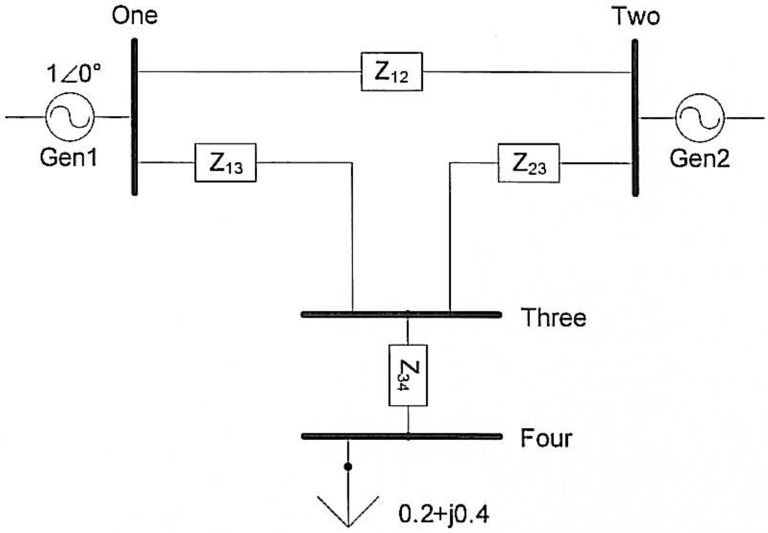
# (A Better) Power Flow Example

Consider the four-bus three-phase network shown below.




**1. Identify the bus types.**

*Identify busses by the information that is specified at each bus.*

Bus 1: Specified information: Generator voltage and angle, |V| θ. (This is the angle reference bus). Thus, this is a slack bus.

Bus 2: Generator bus. Generators must have their real power output specified, and they can either be set to output a certain reactive power OR set to regulate a certain bus voltage (by adjusting reactive power). Most generators are set to regulate their own bus to a commanded voltage. Thus, if no other information is given, we assume the generator P and |V| are set for this bus. Thus, P and |V| are known and this is a PV bus.

Bus 3: This is a connecting bus with zero load. Thus load power is known, and this is a PQ bus.

Bus 4: This is a load bus with P=0.2, Q=0.4 PU both given. Thus this is a PQ bus.

**2. Form the Admittance Matrix**

*In order to write the power flow equations, we first need to catalog all of the system impedances into an organized matrix.*

- The admittance matrix of an n-bus system will be dimensioned nxn. Thus, our matrix will be 4x4.
- Our admittance matrix:

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix}$$

- The on-diagonal terms $Y_{ii}$ describe the self admittance (admittance = 1/impedance) at each bus.
- The off-diagonal terms $Y_{ik}$ describe the mutual admittance connecting the i$^{th}$ and j$^{th}$ bus.

How to form the admittance matrix:

1. ***Note:*** I will be using CAPITAL LETTERS ($Y_{ik}$) to describe terms in the admittance matrix, and LOWER-CASE LETTERS to describe line admittances ($y_{ik} = 1/z_{ik}$).
2. Diagonal terms $Yii = \Sigma$ of all connected line admittances to the $i^{th}$ bus.
   Example at Bus 2:

$$Y_{22} = y_{12} + y_{23} = 1/z_{12} + 1/z_{23}$$

3. Off-diagonal terms $Y_{ik}$ = *negative* line admittance between $i^{th}$ and $k^{th}$ bus.
   Example Bus 2 to Bus 3:

$$Y_{23} = -y_{23} = -1/z_{23}$$

4. Non-connected busses have zero admittance (= infinite impedance) between them.
   Example Bus 1 to Bus 4:

$$Y_{14} = 0$$

5. Admittance matrix is symmetric. $Y_{ik} = Y_{ki}$.
   Example Bus 3 to Bus 2:

$$Y_{32} = Y_{23} = -1/z_{23}$$

Thus, for the network given, we find the admittance matrix to be:

$$Y = \begin{bmatrix} (y_{12} + y_{13}) & -y_{12} & -y_{13} & 0 \\ -y_{12} & (y_{12} + y_{23}) & -y_{23} & 0 \\ -y_{13} & -y_{23} & (y_{13} + y_{23} + y_{34}) & -y_{34} \\ 0 & 0 & -y_{34} & y_{34} \end{bmatrix}$$

where

$$y_{12} = 1/z_{12} \quad y_{13} = 1/z_{13} \quad y_{23} = 1/z_{23} \quad y_{34} = 1/z_{34}.$$

*For more help on the admittance matrix, please see examples 9.1-9.4 in your text. They're easy!*

**3. The Power Flow Equations.**

Before getting into the equations, let's review what the power flow equations mean. In any power system, there are some quantities that are easy to measure (from metering devices out in the field) and some that are difficult to measure and often just determined mathematically.

