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APPLICATION OF WYNN'S EPSILON ALGORITHM TO PERIODIC CONTINUED FRACTIONS

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1. Introduction

The infinite continued fraction

$$[a_0, a_1, \ldots, \dot{a}_n, \ldots, \dot{a}_{n+l-1}] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \tag{1}$$

in which

$$a_{m+l} = a_m \quad \text{for} \quad m \geqq n \geqq 0 \tag{2}$$

is periodic with period l and is equal to a quadratic surd if and only if the partial quotients, a_k, are integers or rational numbers [1]. We shall also assume that they are positive. The transformation discussed below applies only to pure periodic fractions where n is zero. Any fraction with $n>0$ is equivalent [1] to a pure periodic fraction in the sense that numbers x and y are equivalent if

$$x = (ay+b)/(cy+d) \quad \text{where} \quad |ad-bc| = 1, \tag{3}$$

and such a fraction may be included in the discussion provided that it is first replaced by an equivalent pure periodic fraction. The convergents to the continued fraction (1) may be written [1]

$$w_k = p_k/q_k \quad \text{for} \quad k \geqq 0 \tag{4}$$

where p_k and q_k satisfy the recurrence relations

$$u_{k+1} = a_{k+1} u_k + u_{k-1} \tag{5}$$

where u stands for p or q. We may extend the index k to negative values and start these recurrence relations with the pairs of values

$$p_{-2} = 0, \qquad p_{-1} = 1, \qquad q_{-1} = 0, \qquad q_0 = 1, \tag{6}$$

and set w_{-1} to infinity.

Wynn's epsilon algorithm [2] accelerates (in suitably *restricted* cases) the convergence of a sequence. The even transformations $\varepsilon_{2j}(v_k)$ of the sequence $\{v_k\}$ form sequences with the same limit as $\{v_k\}$, again in suitably *restricted* cases. (The limits could differ; for example if $v_k \equiv a^k$ with $|a|>1$ then the limit of the sequence $\{v_k\}$ is infinite while that of the sequence $\{\varepsilon_2(v_k)\}$ is zero. For a fuller discussion of convergence acceleration the reader is referred to the book by Brezinski [3].) We can place these even transformations in a triangular array whose first column contains the original sequence $\{v_k\}$ and whose subsequent columns contain the even transformations, as illustrated in the table.

Table Even transformations of the sequence $\{v_k\}$

$\varepsilon_{-2}(v_0)=\infty$				
$\varepsilon_{-2}(v_1)=\infty$	$\varepsilon_0(v_0)=v_0$			
$\vdots$	$\varepsilon_0(v_1)=v_1$	$\varepsilon_2(v_0)$		
$\vdots$	$\vdots$	$\varepsilon_2(v_1)$	$\varepsilon_4(v_0)$	
$\vdots$	$\vdots$	$\vdots$	$\varepsilon_4(v_1)$	$\varepsilon_6(v_0)$
$\vdots$	$\vdots$	N	$\vdots$	$\varepsilon_6(v_1)$
$\vdots$	W	C	E	$\vdots$
$\vdots$	$\vdots$	S	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Wynn [4] has given the following relation between entries in the array;

$$(E-C)^{-1}+(W-C)^{-1}=(N-C)^{-1}+(S-C)^{-1} \tag{7}$$

where C denotes an entry with neighbours W, N, E, and S, as illustrated. The transformations may be constructed from equation (7), the entries in two adjacent columns being used to generate those in the next column to the right. In order to start the process Wynn introduced an extra column to the left with entries taken as infinite, as illustrated to the left of the broken line in the table. (Wynn's paper [4] is concerned with ratios of power series but the results are applicable to this simpler situation; also our array has the rows and columns of his interchanged). We show below in Section 3 that for the sequence of convergents $\{w_k\}$ to a pure periodic continued fraction

$$\varepsilon_{2j}(w_{rl-1})=w_{(j+1)(j+r)l-1} \quad \text{for} \quad r>0 \tag{8}$$

where the transformation with j taken as zero yields the starting sequence $\{w_{rl-1}\}$ contained within $\{w_k\}$.

We first prove two simple properties of the sequences $\{u_k\}$ and $\{w_k\}$.

2. Two simple properties pertaining to a pure periodic continued fraction

We take n of equation (2) to be zero so that the fraction is pure periodic. We prove that

$$u_{i+2l}=Au_{i+l}+(-1)^{l+1}u_i \tag{9}$$

where A is independent of i and the starting conditions of $\{u_k\}$. Clearly if it is true for i

and $i+1$ then from equations (2) and (5) it is true for $i+2$. Let

$$u_k = X_k u_0 + Y_k u_1 \tag{10}$$

From equations (5) and (10) we see that X_k and Y_k also satisfy equation (5) but with starting conditions

$$X_0 = 1, \qquad X_1 = 0, \qquad Y_0 = 0, \qquad Y_1 = 1, \tag{11}$$

and it follows that

$$X_{k+1} Y_k - Y_{k+1} X_k = (-1)^{k+1} \tag{12}$$

From equation (10) and the periodicity of the partial quotients

$$\begin{aligned} u_l &= X_l u_0 + Y_l u_1 \\ u_{l+1} &= X_{l+1} u_0 + Y_{l+1} u_1 \\ u_{2l} &= X_l u_l + Y_l u_{l+1} \\ u_{2l+1} &= X_{l+1} u_l + Y_{l+1} u_{l+1}. \end{aligned} \tag{13}$$

