Vacation Sheet 1

Ben Eills

December 7, 2011

These sheets will cover a period of about 4 weeks; they are designed to familiarize you with the content of the Edexcel C1 Syllabus. Some questions resemble those you will encounter when sitting the exam, others are written to test and extend your understanding of the topics.

Submit neat answers to all the questions you attempt (include all the working you wish to be marked, e.g. calculations, diagrams). Partial answers should also be handed in: these may be awarded partial credit.

1. This question tests your understanding of basic mathematical language and grammar, as is required in the Syllabus 'Preamble'. Decide which of the following statements are true, and which are false. Justify your answers.

   (a) n is even $\Rightarrow 4n + 3$ is odd

   (b) p is prime $\Rightarrow p + 1$ is not prime

   (c) There is some real number a such that $x^2 + ax + a$ has exactly one root

   (d) Every **surd** of the form $\frac{1}{a+b\sqrt{2}}$ where a and b are integers may be written as a fraction with a rational denominator

2. We review here the basic "rules" of calculus you have learned thus far in the course.

   (a) Find the derivative of a simple power $\frac{d}{dx}Ax^n$

   (b) Find the derivative of a general polynomial, $\frac{d}{dx}[A_0+A_1x^1+...+A_nx^n]$

   (c) We call a point x at which $\frac{dy}{dx} = 0$ a **critical point** or a **turning point**. Find all turning points (if indeed there are any) of the following functions using calculus and indicate where they lie on the graphs of the functions.

      i. $y = x^2 - 2x - 35$

      ii. $y = (x + 1)(x + 2)(x + 3)$

      iii. $y = x/4$

      iv. $y = x^3$

(d) Do the corresponding questions to parts (a) and (b) using integration, rather than differentiation. Find the integrals of the functions in part (c).

3. Consider the quadratic $z = ax^2 + bx + c$, where a, b and c are constants.

   (a) "Complete the square" on this quadratic and deduce the quadratic formula (used to find roots of the equation $z = 0$)

   (b) Write down the **discriminant** of the quadratic, and sketch the graphs corresponding to the three cases:

      i. $discriminant = 0$

      ii. $discriminant < 0$

      iii. $discriminant > 0$

      [You do not need to determine points of intersection, etc.]

   (c) Assuming that the discriminant is positive, write down the two roots of $z = 0$

4. *[This question is more difficult. Try it.]* Let n be some positive integer.

   (a) What can you say about the integers $2n$ and $2n + 1$?

   (b) Let $p(x)$ and $q(x)$ be monic polynomials of degree $2n$ and $2n + 1$ respectively. *(The degree of a polynomial is the highest power of x that appears. e.g.* $3x^2 + 5$ *has degree 2. A* ***monic*** *polynomial has coefficient 1 for the highest power of x. e.g.* $x^3 + 7$ *is monic, but* $2x - 9$ *is not monic.)* A function is ***bounded below*** if you can draw a horizontal line that its graph never goes below. We similarly define ***bounded above***. So, for example, the function x^2 is bounded below, but not above. What can you say about the boundedness of $p(x)$ and $q(x)$?

   (c) Let $r(x)$ be a polynomial that is bounded both above and below. What can you say about $r(x)$? Hence determine $\frac{d}{dx}r(x)$. *(Your answer should be very simple!)*

Vacation Sheet 2

Ben Eills

December 11, 2011

Submit neat answers to all the questions you attempt (include all the working you wish to be marked, e.g. calculations, diagrams). Partial answers should also be handed in: these may be awarded partial credit.

1. Again we test understanding of mathematical language. Decide which of the following statements are true, and which are false. Justify your answers.

   (a) $b^2 - 4ac \geq 0$ implies that $a^2 + bx + c = 0$ has a root

   (b) a and b are odd $\Rightarrow a \times b$ is odd

   (c) $a^2 + bx + c = 0$ has a root implies that $b^2 - 4ac > 0$

2. We give calculus question in the examination style.

   (a) Let $y = (x + 3)(x - 5)$
   Write down the roots of y and the derivative $\dfrac{dy}{dx}$ of y
   Find the gradient of the tangent lines to y at the roots, and hence the equations of both tangent lines.
   Do these lines meet? If so, where?
   Convey the above information in a brief sketch.