- Easy to measure: Complex power and Voltage magnitude (P, Q, |V|). Most substations in a power system will have metering devices installed on each line that measure line voltage and line current in the time-domain. This is enough information to calculate P, Q, and |V|.
  - $P = |V||I|\cos\phi$, where $\phi = \angle V - \angle I$. (The components of current and voltage that are in-phase with each other, multiplied.)

- $Q = |V||I|\sin\phi$. (The components of current and voltage that are completely out of phase with each other, multiplied.)

- Difficult to measure: Voltage angles. $(\theta = \angle V)$. Recall that angles are really *angle differences* with respect to a reference angle (at the slack bus). In order to measure voltage angles, equipment is needed that can compare the zero-crossing of a voltage wave with a standardized, high-accuracy time clock (usually supplied by a GPS receiver). This equipment tends to be expensive, and thus there are very few angle-measuring devices in any power system.

But we want to know the voltages and voltage angles in our system. Once these are known, all other quantities are easily computed. (If I told you all the node voltages in a circuit, you'd have an incredibly easy time calculating currents and powers. As in any circuit problem, the difficult part is *finding* the node voltages.) Voltages and voltage angles (V and θ) are referred to as **the system states**.

Let $x$ be a vector of system states for our four-bus power system:

$$x = \begin{bmatrix} [\theta] \\ [|V|] \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}.$$

In order to make this vector more readable, I used voltages that were understood to be magnitudes: $V \equiv |V|$.

One simplification that you may like: Because $V_1$, $\theta_1$, and $V_2$ are already known ahead of time (Bus 1 is the slack bus; Bus 2 is a PV bus), we do not need to include Bus 1 $V\theta$ or Bus 2 $V$ in our list of unknown states. Thus, our new state list will include Bus 2 through Bus 4:

$$\boxed{x = [\theta_2 \quad \theta_3 \quad \theta_4 \qquad V_3 \quad V_4]^T}.$$

As we said before, often in a power system, we know P, Q, and |V| from measured data. However, if we were going to run a future study (say we want to run a simulation of the Wisconsin power system as it will be in two months), there will be less information we can accurately predict: We would probably have no way of predicting the voltages, but we can predict load (P and Q) with reasonable accuracy. This is because load is tied to people and businesses that operate fairly predictably, especially when averaged over a large demographic. Any power system engineer knows that load is highly dependent on the date, the time, and the temperature. On weekends, businesses are closed, and the load is less. People turn on their lights when it gets dark. On very hot and very cold days, buildings use more electricity.

Thus, when running a future study, P and Q are often the only variables known at load busses (hence bus 4 in our system).

Here's where the power flow equations come in: we need some way to solve for the system states (***x***), given all of the specified pieces of information we have at each bus: Vθ for Bus 1, PV for Bus 2, and PQ for Busses 3 and 4. The power flow equations relate **state variables** to **other variables** in the system.

## 4. Using the Power Flow Equations

In order to solve for our system states ($x$), we should use every piece of information given to us. In particular, we named each bus according to the **known information at that bus**: Vθ for Bus 1, PV for Bus 2, and PQ for Busses 3 and 4.

Let's have a look at the power flow equations: (copied from handout "Power Flow Equations" on the course webpage)

$$P_i^{sp} = P_i(\theta, V) = V_i \sum_{k=1}^{n} V_k(G_{ik} \cos\theta_{ik} + B_{ik} \sin\theta_{ik}) \qquad (1)$$

$$Q_i^{sp} = Q_i(\theta, V) = V_i \sum_{k=1}^{n} V_k(G_{ik} \sin\theta_{ik} - B_{ik} \cos\theta_{ik}) \qquad (2)$$

As you can see, we can write the power injected at any bus in terms of the system state variables.

