.

Note that there is currently no general proof of the assumption that the renormalon poles are the only relevant poles in the Borel plane [79], but the factorial divergence of the perturbative coefficients is a rather commonly accepted assumption [1, 77, 80–85]. We should therefore assume that at least asymptotically, the perturbation series in quantum field theory approximate factorially divergent series. This is also confirmed by the concrete 30-loop calculation presented in [78]. For many factorially divergent series, the delta transformation produces better numerical results than Padé approximants (see, e.g., Ch. 13 of [5] and [6–11]). Therefore, the delta transformation can be expected to provide a competitive alternative to Padé approximants. We again refer to the large number of recent publications on Padé-based extrapolations in quantum field theory [15–41].

We consider here mainly the following asymptotic improvements of perturbative extrapolations

- a-posteriori corrections based on estimates not only for the next higher-order coefficients, but also for the error which is to be expected in the estimation of that coefficient and
- the use of known (renormalon) poles in order to fix the denominator structure of Padé approximants in the context of the Borel-Padé method.

The general idea of a-posteriori corrections is the following. In higher orders of perturbation theory a number of lower-order coefficients are available which may be used in order to construct rational approximants. These coefficients can, apart from being useful for the construction the approximants itself, also be utilized in order to obtain an estimate for the expected error in the perturbative prediction. To this end, the extrapolation procedure is applied to the known lower-order terms in the perturbation series. A comparison of the “predictions” for the known lower-order with their exact results gives lower-order correction factors which may be extrapolated to higher order. This immediately leads to a correction factor for the next higher-order perturbative extrapolation (see Section 4).

Using a-posteriori corrections and the pole structure, we obtain improved results for the perturbative predictions of the model problem studied in [12, 45, 55] (see Table 2). The a-posteriori corrections also lead to an improved estimate for the 10th order anomalous magnetic moment of the muon and bring the a-posteriori corrected prediction in close agreement with an analytically obtained estimate [91]. The renormalization-group analysis can lead to a resummation of certain classes of Feynman diagrams, but it does not lead to an understanding of topologically new effects which occur in higher orders of perturbation theory. This phenomenon has led to problems in perturbative predictions in the past, especially in those cases where these predictions were improved on the basis of a renormalization group analysis. We refer to the analysis by Kataev and Starshenko on the 10th order anomalous magnetic moment of the muon [90] and various investigations on the QCD beta function [36, 40]. Notably, as discussed in Section 4, the a-posteriori corrected prediction appears to be consistent with the topologically new effects observed in 10th order of perturbation theory and calculated approximately in [91].

We would like to stress here again that the concept of a-posteriori corrections is rather general and can be applied to all prediction algorithms mentioned above. Specifically, we refer to the investigation [18] on significant improvements which can be achieved in the context of Padé-based predictions with this technique. The reduction of the magnitude of the a-posteriori correction is an attractive feature of the predictions based on the delta transformation.

In Sections 5 we show that for the more realistic 30-loop series calculated by Broadhurst and Kreimer in [78], even the asymptotically optimized Borel-Padé technique cannot quite match the accuracy obtainable by the plain delta transformation. It is only when an additional pole is explicitly put in by hand in the Padé transformations that the combined, asymptotically improved Borel-Padé technique becomes more accurate than the plain delta transformation. We stress here that the construction of the asymptotically improved Borel-Padé technique requires in itself a knowledge of the large-order asymptotics of the perturbative coefficients. Such information is not available in general cases. Specifically, in those cases where only a small number of coefficients are known the leading asymptotics cannot be reliably inferred from empirical approaches, either.

It has been the purpose of this article to clarify how resummation algorithms and perturbative extrapolations can be improved if additional information is available on a particular input series. We have explained in Sections 3–5 various algorithms by which the resummation of divergent series and the prediction of perturbative coefficients can be improved on the basis of additionally available asymptotic information on a given series; these improvements can be applied to the Borel-Padé based techniques and to the delta transformation techniques [12].

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