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# Adaptive Control of Redundant Multiple Robots in Cooperative Motion

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**Abstract.** A redundant robot has more degrees of freedom than what is needed to uniquely position the robot end-effector. In practical applications the extra degrees of freedom increase the orientation and reach of the robot. Also the load carrying capacity of a single robot can be increased by cooperative manipulation of the load by two or more robots. In this paper, we develop an adaptive control scheme for kinematically redundant multiple robots in cooperative motion.

In a usual robotic task, only the end-effector position trajectory is specified. The joint position trajectory will therefore be unknown for a redundant multi-robot system and it must be selected from a self-motion manifold for a specified end-effector or load motion. In this paper, it is shown that the adaptive control of cooperative multiple redundant robots can be addressed as a reference velocity tracking problem in the joint space. A stable adaptive velocity control law is derived. This controller ensures the bounded estimation of the unknown dynamic parameters of the robots and the load, the exponential convergence to zero of the velocity tracking errors, and the boundedness of the internal forces. The individual robot joint motions are shown to be stable by decomposing the joint coordinates into two variables, one which is homeomorphic to the load coordinates, the other to the coordinates of the self-motion manifold. The dynamics on the self-motion manifold are directly shown to be related to the concept of zero-dynamics. It is shown that if the reference joint trajectory is selected to optimize a certain type of objective functions, then stable dynamics on the self-motion manifold result. The overall stability of the joint positions is established from the stability of two cascaded dynamic systems involving the two decomposed coordinates.

**Key words:** adaptive control, redundant multiple robots, position/force control.

## 1. Introduction

Recently considerable amount of research has focused on the problem of cooperative control and coordination of multiple robots. Interest in multi-robot systems has arisen because several tasks require the use of two or more robots. Examples of such tasks include the joining and securing of large pipes for the construction of space structures, picking up and carrying heavy loads, and grasping odd shaped loads. Cooperative robots may be used in hazardous or unsafe environments such as in space, in deep waters and in radioactive environments. By using more than one robot, the manipulation capability and the workspace of the system are further increased. However multi-robot systems are more difficult to control than single robots. Additional problems arise as the parameters of the robots and the manipulated load may not be known exactly.

Several control schemes have been proposed for cooperative multiple robots with rigid joints manipulating a common load. Zheng and Luh [37] considered the kinematic and dynamic models of the multi-robot system and developed an inverse dynamics scheme for load position control. Hsu et al. [11] developed a control algorithm for the coordinated manipulation of multi-fingered robot hand. Tarn et al. [30] developed a robust nonlinear control scheme using nonlinear transformation techniques. Yun et al. [36] used exact linearization and output decoupling techniques to control multiple robots. Yoshikawa and Zheng [35] developed linearizing tracking control laws for multiple robots; experiments were also reported. Few adaptive control schemes for cooperative robots manipulating a common load have been proposed. Walker et al. [33] developed an algorithm for the control of two robots handling a common load of unknown mass. Zribi and Ahmad [38] proposed a robust adaptive controller for multi-robot systems manipulating a rigid object when bounded disturbances are present. The problem of manipulating a load using multiple robots when the load makes contact with the environment was addressed by Hayati [8], Cole [5] and Hu and Goldenberg [12]. Ahmad and Guo [1] addressed the problem of controlling multiple flexible joint robots with linear dynamics.

There are very few papers in the area of control of multiple redundant robots, these include the paper by Tarn and Bejczy [29] which addressed the zero dynamics issue. The paper by Tao and Luh [28] also addressed the problem of multiple redundant robot control. The reason that there are such a few works in the multiple redundant robot control is primarily because non-redundant robot control schemes cannot be easily extended to control redundant robot systems. This is the case because a redundant robot has more joints than what is required to position the end-effector. Usually the end-effector trajectory is known and thus the joint trajectory cannot be found uniquely. In fact, for a fixed end-effector position there is a self-motion manifold on which joint motions could occur without affecting the end-effector position. In Figure 1, we show a planar redundant robot with three prismatic axes, we see that if the end-effector is stationary the joints may move in a straight line in the joint space without affecting the end-effector. An arbitrary joint trajectory which ensures end-effector position cannot be used as this may not result in stable joint motions on the self-motion manifold and would therefore affect the overall stability of the system. Therefore one cannot extend strategies designed for non-redundant robots to multiple redundant robots. We should note here that the extra joints are extremely useful in real applications as they can be used to configure the manipulator posture, to avoid obstacles in the workspace or to avoid joint singularities.

Initial interest in the control of redundant robots started with the work of Whitney [34] who devised a kinematical resolved motion rate control strategy. Since

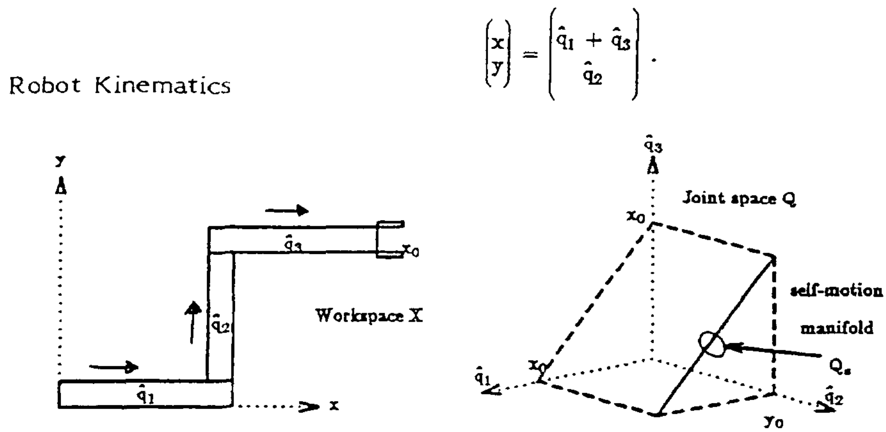


*Figure 1.* Self motion manifold for a three link prismatic joint PPP robot.

then a number of researchers have addressed the joint coordination and control of redundant robots (see Nenchev [20] for a review of those developments). The tutorial review by Siciliano [24] and the tutorial workshop report on the theory and applications of redundant robots at the 1989 IEEE robotics and Automation conference [2] covered some developments in this area (see also [3, 10, 17, 19]). In the area of redundant robot adaptive control, Seraji [23] presented an approach based on the model reference adaptive control theory. He resolved the redundancy problem by adding additional task dependent kinematic constraints to the end-effector kinematics which effectively ensured that the joint solutions were unique. Niemeyer and Slotine [21] applied sliding mode adaptive control to redundant manipulators. They used the passivity principle to prove the stability of the system; they also performed some experiments to demonstrate their control law. Colbaugh et al. [4] proposed an adaptive inverse kinematics algorithm that did not require the knowledge of the kinematics of the robots. However their algorithm required persistent excitation conditions; also their algorithm did not consider the dynamics of the robot. Luo et al. [18] developed an adaptive control law for redundant robots making use of weighted scaling functions and the concept of zero dynamics to show that both the joint motions on the self-motion manifold and the end-effector motions would be stable when their control scheme is used.

To be able to implement control laws for multi-robot systems, one needs to investigate the real-time computational aspects of these systems. Kircanski et al. [15] addressed the problem of numerical computation for multiple robot systems. They introduced different methods of carrying out numerical computations of the dynamic and kinematic models of the system; they showed that the customization of the models results in a considerable savings in the numerical computations.

In this paper, we address the problem of controlling redundant multiple robots manipulating a load cooperatively. We assume that the load mass/inertial parameters and the robots mass/inertial parameters are unknown. We first state the dynamic models of the robots and the load, and give several properties of the multi-robot system. Next the redundancy resolution problem is discussed, and a model for adaptive resolution of the redundancy is established. A controller that leads to the exponential tracking of the velocity tracking errors and the boundedness of the internal forces is then derived. The boundedness of the joint motions and the control torque are proved next. The conclusion can be found in the final section of the paper.

## 2. Multi-Robot System Model

### 2.1. Dynamic Model

The general dynamic model for a cooperative multi-robot system has been investigated thoroughly in the literature, and is also described below for completeness. In Figure 2, we depict the organization of multiple robots grasping a common load which is to be manipulated cooperatively. We first state a few assumptions related to the grasping of the load and the reachability of the trajectory.

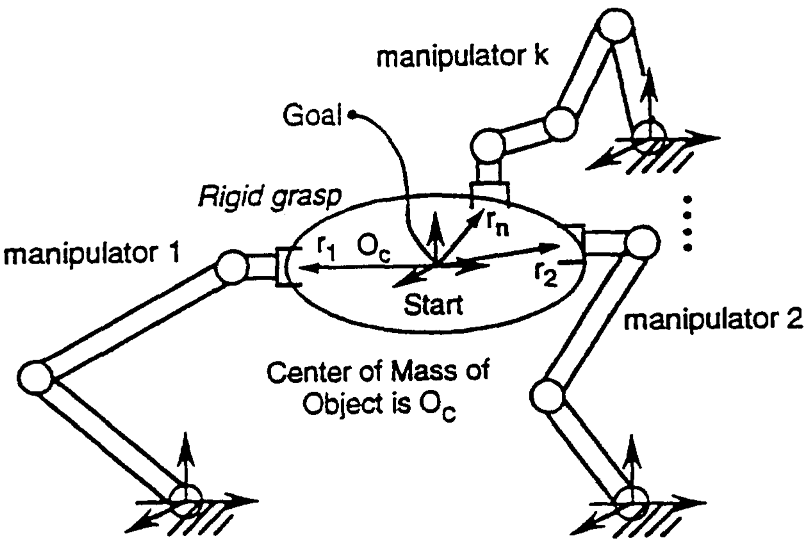


*Figure 2.* Multi-robot system organization, with desired trajectory.

ASSUMPTIONS

(A1) The manipulators are rigidly grasping the load, (i.e., there is no motion between the contact point of the load and the end-effectors).

(A2) The desired trajectory is reachable and the end-effectors can be positioned at the workspace positions without exceeding any joint motion limits.

