PV Bus Modeling in a Holomorphically Embedded Power-Flow Formulation

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***Abstract*—This paper introduces a PV bus model, compatible with the holomorphic embedding (HE) approach, for solving the power-flow bus-power-equilibrium equations (BPEE's). Because the BPEE's in traditional form are nonanalytic due to the presence of the complex-conjugate operator, many powerful tools applicable to the analytic functions cannot be used. Holomorphism is obtained by embedding the BPEE's into a bigger problem in such a way as to render the embedded problem analytic. The effect of the holomorphic embedding is to perform a type of curve following, but the curve followed is that of the embedded function, not the PV curve which is followed by traditional continuation methods. The primary advantage of HE is that it leads to an algorithm that is guaranteed to solve for the stable equilibrium point solution, regardless of starting point and without iteration. In the published literature on the HE approach, there is presently no model for a PV bus. This paper introduces such a model and suggests a remedy for the precision problems that arises with HE in modeling the PV bus.**

I. INTRODUCTION

The objective of the power-flow problem is to find the bus voltage magnitudes and angles and the reactive power injection at each bus in the power system, given the real and reactive power injections at load (PQ) buses and real power injection and voltage magnitude at voltage controlled (generator/PV) buses. Since the bus-power-equilibrium equations are nonlinear, iterative techniques such as Gauss-Seidel, Newton-Raphson and quasi-Newton methods are employed. While these methods and their many variants are ubiquitously used by the power industry and work well for conditions close to nominal conditions, as the system moves into extremis and the voltages move far from nominal, these methods can and do fail to converge. Because of these limitations, advancements in iterative methods have been a topic of research for many years and include improvements in convergence [6]-[7], quantification of the region of convergence [16], [17], calculation of the unstable equilibrium points [18]-[20], and estimating the voltage stability margin [20]-[23].

An N+1-bus network, characterized by the nonlinear BPEE's has 2^N voltage solutions, not all of which may be unique. Starting from a reasonable initial estimate of the voltage profile, if these iterative methods converge, they usually, though not always, converge to what is referred to as the Stable Equilibrium Point (SEP). Particularly troublesome when performing voltage stability analysis, these iterative methods may fail to converge near the bifurcation point. To overcome these drawbacks, research has been conducted on noniterative methods [24]-[27]. One such approach is the recently proposed Holomorphic Embedding Load-Flow Method (HELM) [28], [29]. HELM is guaranteed to find the SEP solution of the BPEE's in a noniterative manner independent of the initial estimate and it is able to obtain the solution even near the bifurcation point. Solving the BPEE's using holomorphic embedding (HE) for a system of all PQ buses (and one slack bus) is discussed in [30]. Proposing a method for including PV buses in the HE formulation is the goal of this paper.

This paper is organized as follows. Section II describes the formulation used in HELM. Section III describes the curve following in the HE formulation and how it differs from the traditional continuation methods. Section IV presents how PV buses can be represented using HE. Section V illustrates solving a two bus problem, with a PV bus using the formulation developed in Section IV. Section VI addresses the precision problems encountered by the HE method when a PV bus is included. Section VII summarizes the results and describes the scope for future work.

II. HOLOMORPHIC EMBEDDING LOAD-FLOW METHOD

A. Holomorphic Functions

A holomorphic function is a complex-valued analytic function. Functions of complex variables that are complex differentiable everywhere in a neighborhood around a point are said to be holomorphic about that point. Since holomorphic functions are analytic, they allow the use of the powerful theorems and techniques applicable only to analytic functions. In real analysis, differentiability in a neighborhood does not guarantee analyticity; however, in complex functions, differentiability guarantees analyticity. i.e., the power series expansion of the function about a point converges to the value of the function at that point [31].

