STÆRÐFRÆÐIGREINING I/IA & II/IIA - FORMÚLUBLAÐ

Benedikt Steinar Magnússon <bsm@hi.is>, https://github.com/benediktmag/formulublad

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Stærðfræðigreining I/IA:

I.1. Tölur og föll

$\sin^2(x) + \cos^2(x) = 1$
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
$\tan(x \pm y) = \frac{\tan(x)\pm\tan(y)}{1\mp\tan(x)\tan(y)}$

$\sin(-x) = -\sin(x)$
$\sin(\pi - x) = \sin(x)$
$\sin(\pi/2 - x) = \cos(x)$
$\sin^2(x) = \frac{1-\cos(2x)}{2}$
$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(-x) = \cos(x)$
$\cos(\pi - x) = -\cos(x)$
$\cos(\pi/2 - x) = \sin(x)$
$\cos^2(x) = \frac{1+\cos(2x)}{2}$
$\cos(2x) = 2\cos^2(x) - 1$

I.3. Afleiður

Taylormargliða:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Skekkjumat: $E_n(x) = \frac{f^{(n+1)}(X)}{(n+1)!}(x-a)^{n+1}$.

I.4. Torræð föll

Andhverfur hornafalla

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

Breiðbogaföll

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\text{arsinh(x)} = \ln\left(x + \sqrt{x^2+1}\right) \qquad \text{arcosh(x)} = \ln\left(x + \sqrt{x^2-1}\right) \qquad \text{artanh(x)} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

I.6. Heildun

Innsetning: $\int f(g(x))g'(x)dx = \int f(u)du$ Hluth.: $\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

I.7. Rúmmál, massi og massamiðja

Kúla: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$.
Sívalningur: $V = \pi r^2 h$, $S = 2\pi r h$.
Keila með grunnflöt A: $V = \frac{1}{3}Ah$.

Snúðar

Snúið um x-ás:

$$V = \pi\int_a^b f(x)^2\,dx, \qquad S = 2\pi\int_a^b |f(x)|\sqrt{1+(f'(x))^2}\,dx.$$

Snúið um y-ás:

$$V = 2\pi\int_a^b xf(x)\,dx, \qquad S = 2\pi\int_a^b |x|\sqrt{1+(f'(x))^2}\,dx.$$

Lengd grafs: $s = \int_a^b \sqrt{1+(f'(x))^2}\,dx$.

Massamiðja plötu

$$\overline{x} = \frac{M_{x=0}}{m} = \frac{\int_a^b x\delta(x)f(x)\,dx}{\int_a^b \delta(x)f(x)\,dx} \qquad \overline{y} = \frac{M_{y=0}}{m} = \frac{\frac{1}{2}\int_a^b \delta(x)f(x)^2\,dx}{\int_a^b \delta(x)f(x)\,dx}.$$

I.8. Diffurjöfnur

Línulegar fyrsta stigs diffurjöfnur, $y' + p(x)y = q(x)$

$$y(x) = e^{-\mu(x)}\int e^{\mu(x)}q(x)\,dx, \qquad \mu(x) = \int p(x)\,dx.$$

Línuleg annars stigs diffurjafna með fastastuðla, $ay'' + by' + cy = 0$

Tilvik I: $y(x) = Ae^{r_1x} + Be^{r_2x}$ ef kennijafnan hefur tvær ólíkar rauntölulausnir r_1 og r_2.
Tilvik II: $y(x) = Ae^{kx} + Bxe^{kx}$ ef kennijafnan hefur eina tvöfalda rauntölulausn $k = -\frac{b}{2a}$.
Tilvik III: $y(x) = Ae^{kx}\cos(\omega x) + Be^{kx}\sin(\omega x)$ ef kennijafnan hefur engar rauntölulausnir, bara tvinntölulausnir $r_1 = k + i\omega$ og $r_2 = k - i\omega$ þar sem $k = -\frac{b}{2a}$ og $\omega = \frac{\sqrt{4ac-b^2}}{2a}$.