The dynamic equation of the $i$th manipulator in cooperative manipulation can be written as,

$$D_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + G_i(q_i) + J_{e_i}(q_i)^T F_{e_i} = \tau_i \quad i = 1, \ldots, k \tag{1}$$

where, $q_i \in R^{n_i}$ is the vector of joint displacements, and $n_i > 6$ is the number of joints of the $i$th robot. The inertia matrix of the $i$th robot is $D_i(q_i) \in R^{n_i \times n_i}$; this is a symmetric positive definite matrix. The matrix of centrifugal and Coriolis forces is $C_i(q_i, \dot{q}_i) \in R^{n_i \times n_i}$. The vector of gravity forces is $G_i(q_i) \in R^{n_i}$; the manipulator Jacobian is $J_{e_i}(q_i) \in R^{6 \times n_i}$. The control torque for the $i$th robot is $\tau_i \in R^{n_i}$. We will define the total number of joints of the $k$ robots as $n$, $n = \sum_{i=1}^{k} n_i$.

The forces/moments applied by the $i$th manipulator on the object at the point of contact are denoted as $F_{e_i} \in R^6; F_{e_i}$ can be written in terms of the contact forces $f_{e_i} \in R^3$ and the contact moments $\eta_{e_i} \in R^3$, (here, six represents the dimension of the Cartesian work space), such that,

$$F_{e_i} = \left[ f_{e_i}^T \; \eta_{e_i}^T \right]^T \quad i = 1, \ldots, k. \tag{2}$$

Now we will group the dynamics of the $k$-robot system to get,

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G_r(q) + J_e^T(q) F_e = \tau. \tag{3}$$

Here, $D \in R^{n \times n}$ is a block diagonal matrix whose diagonal elements are $D_i$. $C \in R^{n \times n}$ is a block diagonal matrix whose diagonal elements are $C_i$ and $J_e \in R^{6k \times n}$ is a block diagonal matrix whose diagonal elements are $J_{e_i}$. Also define the following vectors as,

$$G_r(q, \dot{q}) = \begin{bmatrix} G_1 \\ \cdot \\ \cdot \\ \cdot \\ G_k \end{bmatrix}, \quad q = \begin{bmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_k \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_1 \\ \cdot \\ \cdot \\ \cdot \\ \tau_k \end{bmatrix} \quad \text{and} \quad F_e = \begin{bmatrix} F_{e_1} \\ \cdot \\ \cdot \\ \cdot \\ F_{e_k} \end{bmatrix}. \tag{4}$$

Note that we will drop functional dependencies whenever possible.

Because the object is rigidly grasped, the equations of motion of the object are obtained from the Newton–Euler mechanics as,

$$M_1 \ddot{x}_p + M_1 g_l = \sum_{i=1}^{k} f_{e_i} \tag{5}$$

$$I\dot{\omega} + \omega \times (I\omega) = \sum_{i=1}^{k}(\eta_{e_i} + r_i \times f_{e_i}) \tag{6}$$

where, the position of the center of mass of the object, expressed in the world coordinate frame, is $x_p \in R^3$. The rotational velocity of the the object, expressed in the world coordinate frame, is $\omega \in R^3$. The vector of gravitational forces acting on the object is $M_1 g_l \in R^3$. The mass matrix $M_l \in R^{3\times 3}$ is a diagonal matrix whose diagonal elements are the mass of the load. The matrix $I \in R^{3\times 3}$ is the inertia matrix of the load. The vector $r_i = [r_{ix}, r_{iy}, r_{iz}]^T \in R^3$ represents the translational displacements from the center of mass of the object to the contact point of the object and the $i$th manipulator.

If we let $\dot{x} = [\dot{x}_p^T \; \omega^T]^T$, then the motion of the object expressed by Equations (5) and (6) can be written as,

$$M\ddot{x} + N\dot{x} + G_l = GF_e = F_o. \tag{7}$$

The matrix $M$ and the vectors $N\dot{x}$ and $G_l$ are defined such that,

$$M = \begin{bmatrix} M_l & 0 \\ 0 & I \end{bmatrix}, \qquad N\dot{x} = \begin{bmatrix} 0 \\ \omega \times (I\omega) \end{bmatrix} \quad \text{and} \quad G_l = \begin{bmatrix} M_1 g_l \\ 0 \end{bmatrix}. \tag{8}$$

Note that $G \in R^{6\times 6k}$ is the grasp matrix, and it is defined as,

$$G = [T_1 \; T_2 \; \ldots \; T_k]. \tag{9}$$

The matrix $T_i \in R^{6\times 6}$ is such,

$$T_i = \begin{bmatrix} I_3 & 0 \\ \Omega_i(r_i) & I_3 \end{bmatrix} \quad \text{and} \quad \Omega_i(r_i) = \begin{bmatrix} 0 & -r_{iz} & r_{iy} \\ r_{iz} & 0 & -r_{ix} \\ -r_{iy} & r_{ix} & 0 \end{bmatrix}.$$

The matrix $I_3$ is the 3 by 3 identity matrix.

### 2.2. Kinematic Model

We are interested in controlling the manipulators in some predefined Cartesian task space such that,

$$x_{ei} = K_{e_i}(q_i) \quad i = 1, \ldots, k \tag{10}$$

where, $K_{e_i}(\cdot)$: $R^{n_i} \to R^6$ is the transformation from the joint position space of $q_i$ to the task space containing $x_{e_i}$, and $x_{ei} \in R^6$ is the position and orientation of the point of contact of the $i$th manipulator with the load. If we differentiate Equation (10) with respect to time, and if we define $J_{e_i}(q_i)$ to be the differential map from the $q_i$ space to the $x_{e_i}$ space ($J_{e_i} = \partial K_{e_i}/\partial q_i$), then we can write,

$$\dot{x}_{ei} = J_{e_i}(q_i)\dot{q}_i \quad i = 1, \ldots, k. \tag{11}$$

If these equations are stacked into a single vector by forming the $J_{e_i}$'s into a block diagonal matrix, and concatenating the $\dot{q}_i$'s into one vector $\dot{q}$, we get,

$$v_c = J_e \dot{q} \tag{12}$$

where, $v_c = \begin{bmatrix} \dot{x}_{e1}^T & \dot{x}_{e2}^T & \dots & \dot{x}_{ek}^T \end{bmatrix}^T$ is the vector of end-effector velocities.

Using Equation (7), we can write,

$$F_o = GF_e. \tag{13}$$

Now from the duality between the forces and the velocities, we can write,

$$G^T \dot{x} = v_c \tag{14}$$

where, $\dot{x}$ is the velocity of the object. Thus for the $k$-robot system, we combine Equations (12) and (14) to get,

$$G^T \dot{x} = J_e \dot{q}. \tag{15}$$

Here, $G$ is the grasp matrix defined earlier.

### 2.3. Definition of Internal Forces and Internal Forces Error

The end-effector force of the $i$th manipulator, $F_{e_i}$, can be decomposed into two forces: the motion force and the internal grasping force. The internal grasping forces of all the robots, $F_I = [F_{Ie1}, \dots, F_{Iek}]^T \in R^{6k}$, do not cause any motion to the load. However, we must control these end-effector internal forces in order to prevent excessive compressive forces being applied to the load. We can calculate the internal force $F_I$ from Equation (7); if $F_o$ is known, then,

$$F_e = G^+ F_o + F_I \tag{16}$$

where, $G^+ = G^T (GG^T)^{-1}$ and $GG^+ = I_6$, $I_6$ is the 6 by 6 identity matrix. (For a discussion related to other choices of $G^+$ see [32]; note that other choices of $G^+$ do not affect the derivations presented in this paper). Therefore, we see that $GF_I = 0$, (i.e., the internal forces do not contribute to the motion of the load). The desired internal forces $F_{I,d} \in R^{6k}$ should satisfy the relation $GF_{I,d} = 0$. The internal force error, $e_f = F_I - F_{I,d}$, satisfies the relation $Ge_f = 0$. These properties will be used in the derivation of the control law.

### 2.4. A Few Properties of the Multi-Robot System

In the following we will state several properties which will be used in the derivation of the control law.

(P1)–(P3) PROPERTIES OF THE DYNAMICS OF THE SYSTEM

(P1) $D$ and $M$ are symmetric positive definite matrices.

(P2) $\dot{D} - 2C$ is a skew symmetric matrix, i.e., $\dot{q}^T(\dot{D} - 2C)\dot{q} = 0$. The proof of property P2 can be found in [27].

(P3) $\dot{M} - 2N$ is a skew symmetric matrix, i.e., $\dot{x}^T(\dot{M} - 2N)\dot{x} = 0$. This property can be seen from energy considerations. The total energy of the load is given by $E_L = 1/2\dot{x}^T M\dot{x} + h(x)$, where $h(x)$ is the potential energy and $G_l = \partial h(x)/\partial x$. As the power input to the load is given by, $\dot{E}_L = \dot{x}^T F_o = \dot{x}^T(M\ddot{x} + 1/2\dot{M}\dot{x} + G_l) = \dot{x}^T(M\ddot{x} + N\dot{x} + G_l)$, thus we have the property $\dot{x}^T(\dot{M} - 2N)\dot{x} = 0$.

(P4) LINEAR PARAMETERIZATION OF THE ROBOT DYNAMICS
The linearity of $D$, $C$ and $G_r$ with respect to the manipulators dynamic parameters $P_r \in R^{s_r}$ is now stated. These parameters will be estimated by the proposed adaptive scheme. The robot dynamics can be linearly parametrized [25, 27] such that,

$$Da_r + Cv_r + G_r = Y_r P_r \tag{17}$$

where, $a_r \in R^n$ and $v_r \in R^n$ are vectors which will be defined later on. We will denote $a_r$ as the "reference acceleration of the robots", and $v_r$ as the "reference velocity of the robots". The regressor matrix $Y_r(q, \dot{q}, v_r, a_r) \in R^{n \times s_r}$ represents the structure of the robots dynamics, hence its elements are combinations of the elements of the inertia matrix, centrifugal/Coriolis matrix and the gravity vector.