   (b) y is a quadratic which passes through the point $(1, 18)$ and has derivative $\dfrac{dy}{dx} = 2x - 11$
   Determine y using integration.

   (c) Differentiate and integrate the following functions. *(Hint: π is nothing more than a constant, like any other number)*

      i. $y = x^{100}$
      ii. $y = 0$
      iii. $y = 1 + x + x^2$
      iv. $y = 1 + x + \cdots + x^{100}$
      v. $y = x^{\pi}$

3. This question is about finding solutions to systems of equations. Recall that a **solution** to a list of equations is a value that makes every separate equation true; graphically, a solution is a point where the graphs of every equation intersect. Thus a particular system may have none, one, two, etc. solutions. Some even have infinitely many! Try to find all solutions of the following systems algebraically. Draw graphs if you can.

   (a)
   $$7x + 8y = 10$$
   $$2x + 3y = 5$$

   (b)
   $$y = x^2$$
   $$2y = x + 6$$

4. Sometimes having quadratic equations or other polynomials in the denominator can be a problem (you will see this when you begin to apply calculus to such expressions). We use the method of **partial fractions** to *decompose* the more complicated expression into the sum of simple ones. Use the method to simplify the following.

   (a)
   $$\frac{5x + 19}{(x+3)(x+5)}$$

   (b)
   $$\frac{1}{(x+5)(x+7)}$$

   (c)
   $$\frac{x}{x^2 - 100}$$

Vacation Sheet 3

Ben Eills

December 15, 2011

Submit neat answers to all the questions you attempt (include all the working you wish to be marked, e.g. calculations, diagrams). Partial answers should also be handed in: these may be awarded partial credit. ***This sheet involves no calculus!***

1. We test algebraic manipulation. Simplify the following.

   (a) $\sqrt{lcm(6,4)}\sqrt{\frac{1}{2}+8^{-\frac{2}{3}}}$
   [recall that lcm(a, b) denotes the lowest common multiple of a and b]

   (b) $\dfrac{1}{7+\sqrt{7}}$

   (c) $(7x-9)^2 < 9$

   (d) $(x+5)(x-6) < (x+10)(x-1)$

   (e) $\dfrac{x^3}{10} + \dfrac{x^2}{100} + \dfrac{x}{1000}$

2. Sketch the graphs and indicate the graph transformation(s) involved.

   (a) $y = \dfrac{1}{x}$ **vs.** $y = \dfrac{3}{x}$

   (b) $y = (x+1)(x+2)$ **vs.** $y = (x+3)(x+4)$

   (c) $y = x(x-1)^2$ **vs.** $y = 4(x-2)(x-3)^2$

3. We test coordinate geometry.

   (a) Give the relationship between the gradient of a line and the angle θ it makes with the x-axis (defined to be zero if the line is horizontal), using high school trigonometry.

   (b) We may specify a point in the plane with a minimum of two numbers (the two coordinates of the point). We need a minimum of 6 numbers to specify a triangle in the plane (two for each of the three vertices). Given that a line can be fully specified with any two distinct points lying on it, it is clear that a line can be specified with 4 numbers. Is this the minimum? If not, give a way to specify lines using less numbers.

(c) Given that a line passes through the origin, what is the only other datum needed to fully specify it?

(d) Give the relationship between:

   i. the gradients of two parallel lines

   ii. the gradients of a line and its normal

(e) Suggest a symbol to denote the gradient of a vertical line.