I.9. Runur og raðir

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$$

I.10. Veldaraðir

$$\text{Kvótapróf:}\quad \frac{1}{R} = L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \qquad \text{Rótarpróf:}\quad \frac{1}{R} = L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \qquad \text{fyrir öll } x$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \text{fyrir öll } x$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \text{fyrir öll } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 + \cdots \qquad \text{fyrir } -1 < x < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty}nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad \text{fyrir } -1 < x < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{fyrir } -1 < x \le 1$$

$$\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \text{fyrir } -1 \le x \le 1$$

$$\sinh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad \text{fyrir öll } x$$

$$\cosh x = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \qquad \text{fyrir öll } x$$

Stærðfræðigreining II (STÆ205G+STÆ2013G):

II.1. Ferlar

$$
\begin{aligned}
\mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \\
\mathbf{u}\cdot\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3 \\
\mathbf{u}\times\mathbf{v} &= \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_1 & v_3 \end{bmatrix} \\
\text{lengd } \mathbf{u} = |\mathbf{u}| &= \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2} \\
\text{horn milli vigra } &= \arccos\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) \\
\text{lengd ferils } &= \int_a^b |\mathbf{v}(t)|\,dt
\end{aligned}
$$

Pólhnit

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\tan\theta &= \frac{y}{x} \\
\theta &= 2\arctan\left(\frac{y}{x+r}\right)
\end{aligned}
$$

Frenet-ramminn

Almennt:

$$\mathbf{T} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} \qquad \text{einingarsnertill}$$

$$\mathbf{B} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} \qquad \text{einingartvíþverill}$$

$$\mathbf{N} = \mathbf{B}\times\mathbf{T} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \text{höfuðþverill}$$

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} \qquad \text{krappi}$$

$$\tau = \frac{(\mathbf{v}\times\mathbf{a})\cdot(\frac{d}{dt}\mathbf{a})}{|\mathbf{v}\times\mathbf{a}|^2} \qquad \text{vindingur}$$

$$\rho = \frac{1}{\kappa} \qquad \text{krappageisli}$$

Stikun með bogalengd:

$$\begin{aligned}
\mathbf{T}(s) &= \mathbf{v}(s) \\
\kappa(s) &= \left|\frac{d\mathbf{T}}{ds}\right| \\
\mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s) \\
\mathbf{N}(s) &= \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|} \\
\mathbf{B}(s) &= \mathbf{T}(s) \times \mathbf{N}(s) \\
\mathbf{T}'(s) &= \kappa\mathbf{N} \\
\mathbf{N}'(s) &= -\kappa\mathbf{T} + \tau\mathbf{B} \\
\mathbf{B}'(s) &= -\tau\mathbf{N}
\end{aligned}$$

II.2. Hlutafleiður

Snertiplan: $S(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$.
Hlutafleiður: $f_k(\mathbf{a}) = \lim_{h\to 0} \dfrac{f(\mathbf{a}+h\mathbf{e}_k) - f(\mathbf{a})}{h}$.
Diffranleiki: $\lim_{(h,k)\to(0,0)} \dfrac{f(a+h,b+k) - S(a+h,b+k)}{\sqrt{h^2+k^2}} = 0$.
Jacobi-fylki: $D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$.
Stigull: $\nabla f(x,y) = f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j}$.
Stefnuafleiða: $D_{(u,v)}f(a,b) = \lim_{h\to 0^+} \dfrac{f(a+hu,b+hv) - f(a,b)}{h}$.
Taylor-margliða (n-stigs): $P_{(n)}(x,y) = \sum_{m=0}^{n}\sum_{j=0}^{m} \dfrac{1}{j!(m-j)!} D_1^j D_2^{m-j} f(a,b)(x-a)^j(y-b)^{m-j}$.

II.3. Útgildisverkefni

Lagrange-margfaldari (fyrir $g(x,y)=0$): $L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$.
Lagrange-margfaldarar (fyrir $g=h=0$): $L(x,y,z,\lambda,\mu) = f(x,y,z) + \lambda g(x,y,z) + \mu h(x,y,z)$.

II.4. Margföld heildi

Riemann-summa í tveimur víddum:
$\mathcal{R}(f,P) = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j$.
Almenn breytuskipti fyrir tvöföld heildi: $\iint_D f(x,y)\,dx\,dy = \iint_S g(u,v)\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$.
Breytuskipti í pólhnit: $\iint_D f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.
Almenn breytuskipti fyrir þreföld heildi:
$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$.
Breytuskipti í kúluhnit:
$\iiint_R f(x,y,z)\,dV = \int_\alpha^\beta\int_c^d\int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$.
Yfirborðsflatarmál: $S = \iint_D \sqrt{1 + f_1(x,y)^2 + f_2(x,y)^2}\,dA$.