(P5) LINEAR PARAMETERIZATION OF THE OBJECT DYNAMICS
The load dynamics are linear with respect to the load parameter vector $P_o$,

$$Ma_o + Nv_o + G_l = Y_o P_o \tag{18}$$

where, $a_o \in R^6$ and $v_o \in R^6$ are the "reference acceleration of the load", and the "reference velocity of the load", respectively. We will denote by $P_o \in R^{s_o}$ the vector of $s_o$ load parameters which are constant for a given load. These parameters will be estimated by the proposed adaptive scheme. The regressor matrix $Y_o(x, \dot{x}, v_o, a_o) \in R^{6 \times s_o}$ represents the structure of the load dynamics.

*Remark 1.* Let $\hat{P}_r$ be the vector of the estimates of the parameters of the robots, then the error vector in the estimates of the parameters of the robots is $\tilde{P}_r = \hat{P}_r - P_r$. Similarly, we can write the parameter estimation error vector for the load as $\tilde{P}_o = \hat{P}_o - P_o$. Notice that,

$$\hat{D}a_r + \hat{C}v_r + \hat{G}_r = Y_r \hat{P}_r \tag{19}$$

where, $\hat{D}$ is the estimate of the inertia matrix, $\hat{C}$ is the estimate of the centrifugal/Coriolis matrix, and $\hat{G}_r$ is the estimate of the gravity vector. Also notice that, $\tilde{D}a_r + \tilde{C}v_r + \tilde{G}_r = Y_r \tilde{P}_r$ where $\tilde{D} = \hat{D} - D$, $\tilde{C} = \hat{C} - C$, and $\tilde{G}_r = \hat{G}_r - G_r$. Similarly we can write,

$$\hat{M}a_o + \hat{N}v_o + \hat{G}_l = Y_o P_o \tag{20}$$

where, $\hat{M}$ is the estimate of $M$, $\hat{N}$ is the estimate of $N$, and $\hat{G}_l$ is the estimate of $G_l$. Also notice that $\tilde{M}a_o + \tilde{N}v_o + \tilde{G}_l = Y_o\tilde{P}_o$ where, $\tilde{M}$, $\tilde{N}$ and $\tilde{G}_l$ are the differences between the estimates and the true values.

## 3. Redundancy Resolution Problem

### 3.1. PRELIMINARIES

Consider a kinematically redundant manipulator with the manipulated load center of mass positioned at point $x$ and the joints of the $i$th robot positioned at $q_i$. Then the differentiable kinematic mapping relating $x$ and $q_i$ is $K_i$ such that,

$$x = K_i(q_i) \quad i = 1, \dots, k \tag{21}$$

where, $x \in R^6$ is the position of the load, and $q_i \in R^{n_i}$ is the vector of joint positions of the $i$th manipulator. As $n_i > 6$, the degree of redundancy of the $i$th robot is $r_i = n_i - 6$. As a result of the joint redundancy at the end-effector point $x = x_d$, there exist a set of joint positions, $Q_N^{q_i}$, which lie on the self-motion manifold such that $Q_N^{q_i} = \{q_i \mid x = x_d = K_i(q_i)\}$ (see the example in Figure 1, the self-motion manifold is the line in the joint space). Thus, in order to find a unique joint position $q_i$, additional requirements are necessary; these requirements will be stated later. We will denote the Jacobian of the kinematic map given by Equation (21) by $J_i = T_i^{-T} J_{e_i} \in R^{6\times n_i}$; this equation relates the differential map between the load position kinematics $K_i(\cdot)$ and the $i$th end-effector kinematics $K_{e_i}(\cdot)$. The projection operator onto the null space of $J_i$ is denoted by $P_{J_i}(q_i) = I_{n_i} - J_i^+ J_i$ $(i = 1, \dots, k)$. Let all the columns of the matrix $N_{J_i}$ be the normalized bases of $\ker(J_i)$, which is the null space of $J_i$. Hence we have,

$$J_i P_{J_i} = 0 \quad \text{and} \quad \ker(J_i) = \text{span}(N_{J_i}). \tag{22}$$

The matrix $N_{J_i} \in R^{n_i\times r_i}$ has the following properties that will be used in the text,

$$\begin{aligned} &J_i N_{J_i} = 0 \in R^{6\times r_i}, \qquad N_{J_i}^T J_i^T = 0 \in R^{r_i\times 6}, \\ &N_{J_i}^T J_i^+ = 0 \in R^{r_i\times 6}, \end{aligned} \tag{23}$$

$$\begin{aligned} &N_{J_i}^T P_{J_i} = N_{J_i}^T \in R^{r_i\times n_i}, \qquad N_{J_i}^T N_{J_i} = I_{r_i} \in R^{r_i\times r_i}, \\ &N_{J_i} N_{J_i}^T = P_{J_i} \in R^{n_i\times n_i}. \end{aligned} \tag{24}$$

For any vector $\dot{q}_i \in R^{n_i}$:

$$\text{if} \quad N_{J_i}^T \dot{q}_i = 0 \in R^{r_i} \quad \text{then} \quad P_{J_i}\dot{q}_i = 0 \in R^{n_i}. \tag{25}$$

It should be noted that $\begin{bmatrix} J_i \\ N_{J_i}^T \end{bmatrix}$ is a square matrix of full rank, thus we have,

$$\begin{bmatrix} J_i \\ N_{J_i}^T \end{bmatrix}^{-1} = \begin{bmatrix} J_i^+ & N_{J_i} \end{bmatrix}. \tag{26}$$

These properties show that the pairs $(J_i,\ N_{J_i}^T)$ and $(J_i^+,\ N_{J_i})$ are orthogonal complement operator pairs.

### 3.2. Statement of the Problem of the Redundancy Resolution

The redundancy is usually resolved by the constrained optimization of a performance index $H_i$ $(i = 1, \dots, k)$ which can be used to avoid joint limits, obstacles and singularities. The problem can be formulated as follows: given a desired position $x_d$, find the joint position $q_i$ $(i = 1, \dots, k)$ such that,

$$\min \quad H_i(q_i) \quad \text{subject to} \quad x_d = K_i(q_i) \quad i = 1, \dots, k. \tag{27}$$

We can conclude from the Lagrange multiplier method that the solution of the constrained optimization problem (27) necessarily satisfies the following set of constrained differential equations:

$$P_{J_i} \nabla H_i(q_i) = 0 \quad \text{and} \quad x_d = K_i(q_i) \quad i = 1, \dots, k. \tag{28}$$

We will define the end-effector path tracking error $e$ as,

$$e = K_i(q_i) - x_d = x - x_d \quad i = 1, \dots, k. \tag{29}$$

Our goal is to resolve the "asymptotic resolution of the redundancy problem" such that as $t \to \infty$, we have,

$$e \to 0, \qquad \dot{e} \to 0, \quad \text{and} \quad P_{J_i} \nabla H_i(q_i) \to 0 \quad i = 1, \dots, k. \tag{30}$$

We want to optimize $H_i$ by appropriate joint motion on the self-motion manifold, $Q_N^{q_i}$. At the optimal point, we do not desire further motion on the self-motion manifold. Therefore the projection of the joint velocity on the self-motion manifold must be zero, and $N_{J_i}^T \dot{q}_i \to 0$ as $t \to \infty$. Thus, it is sufficient (not necessary) to write the asymptotic redundancy resolution as $t \to \infty$,

$$\begin{aligned} &\dot{e} + \gamma e \to 0, \\ &N_{J_i}^T(\dot{q}_i - \mu_i \nabla H_i) \to 0 \quad \text{with} \quad N_{J_i}^T \dot{q}_i \to 0 \quad i = 1, \dots, k. \end{aligned} \tag{31}$$

Here, $\gamma > 0$ and $\mu_i \neq 0$. The first equation above can be written as $J_i \dot{q}_i - \dot{x}_d + \gamma e \to 0$. After grouping the terms and using the matrix inversion expressed by (26) we get, as $t \to \infty$,

$$\begin{aligned} &\dot{q}_i - \begin{bmatrix} J_i^+ & N_{J_i} \end{bmatrix} \begin{bmatrix} \dot{x}_d - \gamma e \\ \mu_i N_{J_i}^T \nabla H_i \end{bmatrix} \to 0 \quad \text{with} \\ &q_i \to \{q_i \mid N_{J_i}^T \nabla H_i = 0 \quad \text{and} \quad x_d = K_i(q_i)\}. \end{aligned} \tag{32}$$

Therefore the asymptotic resolution of the redundancy problem can be expressed by the conditions given by (32). These conditions result in the joint velocities approaching their desired values, while the joint positions satisfy a set of constraint equations. Notice that the redundancy resolution problem is characterized by the fact that the desired joint positions are not known in advance. This fact prevents us from directly using the existing adaptive schemes that achieves joint position tracking.

We will denote by $v_{r_i}$ the joint reference velocity for the $i$th robot; we will choose $v_{r_i}$ such that,

$$\begin{aligned} v_{r_i} &= \begin{bmatrix} J_i^+ & N_{J_i} \end{bmatrix} \begin{bmatrix} \dot{x}_d - \gamma e \\ \mu_i N_{J_i}^T \nabla H_i \end{bmatrix} \\ &= J_i^+(\dot{x}_d - \gamma e) + \mu_i P_{J_i} \nabla H_i \quad i = 1, \dots, k. \end{aligned} \tag{33}$$

We will group the $v_{r_i}$ $(i = 1, \dots, k)$ into one vector $v_r$, such that

$$v_r = \begin{bmatrix} v_{r_1}^T & v_{r_2}^T & \dots & v_{r_k}^T \end{bmatrix}^T .$$

We also will denote by $v_o$ the load reference velocity; we will choose $v_o$ such that,

$$v_o = \dot{x}_d - \gamma e. \tag{34}$$

It should be noted that the choice of $v_o$ guarantees that,

$$v_o = J_i v_{r_i} = T_i^{-T} J_{e_i} v_{r_i} \quad i = 1, \dots, k. \tag{35}$$

The asymptotic resolution of the redundancy problem can be solved by a mechanism that ensures that, $\dot{q}_i - v_{r_i} \to 0$ ( $i = 1, \dots, k$), as as $t \to \infty$.