B. Holomorphic Embedding of the Power-Flow Equations

Holomorphic embedding is the technique of embedding a small problem within a large problem with complex variables

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while guaranteeing that the resultant problem is analytic. In the case of the nonanalytic BPEE, the embedding must first eliminate the nonanalyticity of the smaller problem (i.e., the BPEE) due to the complex conjugate operator. Consider an N+1 bus system with the slack bus denoted by the index 0. Let m be the set of PQ buses. The BPEE of a PQ bus, i, is as follows:

$$\sum_{k=0}^{N} Y_{ik} V_k = \frac{S_i^*}{V_i^*}, i \in m \qquad (1)$$

where Y_{ik} is the (i, k) element of the bus admittance matrix, S_i is the complex power injection at bus i, and V_i is the voltage at bus i. The slack bus is denoted by the index 0.

Equation (1) may be holomorphically embedded into a larger problem as follows:

$$\begin{aligned}
&\sum_{k=0}^{N} Y_{ik} V_k(z) - (1-z)\sum_{k=0}^{N} Y_{ik} = \frac{z S_i^*}{V_i^*(z^*)}, \; i \in m \\
&V_k(0) = 1, \forall\, k \\
&V_k(1) = V_k, \forall\, k \\
&V_0(z) = V_0
\end{aligned} \qquad (2)$$

where, z is a complex variable.

Observe the following regarding (2):

i. With the parameter z as a variable, the notation $V(z)$ is used to emphasize that the voltage has become a holomorphic function of the complex parameter z.
ii. The complex conjugate of the voltage, V^* that appears in the BPEE is replaced by, $V^*(z^*)$ and not $V^*(z)$. The presence of z^* rather than z in this term retains the property of holomorphism, and therefore analyticity. (More about this below.)
iii. At z=0, all the injection term (S_i^*) in the power-flow equation vanishes. This represents the reference case where there is no generation and no load. In this case, (2) reduces to

$$\sum_{k=0}^{N} Y_{ik} V_k(0) = \sum_{k=0}^{N} Y_{ik} \qquad (3)$$

Equation (3) is a linear system of equations, with equality occurring when all the bus voltages are equal to 1, provided Y is nonsingular.
iv. At z=1, the BPEE's, (1), are recovered from the embedded system of equations, giving the desired SEP solution.

The solution procedure of (2) is based on representing the voltage function as a power series and then generating a continued fraction equivalent of the series. The subsequent derivations of power series expansion and continued fraction approximation are shown for a general function $V(z)$.

C. Power Series Expansion

Embedding the power-flow equations such that the result is holomorphic guarantees that the voltage can be represented as an equivalent power series using z as the expansion parameter. Thus, the voltage function $V(z)$ in (2) can be expressed as a Maclaurin series as follows:

$$V(z) = \sum_{n=0}^{\infty} V[n](z)^n \qquad (4)$$

where the coefficients $V[n]$ are complex numbers.

The Maclaurin series expansion of the voltage function can be used to prove that the embedding (2) is holomorphic. To be holomorphic, a function must satisfy the Cauchy-Riemann equations. An equivalent condition known as Wirtinger's derivative requires that $\partial V / \partial z^* = 0$ [32].

In ii, it was mentioned that the embedding is holomorphic for $V^*(z^*)$ but not $V^*(z)$. We will prove this statement using the Wirtinger's derivative. The Maclaurin series expansion of the $V^*(z)$ (if it were to exist), and $V^*(z^*)$ are shown below:

$$\begin{aligned}
V^*(z) &= V[0]^* + V[1]^* z^* + \ldots + V[n]^* \left(z^*\right)^n \\
V^*(z^*) &= V[0]^* + V[1]^* z + \ldots + V[n]^* \left(z^n\right)
\end{aligned} \qquad (5)$$

$V^*(z)$ in (5), is a function of z^* and the Wirtinger equations will not be satisfied. The expansion of $V^*(z^*)$ on the other hand is independent of z^* such that $\partial V^*(z^*)/\partial z^* = 0$. Thus the voltage function in (2) is holomorphic.

The power series of the voltage function, (4), when evaluated at z=1, gives the solution to the BPEE's; however, if the power series has a radius of convergence less than 1, then the series will be unbounded. To solve this problem, a technique known as analytic continuation may be applied. This technique allows certain ostensibly unbounded series to converge by effectively increasing the radius of convergence. Analytic continuation uses rational approximants, (i.e., a continued fraction expansion) to achieve this goal. If a solution of the BPEE's exists, a limit to the continued fraction will exist. If the rational approximants do not converge, then the system of BPEE's does not have a solution [29].