4. Consider the simple sequence:

$$(a_n) = 1, 2, 3, 4, \ldots$$

given by the rule $a_n = n$

(a) Write down the terms a_{17}, a_{23} and a_{100}

(b) We use the abbreviation S_n to denote the sum of the first n terms.
So $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{100}$
By simple calculation, $S_5 = 1 + 2 + 3 + 4 + 5 = 15$
Similarly determine S_6, S_7 and S_8
Notice that these are the number of presents given to you by your true love on the 6th, 7th and 8th days of Christmas. Owing to the large number of presents you are receiving this Christmas you are hoping to return all the presents you receive on the 12th day to the store for their cash equivalent. You need to know how many present you will receive on the 12th day, but are unable to do the above arithmetic for this day (the numbers are too large). We look for an easier method.

(c) By representing each sum S_n as a triangle of dots and using the formula for the area of a triangle, determine the number of dots in each triangle of base n, and hence a formula for S_n in terms of n.

(d) Write down the total number of presents you hope to receive on the 12th day of Christmas.
Write down the value of $1 + 2 + \cdots + 100$

(e) By taking out a factor, compute the sum $3 + 6 + 9 + \cdots + 150$

Vacation Sheet 4

Ben Eills

December 17, 2011

Submit neat answers to all the questions you attempt (include all the working you wish to be marked, e.g. calculations, diagrams). Partial answers should also be handed in: these may be awarded partial credit.

1. We test knowledge of transformations.

   (a) Graph the following transformations for a constant a.

      i. $f(x) \longrightarrow f(x+a)$

      ii. $f(x) \longrightarrow af(x)$

      iii. $f(x) \longrightarrow f(ax)$

      iv. $f(x) \longrightarrow f(x)+a$

   (b) Pick some interesting cubic and let this be f(x). Graph f(x) and each of the above transformations when $a = 3$.

2. Consider the two points $P = (a, b)$ and $Q = (c, d)$.

   (a) Suppose that $a \neq c$. Give the equation of the unique line passing through P and Q in the form $y = mx + c$. Why is it important that $a \neq c$?

   (b) Suppose now that $a = c$. Give the equation of the line *(you will not be able to write it in the form $y = mx + c$).*

   (c) Suppose again that $a \neq c$, and that P and Q lie in the upper-right quadrant (i.e. $a, b, c, d \geq 0$). Give the area of the triangle bounded by the line passing through P and Q and the coordinate axes.

3. The Greek mathematician Euclid set forth certain postulates prefacing his work *The Elements*, the most famous of which was the fifth: *"Given a line L and a point P (not on the line), there is exactly one line passing through P that is parallel to L."*

   (a) Let L be the line $y = 3x - 7$, and P the point $(1, 1)$. Give the special line passing through P, parallel to L.

(b) Consider the (Euclidean) lines L and M, given by equations $y = ax+b$ and $y = cx + d$, respectively. Give conditions on a and b for:

   i. L and M to be parallel

   ii. L and M to be perpendicular

(c) Euclid's fifth postulate is only true of "Euclidean Geometry", that is, geometry in planes. In spherical geometry (like on the surface of the globe), "lines" are actually "greater circles" going around the whole sphere at its widest point. Thus the equator and Greenwich Meridian are "lines". Considering the equator and Paris, how many "lines" can be drawn through Paris that are parallel to the Equator?

4. We test the differential and integral calculus.

   (a) Suppose $f(x)$ passes through the origin and has derivative $f'(x) = 3x^2 - 1$. Determine fully $f(x)$. Using the "difference of squares" trick, factorize it as far as possible.

   (b) Suppose that the minimum value of $g(x)$ is -1, and that $g(x)$ has derivative $g'(x) = 2x - 12$. Determine fully $g(x)$, factorizing your answer.

   (c) Differentiate and integrate (the Greek letters are constants):

      i. αx^β

      ii. $\delta + x^{\frac{\sigma}{\phi}}$

   (d) Invent two symbols for your own language, and state the simple differentiation "power rule" in terms of them. **Make them sufficiently complex so that they will not exist in any real-world alphabet!**

Vacation Sheet 5

Ben Eills

December 20, 2011

Submit neat answers to all the questions you attempt (include all the working you wish to be marked, e.g. calculations, diagrams). Partial answers should also be handed in: these may be awarded partial credit.

