STÆRÐFRÆÐIGREINING IIIA - FORMÚLUBLAÐ

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Stærðfræðigreining I/IA:

I.1. Tölur og föll

$\sin^2(x) + \cos^2(x) = 1$
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
$\tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$

$\sin(-x) = -\sin(x)$
$\sin(\pi - x) = \sin(x)$
$\sin(\pi/2 - x) = \cos(x)$
$\sin^2(x) = \frac{1-\cos(2x)}{2}$
$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(-x) = \cos(x)$
$\cos(\pi - x) = -\cos(x)$
$\cos(\pi/2 - x) = \sin(x)$
$\cos^2(x) = \frac{1+\cos(2x)}{2}$
$\cos(2x) = 2\cos^2(x) - 1$

I.3. Afleiður

Taylormargliða:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Skekkjumat: $E_n(x) = \frac{f^{(n+1)}(X)}{(n+1)!}(x-a)^{n+1}$.

I.4. Torræð föll

Andhverfur hornafalla

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Breiðbogaföll

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\text{arsinh(x)} = \ln\left(x + \sqrt{x^2+1}\right) \qquad \text{arcosh(x)} = \ln\left(x + \sqrt{x^2-1}\right) \qquad \text{artanh(x)} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

I.6. Heildun

Innsetning: $\int f(g(x))g'(x)dx = \int f(u)du$ Hluth.: $\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

I.7. Rúmmál, massi og massamiðja

Kúla: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$.
Sívalningur: $V = \pi r^2 h$, $S = 2\pi r h$.
Keila með grunnflöt A: $V = \frac{1}{3}Ah$.

Snúðar

Snúið um x-ás:

$$V = \pi\int_a^b f(x)^2\,dx, \qquad S = 2\pi\int_a^b |f(x)|\sqrt{1+(f'(x))^2}\,dx.$$

Snúið um y-ás:

$$V = 2\pi\int_a^b xf(x)\,dx, \qquad S = 2\pi\int_a^b |x|\sqrt{1+(f'(x))^2}\,dx.$$

Lengd grafs: $s = \int_a^b \sqrt{1+(f'(x))^2}\,dx$.

Massamiðja plötu

$$\overline{x} = \frac{M_{x=0}}{m} = \frac{\int_a^b x\delta(x)f(x)\,dx}{\int_a^b \delta(x)f(x)\,dx} \qquad \overline{y} = \frac{M_{y=0}}{m} = \frac{\frac{1}{2}\int_a^b \delta(x)f(x)^2\,dx}{\int_a^b \delta(x)f(x)\,dx}.$$

I.8. Diffurjöfnur

Línulegar fyrsta stigs diffurjöfnur, $y' + p(x)y = q(x)$

$$y(x) = e^{-\mu(x)}\int e^{\mu(x)}q(x)\,dx, \qquad \mu(x) = \int p(x)\,dx.$$

Línuleg annars stigs diffurjafna með fastastuðla, $ay'' + by' + cy = 0$

Tilvik I: $y(x) = Ae^{r_1x} + Be^{r_2x}$ ef kennijafnan hefur tvær ólíkar rauntölulausnir r_1 og r_2.
Tilvik II: $y(x) = Ae^{kx} + Bxe^{kx}$ ef kennijafnan hefur eina tvöfalda rauntölulausn $k = -\frac{b}{2a}$.
Tilvik III: $y(x) = Ae^{kx}\cos(\omega x) + Be^{kx}\sin(\omega x)$ ef kennijafnan hefur engar rauntölulausnir, bara tvinntölulausnir $r_1 = k + i\omega$ og $r_2 = k - i\omega$ þar sem $k = -\frac{b}{2a}$ og $\omega = \frac{\sqrt{4ac-b^2}}{2a}$.

I.9. Runur og raðir

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$$

I.10. Veldaraðir

Kvótapróf: $\dfrac{1}{R} = L = \lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$ Rótarpróf: $\dfrac{1}{R} = L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \qquad \text{fyrir öll } x$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{fyrir öll } x$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{fyrir öll } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 + \cdots \qquad \text{fyrir } -1 < x < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty}nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad \text{fyrir } -1 < x < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{fyrir } -1 < x \le 1$$

$$\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{fyrir } -1 \le x \le 1$$

$$\sinh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad \text{fyrir öll } x$$

$$\cosh x = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad \text{fyrir öll } x$$

Mathematical Analysis IIIA:

1. Linear Ordinary Differential Equations

First Order Linear Equations

$y' + p(x)y = g(x)$:

$$y(x) = e^{-M(x)} \left(C + \int g(x) e^{M(x)}\, dx \right), \qquad \text{where } M(x) = \int p(x) dx.$$