D. Continued Fraction

Many continued fractions exist which are algebraic approximants to the voltage power series. What is needed is a continued fraction that is the maximal analytic continuation of the power series. Such a fraction exists and may be constructed using the techniques of [30]. One approach is the Viskavatov method [33], which is demonstrated below. Equation (4) can be written as:

$$\begin{aligned}
V(z) &= V[0] + V[1]z + V[2]z^2 + \ldots + V[n]z^n + \ldots \\
&= V[0] + z(V[1] + V[2]z + \ldots + V[n]z^{n-1} + \ldots) \\
&= V[0] + \cfrac{z}{\cfrac{1}{V[1] + V[2]z + \ldots + V[n]z^{n-1} + \ldots}} \\
&= V[0] + \frac{z}{V^{(1)}(z)}
\end{aligned} \qquad (6)$$

The term $V^{(1)}(z)$ in (6) is the reciprocal of another power series given by:

$$\begin{aligned}
V^{(1)}(z) &= \frac{1}{(V[1] + V[2]z + V[3]z^2 + \ldots + V[n]z^{n-1} + \ldots)} \\
&= V^{(1)}[0] + V^{(1)}[1]z + \ldots + V^{(1)}[n-1]z^{n-1} + \ldots
\end{aligned} \qquad (7)$$

While calculating the coefficients of the continued fraction $V^{(1)}(z)$, is ultimately straightforward, showing how this is arrived at is somewhat tedious and is described below. Using the notation of (7), by definition,

$$\begin{aligned}&(V^{(1)}[0]+V^{(1)}[1]z+\ldots+V^{(1)}[n-1]z^{n-1}+\ldots)\\&(V[1]+V[2]z+\ldots+V[n]z^{n-1}+\ldots)=1\end{aligned} \tag{8}$$

Equation (8) is a product of two power series on the LHS. Since this product must equal 1 for any value of z, it must be the case that

$$V^{(1)}[0]=\frac{1}{V[1]} \tag{9}$$

Now, $V^{(1)}[i],\ \forall i=1,2,3,\cdots$ can be calculated from (8) as follows:

i. Assume the coefficients of $V^{(1)}[i]$ have been calculated through index k-1.

ii. Multiply the appropriate terms on the LHS of the two power series, (8), up through the k^{th} term to find the coefficient of z raised to k. (It will be shown later that the coefficients of product of two power series can be determined by the discrete convolution of the two corresponding sequences.)

iii. By equating the coefficients of the power series on both sides of (8), the $V^{(1)}[k]$ term can be calculated.

Applying the technique as described above to the last equation in (6) recursively, yields:

$$V(z)=V[0]+\cfrac{z}{V^{(1)}[0]+\cfrac{z}{\cfrac{1}{V^{(1)}[1]+\ldots+V^{(1)}[n-1]z^{n-2}+\ldots}}} \tag{10}$$

$$V(z)=V[0]+\cfrac{z}{V^{(1)}[0]+\cfrac{z}{V^{(2)}[0]+\cfrac{z}{V^{(3)}[0]+\ldots}}} \tag{11}$$

The continued fraction in (11) can be evaluated directly by replacing z=1 in (11). It can also be evaluated in the form A_n/B_n using the three term recursion relation (12).

$$\begin{aligned}&A_0(z)=V[0],\ A_1(z)=V[0]V^{(1)}[0]+z\\&A_i(z)=V^{(i)}[0]A_{i-1}(z)+zA_{i-2}(z),\ i=2,3,\ldots\\&B_0(z)=1,\ B_1(z)=V^{(1)}[0]\\&B_i(z)=V^{(i)}[0]B_{i-1}(z)+zB_{i-2}(z),\ i=2,3,\ldots\end{aligned} \tag{12}$$

The three-term recursion relation is preferred since it gives flexibility in choosing the number of terms in the continued fraction. In other words, when using the three term recursion relation, *a posteriori* increase in the length of the continued fraction, involves fewer calculations than direct evaluation by starting anew.