1. We test knowledge mathematical language. Decide which of the following are true statements.

   (a) Graph the following transformations for a constant a.

      i. a is positive implies that a^2 is positive

      ii. a^2 is positive implies that a is positive

      iii. Whenever a^2 is odd, $a + 1$ is even

      iv. The discriminant being positive is a sufficient condition for a quadratic to have two distinct roots.

2. Briefly sketch the sine and cosine functions on the interval $[0°, 720°]$.

   (a) Let $f(x)$ be the sine function. Sketch on a separate graph the transformation $f(x + 90°)$

   (b) Comparing your transformation to the cosine function you sketched, write down a relationship between sine and cosine of the form $sin(x + a) = cos(x)$ for some constant a.

   (c) If $f(x) = f(x + b)$ for all x and some constant b, we say that f has **period** b. Intuitively, this means that the graph of $f(x)$ "repeats" itself every b. Only very special functions are **periodic** in this way. Is sine periodic? If so, find its period.

3. We test differentiation and integration technique. Differentiate and integrate the following.

   (a) $f(x) = x^{\frac{1}{1}} + x^{\frac{1}{2}} + \cdots + x^{\frac{1}{100}}$

   (b) $g(x) = x^{\frac{1}{\sqrt{2}}}$

   (c) $h(x) = \dfrac{(x+2)^2}{x^{-\frac{1}{2}}}$

4. Consider the quadratic $z(x) = x^2 + kx + k$

   (a) Write down the interval for k for which:

      i. $z(x)$ has two distinct roots

      ii. $z(x)$ has a repeated root

      iii. $z(x)$ has no real roots

   (b) From now on, suppose that $z(x)$ has two distinct roots. What is the minimum point of $z(x)$?

   (c) Does $z(x)$ have a maximum point?

   (d) We define the **second derivative** of $z(x)$ to be the derivative of the derivative and write it as $\frac{d^2z}{dx^2}$. Therefore, $\frac{d^2z}{dx^2} = \frac{d}{dx}\frac{dz}{dx}$. Find the second derivative of $z(x)$.

Vacation Sheet 6

Ben Eills

December 21, 2011

Submit neat answers to all the questions you attempt (include all the working you wish to be marked, e.g. calculations, diagrams). Partial answers should also be handed in: these may be awarded partial credit.

1. The ancient Babylonians wanted to find a way to compute square roots. If you think about it, you don't really know how to do this, except in the case of "nice" perfect squares (of fractions involving them). Suppose for a minute that we're trying to find the square root of 10 (which lies somewhere between 3 and 4), and that we have a first "approximation" x_0 that we would like to improve. If the approximation is too low, then $\dfrac{10}{x_0}$ is too high, but their average $x_1 = \dfrac{x_0 + \frac{10}{x_0}}{2}$ is closer to the true square root i.e. a better approximation. The Babylonians used this repeatedly to achieve a close estimate to the true root.

   We define the sequence (x_n) as follows (using 2 as our first guess):

   $$x_0 = 2$$
   $$x_{n+1} = \frac{x_n + \frac{10}{x_n}}{2}$$

   (a) Our first approximation ($x_0 = 2$) is not very good. Find the second and third approximations, x_1 and x_2 (without using a calculator).

   (b) Which two integers does $\sqrt{1000}$ lie between?

   (c) Define a sequence (b_n), similar to the above, to find $\sqrt{1000}$ using 20 as your first approximation.

   (d) Write down b_0, b_1, and b_2 (you may use a calculator for this!)

   (e) Find $(b_2)^2$ to determine how good your approximation is. How close is it to 1000?

2. Sketch the circle of radius r, centre origin, given by $x^2 + y^2 = r^2$. Indicate all axis intersections.

(a) Let a be a positive constant. Sketch the 4 basic transformations of this graph using constant a. e.g. $f(x+a)$, $f(ax)$, etc. Indicate all axis intersections.

(b) For each of your transformations, identify its geometric shape and its centre.

