

# STÆRÐFRÆÐIGREINING I/IA & II/IIA - FORMÚLUBLAÐ

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## Stærðfræðigreining I/IA:

### I.1. Tölur og föll

$$\begin{array}{lll} \sin^2(x) + \cos^2(x) = 1 & \sin(-x) = -\sin(x) & \cos(-x) = \cos(x) \\ \sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) & \sin(\pi - x) = \sin(x) & \cos(\pi - x) = -\cos(x) \\ \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) & \sin(\pi/2 - x) = \cos(x) & \cos(\pi/2 - x) = \sin(x) \\ \tan(x \pm y) = \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)} & \sin^2(x) = \frac{1 - \cos(2x)}{2} & \cos^2(x) = \frac{1 + \cos(2x)}{2} \\ & \sin(2x) = 2\sin(x)\cos(x) & \cos(2x) = 2\cos^2(x) - 1 \end{array}$$

### I.3. Afleiður

Taylormargliða:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Skekkjumat:  $E_n(x) = \frac{f^{(n+1)}(X)}{(n+1)!}(x-a)^{n+1}.$

### I.4. Torræð föll

Andhverfur hornafalla

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

Breiðbogaföll

$$\begin{array}{lll} \sinh(x) = \frac{e^x - e^{-x}}{2} & \cosh(x) = \frac{e^x + e^{-x}}{2} & \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \\ \operatorname{arsinh}(x) = \ln(x + \sqrt{x^2 + 1}) & \operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1}) & \operatorname{artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \end{array}$$

### I.6. Heildun

Innsetning:  $\int f(g(x))g'(x)dx = \int f(u)du$       Hluth.:  $\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

### I.7. Rúmmál, massi og massamiðja

Kúla:  $V = \frac{4}{3}\pi r^3$ ,  $S = 4\pi r^2$ .

Sívalningur:  $V = \pi r^2 h$ ,  $S = 2\pi r h$ .

Keila með grunnflöt  $A$ :  $V = \frac{1}{3}Ah$ .

**Snúðar**

Snúðið um  $x$ -ás:

$$V = \pi \int_a^b f(x)^2 dx, \quad S = 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2} dx.$$

Snúðið um  $y$ -ás:

$$V = 2\pi \int_a^b x f(x) dx, \quad S = 2\pi \int_a^b |x|\sqrt{1 + (f'(x))^2} dx.$$

Lengd grafs:  $s = \int_a^b \sqrt{1 + (f'(x))^2} dx.$

## Massamiðja plötu

$$\bar{x} = \frac{M_{x=0}}{m} = \frac{\int_a^b x \delta(x) f(x) dx}{\int_a^b \delta(x) f(x) dx} \quad \bar{y} = \frac{M_{y=0}}{m} = \frac{\frac{1}{2} \int_a^b \delta(x) f(x)^2 dx}{\int_a^b \delta(x) f(x) dx}.$$

## I.8. Diffurjöfnur

Línulegar fyrsta stigs diffurjöfnur,  $y' + p(x)y = q(x)$

$$y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x) dx, \quad \mu(x) = \int p(x) dx.$$

Línuleg annars stigs diffurjafna með fastastuðla,  $ay'' + by' + cy = 0$

Tilvik I:  $y(x) = Ae^{r_1 x} + Be^{r_2 x}$  ef kennijafnan hefur tvær ólíkar rauntölulausnir  $r_1$  og  $r_2$ .

Tilvik II:  $y(x) = Ae^{kx} + Bxe^{kx}$  ef kennijafnan hefur eina tvöfalda rauntölulausn  $k = -\frac{b}{2a}$ .

Tilvik III:  $y(x) = Ae^{kx} \cos(\omega x) + Be^{kx} \sin(\omega x)$  ef kennijafnan hefur engar rauntölulausnir, bara tvinntölulausnir  $r_1 = k + i\omega$  og  $r_2 = k - i\omega$  þar sem  $k = -\frac{b}{2a}$  og  $\omega = \frac{\sqrt{4ac-b^2}}{2a}$ .