In the numerical implementation of the above mentioned steps, the power series of voltage function, (4), and hence the continued fraction expansion, (11), are evaluated only for a finite number of terms. Despite using double precision, beyond 60 terms the power series coefficients cannot be represented accurately due to the accumulation of round-off error [29]. In the subsequent examples, 40 terms were used to arrive at the SEP solution.

III. Curve Following in HELM

As stated previously, holomorphic embedding is different from the traditional continuation methods used for solving the power-flow problem in so many ways that the two are essentially unrelated. In (2), as the parameter z is increased from 0 to 1, the load at the PQ bus, i, increases linearly from 0 to S_i. It is tempting to conclude that the HE formulation also does curve following similar to the continuation methods.

On the contrary, equation (2) represents the BPEE's accurately only at z=1. If shunt reactance exists at bus i in the system, ΣY_{ik} will be unequal to zero; hence, the "(1-z)" term in (2) will be nonzero for all $z\neq1$, making solution of the HE formulation in (2) different from that of (1). In this section, this difference will be demonstrated for a two-bus system with shunt reactance connected to the load bus as shown in Fig. 1.

The BPEE for the PQ bus in Fig. 1 can be written as:

$$\frac{V-V_0}{Z}=\frac{S^*}{V^*}+\frac{V}{X_C} \tag{13}$$

where the variable V is the voltage at the PQ bus, V_0 is the slack-bus voltage, Z is the line impedance, S=-(P_L+jQ_L) is the complex power load at the PQ bus, and X_C is the shunt reactance at the PQ bus.

The system parameters for the two bus model are specified using a 100 MVA base in Fig. 1.

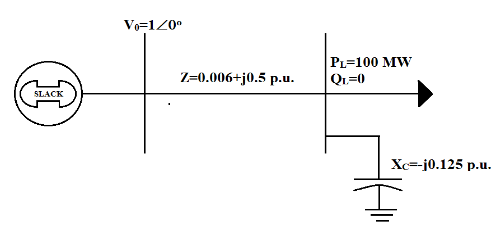

Fig. 1. Two-bus model–curve following in holomorphic embedding

In order to demonstrate the difference between the HE solution approach and a continuation method for solving the BPEE's, a comparison is made between the power voltage (PV) curve generated from (1) and the HE voltage-function curve generated from (2). For purposes of distinction, the HE voltage-function curve is referred as the parameter z vs. voltage (zV) curve.

In Fig. 2, the red line represents the plot of the true (not HE) PV curve for the load bus in the two bus system obtained using a continuation method. The HE voltage function, $V(z)$, defined by (2) is plotted as a function of the parameter z, shown by the blue line in Fig. 2.

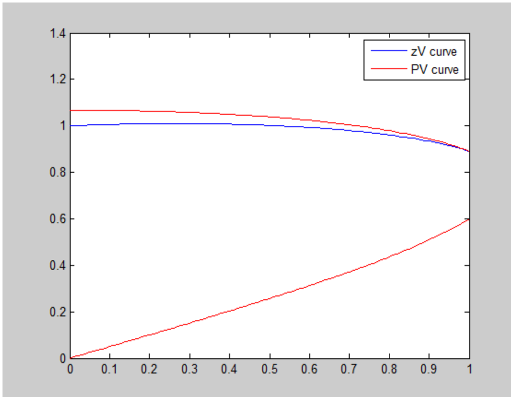

Fig. 2. Two bus model–curve following in holomorphic embedding

Observe in Fig. 2 that a difference exists between the PV and zV curves except at z=1, when the (1-z) term vanishes. At the z=1 point, the solution of the load-bus BPEE's is identical to that of the HE formulation.