In order to proceed further, we will state three more assumptions which will be needed in the development of the control law.

ASSUMPTIONS-CONTINUED

(A3) The desired paths $x_d(t)$, $\dot{x}_d(t)$ and $\ddot{x}_d(t)$ are bounded for all time $t$.

(A4) The Jacobian $J_i(q_i)$ is a full rank, continuously differentiable function matrix, that is, $J_i(q_i)$ is of class $C^r$, $r \geqslant 2$, (i.e., at least twice differentiable).

(A5) The cost function $H_i(q_i)$ $(i = 1, \dots, k)$ given in (27) is a twice differentiable real valued function.

In Assumption (A4), the full rank restriction on $J_i(q_i)$ requires that all possible joint motions $q_i$, do not pass through any singularity configuration of $J_i(q_i)$; this will be shown to be possible with the control law derived in the paper. If $J_i(q_i)$ is continuous and full rank in some subset $S_{J_i}$, then $J_i^+ = J_i^T(J_i J_i^T)^{-1}$, $P_{J_i} = I_{n_i} - J_i^+ J_i$ and $N_{J_i}$ are continuous in $S_{J_i}$. The matrices $J_i$, $J_i^+$, $P_{J_i}$ and $N_{J_i}$ are shift varying linear operators. It is easy to show that any continuous linear operator is bounded, hence $J_i$, $J_i^+$, $P_{J_i}$ and $N_{J_i}$ are bounded in $S_{J_i}$, (i.e., the induced norm of $J_i$, $J_i^+$, $P_{J_i}$ and $N_{J_i}$ are finite in $S_{J_i}$). Furthermore, if $\dot{J}_i$ is continuous in $S_{J_i}$, then $\dot{J}_i^+$ and $\dot{P}_{J_i}$ are continuous on any path with continuous $\dot{q}_i$ in $S_{J_i}$.

## 4. Design of the Control and Update Laws

### 4.1. Design of the Control Law

Our goal is to design an adaptive control law that guarantees the asymptotic convergence of the load tracking error to zero, the convergence of the internal forces to their desired values and the redundancy resolution. We will start by defining few variables needed for the development. The weighted reference velocity error for the $i$th robot is defined as,

$$\rho_{r_i} = w_t(\dot{q}_i - v_{r_i}) \quad i = 1, \ldots, k. \tag{36}$$

The scalar weighting function $w_t$ will be chosen as $w_t = e^{\lambda t}$, where $\lambda$ is a positive constant, (see [13] and [26]). We will group $\rho_{r_i}$ $(i = 1, \ldots, k)$ into one vector $\rho_r$ such that $\rho_r = \left[\rho_{r_1}^T \; \rho_{r_2}^T \; \cdots \; \rho_{r_k}^T\right]^T$. The weighted reference velocity error for the load is defined as,

$$\rho_o = w_t(\dot{x} - v_o). \tag{37}$$

It is easy to show that,

$$\rho_o = J_i \rho_{r_i} \quad i = 1, \ldots, k. \tag{38}$$

Note that $T_i^T \rho_o = J_{e_i} \rho_{r_i}$ $(i = 1, \ldots, k)$. If these equations are stacked into a single vector by forming the $J_{e_i}$'s into a block diagonal matrix, then we can write,

$$G^T \rho_o = J_e \rho_r. \tag{39}$$

As $GG^+ = I_6$, then $\rho_o = (G^+)^T J_e \rho_r$. We will choose $\dot{\rho}_r$ and $\dot{\rho}_o$ such that,

$$\dot{\rho}_{r_i} = w_t(\ddot{q}_i - a_{r_i}) \quad i = 1, \ldots, k \tag{40}$$

$$\dot{\rho}_o = w_t(\ddot{x} - a_o). \tag{41}$$

The choice of $\rho_{r_i}$ given by Equation (36), and the choice of $\dot{\rho}_{r_i}$ given by Equation (40) will result in the following value for $a_{r_i}$,

$$a_{r_i} = \dot{v}_{r_i} + \lambda(v_{r_i} - \dot{q}_i) \quad i = 1, \ldots, k. \tag{42}$$

We can group the $a_{r_i}$ $(i = 1, \ldots, k)$ into one vector $a_r$ such that

$$a_r = \left[a_{r_1}^T \; a_{r_2}^T \; \cdots \; a_{r_k}^T\right]^T,$$

and $a_r = \dot{v}_r + \lambda(v_r - \dot{q})$. Note that $\dot{\rho}_r = w_t(\ddot{q} - a_r)$. Similarly, the choice of $\rho_o$ given by Equation (37), and the choice of $\dot{\rho}_o$ given by Equation (41) will result in the following value for $a_o$,

$$a_o = \dot{v}_o + \lambda(v_o - \dot{x}). \tag{43}$$

It should be noted that $v_r$ and $v_o$ are independent of $\dot{q}$ and $\dot{x}$, hence $a_r$ and $a_o$ are not functions of $\ddot{q}$ and $\ddot{x}$. Therefore the proposed adaptive scheme does not require the measurements of the accelerations $\ddot{q}$ and $\ddot{x}$.

THEOREM 1. *Let the matrices $K_o$, $K_r$, $\Gamma_r$ and $\Gamma_o$ be positive definite matrices, and $K_f$ be a positive semi-definite diagonal matrix. The adaptive control law given by Equations* (44)–(47) *ensures that $\rho_r$, $\rho_o \in L^2 \cap L^\infty$ and $\tilde{P}_r$, $\tilde{P}_o \in L^\infty$.*

$$\begin{aligned}\tau &= \hat{D}a_r + \hat{C}v_r + \hat{G}_r - K_r(\dot{q} - v_r) + \\ &\quad + J_e^T G^+(\hat{M}a_o + \hat{N}v_o + \hat{G}_l - K_o(\dot{x} - v_o)) + J_e^T \tau_f \\ &= Y_r\hat{P}_r - K_r(\dot{q} - v_r) + J_e^T G^+(Y_o\hat{P}_o - K_o(\dot{x} - v_o)) + J_e^T\tau_f.\end{aligned} \tag{44}$$

The force torque $\tau_f$ is given by,

$$\tau_f = F_{I,d} - K_f \int e_f. \tag{45}$$

The parameters update laws are such,

$$\dot{\hat{P}}_r = -\Gamma_r^{-1} Y_r^T \rho_r w_t, \tag{46}$$

$$\dot{\hat{P}}_o = -\Gamma_o^{-1} Y_o^T \rho_o w_t. \tag{47}$$

Recall that $w_t = \mathrm{e}^{\lambda t}$, where $\lambda$ is a positive constant. Also note that a vector function $\nu$ belongs to $L^p$ (i.e., $\nu \in L^p$) where $p \in [1, \infty)$, if the integral $\|\nu\|_p = (\int_o^\infty \|\nu(u)\|^p\,\mathrm{d}u)^{1/p}$ exists. Also $\|\nu\|_\infty = \max\sup|\nu_i(t)|$ for $t \geqslant 0$ (where, sup denotes the supremum).

PRELIMINARIES TO THE PROOF

Before proving Theorem 1, we will derive the equation of the closed loop system. We can solve for the force from Equation (7), thus we get,

$$F_e = G^+(M\ddot{x} + N\dot{x} + G_l) + F_I. \tag{48}$$

If we combine Equations (3) and (48), we get,

$$D\ddot{q} + C\dot{q} + G_r + J_e^T G^+(M\ddot{x} + N\dot{x} + G_l) + J_e^T F_I = \tau. \tag{49}$$

Replacing $\tau$ by its value from Equation (44), we get,

$$\begin{aligned}&D\ddot{q} + C\dot{q} + G_r + J_e^T G^+(M\ddot{x} + N\dot{x} + G_l) + J_e^T F_I \\ &\quad = Y_r\hat{P}_r - K_r(\dot{q} - v_r) + J_e^T G^+(Y_o\hat{P}_o - K_o(\dot{x} - v_o)) + J_e^T\tau_f.\end{aligned} \tag{50}$$

Using the facts that $\hat{P}_r = \tilde{P}_r + P_r$ and $\hat{P}_o = \tilde{P}_o + P_o$, and replacing $Y_r P_r$ and $Y_o P_o$ by their values from Equations (17) and (18), we obtain,

$$\begin{aligned} &D(\ddot{q} - a_r) + C(\dot{q} - v_r) + K_r(\dot{q} - v_r) + \\ &\quad + J_e^T G^+ \left(M(\ddot{x} - a_o) + N(\dot{x} - v_o) + K_o(\dot{x} - v_o)\right) \\ &= Y_r \tilde{P}_r + J_e^T G^+ Y_o \tilde{P}_o + J_e^T (\tau_f - F_I). \end{aligned} \tag{51}$$

Using Equations (36), (37), (40), (41) and (45), we obtain,

$$\begin{aligned} &(D\dot{\rho}_r + C\rho_r + K_r\rho_r + J_e^T G^+ (M\dot{\rho}_o + N\rho_o + K_o\rho_o))\mathrm{e}^{-\lambda t} \\ &\quad = Y_r \tilde{P}_r + J_e^T G^+ Y_o \tilde{P}_o - J_e^T \Big(e_f + K_f \int e_f\Big). \end{aligned} \tag{52}$$

Multiplying the above equation by $G(J_e^T)^+$, we get,

$$\begin{aligned} &G(J_e^T)^+ (D\dot{\rho}_r + C\rho_r + K_r\rho_r)\mathrm{e}^{-\lambda t} + (M\dot{\rho}_o + N\rho_o + K_o\rho_o)\mathrm{e}^{-\lambda t} - \\ &\quad - G(J_e^T)^+ Y_r \tilde{P}_r - Y_o \tilde{P}_o = 0. \end{aligned} \tag{53}$$