## I.9. Runur og raðir

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

## I.10. Veldaraðir

$$\text{Kvótapróf: } \frac{1}{R} = L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{Rótarpróf: } \frac{1}{R} = L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \quad \text{fyrir öll } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{fyrir öll } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{fyrir öll } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad \text{fyrir } -1 < x < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots \quad \text{fyrir } -1 < x < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{fyrir } -1 < x \leq 1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad \text{fyrir } -1 \leq x \leq 1$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \quad \text{fyrir öll } x$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \quad \text{fyrir öll } x$$

Stærðfræðigreining II (STÆ205G+STÆ2013G):

II.1. Ferlar

$$\begin{aligned}\mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \\ \mathbf{u} \cdot \mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3 \\ \mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\ \text{lengd } \mathbf{u} = |\mathbf{u}| &= \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2} \\ \text{horn milli vigra} &= \arccos \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \right) \\ \text{lengd ferils} &= \int_a^b |\mathbf{v}(t)| \, dt\end{aligned}$$

Pólhnit

$$\begin{aligned}r &= \sqrt{x^2 + y^2} \\ \tan \theta &= \frac{y}{x} \\ \theta &= 2 \arctan \left( \frac{y}{x + r} \right)\end{aligned}$$

Frenet-ramminn

Almennt:

$\mathbf{T} = \frac{\mathbf{v}(t)}{ \mathbf{v}(t) }$	einingarsnertill
$\mathbf{B} = \frac{\mathbf{v} \times \mathbf{a}}{ \mathbf{v} \times \mathbf{a} }$	einingartvíþverill
$\mathbf{N} = \mathbf{B} \times \mathbf{T} = \frac{d\mathbf{T}/dt}{ d\mathbf{T}/dt }$	höfuðþverill
$\kappa = \frac{ \mathbf{v} \times \mathbf{a} }{ \mathbf{v} ^3}$	krappi
$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot (\frac{d}{dt}\mathbf{a})}{ \mathbf{v} \times \mathbf{a} ^2}$	vindingur
$\rho = \frac{1}{\kappa}$	krappageisli

Stikun með bogalengd:

$$\begin{aligned}\mathbf{T}(s) &= \mathbf{v}(s) \\ \kappa(s) &= \left| \frac{d\mathbf{T}}{ds} \right| \\ \mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s) \\ \mathbf{N}(s) &= \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|} \\ \mathbf{B}(s) &= \mathbf{T}(s) \times \mathbf{N}(s) \\ \mathbf{T}'(s) &= \kappa\mathbf{N} \\ \mathbf{N}'(s) &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}'(s) &= -\tau\mathbf{N}\end{aligned}$$

## II.2. Hlutfleiður

Snertiplan:  $S(x, y) = f(a, b) + f_1(a, b)(x - a) + f_2(a, b)(y - b)$ .

Hlutfleiður:  $f_k(\mathbf{a}) = \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{e}_k) - f(\mathbf{a})}{h}$ .

Diffrauleiki:  $\lim_{(h,k) \rightarrow (0,0)} \frac{f(a+h, b+k) - S(a+h, b+k)}{\sqrt{h^2 + k^2}} = 0$ .

Jacobi-fylki:  $D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \dots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$ .

Stigull:  $\nabla f(x, y) = f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}$ .

Stefnuafleiða:  $D_{(u,v)}f(a, b) = \lim_{h \rightarrow 0^+} \frac{f(a + hu, b + hv) - f(a, b)}{h}$ .

Taylor-margliða ( $n$ -stigs):  $P_{(n)}(x, y) = \sum_{m=0}^n \sum_{j=0}^m \frac{1}{j!(m-j)!} D_1^j D_2^{m-j} f(a, b)(x-a)^j(y-b)^{m-j}$ .

## II.3. Útgildisverkefni

Lagrange-margfaldari (fyrir  $g(x, y) = 0$ ):  $L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$ .

Lagrange-margfaldarar (fyrir  $g = h = 0$ ):  $L(x, y, z, \lambda, \mu) = f(x, y, z) + \lambda g(x, y, z) + \mu h(x, y, z)$ .

## II.4. Margföld heildi

Riemann-summa í tveimur víddum:

$$\mathcal{R}(f, P) = \sum_{i=1}^m \sum_{j=1}^n f(x_i^*, y_j^*) \Delta x_i \Delta y_j.$$

Almenn breytuskipti fyrir tvöföld heildi:  $\iint_D f(x, y) dx dy = \iint_S g(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$ .

Breytuskipti í pólhnit:  $\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$ .

