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On the Solution of Systems of Equations by the Epsilon Algorithm of Wynn

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Abstract. The ϵ-algorithm has been proposed by Wynn on a number of occasions as a convergence acceleration device for vector sequences; however, little is known concerning its effect upon systems of equations. In this paper, we prove that the algorithm applied to the Picard sequence $\mathbf{x}_{i+1} = F(\mathbf{x}_i)$ of an analytic function F: $\mathbf{R}^n \supset D \to \mathbf{R}^n$ provides a quadratically convergent iterative method; furthermore, no differentiation of F is needed. Some examples illustrate the numerical performance of this method and show that convergence can be obtained even when F is not contractive near the fixed point. A modification of the method is discussed and illustrated.

1. Introduction. The ϵ-algorithm is a nonlinear method for accelerating the convergence of sequences; in its simplest form, it is identical with the δ^2 transformation of Aitken [1]. The determinantal formulae upon which it is based were given by Jacobi [6], Schmidt [11], and Shanks [12]; Wynn [13] developed it and examined it thoroughly in connection with various sequences and series [14]–[17]. The ϵ-algorithm provides higher (integer) order methods for the computation of a fixed point of an analytic function f: $\mathbf{C} \supset D \to \mathbf{C}$ [4]. Using the generalized matrix inverse of Moore [8] and Penrose [9], the method has recently been applied to sequences of matrices and vectors as they arise, for example, in the solution of linear systems of equations [5], [7], [10], [18], [21], [22], [23]. Wynn points out that the algorithm also provides good results in the numerical solution of nonlinear systems [18], [19], [21], [22]. But, until now, nothing is known concerning convergence. In this paper, we examine the behaviour of the ϵ-algorithm when applied to the Picard sequence of an analytic function F: $\mathbf{R}^n \supset D \to \mathbf{R}^n$ with fixed point $\mathbf{z}$. With the help of a theorem of McLeod [7], we show that the algorithm, used in a manner similar to Steffensen's method, is a quadratically convergent iterative method for the computation of $\mathbf{z}$ (compare also Brezinski [2]*). Because of the complicated recursive relationships, the convergence considered is of local nature, and Landau symbols are used in the proof. A short discussion of numerical properties of the method follows at the end of the paper.

We use certain standard notations: $i \in \mathbf{N}$ means that i is a nonnegative integer; lower (upper) case bold face letters denote vectors (matrices); $||\mathbf{x}||$ is the Euclidean norm $(\mathbf{x}^*\mathbf{x})^{1/2}$ of the n-dimensional column vector $\mathbf{x} \in \mathbf{C}^n$; $O(||\mathbf{x}||^i)$ denotes a vector-valued function of the vector $\mathbf{x}$ whose norm remains bounded as $||\mathbf{x}|| \to 0$ after division by $||\mathbf{x}||^i$; $O\{||\mathbf{x}||^i\}$ denotes a real valued function with the same properties.

We also make use of the concept of an analytic function of a vector and of a vector-valued Taylor series. Let D be an open subset of $\mathbf{R}^n$, then F: $\mathbf{R}^n \supset D \to \mathbf{R}^n$ is called

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analytic if, for every point $\mathbf{a} \in D$, there is an open polycylinder $P = \{\mathbf{x} \in \mathbf{R}^n, |x_i - a_i| < r_i, 0 < r_i, 1 \leqq i \leqq n\} \subset D$, such that in P, $F(\mathbf{x})$ is equal to the sum of an absolutely summable power series in the n variables $x_i - a_i$ $(1 \leqq i \leqq n)$. An analytic function is indefinitely differentiable, and, if the segment joining $\mathbf{x}$ and $\mathbf{x} + \mathbf{y}$ is in D, we have, for $r \in \mathbf{N}$,

$$F(\mathbf{x} + \mathbf{y}) = F(\mathbf{x}) + \sum_{k=1}^{r-1} \frac{1}{k!} F^{(k)}(\mathbf{x}) \cdot \mathbf{y}^{(k)} + \left(\int_0^1 \frac{(1-t)^{r-1}}{(r-1)!} F^{(r)}(\mathbf{x} + t\mathbf{y})\, dt\right) \cdot \mathbf{y}^{(r)},$$

where $\mathbf{y}^{(k)}$ stands for $(\mathbf{y}, \mathbf{y}, \cdots, \mathbf{y})$ (k times). For further details, we refer to the famous book of Dieudonné [3].