The $G_{ik}$ and $B_{ik}$ in the equations above come from the $i^{th}$ $k^{th}$ terms in the admittance matrix:

$$G_{ik} = real\{Y_{ik}\} \qquad\qquad B_{ik} = imag\{Y_{ik}\} \qquad (3)$$

Q: Which busses in our system will we write power flow equations for?
A: We will write P and Q equations at every bus where we can match these equations to known quantities. We should have equations for

- $P_2$ since Bus 2 is a **PV** bus.
- $P_3$ and $Q_3$ since Bus 3 is a **PQ** bus.
- $P_4$ and $Q_4$ since Bus 4 is a **PQ** bus.
- You can write other equations if you want to solve for unknown powers, but these introduce no new information and won't help you solve for the state variables.

For equations (1) and (2) above,

- $P_i(\theta,V)$ and $Q_i(\theta,V)$ are the equations that express power in terms of our unknown state variables (θ,V), and
- $P_i^{sp}$ and $Q_i^{sp}$ are known quantities that would normally be given to us.
  - $P_2$, $P_3$, $Q_3$, $P_4$, and $Q_4$ would normally be given to us in the problem description.
  - We will call these *specified* quantities:
    $P_2^{sp} = P_2$, $P_3^{sp} = P_3$, $Q_3^{sp} = Q_3$, $P_4^{sp} = P_4$, and $Q_4^{sp} = Q_4$.

Let's write an equation for the real power at Bus 2. Using (1) above,

$$P_2(\theta,V) = V_2 \sum_{k=1}^{n} V_k(G_{2k} \cos\theta_{2k} + B_{2k} \sin\theta_{2k}) \qquad (4)$$

Note that although we are summing k=1 to n, Bus 4 has no direct connection with Bus 2, so $Y_{24} = Y_{42} = 0$. Thus $B_{24} = 0$ and $G_{24} = 0$, and we do not need to include Bus 4 in the summation. Thus,

$$P_2 = V_2\left[V_1\left(G_{21}\cos\theta_{21} + B_{21}\sin\theta_{21}\right) + V_2\left(G_{22}\cos\theta_{22} + B_{22}\sin\theta_{22}\right) + V_3\left(G_{23}\cos\theta_{23} + B_{23}\sin\theta_{23}\right)\right] \tag{5}$$

where

$$\theta_{ik} = \theta_i - \theta_k \qquad \text{(difference in voltage angles between Bus i and Bus k).} \tag{6}$$

Note that $\theta_{22} = \theta_2 - \theta_2 = 0$. Thus, we can simplify:

$$P_2^{sp} = V_2\left[V_1\left(G_{21}\cos\theta_{21} + B_{21}\sin\theta_{21}\right) + V_2\left(G_{22}\right) + V_3\left(G_{23}\cos\theta_{23} + B_{23}\sin\theta_{23}\right)\right] \tag{7}$$

This is the real power equation at Bus 2. We can now plug in known quantities:

- we know all G and B variables if we've computed the admittance matrix.
- $V_1 = 1$. (Recall that $V_1$ is the voltage ***magnitude*** at Bus 1.)
- $\theta_1 = 0°$. (Bus 1 is the slack bus with angle reference set.) (Thus $\theta_{21} = \theta_2$.)
- $V_2$ is known because Bus 2 is a PV bus.

Now, repeat similarly, writing equations for the known powers at other busses: PQ at Busses 3 and 4.

Once we have all of the specified power equations written, we should set each equation equal to the known quantity. Example at Bus 4 the load is known (see figure):

$$P_4^{sp} = -0.2 \qquad\qquad Q_4^{sp} = -0.4 \tag{8}$$

Setting the specified variables equal to the power equations, we write:

$$P_4^{sp} = -0.2 = P_4(\theta, V) \qquad\qquad Q_4^{sp} = -0.4 = Q_4(\theta, V) \tag{9}$$

Although our problem is completely specified, with 5 equations and 5 unknowns, we cannot solve the problem directly because the power flow equations are **implicit** equations in terms of θ and V. This is not solvable directly by hand, but rather we must use iteration.