*Proof of Theorem 1.* Consider the following Lyapunov function candidate,

$$V = 1/2\rho_r^T D\rho_r + 1/2\tilde{P}_r^T \Gamma_r \tilde{P}_r + 1/2\rho_o^T M\rho_o + 1/2\tilde{P}_o^T \Gamma_o \tilde{P}_o. \tag{54}$$

Now if we differentiate $V$ with respect to time and use Properties (P1)–(P3), we get,

$$\dot{V} = \rho_r^T (D\dot{\rho}_r + C\rho_r) + \tilde{P}_r^T \Gamma_r \dot{\tilde{P}}_r + \rho_o^T (M\dot{\rho}_o + N\rho_o) + \tilde{P}_o^T \Gamma_o \dot{\tilde{P}}_o. \tag{55}$$

Using the fact that $\rho_o = (G^+)^T J_e \rho_r$, the above equation yields,

$$\begin{aligned} \dot{V} &= \rho_r^T (D\dot{\rho}_r + C\rho_r + J_e^T G^+ (M\dot{\rho}_o + N\rho_o)) + \\ &\quad + \tilde{P}_r^T \Gamma_r \dot{\tilde{P}}_r + \tilde{P}_o^T \Gamma_o \dot{\tilde{P}}_o. \end{aligned} \tag{56}$$

Using Equation (52), we get,

$$\begin{aligned} \dot{V} &= \rho_r^T \Big( - K_r\rho_r - J_e^T G^+ K_o\rho_o + Y_r \tilde{P}_r w_t + J_e^T G^+ Y_o \tilde{P}_o w_t - \\ &\qquad - J_e^T \Big(e_f + K_f \int e_f\Big) w_t \Big) + \tilde{P}_r^T \Gamma_r \dot{\tilde{P}}_r + \tilde{P}_o^T \Gamma_o \dot{\tilde{P}}_o. \end{aligned} \tag{57}$$

Recall the facts that $\rho_r^T J_e^T = \rho_o^T G$ and $GG^+ = I_6$, we obtain,

$$\begin{aligned} \dot{V} &= -\rho_r^T K_r \rho_r - \rho_o^T K_o \rho_o + \rho_r^T Y_r \tilde{P}_r w_t + \rho_o Y_o \tilde{P}_o w_t - \\ &\quad - \rho_o^T G \Big(e_f + K_f \int e_f\Big) w_t + \tilde{P}_r^T \Gamma_r \dot{\tilde{P}}_r + \tilde{P}_o^T \Gamma_o \dot{\tilde{P}}_o. \end{aligned} \tag{58}$$

Using the update laws given by Equations (46) and (47), and the fact that $G(e_f + K_f \int e_f) = 0$, we get,

$$\dot{V} = -\rho_r^T K_r \rho_r - \rho_o^T K_o \rho_o. \tag{59}$$

Hence $V > 0$ and $\dot{V} \leqslant 0$. Thus we can conclude that $\rho_r, \rho_o \in L^2 \cap L^\infty$ and that $\tilde{P}_r, \tilde{P}_o \in L^\infty$.

COROLLARY 1. *$\dot{q}_i - v_{r_i} \to 0$ $(i = 1, \ldots, k)$ and $\dot{x} - v_o \to 0$ at the rate of* $\mathrm{e}^{-\lambda t}$.

*Proof.* From Equation (36), we can write $\dot{q}_i - v_{r_i} = \rho_{r_i}\mathrm{e}^{-\lambda t}$. Because $\rho_{r_i}$ is bounded, $\dot{q}_i - v_{r_i} \to 0$ at the rate of $\mathrm{e}^{-\lambda t}$. Similarly, from Equation (37), we can write $\dot{x} - v_o = \rho_o\mathrm{e}^{-\lambda t}$. Because $\rho_o$ is bounded, $\dot{x} - v_o \to 0$ at the rate of $\mathrm{e}^{-\lambda t}$.

Hence we can conclude from Equation (52) that,

$$D\dot{\rho}_r\mathrm{e}^{-\lambda t} + J_e^T G^+ M\dot{\rho}_o\mathrm{e}^{-\lambda t} - Y_r\tilde{P}_r - J_e^T G^+ Y_o\tilde{P}_o + \\ + J_e^T\left(e_f + K_f\int e_f\right) \to 0 \quad \text{as} \quad t \to \infty \tag{60}$$

provided that the joint position $q_i$ is bounded and $v_{r_i}$ is bounded. We will show through the analysis of the perturbed dynamical system $\dot{q}_i - v_{r_i} = w_t^{-1}\rho_{r_i}$ that $q_i$ is bounded and stable for an appropriate choice of $v_{r_i}$. In fact the boundedness of $q_i$ and the boundedness of $J_e^T G^+$ depend on the stability of the system $\dot{q}_i = v_{r_i}$ and therefore on the choice of $v_{r_i}$; this fact will be shown in the final sections.

### 4.2. Boundedness of the Internal Forces

THEOREM 2. *The adaptive control law given by Equations* (44)–(47) *ensure the boundedness of the internal forces.*

*Proof.* We have proved previously that $\tilde{P}_r$ and $\tilde{P}_o$ are bounded. Also, it can be seen from Equation (53) that $\dot{\rho}_r\mathrm{e}^{-\lambda t}$ and $\dot{\rho}_o\mathrm{e}^{-\lambda t}$ are bounded. Using Equation (60), we can conclude that, $e_f + K_f \int e_f$ is bounded as $J_e^T$ is a full rank matrix.

## 5. Boundedness of the Joint Motions and Control Torques

In this section we will prove the boundedness of $q_i$, $\dot{q}_i$ $(i = 1, \ldots, k)$, and the control torque $\tau$ based on a perturbation model. Note that Equation (36) can be written as a decayed perturbation system,

$$\dot{q}_i = v_{r_i}(q_i) + \delta_i(q_i, t) \quad i = 1, \ldots, k. \tag{61}$$

Recall from Corollary 1 that $\delta_i = \dot{q}_i - v_{r_i} \to 0$ as $t \to \infty$, thus the perturbation $\delta_i = w_t^{-1}\rho_{r_i}$ $(i = 1, \ldots, k)$ is bounded and tends to zero as $t \to \infty$.

We will prove the boundedness of $q_i$ in the perturbed system, described by Equation (61), by ensuring the boundedness of $q_i$ in the unperturbed system $\dot{q}_i = v_{r_i}(q_i)$. In the following, we will consider several lemmas that establish the relationship between the boundedness of the perturbed and unperturbed systems. The first important lemma which is stated without proof is the result of Markus and Opial [7]. Recall that the set $S$ is said to be invariant if each solution starting in $S$ remains in $S$ for all $t$ [7]. Specifically, for a continuous time system, $S$ is said to be an invariant set under the vector field $\dot{z} = f(z)$ if for any $z(0) = z^0 \in S$, we have $z(t) \in S$ for all $t \in R^+$.

LEMMA 1 (Stability of a perturbed system [7]). *Consider the perturbed differential equation with* $z_\delta \in R^{n_i}$,

$$\dot{z}_\delta = f(z_\delta) + \delta(z_\delta, t) \quad with \quad z_\delta(0) = z^0. \tag{62}$$

This system is called "asymptotically autonomous" if
(1) $\delta(z_\delta, t) \to 0$ as $t \to \infty$ uniformly for $z_\delta$ in an arbitrary compact set of $\Omega$ in $R^{n_i}$, or,
(2) $\delta(z_\delta, t) \in L^1$ for all $z_\delta$ which are bounded and continuous on $\Omega$ for $t \geqslant 0$.

Then, the positive limit sets (i.e., the sets with $t \in R^+$ and $t \to \infty$) of the solutions of (62) are invariant sets of the original stable differential equation,

$$\dot{z} = f(z) \quad \text{with} \quad z(0) = z^0. \tag{63}$$

Notice that because of the choice of $w_t$, the redundancy resolution Equation (61) modeled as a perturbed system is indeed asymptotically autonomous, since the perturbed term $\delta_i$ is integrable as $\int_0^\infty \|\delta_i\| \, dt \leqslant B_{\delta_i}/\lambda$, where $B_{\delta_i}$ is a positive constant.

LEMMA 2 (Asymptotic stability of the perturbed system). *Assume that the perturbed system* (62) *is an asymptotically autonomous system. Then the limit solution set of* (62) *is the limit solution set of* (63). *If the positive limit set of* (63) *is bounded, then* $\|z_\delta - z\|$ *is bounded as* $t \to \infty$.

*Proof.* Let $V_z$ be a continuous Lyapunov function defined on the set $G_s \subset R^{n_i}$. We define $E$ to be the set of all points in the closure [22] of $G_s$, (the closure of $G_s$ will be denoted as $\bar{G}_s$), where $\dot{V}_z = 0$, that is,

$$E = \{z \mid \dot{V}_z = 0, \; z \in \bar{G}_s\}. \tag{64}$$

Let $M_s$ be the largest invariant set in $E$, then LaSalle's invariance theorem [16] asserts that every solution of (63) approaches $M_s$ as $t \to \infty$. Thus the result of Lemma 1 yields that the positive limit set of (62) is the positive limit set of (63), hence $z_\delta$ tends to some limit points of the unperturbed system. Moreover, if the positive limit set of (63) is bounded, then $\|z_\delta - z\|$ is bounded as $t \to \infty$.

We should note that the asymptotic convergence to the positive limit set is a local behavior. Lemma 2 tells us that if $\Delta_h$ is the measure of the limit set of (63) (i.e., $\|z_\delta - z\| < \Delta_h$ as $t \to \infty$), then given any number $h_z > \Delta_h$, we can always find a time $t_h$, such that for $t > t_h$ we have $\|z_\delta(t) - z(t)\| < h_z$.

In the next lemma, we will show that the trajectory of (62) is bounded in $t \in [0, \; t_h]$.