2. Picard Sequences. We consider some iterative schemes for determining a fixed point $\mathbf{z}$ of the equation $\mathbf{x} = F(\mathbf{x})$. If $\mathbf{s}_p$ $(p \in \mathbf{N}, 0 \leqq p)$ is near $\mathbf{z}$, we have, using a Taylor expansion for $F(\mathbf{z})$,

$$\mathbf{z} = F(\mathbf{s}_p) + F'(\mathbf{s}_p)(\mathbf{z} - \mathbf{s}_p) + O(\|\mathbf{z} - \mathbf{s}_p\|^2). \tag{1}$$

Thus, when using the simple iteration scheme

$$\mathbf{s}_{p+1} = F(\mathbf{s}_p) \qquad (0 \leqq p), \tag{2}$$

we have

$$\mathbf{z} - \mathbf{s}_{p+1} = F'(\mathbf{s}_p)(\mathbf{z} - \mathbf{s}_p) + O(\|\mathbf{z} - \mathbf{s}_p\|^2).$$

Hence, the simple scheme (2) is, in general, at best linearly convergent; whether it converges or not depends upon the magnitudes of the eigenvalues of the Jacobian matrices $F'(\mathbf{s}_p)$ $(0 \leqq p)$ in the neighbourhood of $\mathbf{z}$. We can, however, devise a quadratically convergent scheme based upon the solution of the linear system

$$\hat{\mathbf{s}}_{p+1} = F(\hat{\mathbf{s}}_p) + F'(\hat{\mathbf{s}}_p)(\hat{\mathbf{s}}_{p+1} - \hat{\mathbf{s}}_p) \qquad (0 \leqq p)$$

or

$$(\mathbf{I} - F'(\hat{\mathbf{s}}_p))\hat{\mathbf{s}}_{p+1} = F(\hat{\mathbf{s}}_p) - F'(\hat{\mathbf{s}}_p)\hat{\mathbf{s}}_p \qquad (0 \leqq p) \tag{3}$$

for $\hat{\mathbf{s}}_{p+1}$. For, replacing $\mathbf{s}_p$ in formula (1) by $\hat{\mathbf{s}}_p$, we now have

$$\mathbf{z} - \hat{\mathbf{s}}_{p+1} = F'(\hat{\mathbf{s}}_p)(\mathbf{z} - \hat{\mathbf{s}}_{p+1}) + O(\|\mathbf{z} - \hat{\mathbf{s}}_p\|^2) \qquad (0 \leqq p),$$

i.e.,

$$(\mathbf{I} - F'(\hat{\mathbf{s}}_p))(\mathbf{z} - \hat{\mathbf{s}}_{p+1}) = O(\|\mathbf{z} - \hat{\mathbf{s}}_p\|^2) \qquad (0 \leqq p)$$

or, again subject to certain assumptions concerning the eigenvalues of $F'(\mathbf{x})$ in the neighbourhood of $\mathbf{z}$,

$$\mathbf{z} - \hat{\mathbf{s}}_{p+1} = O(\|\mathbf{z} - \hat{\mathbf{s}}_p\|^2) \qquad (0 \leqq p).$$

The second scheme, although yielding quadratic convergence, involves evaluation of a Jacobian matrix and the solution of a linear system at each stage. However, by use of the ϵ-algorithm one can, as we shall show, obtain quadratic convergence without the computation of the derivatives occurring in the Jacobian matrix, and without the solution of a linear system.

3. The Algorithm. The ϵ-algorithm [13], [22] is a computational procedure in which successive columns of an array $(\epsilon_q^{(p)})_{0\leq p, 0\leq q}$ with row index p are obtained by use of the formula

$$\epsilon_{q+1}^{(p)} = \epsilon_{q-1}^{(p+1)} + (\epsilon_q^{(p+1)} - \epsilon_q^{(p)})^{-1} \qquad (0 \leqq p, 0 \leqq q), \tag{4}$$

starting from the initial conditions

$$\epsilon_{-1}^{(p)} = 0, \qquad \epsilon_0^{(p)} = s_p \qquad (0 \leqq p). \tag{5}$$

If the inverse of a nonzero vector $\mathbf{x} \in \mathbf{C}^n$ is defined, by [8], [9],

$$\mathbf{x}^{-1} = (\mathbf{x}^*\mathbf{x})^{-1}\bar{\mathbf{x}}, \tag{6}$$

then we can apply the algorithm to sequences $\{\mathbf{s}_p\}_{0\leq p}$ of vectors and have the fundamental theorem [7], [23] which we need later:

THEOREM 1. *Let* $\{\mathbf{s}_p\}_{0\leq p}$ *be a sequence of vectors with complex coefficients which satisfy the irreducible linear recursion*

$$\sum_{r=0}^{m} c_r \mathbf{s}_{p+r} = \left(\sum_{r=0}^{m} c_r\right)\mathbf{s} \qquad (0 \leqq p), \tag{7}$$

where $\mathbf{s}$ *is fixed and*

$$\sum_{r=0}^{m} c_r \neq 0, \qquad c_r \in \mathbf{R}. \tag{8}$$

If then the elements of the array $(\boldsymbol{\varepsilon}_q^{(p)})$ *are determined by using* (4), (5), *and* (6), *and if all* $\boldsymbol{\varepsilon}_q^{(p)}$ *with* $p + q \leqq 2m$ *exist, then*

$$\boldsymbol{\varepsilon}_{2m}^{(0)} = \mathbf{s}.$$

Following a conjecture of Wynn [24] and Greville [5], Theorem 1 remains true if relations (7), (8) hold for complex scalars only, but this has not yet been proved. In conclusion, we get