IV. PV Bus Modeling

Current literature does not contain a HE equation for modeling the PV/voltage controlled bus. This section proposes one such equation. Consider a system with N+1 buses, the slack bus being denoted by the index 0. Let p denote the set of PV buses. The traditional defining equations for a PV bus, i, are:

$$P_i = \mathrm{Re}(V_i \sum_{k=0}^{N} Y_{ik}^* V_k^*), \forall i \in p \tag{14}$$

$$|V_i| = V_{sp,i}, \forall i \in p \tag{15}$$

where P_i denotes the real power injection and $V_{sp,i}$, the voltage magnitude at bus i.

Unlike the PQ bus where complex equality constraints are used, for the PV bus equality constraints needed are only on the real part of the complex power injection in the traditional formulation. Since the HE formulation requires complex equality constraints, manipulation of (14) is necessary, as shown below.

Real power injection P_i can be expressed in terms of complex power S_i as follows:

$$\begin{aligned} &S_i + S_i^* = 2P_i \\ &(V_i \sum_{k=0}^{N} Y_{ik}^* V_k^*) + (V_i^* \sum_{k=0}^{N} Y_{ik} V_k) = 2P_i \end{aligned} \tag{16}$$

Multiplying both sides of (16) by V_i gives,

$$(V_i^2 \sum_{k=0}^{N} Y_{ik}^* V_k^*) + (|V_i|^2 \sum_{k=0}^{N} Y_{ik} V_k) = 2P_i V_i \tag{17}$$

For a PV bus, the voltage magnitude at the bus is specified in (15). Therefore (17) can be written as:

$$|V_{sp}|^2 \sum_{k=0}^{N} Y_{ik} V_k = 2P_i V_i - (V_i^2 \sum_{k=0}^{N} Y_{ik}^* V_k^*) \tag{18}$$

The reformulated BPEE's, (18), for the PV bus is then holomorphically embedded using a complex parameter z.

$$\begin{aligned} &|V_{sp}|^2 \sum_{k=0}^{N} Y_{ik} V_k(z) \\ &= z2P_i V_i(z) - z\,(V_i(z)^2 \sum_{k=0}^{N} Y_{ik}^* V_k^*(z^*) + (1-z)|V_{sp}|^2 \sum_{k=0}^{N} Y_{ik} \\ &V_k(0) = 1,\ V_k(1) = V_k, \forall\, k \\ &V_0(z) = V_0 \end{aligned} \tag{19}$$

Equation (19) represents voltage as an implicit function of the complex parameter z. The slack bus voltage term $|V_{sp}|^2\ Y_{i0} V_0$ can be moved to the RHS in (19) to give:

$$\begin{aligned} |V_{sp}|^2 \sum_{k=1}^{N} Y_{ik} V_k(z) &= z2P_i V_i(z) - z(V_i(z)^2 \sum_{k=0}^{N} Y_{ik}^* V_k^*(z^*)) \\ &+ (1-z)|V_{sp}|^2 \sum_{k=0}^{N} Y_{ik} - |V_{sp}|^2 Y_{i0} V_0 \end{aligned} \tag{20}$$

In (20), the term $(V_i(z)^2 \sum_{k=0}^{N} Y_{ik}^* V_k^*(z^*))$ involves product of three power series. Calculating the product of two power series is demonstrated below, the product of three power series being an extension of the same. Consider the Maclaurin series expansion of two functions:

$$a(z) = \sum_{n=0}^{\infty} a[n] z^n,\ b(z) = \sum_{n=0}^{\infty} b[n] z^n \tag{21}$$

The product of the two power series, say $c(z)$ is of the form, $c(z) = \sum_{n=0}^{\infty} c[n] z^n$, where the coefficients $c[n]$ is given by:

$$c[n] = \sum_{j+k=n} a[j] b[k] = \sum_{j=0}^{n} a[j] b[n-j] \tag{22}$$

i.e., the coefficients of product of two power series $a(z)$, $b(z)$ is equal to the discrete convolution of the two original sequences $a[n]$, $b[n]$. More often in the algorithm we require the individual coefficients of z raised to a particular power rather than the whole series; hence convolution of the sequences is used. The product of three power series in (20) can be evaluated similarly through convolutions. Let

$$\begin{aligned} &W_i(z) = (V_i(z)\ \ V_i(z)\ \sum_{k=0}^{N} Y_{ik}^* V_k^*(z^*)) \\ &\text{where, } W_i(z) = \sum_{n=0}^{\infty} W_i[n] z^n \end{aligned} \tag{23}$$