LEMMA 3 (Boundedness of the perturbed system). *Consider the perturbed differential equation* (62) *and suppose that the mapping $f\colon R^{n_i} \to R^{n_i}$ has a Lipschitz constant $c_L > 0$, also suppose that the perturbation $\delta(z_\delta, t)$ along the trajectory $z_\delta$ has a bounded $L^1$ norm, (i.e. $\int_0^\infty \|\delta\| \,\mathrm{d}t \leqslant B_\delta$, where $B_\delta$ is a positive constant). Then the trajectory $z_\delta(t)$ is bounded up to a given time $t_h$ if the differential equation* (63) *is stable.*

*Proof.* It is sufficient to show that $\|z_\delta - z\|$ is bounded for all $t \in [0, \infty)$, since $z(t)$ is bounded by the assumption of the stability of (63).

The solution curve of (62) can be written as,

$$z_\delta(t) - z^0 = \int_{u=0}^{t} f(z_\delta)\,\mathrm{d}u + \int_{u=0}^{t} \delta(z_\delta, u)\,\mathrm{d}u.$$

Similarly for the unperturbed system (63), we have,

$$z(t) - z^0 = \int_{u=0}^{t} f(z)\,\mathrm{d}u.$$

Combining these two equations, we get,

$$z_\delta(t) - z(t) = \int_{u=0}^{t} (f(z_\delta) - f(z))\,\mathrm{d}u + \int_{u=0}^{t} \delta(z_\delta, u)\,\mathrm{d}u.$$

As $f(\cdot)$ is Lipschitz by assumption, hence,

$$\|z_\delta - z\| \leqslant B_\delta + \int_{u=0}^{t} c_L \|z_\delta - z\|\,\mathrm{d}u.$$

Using the Bellman–Gronwall lemma [9], we have,

$$\|z_\delta - z\| \leqslant B_\delta \mathrm{e}^{c_L t_h} \tag{65}$$

for $t = t_h$. Hence the stability of the unperturbed system (63) ensures the boundedness of $z$, then $z_\delta$ is bounded in $t \in [0, t_h]$ for any given $t_h \geqslant 0$.

Using Lemma 2 and Lemma 3 to solve the asymptotic redundancy resolution problem, we can then state the following propositions.

PROPOSITION 1 (The boundedness of the joint positions and the parameter estimates). *If we assume that the function $v_{r_i}$ $(i = 1, \ldots, k)$ in* (33) *is Lipschitz, then we can find a set $R_{q_i^0}$ (the set of initial $q_i$) such that the solutions of the adaptive control system (i.e. the parameter estimates and the joint positions) are bounded for any time $t$. Therefore with the adaptive control law given by Equations* (44)–(47), *the solution of* (36) *is bounded for any time $t$, if the solution of the unperturbed system,*

$$J_i \dot{q}_i = \dot{x}_d + \gamma e \quad \textit{and} \quad N_{J_i}^T \dot{q}_i = \mu_i N_{J_i}^T \nabla H_i \quad i = 1, \ldots, k \tag{66}$$

*is bounded in $R_{q_i^0}$.*

*Proof.* The adaptive system given by Equations (3), (7), (44)–(47) is an asymptotically autonomous system because we have shown that the perturbation term is a uniformly bounded time decreasing function. The set $\{q_i \mid \|\dot{q}_i - v_{r_i}\| \leqslant B_{\delta_i}\}$ can be taken as the compact set $\Omega$ in Lemma 1. Thus Lemma 2 and Lemma 3 guarantee the boundedness of the adaptive system for all $t$ if $q_i$, the solution of (66), is bounded.

The boundedness of the unperturbed system will be studied in the next section.

PROPOSITION 2 (The boundedness of the joint velocities). *Based on Assumptions* (A3), (A4) *and* (A5), *the boundedness of the joint position* $q_i$ $(i = 1, \dots, k)$ *ensures the boundedness of the joint velocity* $\dot{q}_i$ $(i = 1, \dots, k)$.

*Proof.* The joint reference velocity $v_{r_i}$ $(i = 1, \dots, k)$ given by (33) is a function of $x_d$, $\dot{x}_d$, and $q_i$. By Assumption (A3), the boundedness of $q_i$ yields the boundedness of $\dot{x}_d - \gamma e$. By Assumptions (A4) and (A5), the boundedness of $q_i$ yields the boundedness of $J_i^+(q_i)$, $P_{J_i}(q_i)$ and $\nabla H_i(q_i)$. Hence $v_{r_i}(q_i)$ is bounded for all bounded $q_i$.

Therefore the boundedness of $\|\dot{q}_i - v_{r_i}\|$ in the adaptive system leads us to the boundedness of $\dot{q}_i$ $(i = 1, \dots, k)$, provided that $q_i$ is bounded.

PROPOSITION 3 (The boundedness of the control torque). *Based on Assumptions* (A3)–(A5), *if* $q_i$ *and* $\dot{q}_i$ $(i = 1, \dots, k)$ *are bounded, then the adaptive control torque defined by Equation* (44) *is bounded.*

*Proof.* Based on Assumptions (A3)–(A5) and the boundedness of $q_i$ and $\dot{q}_i$, the reference velocities $v_{r_i}$ $(i = 1, \dots, k)$ and $v_o$, and the reference accelerations $a_{r_i}$ $(i = 1, \dots, k)$ and $a_o$ are bounded. Therefore the control torque is bounded.

## 6. The Stability of the Unperturbed System

The trajectories $q_i$ $(i = 1, \dots, k)$ of the unperturbed system are bounded if $q_i$ $(i = 1, \dots, k)$ on the self-motion manifold are bounded. The dynamics of $q_i$ on the self-motion manifold have to be shown to result in joint positions which are bounded. We will show that the quadratic cost function $H_i(q_i)$ $(i = 1, \dots, k)$ is a special choice which guarantees the boundedness of $q_i$.

In the below, we will examine the boundedness of the unperturbed system by using a homeomorphic transformation of the coordinates. The mapping $F$: $S_1 \to S_2$ is continuous if the inverse image of every open set of $S_2$ is an open set of $S_1$; the mapping $F$ is open if the image of an open set of $S_1$ is an open set of $S_2$. The mapping $F$ is a homeomorphism if it is a bijection and both continuous and open [14]. Also a homeomorphism preserves the topological properties such as the openness, connectedness, and the convergence of a set [22]. We will find a homeomorphism which transforms the joint coordinates of $q_i$ into a decomposable coordinates $\xi_i$ and $\zeta_i$. Here $\xi_i$ is homeomorphic to the workspace coordinates $x$;

the variable $\zeta_i$ will be used to represent the dynamics on the self-motion manifold. Hence the unperturbed system $\dot{q}_i = v_{r_i}(q_i)$ $(i = 1, \dots, k)$ is transformed into a cascaded system,

$$\dot{\zeta}_i = v_{\zeta_i}(\zeta_i, \xi_i), \qquad \dot{\xi}_i = v_{\xi_i}(\xi_i) \quad i = 1, \dots, k. \tag{67}$$

The boundedness of $q_i$ $(i = 1, \dots, k)$ will be deduced from the boundedness of $\xi_i$ and $\zeta_i$. To find the homeomorphism, we will adopt the method used to prove the sufficiency of the Frobenius' theorem [14]. We will construct a diffeomorphism which is based on the self-motion manifold. For any given $x$, all the $q_i$ such that $x = K_i(q_i)$ lie on a leaf of the self-motion manifold which will be denoted as $Q_N^{q_i^0}$, (a leaf of the self-motion manifold is a sub-manifold). Note that a mapping $F$ is a diffeomorphism if $F$ is bijective and both $F$ and $F^{-1}$ are smooth mappings [14]. The sub-manifold $Q_N^{q_i^0}$ is a connected region. $N_{J_i}(q_i)$ is nonsingular by assumption, thus the distribution $\Delta_i = \ker(J_i) = \mathrm{span}(N_{J_i})$ is nonsingular. The null space of a Jacobian matrix is always completely integrable, hence $\Delta_i$ is involutive. The distribution $\Delta_i = \ker(J_i)$ has an annihilator $\Delta_i^{\perp}$ which is spanned by $J_i$. The integrability of $\Delta_i$ allows us to construct the integral manifold by piecewise integrating every column of $N_{J_i}$.

Let $\Phi_t^{f_i}$ denote the flow of the vector field $f_i$, such that $q_i(t) = \Phi_t^{f_i}(q_i^0)$ solves the ordinary differential equation $\dot{q}_i = f_i(q_i)$ with initial condition $q_i^0$. The transition mapping $\Phi_t^{f_i}$ which maps $q_i^0$ to $q_i(t)$ is a diffeomorphism, and has the property $\partial\Phi_t^{f_i}(q_i^0)/\partial t = f_i(q_i(t))$ [9, 14]. The flow of each vector field, represented by a column of $N_{J_i} = [N_{J_i^1}, \dots, N_{J_i^{r_i}}]$, is the solution of the following differential equations,

$$\dot{q}_i = N_{J_i^l}(q_i) \quad \text{with} \quad q_i(0) = q_i^0 \quad l = 1, \dots, r_i \tag{68}$$

and can be written as, $q_i(t = \zeta_i^l) = \Phi_{\zeta_i^l}^{N_{J_i^l}}(q_i^0)$. Thus we have,

$$\frac{\partial \Phi_{\zeta_i^l}^{N_{J_i^l}}(q_i^0)}{\partial \zeta_i^l} = N_{J_i^l}(q_i).$$

LEMMA 4 (The parameterized equation of the self-motion manifold). *Given a kinematic mapping $x = K_i(q_i)$. The composite mapping such that,*

$$(\zeta_i^1, \dots, \zeta_i^{r_i}) \to q_i(t) = \Phi_{\zeta_i^{r_i}}^{N_{J_i^{r_i}}} \circ \cdots \circ \Phi_{\zeta_i^1}^{N_{J_i^1}}(q_i^0)$$

*and*

$$t = \zeta_i^1 + \cdots + \zeta_i^{r_i} \tag{69}$$

*is a locally parametrized equation of the manifold* $Q_N^{q_i^0} = \{q_i \in C_s(q_i^0) \mid x_0 = K_i(q_i) = K_i(q_i^0)\}$*, which passes through* $q_i^0$*. Here* $C_s(q_i^0)$ *is used to denote the connected regions of the self-motion manifold and* $C_s(q_i^0)$ *passes through the initial configuration* $q_i^0$*.*

*Proof.* We shall show that for $t = \zeta_i^1 + \cdots + \zeta_i^{r_i}$, we have $K_i(q_i(t)) = K_i(q_i^0)$. Since $x = K_i(q_i)$, it suffices to show that $x$ is unchanged whenever $\zeta_i$ varies locally, (i.e., $\partial x/\partial \zeta_i^l = 0$ for $l = 1, \ldots, r_i$).