COROLLARY. *Let* $\mathbf{z}$ *be the unique solution of the linear system* $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{c}$ *with real coefficients and let* m *be the degree of the minimal polynomial of the matrix* $\mathbf{A}$ *for* $\mathbf{y} = \mathbf{x}_0 - \mathbf{z}$. *If the* ϵ*-algorithm is applied to the Picard sequence* $\{\mathbf{x}_p;\ \mathbf{x}_{p+1} = \mathbf{A}\mathbf{x}_p + \mathbf{c}\}_{0\leq p}$ *and if all* $\boldsymbol{\varepsilon}_q^{(p)}$ *with* $p + q \leqq 2m$ *exist, then*

$$\boldsymbol{\varepsilon}_{2m}^{(0)} = \mathbf{z}.$$

Proof. Let $p(x) = \sum_{r=0}^{m} a_r x^r$ be the minimal polynomial of $\mathbf{A}$ for $\mathbf{y}$, then

$$\sum_{r=0}^{m} a_r \mathbf{x}_{p+r} = \left(\sum_{r=0}^{m} a_r\right)\mathbf{z} + \left(\sum_{r=0}^{m} a_r \mathbf{A}^{p+r}\right)\mathbf{y} = \left(\sum_{r=0}^{m} a_r\right)\mathbf{z},$$

because $\mathbf{x}_p = \mathbf{z} + \mathbf{A}^p\mathbf{y}$ holds. By assumption, we have $\sum_{r=0}^{m} a_r \neq 0$, since 1 is not eigenvalue of $\mathbf{A}$ (the equation $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{c}$ has a unique solution), and the Corollary results from Theorem 1.

4. The Application of the Epsilon Algorithm to Picard Sequences. The general strategy adopted in deriving our main result is this: we first consider the behaviour of the vectors $\tilde{\boldsymbol{\varepsilon}}_q^{(p)}$ ($p + q \leqq 2n$) derived by means of the ϵ-algorithm from the sequence $\tilde{\mathbf{s}}_p = \mathbf{z} + \mathbf{A}^p\mathbf{y}$ ($0 \leqq p$), where $\mathbf{y}, \mathbf{z} \in \mathbf{R}^n$ and $\mathbf{A}$ is a real $n \times n$ matrix, for small values of $||\mathbf{y}||$ (we know from the above Corollary that, subject to certain conditions, $\tilde{\boldsymbol{\varepsilon}}_{2n}^{(0)} = \mathbf{z}$). We then consider the behaviour of corresponding vectors derived from the sequence $\mathbf{s}_p = \tilde{\mathbf{s}}_p + \boldsymbol{\delta}_p$, where $\boldsymbol{\delta}_p = O(||\mathbf{y}||^2)$ ($0 \leqq p$). Finally, we use these results with $\mathbf{A} = F'(\mathbf{z})$ and

$$\mathbf{s}_{p+1} = F(\mathbf{s}_p) = \mathbf{z} + F'(\mathbf{z})(\mathbf{s}_p - \mathbf{z}) + O(||\mathbf{s}_p - \mathbf{z}||^2) \qquad (0 \leqq p)$$

to examine the behaviour of the vectors $\boldsymbol{\varepsilon}_q^{(p)}$ produced from this iterative scheme when $\mathbf{s}_0$ is near a fixed point $\mathbf{z}$ and, in particular, to show that repeated use of the vector $\boldsymbol{\varepsilon}_{2n}^{(0)}$ in place of $\mathbf{s}_0$ results in a quadratically convergent process for determining the fixed point in question. *In the sequel, let $Q_m(\mathbf{A}) \subset \mathbf{R}^n$ be the set of vectors $\mathbf{x}$ for which m is the degree of the minimal polynomial of* $\mathbf{A}$.

LEMMA 1. *For a given* $\mathbf{z}$, *let $\tilde{\boldsymbol{\varepsilon}}_q^{(p)}$ be the vectors obtained by means of the ϵ-algorithm from the sequence $\{\tilde{\mathbf{s}}_p; \tilde{\mathbf{s}}_p = \mathbf{z} + \mathbf{A}^p\mathbf{y}\}_{0 \leq p}$. If there is a neighbourhood U of* $\mathbf{0}$ *such that all $\tilde{\boldsymbol{\varepsilon}}_q^{(p)}$ with $p + q \leqq 2m$ exist for all $\mathbf{y} \in U \cap Q_m(\mathbf{A})$, then*

$$\tilde{\boldsymbol{\varepsilon}}_q^{(p)} = \mathbf{z} + O(||\mathbf{y}||), \qquad q \textit{ even},$$

$$\tilde{\boldsymbol{\varepsilon}}_q^{(p)} = O(||\mathbf{y}||^{-1}), \qquad q \textit{ odd},$$

for $\mathbf{y} \in Q_m(\mathbf{A})$ and $p + q \leqq 2m$.