Almenn breytuskipti fyrir þreföld heildi:

$$\iiint_R f(x, y, z) dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw.$$

Breytuskipti í kúluhnit:

$$\iiint_R f(x, y, z) dV = \int_{\alpha}^{\beta} \int_c^d \int_a^b f(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) \rho^2 \sin \varphi d\rho d\varphi d\theta.$$

Yfirborðsflatarmál:  $S = \iint_D \sqrt{1 + f_1(x, y)^2 + f_2(x, y)^2} dA$ .

## II.5. Vigursvið

Mætti ( $\phi$ ) fyrir  $\mathbf{F}$ :  $\mathbf{F}(x, y, z) = \nabla \phi(x, y, z)$ .

Heildi falls yfir feril:  $\int_C f(x, y) ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} dt$ .

Heildi vigursviðs eftir ferli:  $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$ .

Flatarmál stikaflatar  $\mathbf{r}$ :  $A = \iint_D dS = \iint_D \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv$ .

Heildi falls  $f$  yfir stikaflöt:  $\iint_S f dS = \iint_D f(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv$ .

Heildi vigursviðs  $\mathbf{F}$  yfir stikaflöt:  $\iint_S \mathbf{F} \cdot \mathbf{N} dS = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot \left( \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du dv$ .

## II.6. Diffur- og heildarreikningur vigursviða

$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$ .

Stigull:  $\mathbf{grad} \varphi = \nabla \varphi = \frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} + \frac{\partial \varphi}{\partial z} \mathbf{k}$

Sundurleitni:  $\mathbf{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ .

Rót:  $\mathbf{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}$ .

Sundurleitnisetning I:  $\mathbf{div} \mathbf{F}(P) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{V_\varepsilon} \iint_{S_\varepsilon} \mathbf{F} \cdot \mathbf{N} dS$ .

Setning Stokes:  $\mathbf{N} \cdot \mathbf{curl} \mathbf{F}(P) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{A_\varepsilon} \oint_{C_\varepsilon} \mathbf{F} \cdot d\mathbf{r}$ .

Setning Green:  $\oint_C F_1(x, y) dx + F_2(x, y) dy = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA$ .

Sundurleitnisetningin í tveimur víddum:  $\iint_R \mathbf{div} \mathbf{F} dA = \oint_C \mathbf{F} \cdot \mathbf{N} ds$ .

Sundurleitnisetningin, setning Gauss:  $\iiint_D \mathbf{div} \mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{N} dS$ .

$\iiint_D \mathbf{curl} \mathbf{F} dV = - \iint_S \mathbf{F} \times \mathbf{N} dS$ .

$\iiint_D \mathbf{grad} \varphi dV = \iint_S \varphi \mathbf{N} dS$ .

Setning Stokes:  $\iint_S \mathbf{curl} \mathbf{F} \cdot \mathbf{N} dS = \oint_C \mathbf{F} \cdot \mathbf{T} ds$ .

## Reiknireglur

$$(i) \quad \nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$$

$$(ii) \quad \nabla \cdot (\phi\mathbf{F}) = \phi\nabla \cdot \mathbf{F} + (\nabla\phi) \cdot \mathbf{F}$$

$$(iii) \quad \nabla(f \circ \phi) = (f' \circ \phi)\nabla\phi$$

$$(iv) \quad \Delta(f \circ \phi) = (f' \circ \phi)\Delta\phi + (f'' \circ \phi)\|\nabla\phi\|^2$$

$$(v) \quad \nabla(\chi \circ \mathbf{F}) = (((\nabla\chi) \circ \mathbf{F})\partial_1\mathbf{F})\mathbf{e}_1 + \cdots + (((\nabla\chi) \circ \mathbf{F})\partial_n\mathbf{F})\mathbf{e}_n$$

$$(i) \quad \mathbf{curl grad} \phi = \nabla \times \nabla\phi = 0$$

$$(ii) \quad \mathbf{div curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0$$

$$(iii) \quad \nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$$

$$(iv) \quad \nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

$$(v) \quad \nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} - (\mathbf{F} \cdot \nabla)\mathbf{G}$$

$$(vi) \quad \nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$$

$$(vii) \quad \mathbf{curl curl} \mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F} = \mathbf{grad div} \mathbf{F} - \nabla^2\mathbf{F}$$