First, consider the rightmost integral $\Phi_{\zeta_i^1}^{N_{J_i^1}}$ in (69). Let $q_{\zeta_i^1} = \Phi_{\zeta_i^1}^{N_{J_i^1}}(q_i^0)$. Then

$$\frac{\partial x}{\partial \zeta_i^1} = \frac{\partial K_i}{\partial \zeta_i^1}\bigg|_{q_{\zeta_i^1}} = \frac{\partial K_i}{\partial q_i}\bigg|_{q_{\zeta_i^1}} \frac{\partial q_{\zeta_i^1}}{\partial \zeta_i^1} = J_i(q_{\zeta_i^1}) N_{J_i^1}(q_{\zeta_i^1}) = 0. \tag{70}$$

Hence $q_{\zeta_i^1} \in Q_N^{q_i^0}$ when $q_i^0 \in Q_N^{q_i^0}$. Similarly, we have, $\partial x/\partial \zeta_i^l = 0$ for $l = 2, \ldots, r_i$. Then for the $l$th transition, we have $q_i(t = \zeta_i^1 + \cdots + \zeta_i^l) = \Phi_{\zeta_i^l}^{N_{J_i^l}}(q_i(t = \zeta_i^1 + \cdots + \zeta_i^{l-1}))$. Thus $q_i(t = \zeta_i^1 + \cdots + \zeta_i^l) \in Q_N^{q_i^0}$. Moreover these $q_i$'s are connected since $\Phi_{\zeta_i^l}^{N_{J_i^l}}$ $(l = 1, \ldots, r_i)$ are continuous mapping with respect to $\zeta_i^l$. Therefore (69) maps $\zeta_i$ to $q_i(t) \in Q_N^{q_i^0}$. This is a diffeomorphism because it is a composition of diffeomorphisms $\Phi_{\zeta_i^l}^{N_{J_i^l}}$. Hence this mapping satisfies the conditions to be a parameterization of the manifold.

LEMMA 5 (Decomposition of the coordinates). *Given a kinematic mapping* $x = K_i(q_i)$ $(i = 1, \ldots, k)$*, and let* $U_i$ *be the image of the joint space* $Q_i$*. At any point* $q_i \in Q_i$*, there exists a diffeomorphism* $F_i^{-1}$*, which decomposes* $q_i$ *into* $\zeta_i \in R^{r_i}$ *and* $\xi_i \in R^6$*, such that* $F_i^{-1}(q_i) = \begin{bmatrix} \zeta_i \\ \xi_i \end{bmatrix}$*. The mapping* $\zeta_i(q_i)$ *maps* $q_i$ *on the self-motion manifold* $Q_N^{q_i}$ *into* $\zeta_i$*.*

*Proof.* We will construct the desired diffeomorphism on the given leaf of the self-motion manifold. Recall That $N_{J_i}$ is the orthogonal complement of $J_i^+$. The matrix $J_i$ is assumed to be full rank and has the right inverse $J_i^+ = J_i^T(J_i J_i^T)^{-1}$. Then the range space of $J_i^+$ and the range space of $J_i^T$ are equal. The domain space of any matrix is the direct-sum of its row space and its null space, hence the domain of $J_i$ is $R^{n_i}$. Thus we have, $\operatorname{rank}([N_{J_i},\ J_i^+]) = n_i$.

Consider the composite mapping $F_i$ such that,

$$\begin{aligned} &(\zeta_i^1, \ldots, \zeta_i^{r_i}, \xi_i^1, \ldots, \xi_i^6) \longrightarrow q_i(t) \\ &= \Phi_{\xi_i^6}^{J_{i,6}^+} \circ \cdots \circ \Phi_{\xi_i^1}^{J_{i,1}^+} \circ \Phi_{\zeta_i^{r_i}}^{N_{J_i^{r_i}}} \circ \cdots \circ \Phi_{\zeta_i^1}^{N_{J_i^1}}(q_i^0). \end{aligned} \tag{71}$$

The mapping $F_i$ is a diffeomorphism, since the composition of diffeomorphisms is a diffeomorphism. Therefore, the inverse of $F_i$, $F_i^{-1}$, exists and it is a smooth mapping. Thus,

$$\begin{bmatrix} \zeta_i \\ \xi_i \end{bmatrix} = F_i^{-1}(q_i) \quad i = 1, \ldots, k \tag{72}$$

where, $\zeta_i = (\zeta_i^1, \ldots, \zeta_i^{r_i})^T$ and $\xi_i = (\xi_i^1, \ldots, \xi_i^6)^T$ are real functions. We have, $(\zeta_i,\ \xi_i) = F_i^{-1} \circ F_i(\zeta_i, \xi_i)$. Then the Jacobian matrices of $F_i^{-1}$ and $F_i$ should satisfy the following equation,

$$\begin{bmatrix} \dfrac{\partial \zeta_i}{\partial q_i} \\ \dfrac{\partial \xi_i}{\partial q_i} \end{bmatrix} \begin{bmatrix} \dfrac{\partial F_i}{\partial \zeta_i} & \dfrac{\partial F_i}{\partial \xi_i} \end{bmatrix} = I_{n_i} \quad i = 1, \ldots, k. \tag{73}$$

As the distribution $\Delta_i = \ker(J_i)$ is involutive, the diffeomorphism $F_i$ has the property that for every $q_i$, the $r_i$ columns of the Jacobian matrix $\partial F_i / \partial \zeta_i$ are linearly independent vectors in the distribution $\Delta_i$ [14].

In the next lemma we will find the relationships between the derivatives of $(\zeta_i,\ \xi_i)$ and that of $q_i$ $(i = 1, \ldots, k)$.

LEMMA 6 (The time derivatives of the transformed coordinates). *The transformation $F_i$ given in Lemma 5 allows us to write,*

$$\dot{\xi}_i = M_{J_i} J_i \dot{q}_i \tag{74}$$

$$\dot{\zeta}_i = M_{N_i}^{-1} N_{J_i}^T \dot{q}_i. \tag{75}$$

*Proof.* We can always find a nonsingular $r_i \times r_i$ matrix $M_{N_i}$, which expresses $\partial F_i / \partial \zeta_i$ as a linear combination of the columns of $N_{J_i}$, thus,

$$\frac{\partial F_i}{\partial \zeta_i} = N_{J_i} M_{N_i}. \tag{76}$$

From (73) we have $\frac{\partial \xi_i}{\partial q_i} \frac{\partial F_i}{\partial \zeta_i} = 0$, thus, $\frac{\partial \xi_i}{\partial q_i} N_{J_i} M_{N_i} = 0 \in R^{6 \times r_i}$. Hence $N_{J_i}$ annihilates $\partial \xi_i / \partial q_i$. Recall that $J_i N_{J_i} = 0$, thus each row of $\partial \xi_i / \partial q_i$ must be a linear combination of the rows of $J_i$. Hence,

$$\frac{\partial \xi_i}{\partial q_i} = M_{J_i} J_i. \tag{77}$$

Here $M_{J_i}$ is a nonsingular $6 \times 6$ matrix. Therefore $\dot{\xi}_i = \frac{\partial \xi_i}{\partial q_i} \dot{q}_i = M_{J_i} J_i \dot{q}_i$, which corresponds to Equation (74). From (73) we have, $\frac{\partial \xi_i}{\partial q_i} \frac{\partial F_i}{\partial \xi_i} = I_6$; combining this equation with (77) yields,

$$\frac{\partial F_i}{\partial \xi_i} = J_i^+ M_{J_i}^{-1} \tag{78}$$

because the nonsingular matrix $J_i$ has a unique pseudo-inverse $J_i^+$ such that $J_i J_i^+ = I_6$. We can write, $\dot{q}_i = \frac{\partial F_i}{\partial \zeta_i}\dot{\zeta}_i + \frac{\partial F_i}{\partial \xi_i}\dot{\xi}_i$. Thus we have,

$$\frac{\partial F_i}{\partial \zeta_i}\dot{\zeta}_i = \Big(I_{n_i} - \frac{\partial F_i}{\partial \xi_i} M_{J_i} J_i\Big)\dot{q}_i = (I_{n_i} - J_i^+ J_i)\dot{q}_i = P_{J_i}\dot{q}_i. \tag{79}$$

To obtain (75), we substitute (76) into the above equation and pre-multiply both sides by $N_{J_i}^T$. Notice that $N_{J_i}^T N_{J_i} = I_{r_i}$, since each column of $N_{J_i}$ is a normalized basis vector.

*Remark* 2. Equation (74) implies that $\dot{\xi}_i = M_{J_i}\dot{x}$ and $\partial\xi_i/\partial x = M_{J_i}$. From the implicit mapping theorem, the non-singularity of $M_{J_i}$ ensures that $\xi_i$ is homeomorphic to $x$.

LEMMA 7 (The decomposition of the unperturbed system). *Using the transformation $F_i$ given by Lemma* 5, *we can write the unperturbed system $\dot{q}_i = v_{r_i}(q_i)$ ($v_{r_i}$ is expressed by Equation* (33)) *as a cascaded system in the following form,*

$$\dot{\zeta}_i = \mu_i M_{N_i}^{-1}(N_{J_i}^T \nabla H_i)(q_i(\zeta_i, e)), \tag{80}$$

$$\dot{e} + \gamma e = 0. \tag{81}$$

The notation used in (80) means that $N_{J_i}^T$ and $\nabla H_i$ are functions of $(\zeta_i,\ e)$ through dependency on the joint variable $q_i$.