Proof. Let $m > 0$, $p \leqq 2m - q$, and $\Delta_p\tilde{\boldsymbol{\varepsilon}}_q^{(p)} = \tilde{\boldsymbol{\varepsilon}}_q^{(p+1)} - \tilde{\boldsymbol{\varepsilon}}_q^{(p)}$. For $q = 1$, we get $\Delta_p\tilde{\boldsymbol{\varepsilon}}_0^{(p)} = \mathbf{A}^p(\mathbf{A} - \mathbf{I})\mathbf{y} = \mathbf{B}_p\mathbf{y}$, and $\mathbf{B}_p\mathbf{y} \neq \mathbf{0}$ for $\mathbf{y} \in Q_m(\mathbf{A})$, by assumption. Hence,

$$\begin{aligned} ||\tilde{\boldsymbol{\varepsilon}}_1^{(p)}|| &= ||(\mathbf{y}^*\mathbf{B}_p^*\mathbf{B}_p\mathbf{y})^{-1}\mathbf{B}_p\mathbf{y}|| \\ &= \frac{1}{||\mathbf{y}||}\,\frac{\mathbf{y}^*\mathbf{y}}{\mathbf{y}^*\mathbf{B}_p^*\mathbf{B}_p\mathbf{y}}\,\frac{1}{||\mathbf{y}||}\,||\mathbf{B}_p\mathbf{y}|| \leqq \frac{1}{||\mathbf{y}||}\,\frac{||\mathbf{B}_p||}{\lambda_{\min}}, \end{aligned}$$

where $0 < \lambda_{\min}$ is the smallest eigenvalue of $\mathbf{B}_p^*\mathbf{B}_p$. Let now $k \in \mathbf{N}$, $k < m$, $\mathbf{y} \in Q_m(\mathbf{A})$, and let the statement be true for all $q \leqq 2k$. By assumption, we have $\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k}^{(p)} = O(||\mathbf{y}||) \neq \mathbf{0}$, thus

$$(\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k}^{(p)})^*(\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k}^{(p)}) = O\{||\mathbf{y}||^2\},$$

$$\begin{aligned} \tilde{\boldsymbol{\varepsilon}}_{2k+1}^{(p)} &= \tilde{\boldsymbol{\varepsilon}}_{2k-1}^{(p+1)} + [(\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k}^{(p)})^*(\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k}^{(p)})]^{-1}\,\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k}^{(p)} \\ &= O(||\mathbf{y}||^{-1}) + O\{||\mathbf{y}||^{-2}\}O(||\mathbf{y}||) = O(||\mathbf{y}||^{-1}). \end{aligned}$$

$\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k+1}^{(p)} \neq \mathbf{0}$, since, by assumption, all $\tilde{\boldsymbol{\varepsilon}}_q^{(p)}$ which contribute to $\tilde{\boldsymbol{\varepsilon}}_{2m}^{(0)}$ exist. Therefore,

$$(\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k+1}^{(p)})^*(\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k+1}^{(p)}) = O\{||\mathbf{y}||^{-2}\},$$

and

$$\begin{aligned} \tilde{\boldsymbol{\varepsilon}}_{2k+2}^{(p)} &= \tilde{\boldsymbol{\varepsilon}}_{2k}^{(p+1)} + [(\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k+1}^{(p)})^*(\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k+1}^{(p)})]^{-1}\,\Delta_p\tilde{\boldsymbol{\varepsilon}}_{2k+1}^{(p)} \\ &= \mathbf{z} + O(||\mathbf{y}||) + O\{||\mathbf{y}||^2\}O(||\mathbf{y}||^{-1}) = \mathbf{z} + O(||\mathbf{y}||), \end{aligned}$$

and the assertion of the lemma follows by induction.

LEMMA 2. *Let $\{\delta_p\}_{0 \leq p}$ be a sequence of analytic functions $\boldsymbol{\delta}_p(\mathbf{y}) = O(||\mathbf{y}||^2)$. For a given* $\mathbf{z}$, *let $\boldsymbol{\varepsilon}_q^{(p)}$ be the vectors obtained by means of the ϵ-algorithm from the sequence*

$\{s_p;\ s_p = z + A^p y + \delta_p(y)\}_{0\le p}$. *If there is a neighbourhood U of* 0 *such that all* $\varepsilon_q^{(p)}$, $\bar{\varepsilon}_q^{(p)}$ *with $p + q \leqq 2m$ exist for all* $y \in U \cap Q_m(A)$, *then*

$$\varepsilon_q^{(p)} = \bar{\varepsilon}_q^{(p)} + O(\|y\|^2), \qquad q \text{ even},$$

$$\varepsilon_q^{(p)} = \bar{\varepsilon}_q^{(p)} + O(1), \qquad q \text{ odd},$$

for $y \in Q_m(A)$ *and* $p + q \leqq 2m$.