Thus, we must solve using an **iterative** technique:

1. Guess initial values for the states. (It is common to use a "flat start", where all busses are $1\angle 0°$.)
2. Solve the power flow equations and realize there is a mismatch because you did not guess the right answer. (E.g., $P_4(\theta, V) \neq P_4^{sp} = -0.2$)
3. Find the error (the difference between the power you wanted, and the power you got)
   Error = $f_4(x) = P_4(\theta, V) - P_4^{sp} = P_4(\theta, V) + 0.2$
   where we want the error = 0.
4. Use some method to adjust your guess (state variables $\boldsymbol{x}$) based on the error. One common method is to use partial derivatives in the Newton Raphson process.

5. Keep iterating and adjusting you state variables, until you converge to the correct answer and error $\approx 0$.

One quick note: it is best to use an **error function** because we cannot satisfy the equality $P_i^{sp} = P_i(\theta, V)$ and $Q_i^{sp} = Q_i(\theta, V)$ initially (because our initial guess was wrong), but we should be able to adjust our guess to approximately satisfy the equality later on.

The error function:

$$f(x) = \begin{bmatrix} P_2(\theta,V) \\ P_3(\theta,V) \\ P_4(\theta,V) \\ Q_3(\theta,V) \\ Q_4(\theta,V) \end{bmatrix} - \begin{bmatrix} P_2^{sp} \\ P_3^{sp} \\ P_4^{sp} \\ Q_3^{sp} \\ Q_4^{sp} \end{bmatrix} = \begin{bmatrix} P_2(\theta,V) \\ P_3(\theta,V) \\ P_4(\theta,V) \\ Q_3(\theta,V) \\ Q_4(\theta,V) \end{bmatrix} - \begin{bmatrix} P_2 given \\ 0 \\ -0.2 \\ 0 \\ -0.4 \end{bmatrix}$$

, which we desire *f(x) = 0.*

## 5. Partial Derivatives and the Jacobian

In step 4 on the previous page, we suggested that we could adjust our guess using partial derivatives. The idea here is that partials ($\partial S / \partial x$) allow us to relate our state variables (***x***) to the power equations (***S*** $=$ ***P*** $+$ ***jQ***). Thus, after making an initial 'flat start' guess, if our power equations do not equal the specified power (i.e. there is a non-zero error), then we could use partial derivatives to adjust our state variables in order to correct the error. This is the idea behind the **Newton Raphson** technique.

***A quick look at the Jacobian…***

The Jacobian is just a matrix of partial derivatives of powers with respect to state variables. The matrix will correspond *only* to the power equations we have written and the unknown state variables we are solving for. Thus, for our four bus system,

$$J(x) = \begin{bmatrix} \frac{\partial P_2}{\partial \theta_2} & \frac{\partial P_2}{\partial \theta_3} & \frac{\partial P_2}{\partial \theta_4} & \frac{\partial P_2}{\partial V_3} & \frac{\partial P_2}{\partial V_4} \\ \frac{\partial P_3}{\partial \theta_2} & \frac{\partial P_3}{\partial \theta_3} & \frac{\partial P_3}{\partial \theta_4} & \frac{\partial P_3}{\partial V_3} & \frac{\partial P_3}{\partial V_4} \\ \frac{\partial P_4}{\partial \theta_2} & \frac{\partial P_4}{\partial \theta_3} & \frac{\partial P_4}{\partial \theta_4} & \frac{\partial P_4}{\partial V_3} & \frac{\partial P_4}{\partial V_4} \\ \frac{\partial Q_3}{\partial \theta_2} & \frac{\partial Q_3}{\partial \theta_3} & \frac{\partial Q_3}{\partial \theta_4} & \frac{\partial Q_3}{\partial V_3} & \frac{\partial Q_3}{\partial V_4} \\ \frac{\partial Q_4}{\partial \theta_2} & \frac{\partial Q_4}{\partial \theta_3} & \frac{\partial Q_4}{\partial \theta_4} & \frac{\partial Q_4}{\partial V_3} & \frac{\partial Q_4}{\partial V_4} \end{bmatrix}$$

Q. Which terms are zero?

A. The partial derivatives are zero for every term corresponding to two busses that are not connected. Example:

No line connects Bus 4 and Bus 2, thus

$$\frac{\partial P_2}{\partial \theta_4} = \frac{\partial P_2}{\partial V_4} = \frac{\partial P_4}{\partial \theta_2} = \frac{\partial Q_4}{\partial \theta_2} = 0 .$$