

# STÆRÐFRÆÐIGREINING IIIA - FORMÚLUBLAÐ

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## Stærðfræðigreining I/IA:

### I.1. Tölur og föll

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1 \\ \sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y) \\ \cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y) \\ \tan(x \pm y) &= \frac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}\end{aligned}$$

$$\begin{aligned}\sin(-x) &= -\sin(x) & \cos(-x) &= \cos(x) \\ \sin(\pi - x) &= \sin(x) & \cos(\pi - x) &= -\cos(x) \\ \sin(\pi/2 - x) &= \cos(x) & \cos(\pi/2 - x) &= \sin(x) \\ \sin^2(x) &= \frac{1-\cos(2x)}{2} & \cos^2(x) &= \frac{1+\cos(2x)}{2} \\ \sin(2x) &= 2\sin(x)\cos(x) & \cos(2x) &= 2\cos^2(x) - 1\end{aligned}$$

### I.3. Afleiður

Taylormargliða:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Skekjkjumat:  $E_n(x) = \frac{f^{(n+1)}(X)}{(n+1)!}(x-a)^{n+1}.$

### I.4. Torræð föll

#### Andhverfur hornafalla

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arccos(x) = \frac{-1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

#### Breiðbogaföll

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\text{arsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right) \quad \text{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \quad \text{artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

### I.6. Heildun

Innsetning:  $\int f(g(x))g'(x)dx = \int f(u)du$       Hluth.:  $\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$

### I.7. Rúmmál, massi og massamiðja

Kúla:  $V = \frac{4}{3}\pi r^3$ ,  $S = 4\pi r^2$ .

Sívalningur:  $V = \pi r^2 h$ ,  $S = 2\pi r h$ .

Keila með grunnflót  $A$ :  $V = \frac{1}{3}Ah$ .

#### Snúðar

Snúið um  $x$ -ás:

$$V = \pi \int_a^b f(x)^2 dx, \quad S = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2} dx.$$

Snúið um  $y$ -ás:

$$V = 2\pi \int_a^b x f(x) dx, \quad S = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2} dx.$$

Lengd grafs:  $s = \int_a^b \sqrt{1 + (f'(x))^2} dx$ .

## Massamiðja plötu

$$\bar{x} = \frac{M_{x=0}}{m} = \frac{\int_a^b x\delta(x)f(x) dx}{\int_a^b \delta(x)f(x) dx} \quad \bar{y} = \frac{M_{y=0}}{m} = \frac{\frac{1}{2}\int_a^b \delta(x)f(x)^2 dx}{\int_a^b \delta(x)f(x) dx}.$$

## I.8. Diffurjöfnur

Línulegar fyrsta stigs diffurjöfnur,  $y' + p(x)y = q(x)$

$$y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x) dx, \quad \mu(x) = \int p(x) dx.$$

Línuleg annars stigs diffurjafna með fastastuðla,  $ay'' + by' + cy = 0$

Tilvik I:  $y(x) = Ae^{r_1 x} + Be^{r_2 x}$  ef kennijafnan hefur tvær ólíkar rauntölulausnir  $r_1$  og  $r_2$ .

Tilvik II:  $y(x) = Ae^{kx} + Bxe^{kx}$  ef kennijafnan hefur eina tvöfalda rauntölulausn  $k = -\frac{b}{2a}$ .

Tilvik III:  $y(x) = Ae^{kx} \cos(\omega x) + Be^{kx} \sin(\omega x)$  ef kennijafnan hefur engar rauntölulausnir, bara tvinntölulausnir  $r_1 = k + i\omega$  og  $r_2 = k - i\omega$  þar sem  $k = -\frac{b}{2a}$  og  $\omega = \frac{\sqrt{4ac-b^2}}{2a}$ .

## I.9. Runur og raðir

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

## I.10. Veldaraðir

$$\text{Kvótapróf: } \frac{1}{R} = L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \text{Rótarpróf: } \frac{1}{R} = L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \quad \text{fyrir öll } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{fyrir öll } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{fyrir öll } x$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad \text{fyrir } -1 < x < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \dots \quad \text{fyrir } -1 < x < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{fyrir } -1 < x \leq 1$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad \text{fyrir } -1 \leq x \leq 1$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \quad \text{fyrir öll } x$$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \quad \text{fyrir öll } x$$

# Mathematical Analysis IIIA:

## 1. Linear Ordinary Differential Equations

### First Order Linear Equations

$y' + p(x)y = g(x)$ :

$$y(x) = e^{-M(x)} \left( C + \int g(x)e^{M(x)} dx \right), \quad \text{where } M(x) = \int p(x)dx.$$

### Reduction of order

$y'' + p(x)y' + q(x)y = 0$ :

$$u_2(x) = u_1(x) \int \frac{e^{-P(t)}}{u_1^2(t)} dt, \quad \text{where } P(x) = \int p(x) dx.$$

### Shift rule

$$P(D)(e^{\lambda x} f(x)) = e^{\lambda x} P(D + \lambda) f(x)$$

### Green function

Solution to  $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0y = g(x)$ ,  $U_1(y) = U_2(y) = \dots = U_n(y) = 0$ ,

$$y(x) = \int_a^b G(x, \xi)g(\xi) d\xi.$$

## 2. Separation of Variables

### Newton's equation

Equation:  $x'' = F(x)$  or  $(\{x' = y, y' = F(x)\})$ .