Proof. Let $m > 0$ and $p \leqq 2m - q$. For $q = 1$, we have $\Delta_p\varepsilon_0^{(p)} = \Delta_p\bar{\varepsilon}_0^{(p)} + O(\|y\|^2) \neq 0$ and $\Delta_p\bar{\varepsilon}_0^{(p)} \neq 0$ for $y \in Q_m(A)$, by assumption. Then

$$\begin{aligned}(\Delta_p\varepsilon_0^{(p)})^*(\Delta_p\varepsilon_0^{(p)}) &= (\Delta_p\bar{\varepsilon}_0^{(p)})^*(\Delta_p\bar{\varepsilon}_0^{(p)}) + 2(\Delta_p\bar{\varepsilon}_0^{(p)})^*O(\|y\|^2) + O\{\|y\|^4\}\\ &= (\Delta_p\bar{\varepsilon}_0^{(p)})^*(\Delta_p\bar{\varepsilon}_0^{(p)})\left[1 + 2\frac{(\Delta_p\bar{\varepsilon}_0^{(p)})^*O(\|y\|^2)}{(\Delta_p\bar{\varepsilon}_0^{(p)})^*(\Delta_p\bar{\varepsilon}_0^{(p)})} + \frac{O\{\|y\|^4\}}{(\Delta_p\bar{\varepsilon}_0^{(p)})^*(\Delta_p\bar{\varepsilon}_0^{(p)})}\right].\end{aligned}$$

$\Delta_p\bar{\varepsilon}_0^{(p)} = O(\|y\|)$ and hence,

$$(\Delta_p\varepsilon_0^{(p)})^*(\Delta_p\varepsilon_0^{(p)}) = (\Delta_p\bar{\varepsilon}_0^{(p)})^*(\Delta_p\bar{\varepsilon}_0^{(p)})(1 + O\{\|y\|\}).$$

Since $\Delta_p\varepsilon_0^{(p)}$ is an analytic function, we get

$$[(\Delta_p\varepsilon_0^{(p)})^*(\Delta_p\varepsilon_0^{(p)})]^{-1} = [(\Delta_p\bar{\varepsilon}_0^{(p)})^*(\Delta_p\bar{\varepsilon}_0^{(p)})]^{-1}[1 + O\{\|y\|\}]$$

and

$$\begin{aligned}\varepsilon_1^{(p)} &= \bar{\varepsilon}_1^{(p)} + [(\Delta_p\bar{\varepsilon}_0^{(p)})^*(\Delta_p\bar{\varepsilon}_0^{(p)})]^{-1}O\{\|y\|\}\Delta_p\bar{\varepsilon}_0^{(p)}\\ &\quad + [(\Delta_p\bar{\varepsilon}_0^{(p)})^*(\Delta_p\bar{\varepsilon}_0^{(p)})]^{-1}[1 + O\{\|y\|\}]O(\|y\|^2)\\ &= \bar{\varepsilon}_1^{(p)} + O(1).\end{aligned}$$

Let now $k \in \mathbf{N}$, $k < m$, $y \in Q_m(A)$, and let the statement be true for all $q \leqq 2k$. By assumption, we have $\Delta_p\varepsilon_{2k}^{(p)} = \Delta_p\bar{\varepsilon}_{2k}^{(p)} + O(\|y\|^2) \neq 0$ and $\Delta_p\bar{\varepsilon}_{2k}^{(p)} \neq 0$. According to the proof for $q = 1$, we get, by use of Lemma 1,

$$[(\Delta_p\varepsilon_{2k}^{(p)})^*(\Delta_p\varepsilon_{2k}^{(p)})]^{-1}\,\Delta_p\varepsilon_{2k}^{(p)} = [(\Delta_p\bar{\varepsilon}_{2k}^{(p)})^*(\Delta_p\bar{\varepsilon}_{2k}^{(p)})]^{-1}\,\Delta_p\bar{\varepsilon}_{2k}^{(p)} + O(1)$$

and hence,

$$\varepsilon_{2k+1}^{(p)} = \bar{\varepsilon}_{2k+1}^{(p)} + O(1).$$

$\Delta_p\varepsilon_{2k+1}^{(p)} = \Delta_p\bar{\varepsilon}_{2k+1}^{(p)} + O(1)$ and $\Delta_p\bar{\varepsilon}_{2k+1}^{(p)}$ are equally supposed to be different from zero and, therefore, we get, by use of Lemma 1,