Substituting (23) into (20), and then expanding $V_i(z)$, $V_k(z)$ as a Maclaurin series, an expression for the coefficients of the power series, $V_k[n]$, is obtained as shown in (24).

$$|V_{sp}|^2 \sum_{k=1}^{N} Y_{ik} V_k[n] = (2P_i V_i[n-1] - W_i[n-1])$$

$$- \delta_{n0} |V_{sp}|^2 (Y_{i0} V_0) + (\delta_{n0} - \delta_{n1}) |V_{sp}|^2 \sum_{k=0}^{N} Y_{ik} \quad (24)$$

where, $\delta_{n0} = \begin{Bmatrix} 1, n=0 \\ 0, n \neq 0 \end{Bmatrix}$, $\delta_{n1} = \begin{Bmatrix} 1, n=1 \\ 0, n \neq 1 \end{Bmatrix}$

A recursive relation for $V_k[n]$ coefficient is obtained in terms of $V_i[n\text{-}1]$, and $W_i[n\text{-}1]$ due to the product with the 'z' term on the RHS in (20). Observe that the "(1-z)" term in the RHS in (20), upon expansion, has coefficients for powers of z raised to 0, and 1 only; hence, in the relation for finding $V_k[n]$ in (24) the terms associated with (1-z) exist only for n=0, 1 as indicated by δ_{n0}, δ_{n1} respectively.

In calculating the power series coefficients using (24), coefficients up through $V_i[n\text{-}1]$, $W_i[n\text{-}1]$ are known from previous iterations; hence the RHS in (24) becomes a constant. Thus (24) becomes a linear system of equations which is then solved using traditional methods. The power series for the voltages at the PV buses is then evaluated using the same procedure as that for the PQ buses.

V. Two Bus Example

Consider the simple lossless two bus model with a PV bus as shown in Fig. 3. The parameters in this figure are specified using a 100 MVA base. Real power injection at the PV bus is P_1=P_G-P_L= -1 p.u.

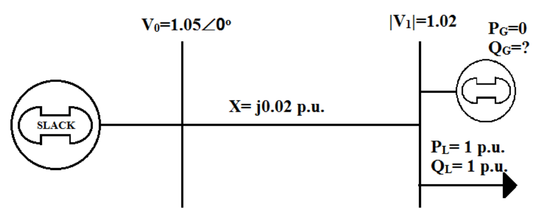

Fig. 3. Two bus model with voltage controlled bus

The solution voltage of the PV bus in the system as described in Fig. 3 is found from PowerWorld to be V_1=1.0198-j0.019=1.02∠-1.07°.

The formulation for evaluating the power series coefficients of the voltage function from (24) is:

$$Y_{22} V_1[n] = \frac{1}{|V_{sp}|^2}((2P_1 V_1[n-1]) - W_1[n-1])$$

$$-(\delta_{n0} Y_{10} V_{slack}) + ((\delta_{n0} - \delta_{n1}) \sum_{k=0,1} Y_{1k}) \quad (25)$$

The power series coefficients are first evaluated from (25). This is followed by evaluation of the continued fraction expansion via the three term recursion relation. At z=1, the solution is found to be V_1= 1.0198-j0.0191 =1.02∠-1.07°, which is the desired SEP solution.

Thus the proposed HE PV bus model is validated for this simple system and the proposed solution procedure is seen to arrive at the SEP solution with sufficient precision.

VI. Precision Problems in the Two Bus Example

Unlike solving the PQ bus BPEE's, precision problems are encountered when solving the voltage controlled bus BPEE's using the HE formulation. Precision problems can be attributed to the multiple recursive convolutions that arise in the PV bus formulation. For example, the calculation of $W_i[n\text{-}1]$ in (23) involves the product of three power series, which is evaluated as a convolution of the corresponding sequences. The recursive nature of the calculation exacerbates the round-off error problem and hence this algorithm is more sensitive to precision limitations.