Reduction of order

$y'' + p(x)y' + q(x)y = 0$:

$$u_2(x) = u_1(x) \int \frac{e^{-P(t)}}{u_1^2(t)}\, dt, \qquad \text{where } P(x) = \int p(x)\, dx.$$

Shift rule

$$P(D)(e^{\lambda x} f(x)) = e^{\lambda x} P(D+\lambda) f(x)$$

Green function

Solution to $y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0 y = g(x)$, $U_1(y) = U_2(y) = \cdots = U_n(y) = 0$,

$$y(x) = \int_a^b G(x,\xi) g(\xi)\, d\xi.$$

2. Separation of Variables

Newton's equation

Equation: $x'' = F(x)$ or $(\{x' = y, y' = F(x)\})$.
Potential: $U(x) := -\int F(x)\, dx$.
Energy: $E(x,y) := U(x) + \frac{y^2}{2}$.
Period of small oscillations: $\lim_{\epsilon \to 0} L_\epsilon = \frac{2\pi}{\sqrt{U''(x_0)}}$.

3. Series Solutions of Linear Equations

Recursive formula for the coefficients at an ordinary point

$$(k+2)(k+1)c_{k+2} + \sum_{j=0}^{k} (k-j+1) a_j c_{k-j+1} + \sum_{j=0}^{k} b_j c_{k-j} = 0.$$

Recursive formula for the coefficients at a regular singular point

$$P(k+\gamma)c_k + \sum_{j=0}^{k-1} ((j+\gamma)a_{k-j} + b_{k-j}) c_j = 0, \qquad k \in \mathbb{N},$$

with the understanding that for $k = 0$ the sum is empty, and the indicial polynomial P is given by

$$P(X) = X(X-1) + a_0 X + b_0.$$

4. Existence Theory

Picard iteration

$$\phi_{m+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_m(t))\, dt.$$

5. The Exponential of a Matrix

Solution to $y' = A(x)y + h(x)$, $y(x_0) = \eta$ is $y(x) = e^{(x-x_0)A}\eta + \int_{x_0}^{x} e^{(x-t)A}h(t)\,dt$.

Newton divided differences

$$F[\lambda_l] = F(\lambda_l), \qquad F[\lambda_l, \ldots, \lambda_{l+k}] = \frac{F[\lambda_l + 1, \ldots, \lambda_{l+k}] - F[\lambda_l, \ldots, \lambda_{l+k-1}]}{\lambda_{l+k} - \lambda_l}.$$

If $\lambda_l = \cdots = \lambda_{k+l}$, then $F[\lambda, \ldots, \lambda_{l+k}] = F^{(k)}(\lambda_l)/k!$.

7. Sturm-Liouville Theory

Sturm-Liouville form

$$Ly = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = g(x).$$

Inner product, norm and Bessel inequality

$$\langle u, v\rangle = \int_a^b u(x)\overline{v(x)}\,dx, \qquad \|u\|_2 = \sqrt{\langle u, u\rangle}, \qquad \sum_{n=1}^{\infty} |\langle f, \phi_n\rangle|^2 \le \|f\|_2^2.$$

Solutions to non-homogeneous equations

$$Ly - \mu y = h(x),\ U_1(y) = 0,\ U_2(y) = 0.$$

If μ not an eigenvalue: $y(x) = \sum_{n=1}^{\infty} (\lambda_n - \mu)^{-1}\langle h, \phi_n\rangle\phi_n(x)$.
If μ is an eigenvalue: $y(x) = \sum_{\lambda_n \ne \mu}^{\infty} (\lambda_n - \mu)^{-1}\langle h, \phi_n\rangle\phi_n(x)$.

Fourier Analysis

Fourier series f on $[-L, L]$: Exponential form:

$$f(x) \sim \frac{c_0}{2} + \sum_{n=-\infty}^{\infty} c_n e^{\frac{inx\pi}{L}},$$
$$c_n = \hat{f}(n) = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\,dx, \qquad n \in \mathbb{Z}.$$

Fourier series f on $[-L, L]$: Trigonometric form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad n \ge 0. \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad n \ge 1.$$

Connections between coefficients:

$c_0 = a_0$,		$a_0 = c_0$,	
$c_n = (a_n - ib_n)/2$,	for $n > 0$,	$a_n = c_n + c_{-n}$,	for $n > 0$,
$c_n = (a_{-n} + ib_{-n})/2$,	for $n < 0$.	$b_n = i(c_n - c_{-n})$,	for $n > 0$.

Solving Ordinary Differential Equations

If $P(D)u = f$, then $P(in)\hat{u}(n) = \hat{f}(n)$ and

$$u(x) = \sum_{n=-\infty}^{\infty} \frac{\hat{f}(n)}{P(in)} e^{inx}.$$