$$\begin{aligned}&(\Delta_p\varepsilon_{2k+1}^{(p)})^*(\Delta_p\varepsilon_{2k+1}^{(p)})\\ &\quad = (\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})^*(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})\left[1 + 2\frac{(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})^*O(1)}{(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})^*(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})} + \frac{O\{1\}}{(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})^*(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})}\right]\\ &\quad = (\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})^*(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})(1 + O\{\|y\|\}).\end{aligned}$$

$$\begin{aligned}\varepsilon_{2k+2}^{(p)} &= \bar{\varepsilon}_{2k}^{(p+1)} + O(\|y\|^2)\\ &\quad + [(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})^*(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})]^{-1}[1 + O\{\|y\|\}][\Delta_p\bar{\varepsilon}_{2k+1}^{(p)} + O(1)]\\ &= \bar{\varepsilon}_{2k+2}^{(p+1)} + O(\|y\|^2) + [(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})^*(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})]^{-1}O\{\|y\|\}\,\Delta_p\bar{\varepsilon}_{2k+1}^{(p)}\\ &\quad + [(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})^*(\Delta_p\bar{\varepsilon}_{2k+1}^{(p)})]^{-1}[1 + O\{\|y\|\}]O(1)\\ &= \bar{\varepsilon}_{2k+2}^{(p)} + O(\|y\|^2).\end{aligned}$$

In conclusion, we have the following result:

THEOREM 2. *Let* $F: \mathbf{R}^n \supset D \to \mathbf{R}^n$ *be an analytic function with fixed point* $\mathbf{z} \in \mathring{D}$ *and let* $Q_m(F'(\mathbf{z})) \subset \mathbf{R}^n$ *be the set of vectors* $\mathbf{x}$ *for which m is the degree of the minimal polynomial of* $F'(\mathbf{z})$. *Further, let* $\boldsymbol{\varepsilon}_q^{(p)}$ *and* $\tilde{\boldsymbol{\varepsilon}}_q^{(p)}$ *be the vectors obtained by means of the* ϵ*-algorithm from the sequences*

$$\{\mathbf{s}_p;\ \mathbf{s}_{p+1} = F(\mathbf{s}_p)\}_{0 \leq p}, \quad \text{and} \quad \{\tilde{\mathbf{s}}_p;\ \tilde{\mathbf{s}}_p = \mathbf{z} + (F'(\mathbf{z}))^p\,(\mathbf{s}_0 - \mathbf{z})\}_{0 \leq p},$$

respectively. Assume that

(i) 1 *is not an eigenvalue of* $F'(\mathbf{z})$,

(ii) *the vectors* $\boldsymbol{\varepsilon}_q^{(p)}$, $\tilde{\boldsymbol{\varepsilon}}_q^{(p)}$, $p + q \leqq 2m$, *exist for all* $\mathbf{s}_0$ *sufficiently close to* $\mathbf{z}$ *with* $\mathbf{s}_0 - \mathbf{z} \in Q_m(F'(\mathbf{z}))$.

Set

$$\boldsymbol{\varepsilon}_{2m}^{(0)} = G(\mathbf{s}_0, \cdots, \mathbf{s}_{2m}) = H_F(\mathbf{s}_0), \tag{9}$$

then the computational procedure

$$\mathbf{x}_{i+1} = H_F(\mathbf{x}_i) \qquad (0 \leqq i)$$

is, for $\mathbf{x}_0$ *sufficiently close to* $\mathbf{z}$ *and* $\mathbf{x}_0 - \mathbf{z} \in Q_m(F'(\mathbf{z}))$, *a quadratically convergent iterative method for the computation of* $\mathbf{z}$.

Proof. By the corollary and Lemma 2, we have

$$H_F(\mathbf{x}_0) = \boldsymbol{\varepsilon}_{2m}^{(0)} = \mathbf{z} + O(\|\mathbf{x}_0 - \mathbf{z}\|^2)$$

for $\mathbf{x}_0 - \mathbf{z} \in Q_m(F'(\mathbf{z}))$.

5. A Modification of the Method. When a system of equations $\mathbf{x} = F(\mathbf{x})$ of order n is to be solved by the ϵ-algorithm, the way of doing this is normally to put $m = n$. Then, we need, for each step of iteration, $4n^3 + 2n^2$ multiplications, $2n^2 + n$ divisions, $6n^3 - n^2$ additions/subtractions and the computation of $\mathbf{s}_p = F(\mathbf{s}_{p-1})$ for $1 \leqq p \leqq 2n$. The computation of the vectors $\mathbf{s}_p$ rather quickly produces a characteristic overflow if the eigenvalues of the Jacobian matrix $F'(\mathbf{x})$ are greater in absolute value than unity near the fixed point $\mathbf{z}$. This disadvantage can possibly be eliminated by replacing the Picard sequence $\mathbf{s}_{p+1} = F(\mathbf{s}_p)$ by

$$\mathbf{s}_{p+1} = F_\alpha(\mathbf{s}_p) = (1 - \alpha)\mathbf{s}_{p-1} + \alpha F(\mathbf{s}_p) \qquad (0 \leqq p)$$