Using the same two-bus model shown in Fig. 3, the real power demand at the PV bus is gradually increased and the voltage solution is found using PowerWorld and the HE formulation. To determine the effect of precision on the solution, the HE method was implemented, and tested with single and double precision. Using the HE formulation, as the real power increases, the solved voltage magnitude at the PV bus deviates from the specified bus voltage. The absolute value of the error between the specified voltage magnitude and the calculated voltage magnitude using the HE formulation is plotted against the absolute value of the bus voltage angle, δ, in Fig. 4 in a logarithmic scale. The bus angle is chosen since an increase in the bus angle is reflective of the increase in real power. As the PV bus voltage angle increases, the voltage magnitude error increases as shown in Fig. 4. It was observed that the HE method became numerically unstable beyond a particular δ value The numerical instability was found to occur at a smaller value of δ in case of single precision when compared to double precision.

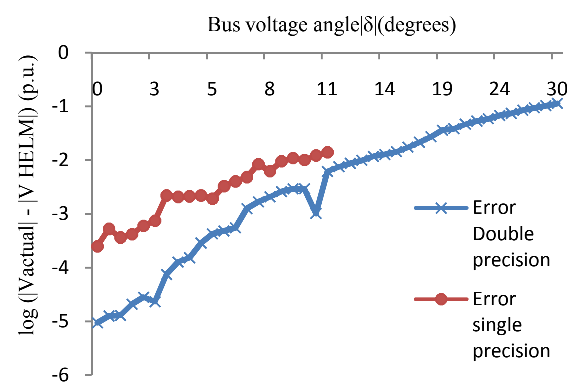

Fig. 4. Plot of absolute error in per unit voltage magnitude vs. bus angle

Mismatch in real power is also evaluated by substituting the voltage solution obtained from PowerWorld, and HE formulations (using both single and double precision) into the BPEE (14). Fig. 5 shows the mismatch in real power, obtained for each method, against the real power in per unit.

The real power mismatch using the voltage solution from PowerWorld is essentially zero. However, in the HE method, the real power mismatch increases with the increase in real power load.

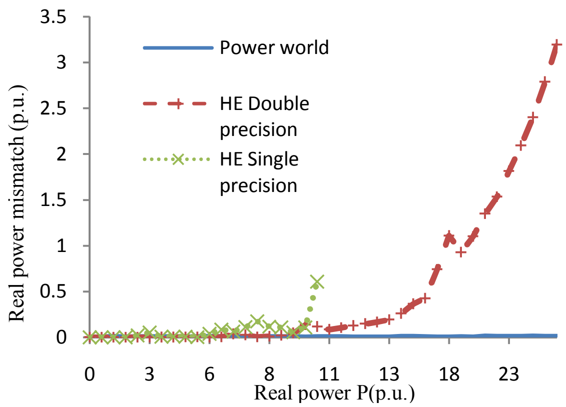

Fig. 5. Plot of mismatch in real power vs. real power in p.u

From Fig. 4 and Fig. 5, it can be seen that, when using the HE formulation for solving the power-flow problem with PV buses, the mismatch in real power injection and voltage magnitude obtained using double precision is significantly smaller than when using single precision. Hence, we believe that greater than double precision is needed to obtain accurate solutions of the BPEE. Further research is being carried out in this direction.

VII. Conclusion

The principles of HE were developed and the solution techniques needed, namely convolution, analytic continuation, and continued fraction expansion, were presented. The relationship between the PV curve of the bus-power-equilibrium equations and the zV curve of an equivalent holomorphic embedding was presented.

A PV bus model consistent with the principles of holomorphic embedding method was introduced and applied to a simple two-bus problem. The precision problem that arises in solving the PV bus problem was demonstrated. It is suspected that precision problems arise due to the multiple and recursive convolutions required by the algorithm. Consequently, greater than double precision is needed to accurately calculate the power series, the associated continued fraction terms and to reliably solve problems involving PV buses. Further exploration of precision issues is ongoing.

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