*Proof.* The unperturbed system is now given by,

$$\dot{q}_i = J_i^+(\dot{x}_d - \gamma e) + \mu_i P_{J_i}\nabla H_i \quad i = 1,\ldots,k. \tag{82}$$

Equation (81) is obtained by pre-multiplying both sides of (82) by $J_i$ and recalling that $J_i P_{J_i} = 0$. Similarly, Equation (80) is obtained by pre-multiplying both sides of (82) by $M_{N_i}^{-1} N_{J_i}^T$ and recalling that $N_{J_i}^T J_i^+ = 0$. Notice that $q_i$ can be decomposed into $(\zeta_i,\ \xi_i)$ by $F_i^{-1}$ given by (72). Also notice that $\xi_i$ is homeomorphic to $x$. Thus $\xi_i$ is homeomorphic to $e$ because there is one to one mapping between $x$ and $e$. Therefore $e$ is independent of $\zeta_i$, so $q_i$ can be decomposed into $(\zeta_i, e)$.

LEMMA 8 (The stability of the cascaded system). *Consider the system* (80) *and* (81) *in hierarchical form,*

$$\dot{\zeta}_i = f_i(\zeta_i, \xi_i) \quad \textit{and} \quad \dot{\xi}_i = g_i(\xi_i). \tag{83}$$

*If the functions $f_i$ and $g_i$ are continuously differentiable, then $(\zeta_i, \xi_i) = (0,0)$ is a locally asymptotically stable equilibrium of the system, if and only if $\xi_i = 0$ is a locally asymptotically stable equilibrium of $g_i(\xi_i)$ and $\zeta_i = 0$ is a locally asymptotically stable equilibrium of $f_i(\zeta_i, 0)$.*

The proof of this lemma can be found in Vidyasagar [31].

The equilibrium point of the cascaded system given in Lemma 7 is $e = 0$, $\zeta_i = \zeta_i^*$. Here $\zeta_i^*$ is the coordinates such that $(N_{J_i}^T \nabla H_i)(q_i(\zeta_i^*, 0)) = 0$ $(i = 1, \ldots, k)$. The equilibrium joint position $q_i^*$ is then, $q_i^* = F_i(\zeta_i^*, 0)$ $(i = 1, \ldots, k)$.

*Remark 3.* Setting $e = 0$ in (80) gives us the zero-dynamics [6],

$$\dot{\zeta}_i = \mu_i M_{N_i}^{-1} (N_{J_i}^T \nabla H_i)(q_i(\zeta_i, 0)) \tag{84}$$

of the unperturbed system. Note that the zero dynamics is defined on the manifold $R^{r_i}$. Equations (75) and (84) lead to,

$$N_{J_i}^T \dot{q}_i = \mu_i (N_{J_i}^T \nabla H_i)(q_i(\zeta_i, 0)) \quad \text{or} \quad P_{J_i} \dot{q}_i = \mu_i (P_{J_i} \nabla H_i)(q_i(\zeta_i, 0))$$
$$\text{for } i = 1, \ldots, k. \tag{85}$$

Notice that $q_i(\zeta_i, 0) \in Q_N^{q_i}$. Equation (85) is defined on the manifold $\{q_i = F_i(\zeta_i, \xi_i) \mid \zeta_i \in R^{r_i} \text{ and } e = 0\}$. This zero-dynamics manifold is also expressed by,

$$Q_N^{q_i} = \{q_i \mid x_d = K_i(q_i) \text{ and } J_i \dot{q}_i = 0\} \quad i = 1, \ldots, k \tag{86}$$

and it is indeed the self-motion manifold over $x_d$. We observe that the identity $\dot{q}_i = (J_i^+ J_i + P_{J_i})\dot{q}_i$ is satisfied on any $q_i \in Q_i$. However for motions on the self-motion manifold $\dot{x} = J_i(q_i)\dot{q}_i = 0$, and thus for motions on the self-motion manifold we also have $\dot{q}_i = P_{J_i}\dot{q}_i$. Equation (85) can be rewritten as,

$$\dot{q}_i = \mu_i (P_{J_i} \nabla H_i)(q_i) \quad \text{for all} \quad q_i \in Q_N^{q_i}. \tag{87}$$

Equation (87) will be called the "quivalent zero dynamics" expressed in the joint space and defined on the manifold $Q_N^{q_i}$.

PROPOSITION 4 (The boundedness of the unperturbed system). *The equilibrium point $q_i^*$ $(i = 1, \ldots, k)$ of the unperturbed system is asymptotically stable if the equilibrium point $(\zeta_i^*, 0)$ of the zero-dynamics* (87) *is asymptotically stable. The trajectory $q_i(t)$ of the unperturbed system starting from any finite initial configuration $q_i^0$ is bounded if the solution trajectory of the zero dynamics defined on a leaf of the self-motion manifold $Q_N^{q_i^0} = \{q_i \in C_s(q_i^0) \mid K_i(q_i) = K_i(q_i^0) = x_0\}$ is bounded.*

*Proof.* Lemma 7 asserts that the unperturbed system given by (82) can be decomposed into a cascaded system, then the asymptotic stability results are obtained immediately from Lemma 8.

PROPOSITION 5 (The boundedness of $q_i$ is guaranteed with the choice of $H_i$). *Let the cost function $H_i(q_i)$ be a quadratic of the form:*

$$H_i(q_i) = 1/2(q_i - q_{c_i})^T M_{h_i} (q_i - q_{c_i}) \quad i = 1, \ldots, k \tag{88}$$

*where $q_{c_i}$ is fixed, and $M_{h_i}$ is a symmetric positive definite matrix. Further let $q_{c_i}$ be given in a set of isolated points. Consider the zero-dynamics,*

$$\dot{q}_i = \mu_i(P_{J_i}\nabla H_i)(q_i) = \mu_i P_{J_i} M_{h_i}(q_i - q_{c_i}) \quad \textit{with} \quad q_i \in Q_N^{q_i}. \tag{89}$$

*The vector $q_i$ is bounded and $q_i \to q_i^*$ as $t \to \infty$ for every fixed $q_{c_i}$. Here $q_i^*$ is the optimal solution of the problem given by* (27).

*Proof.* Let the Lyapunov function candidate $V_i$ be,

$$V_i = 1/2(q_i - q_{c_i})^T M_{h_i}(q_i - q_{c_i}) \quad q_i \in Q_N^{q_i}. \tag{90}$$

The derivative of $V_i$ is,

$$\begin{aligned} \dot{V}_i &= \mu_i(q_i - q_{c_i})^T M_{h_i} P_{J_i} M_{h_i}(q_i - q_{c_i}) \\ &= \mu_i \|P_{J_i} M_{h_i}(q_i - q_{c_i})\|^2 \leqslant 0 \quad \text{for} \quad \mu_i < 0. \end{aligned} \tag{91}$$

Here the fact that $P_{J_i}$ is a projector was used. Hence $q_i - q_{c_i} \in L^\infty$. In addition, because of the boundedness of $q_{c_i}$ we have $q_i \in L^\infty$. Notice that the set $E_i = \{q_i \mid \dot{V}_i = 0\}$ is the the set of equilibrium points of (89), and is therefore an invariant set. From LaSalle's extension of Lyapunov direct method [16], $q_i \to q_i^*$ $(i = 1, \dots, k)$ as $t \to \infty$ because $q_i$ is in a bounded set.

*Remark 4.* We see from the last proposition that the choice $q_{c_i}$ and $M_{h_i}$ $(i = 1, \dots, k)$ can be used to ensure that $q_i^*$ is far from the singular configurations. Thus ensuring that the joints of the robots do not go through singular configurations. It should be noted that the exact value of the joint position $q_i^* \in Q_N^{q_i}$ $(i = 1, \dots, k)$ can be obtained by simulation of Equation (89).

*Remark 5.* The quadratic performance function defined in (88) ensures that the function $v_{r_i}$ $(i = 1, \dots, k)$ is locally Lipschitz.

$$v_{r_i} = J_i^+(\dot{x}_d - \gamma e) + \mu_i P_{J_i} M_{h_i}(q_i - q_{c_i}) \quad i = 1, \dots, k. \tag{92}$$

The matrices $J_i^+$ and $P_{J_i}$ are differentiable because of Assumption (A4). A continuously differentiable function is locally Lipschitz. Also notice that $M_{h_i}$ is a constant matrix. Hence the function given in (92) is differentiable with respect to $q_i$ and is therefore Lipschitz.

## 7. Conclusion

In this paper, we addressed the problem of controlling redundant multiple robots manipulating a load cooperatively. We proposed an adaptive controller that ensures the exponential tracking of the load position to its desired value and the boundedness of the internal forces. The control law also guarantees that the errors of the parameters remain bounded, and that the redundancy resolution error is asymptotically stable. Measurements of the joints or load accelerations are not required. The concepts of zero dynamics and stability of perturbed nonlinear dynamical systems were used to prove the stability of the adaptive system, particularly the stability of the joint motions on the self-motion manifold. The overall stability of the system is established for a certain class of optimization functions used for redundancy resolution.

Further work can be done to simplify the control law calculations, as the control law is rather complex. It is also possible to use the results of [15] to customize the dynamic and kinematic models and the control law, and therefore obtain significant savings in the numerical computations. It should be noted that in general, real-time computations needed for control laws for redundant robots are more intensive than for non-redundant robots. Other possible areas of future developments can address actuator dynamics, the effects of joint flexibility and effects of bounded torques. At this stage experimental work should be carried out to verify the effectiveness of the control law proposed in this paper. In such an experiment, the workspace trajectory must be selected to be reachable and the torques must be sufficient to ensure that the desired load trajectories are feasible. If such a desired trajectory is found, then the collisions between the robots and the singularities may be avoided by an appropriate choice of $H_i$.

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