Potential:  $U(x) := - \int F(x) dx$ .

Energy:  $E(x, y) := U(x) + \frac{y^2}{2}$ .

Period of small oscillations:  $\lim_{\epsilon \rightarrow 0} L_\epsilon = \frac{2\pi}{\sqrt{U''(x_0)}}$ .

## 3. Series Solutions of Linear Equations

### Recursive formula for the coefficients at an ordinary point

$$(k+2)(k+1)c_{k+2} + \sum_{j=0}^k (k-j+1)a_j c_{k-j+1} + \sum_{j=0}^k b_j c_{k-j} = 0.$$

### Recursive formula for the coefficients at a regular singular point

$$P(k+\gamma)c_k + \sum_{j=0}^{k-1} ((j+\gamma)a_{k-j} + b_{k-j})c_j = 0, \quad k \in \mathbb{N},$$

with the understanding that for  $k = 0$  the sum is empty, and the indicial polynomial  $P$  is given by

$$P(X) = X(X-1) + a_0X + b_0.$$

## 4. Existence Theory

### Picard iteration

$$\phi_{m+1}(x) = y_0 + \int_{x_0}^x f(t, \phi_m(t)) dt.$$

## 5. The Exponential of a Matrix

Solution to  $y' = A(x)y + h(x)$ ,  $y(x_0) = \eta$  is  $y(x) = e^{(x-x_0)A}\eta + \int_{x_0}^x e^{(x-t)A}h(t) dt$ .

### Newton divided differences

$$F[\lambda_l] = F(\lambda_l), \quad F[\lambda_l, \dots, \lambda_{l+k}] = \frac{F[\lambda_l + 1, \dots, \lambda_{l+k}] - F[\lambda_l, \dots, \lambda_{l+k-1}]}{\lambda_{l+k} - \lambda_l}.$$

If  $\lambda_l = \dots = \lambda_{k+l}$ , then  $F[\lambda, \dots, \lambda_{l+k}] = F^{(k)}(\lambda_l)/k!$ .

## 7. Sturm-Liouville Theory

### Sturm-Liouville form

$$Ly = \frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y = g(x).$$

### Inner product, norm and Bessel inequality

$$\langle u, v \rangle = \int_a^b u(x)\overline{v(x)} dx, \quad \|u\|_2 = \sqrt{\langle u, u \rangle}, \quad \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2.$$

### Solutions to non-homogeneous equations

$$Ly - \mu y = h(x), \quad U_1(y) = 0, \quad U_2(y) = 0.$$

If  $\mu$  not an eigenvalue:  $y(x) = \sum_{n=1}^{\infty} (\lambda_n - \mu)^{-1} \langle h, \phi_n \rangle \phi_n(x)$ .

If  $\mu$  is an eigenvalue:  $y(x) = \sum_{\lambda_n \neq \mu} (\lambda_n - \mu)^{-1} \langle h, \phi_n \rangle \phi_n(x)$ .

## Fourier Analysis

### Fourier series $f$ on $[-L, L]$ : Exponential form:

$$f(x) \sim \frac{c_0}{2} + \sum_{n=-\infty}^{\infty} c_n e^{\frac{inx\pi}{L}},$$

$$c_n = \hat{f}(n) = \frac{1}{2L} \int_{-L}^L f(x) e^{-\frac{inx\pi}{L}} dx, \quad n \in \mathbb{Z}.$$

### Fourier series $f$ on $[-L, L]$ : Trigonometric form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0. \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

### Connections between coefficients:

$$\begin{aligned} c_0 &= a_0, & a_0 &= c_0, \\ c_n &= (a_n - ib_n)/2, & \text{for } n > 0, & a_n &= c_n + c_{-n}, & \text{for } n > 0, \\ c_n &= (a_{-n} + ib_{-n})/2, & \text{for } n < 0. & b_n &= i(c_n - c_{-n}), & \text{for } n > 0. \end{aligned}$$

## Solving Ordinary Differential Equations

If  $P(D)u = f$ , then  $P(in)\hat{u}(n) = \hat{f}(n)$  and

$$u(x) = \sum_{n=-\infty}^{\infty} \frac{\hat{f}(n)}{P(in)} e^{inx}.$$