with a suitable α, $0 < \alpha < 1$; in this way, the rate of growth of the components of the vectors $\mathbf{s}_p$ is reduced. If we have, for example, $\rho(F'(\mathbf{z})) = 2$ for the spectral radius ρ of $F'(\mathbf{z})$, we get $\rho(F_\alpha'(\mathbf{z})) = 3/2$ for $\alpha = 1/2$. Those eigenvalues λ of $F'(\mathbf{z})$ for which $|\lambda| < 1$ are thereby increased, but they remain smaller than one in absolute value. Apart from this, convergence is slow if the eigenvalues of $F'(\mathbf{x})$ approach one near $\mathbf{z}$.

The rounding errors affect the computation severely. Perhaps, it is possible that the numerical properties can be improved if a modification proposed by Wynn [20] is applied. If the eigenvalues λ of $F'(\mathbf{x})$ with $|\lambda| < 1$ predominate, we can indicate a modification of the method, by giving up the (theoretic) quadratic convergence, which considerably reduces the amount of work. To achieve this, we replace $2m$ by $2[(m + 1)/2]$ in (9) and obtain for the basic formula of the algorithm

$$\boldsymbol{\varepsilon}_n^{(0)} = G(\mathbf{s}_0, \cdots, \mathbf{s}_n) = H_F^*(\mathbf{s}_0) \tag{9*}$$

in the case $m = n$ even. We need now, per step of iteration, only

$$(n^3 + 8n^2 - 4n)/8$$

multiplications/divisions,

$$(6n^3 - 2n^2)/8$$

additions/subtractions and the computation of $\mathbf{s}_p = F(\mathbf{s}_{p-1})$ for $1 \leqq p \leqq n$.

6. Numerical Examples. Let $F\colon \mathbf{R}^4 \to \mathbf{R}^4$. In order to illustrate the method of Theorem 2 and its modifications, we consider some systems of quadratic equations $\mathbf{x} = F(\mathbf{x})$ with fixed point $\mathbf{z} = (1, 1, 1, 1)^T$:

$$F(\mathbf{x}) = \mathbf{z} + F'(\mathbf{z})(\mathbf{x} - \mathbf{z}) + \tfrac{1}{2}F''(\mathbf{z})(\mathbf{x} - \mathbf{z})^{(2)}. \tag{10}$$

For the Taylor series (10), we write briefly

$$F(\mathbf{x}) = \mathbf{z} + \mathbf{A}(\mathbf{x} - \mathbf{z}) + Q(\mathbf{x} - \mathbf{z}) \tag{11}$$

and choose for $\mathbf{A}$ (linear) and Q various mappings. The fixed point $\mathbf{z}$ of the systems given in that manner is computed by means of single-precision arithmetic with ten decimal digits. In detail, let $P^{(i)}(\mathbf{x}) = (p_1^{(i)}(\mathbf{x}), \cdots, p_4^{(i)}(\mathbf{x}))^T$ and

$$\begin{aligned}
p_1^{(1)}(\mathbf{x}) &= -(x_1^2 + x_1x_4)/2, & p_1^{(2)}(\mathbf{x}) &= -x_1^2/4,\\
p_2^{(1)}(\mathbf{x}) &= -x_2^2/2, & p_2^{(2)}(\mathbf{x}) &= -x_2^2/4,\\
p_3^{(1)}(\mathbf{x}) &= -x_3^2/2, & p_3^{(2)}(\mathbf{x}) &= -x_3^2/4,\\
p_4^{(1)}(\mathbf{x}) &= -(x_4x_1 + x_4^2)/2, & p_4^{(2)}(\mathbf{x}) &= -x_4^2/4.
\end{aligned}$$

Furthermore, let

$$\begin{aligned}
\mathbf{D}_1 &= (0.9, 0.8, 0.7, 0.6),\\
\mathbf{D}_2 &= (1.5, 0.8, 0.7, 0.6),\\
\mathbf{D}_3 &= (2.0, 0.8, 0.7, 0.6)
\end{aligned}$$

be diagonal matrices and

$$\mathbf{U}_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1 \end{pmatrix}, \qquad \mathbf{U}_2 = \begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 2 & 3 & 4\\ 1 & 3 & 6 & 10\\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

We remark that $\mathbf{U}_1$ is orthogonal, whereas $\mathbf{U}_2$ is the ill-conditioned Pascal matrix of order four having an integer-valued inverse. It should be pointed out that

$$\left.\left(\frac{\partial P^{(i)}(\mathbf{x} - \mathbf{z})}{\partial \mathbf{x}}\right)\right|_{\mathbf{x}=\mathbf{z}} = 0 \quad (\text{Matrix}) \qquad (j = 1, 2);$$

hence, choosing $Q = P^{(i)}$ in eq. (11), we get, indeed, $F'(\mathbf{z}) = \mathbf{A}$. Now, if $\mathbf{A} = \mathbf{U}_m\mathbf{D}_l\mathbf{U}_m^{-1}$ ($l = 1, 2, 3$; $m = 1, 2$), then $\mathbf{D}_l$ is the matrix of eigenvalues and $\mathbf{U}_m$ is the matrix of eigenvectors of $F'(\mathbf{z})$.