Elementary manipulation of equations (12) and (13) yields equation (9) for $i=0$ and 1 with A identified as $X_l + Y_{l+1}$, which is independent of i and the starting values of $\{u_k\}$. Thus we have proved equation (9). It is satisfied by $\{p_k\}$ and $\{q_k\}$.

From equation (9) with the period replaced by a multiple of itself sl, say, and equations (6) we find

$$\begin{aligned} p_{(r+1)sl-1} q_{rsl-1} - q_{(r+1)sl-1} p_{rsl-1} &= (-1)^{sl} [p_{rsl-1} q_{(r-1)sl-1} - q_{rsl-1} p_{(r-1)sl-1}] \\ &= \cdots = -(-1)^{rsl} q_{sl-1} \quad \text{for} \quad r \geqq 0. \end{aligned} \tag{14}$$

$$\begin{aligned} & q_{(r+1)sl-1} q_{rsl-1} - (-1)^{sl} q_{rsl-1} q_{(r-1)sl-1} \\ & = q_{(r+2)sl-1} q_{(r-1)sl-1} - (-1)^{sl} q_{(r+1)sl-1} q_{(r-2)sl-1} \\ & = \cdots = q_{2rsl-1} q_{sl-1} \quad \text{for} \quad r > 0. \end{aligned} \tag{15}$$

Because the partial quotients are positive, q_{sl-1} is non zero and hence from equations (4), (14) and (15) we have (taking w_{-1} as infinite)

$$[w_{(r+1)sl-1} - w_{rsl-1}]^{-1} + [w_{(r-1)sl-1} - w_{rsl-1}]^{-1} = -(-1)^{rsl} q_{2rsl-1} \quad \text{for} \quad r > 0 \tag{16}$$

Thus the left hand side of equation (16) depends only on the index $rsl-1$ of the central of the three equally spaced terms $w_{(r\pm 1)sl-1}$ and w_{rsl-1} of the sequence $\{w_k\}$ and is independent of the spacing; this index must however be one less than an exact multiple of the spacing (otherwise equations (6) would be inapplicable). This left hand side of equation (16) is also a special case of either side of equation (7) with the appropriate values of C, W, N, E and S.

3. Application of the epsilon algorithm

We construct an array such as that in the table in which the infinite values of the left most column ε_{-2} assigned by Wynn are interpreted as the values of w_{-1} and the column ε_0 contains the values w_{rl-1}. Consider an entry $C=w_{rl-1}$ in column ε_0. The entry immediately above it is $N=w_{(r-1)l-1}$ while that immediately below it is $S=w_{(r+1)l-1}$. The entries in column ε_0 are equally spaced in index with spacing l; thus the entries N, C and S are equally spaced in index. The entry immediately left of C is $W=w_{-1}$ (in column ε_{-2}). Equations (7) and (16) show that equal vertical spacing (in N, C and S) implies equal horizontal spacing (in W, C and E, E being the entry immediately right of C). The horizontal spacing in index from W to C is equal to rl and therefore that from C to E is also equal to rl. Hence $E=w_{2rl-1}$ and in this way we construct the entries for column ε_2. These are equally spaced with spacing $2l$ and we repeat the process to obtain the entries in columns ε_4, ε_6 etc., eventually to derive equation (8).

The column ε_2 contains the results of one application of Aitken's transformation. We may obtain the result of repeated application of Aitken's transformation by setting N, C and S equal to the values obtained from one application of Aitken's transformation, setting $W=w_{-1}$ again and repeating the process to find the result for j applications as $w_{2^j(j+r)l-1}$ for $r>0$. Phillips's result [5] concerning Aitken's transformation and the Fibonacci series is a special case of this for unit period.

The reader who finds the above treatment of the infinities somewhat casual may prefer to evaluate the entries in column ε_2 explicitly first from Aitken's transformation which in this case is equation (7) with the term in $C-W$ omitted.

We see that application of Wynn's algorithm or multiple applications of Aitken's transformation always yields a convergent to the continued fraction. Each new column of the array contains an equally spaced (in the index) sequence of convergents each of whose index is always one less than an exact multiple of the period. Hence if we apply Wynn's algorithm for $j=j'$, say, use the values obtained as new starting values and apply the algorithm again for $j=j''$, say, we still obtain a convergent to the continued fraction. Indeed application of any series of Wynn's algorithm yields a convergent to the continued fraction.

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