II.5. Vigursvið

Mætti (ϕ) fyrir $\mathbf{F}$: $\mathbf{F}(x,y,z) = \nabla\phi(x,y,z)$.

Heildi falls yfir feril: $\displaystyle\int_{\mathcal{C}} f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t),y(t))\,\sqrt{x'(t)^2+y'(t)^2}\,dt$.

Heildi vigursviðs eftir ferli: $\displaystyle\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt$.

Flatarmál stikaflatar $\mathbf{r}$: $\displaystyle A = \iint_D dS = \iint_D \left|\tfrac{\partial\mathbf{r}}{\partial u}\times\tfrac{\partial\mathbf{r}}{\partial v}\right| dudv$.

Heildi falls f yfir stikaflöt: $\displaystyle\iint_{\mathcal{S}} f\,dS = \iint_D f(\mathbf{r}(u,v))\left|\tfrac{\partial\mathbf{r}}{\partial u}\times\tfrac{\partial\mathbf{r}}{\partial v}\right| dudv$.

Heildi vigursviðs $\mathbf{F}$ yfir stikaflöt: $\displaystyle\iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\,dS = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right) du\,dv$.

II.6. Diffur- og heildarreikningur vigursviða

$\displaystyle\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$.

Stigull: $\displaystyle\mathbf{grad}\,\varphi = \nabla\varphi = \frac{\partial\varphi}{\partial x}\,\mathbf{i} + \frac{\partial\varphi}{\partial y}\,\mathbf{j} + \frac{\partial\varphi}{\partial z}\,\mathbf{k}$

Sundurleitni: $\displaystyle\mathbf{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

Rót: $\displaystyle\mathbf{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$.

Sundurleitnisetning I: $\displaystyle\mathbf{div}\,\mathbf{F}(P) = \lim_{\varepsilon\to 0^+} \frac{1}{V_\varepsilon}\iint_{\mathcal{S}_\varepsilon} \mathbf{F}\cdot\mathbf{N}\,dS$.

Setning Stokes: $\displaystyle\mathbf{N}\cdot\mathbf{curl}\,\mathbf{F}(P) = \lim_{\varepsilon\to 0^+} \frac{1}{A_\varepsilon}\oint_{C_\varepsilon} \mathbf{F}\cdot d\mathbf{r}$.

Setning Green: $\displaystyle\oint_{\mathcal{C}} F_1(x,y)\,dx + F_2(x,y)\,dy = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA$.

Sundurleitnisetningin í tveimur víddum: $\displaystyle\iint_R \mathbf{div}\,\mathbf{F}\,dA = \oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{N}\,ds$.

Sundurleitnisetningin, setning Gauss: $\displaystyle\iiint_D \mathbf{div}\,\mathbf{F}\,dV = \iint_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\,dS$.

$\displaystyle\iiint_D \mathbf{curl}\,\mathbf{F}\,dV = -\iint_{\mathcal{S}} \mathbf{F}\times\mathbf{N}\,dS$.

$\displaystyle\iiint_D \mathbf{grad}\,\varphi\,dV = \iint_{\mathcal{S}} \varphi\mathbf{N}\,dS$.

Setning Stokes: $\displaystyle\iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F}\cdot\mathbf{N}\,dS = \oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds$.

Reiknireglur

(i) $\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$

(ii) $\nabla \cdot (\phi\mathbf{F}) = \phi\nabla \cdot \mathbf{F} + (\nabla\phi) \cdot \mathbf{F}$

(iii) $\nabla(f \circ \phi) = (f' \circ \phi)\nabla\phi$

(iv) $\Delta(f \circ \phi) = (f' \circ \phi)\Delta\phi + (f'' \circ \phi)\|\nabla\phi\|^2$

(v) $\nabla(\chi \circ \mathbf{F}) = (((\nabla\chi) \circ \mathbf{F})\partial_1\mathbf{F})\mathbf{e}_1 + \cdots + (((\nabla\chi) \circ \mathbf{F})\partial_n\mathbf{F})\mathbf{e}_n$

(i) $\mathbf{curl}\,\mathbf{grad}\,\phi = \nabla \times \nabla\phi = 0$

(ii) $\mathbf{div}\,\mathbf{curl}\,\mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$

(iii) $\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$

(iv) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(v) $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

(vi) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$

(vii) $\mathbf{curl\ curl\ F} = \nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F} = \mathbf{grad}\,\mathbf{div}\,\mathbf{F} - \nabla^2\mathbf{F}$