In Examples I–VI, $\mathbf{z}$ is computed by the method proposed in Theorem 2.

I	2.0	$1.2\cdot10^{-2}$	$1.0\cdot10^{-5}$					
II	$7.4\cdot10^{-1}$	$6.6\cdot10^{-1}$	$4.5\cdot10^{-1}$	$1.4\cdot10^{-1}$	$6.8\cdot10^{-2}$	$8.4\cdot10^{-3}$	$7.5\cdot10^{-5}$	$2.5\cdot10^{-8}$
III	$6.0\cdot10^{-1}$	$5.4\cdot10^{-5}$						
IV	0.9	$8.2\cdot10^{-2}$	$2.7\cdot10^{-6}$					
V	2.0	$9.9\cdot10^{-1}$	$2.8\cdot10^{-6}$					
VI	1.8	$1.5\cdot10^{-1}$	$3.7\cdot10^{-2}$	$1.1\cdot10^{-2}$	$4.4\cdot10^{-3}$	$2.2\cdot10^{-8}$	$1.1\cdot10^{-3}$	$3.6\cdot10^{-4}$
VII	1.9	$8.6\cdot10^{-2}$	$5.5\cdot10^{-3}$	$5.0\cdot10^{-5}$	$5.2\cdot10^{-8}$			
VIII	$7.9\cdot10^{-1}$	$6.5\cdot10^{-1}$	$4.3\cdot10^{-1}$	$1.3\cdot10^{-1}$	$4.6\cdot10^{-2}$	$1.6\cdot10^{-3}$	$5.1\cdot10^{-5}$	$4.8\cdot10^{-8}$
IX	$6.0\cdot10^{-1}$	$6.0\cdot10^{-3}$	$4.0\cdot10^{-6}$					
X	$8.9\cdot10^{-1}$	$1.1\cdot10^{-1}$	$3.2\cdot10^{-4}$	$2.4\cdot10^{-7}$				
XI	$3.8\cdot10^{-1}$	$5.1\cdot10^{-1}$	$1.1\cdot10^{-1}$	$3.8\cdot10^{-4}$	$1.1\cdot10^{-6}$			
XII	1.5	$3.2\cdot10^{-1}$	$1.1\cdot10^{-1}$	$4.6\cdot10^{-2}$	$2.1\cdot10^{-2}$	$1.0\cdot10^{-2}$	$9.5\cdot10^{-4}$	$6.5\cdot10^{-4}$

Example I: $F'(\mathbf{z}) = \mathbf{U}_1\mathbf{D}_1\mathbf{U}_1^{-1}$, $Q = P^{(1)}$, initial vector $\mathbf{x}_0 = 2\mathbf{z}$;
Example II: $F'(\mathbf{z}) = \mathbf{U}_1\mathbf{D}_2\mathbf{U}_1^{-1}$, $Q = P^{(1)}$, $\mathbf{x}_0 = \mathbf{0}$;
Example III: as Example II but using $\mathbf{x}_0 = 2\mathbf{z}$;
Example IV: $F'(\mathbf{z}) = \mathbf{U}_2\mathbf{D}_2\mathbf{U}_2^{-1}$, $Q = P^{(2)}$, $\mathbf{x}_0 = 0.5\mathbf{z}$;
Example V: as Example IV but using $\mathbf{x}_0 = 1.5\mathbf{z}$;
Example VI: $F'(\mathbf{z}) = \mathbf{U}_1\mathbf{D}_3\mathbf{U}_1^{-1}$, $Q = P^{(1)}$, $\mathbf{x}_0 = 2\mathbf{z}$, using the modified Picard sequence $\mathbf{s}_{p+1} = F_\alpha(\mathbf{s}_p)$ with $\alpha = 1/2$.

The Examples VII–XII are the same as Examples I–VI, respectively, but $\mathbf{z}$ is computed using formula (9*) instead of (9).

The above table contains in column i ($1 \leqq i \leqq 8$) the values $\|\mathbf{x}_i - \mathbf{x}_{i-1}\|$ (compare Theorem 2) with rounded mantissae; values for which $\|\mathbf{z} - \mathbf{x}_i\| < 5.0 \cdot 10^{-9}$ (the process has then terminated) are omitted. Generally speaking, we have found that the algorithm produces better results if the Jacobian matrix of the given system $\mathbf{x} = F(\mathbf{x})$ is symmetric. Finally, it should be mentioned that it seems to be impossible at the moment to say more about the error than that it is of quadratic order.

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