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3. We test integration technique. Suppose that $\int f(x)dx = g(x)$.
Determine the following, simplifying fully. (Your answers should be in terms of $g(x)$)

   (a) $\int (t + t^3 + t^5)dt$

   (b) $\int 4f(x)dx$

   (c) $\int (x^{1/2} + x^{1/3} - f(x))dx$

4. Let P = (3, 5) and Q = (5, 7) be pints in the Euclidean plane.

   (a) Find the length of the line segment PQ, and its midpoint C.

   (b) What is the circumference of the circle having centre C that passes through both P and Q? And its area?

   (c) We want to find a quadratic of the form $y(x) = ax^2 + bx$ that passes through both P and Q. So y must satisfy $y(3) = 5$ and $y(5) = 7$. We set up the system of equations:

$$5 = 3^2a + 3b$$

$$7 = 5^2a + 5b$$

   Solve the system to find a and b and hence give the required quadratic $y(x)$.

Vacation Sheet 7

Ben Eills

December 22, 2011

Submit neat answers to all the questions you attempt (include all the working you wish to be marked, e.g. calculations, diagrams). Partial answers should also be handed in: these may be awarded partial credit.

1. So far, you have been performing what is known as **integration in elementary terms**, or **primitive-finding**. Although the use of the derivative is clear to you (it gives you the gradient function), the value of the integral has not been made clear.

   The integral was first formulated in terms of finding area and was only later (surprisingly) discovered to be the reverse operation of differentiation.

   Suppose that $\int f(x)dx = F(x)$, and that we want to find the area A under the graph of f(x) between two points a and b. We define the **definite integral** as follows:

   $$A = \int_a^b f(x)dx = F(b) - F(a)$$

   Find the specified areas (try to draw graphs where possible). The first example has been done for you.

   (a) **Find the area under** $f(x) = (x + 2)(x - 1)$ **between 2 and 3.**

   We find the integral of $f(x)$.

   $$\begin{aligned} F(x) &= \int f(x)dx \\ &= \int (x+2)(x-1)dx \\ &= \int (x^2 + x - 2)dx \\ &= \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x \end{aligned}$$

We find the area under $f(x)$ between 2 and 3.

$$\begin{aligned} Area &= \int_2^3 f(x)dx \\ &= F(3) - F(2) \\ &= (\frac{1}{3}3^3 + \frac{1}{2}3^2 - 2 \cdot 3) - (\frac{1}{3}2^3 + \frac{1}{2}2^2 - 2 \cdot 2) \\ &= \frac{15}{2} - \frac{2}{3} \\ &= \frac{41}{6} \end{aligned}$$

(b) Find the area under $f(x) = 2x$ between 0 and 5. Check your answer with the formula for the area of a triangle.

(c) Find the area of the shape bounded by $f(x) = -(x-3)(x+3)$ and the x-axis (i.e. between -3 and +3)

(d) Find $\int_1^9 x^{\frac{1}{2}} dx$

2. Solve the following inequalities, giving your answers in the simplest form possible.

   (a) $x^2 + 5x - 5 < 3x + 10$

   (b) $(x-4)(x-6) < 0$ and $2x - 7 \leq 3$

   (c) i. $z(z+7) > 0$

       ii. $z(z+7) = 0$

       iii. $z(z+7) < 0$

3. Let $\phi(t) = t^2 - 5t - 11$ and $\theta(t) = t^2 + kt - k$

   (a) What are the discriminants Δ_ϕ and Δ_θ of $\phi(t)$ and $\theta(t)$, respectively?

   (b) Give the intervals where $\phi(t)$ is positive, negative and zero.

   (c) Similarly, do the above for $\theta(t)$ (giving your answer in terms of k).

   (d) Write down some value of k so that $\theta(t)$ has two roots. Give the roots of $\phi(t)$ and $\theta(t)$.

4. Let P = (3, 5), Q = (9, 5) and R = (4, 7) be points in the Euclidean plane.

   (a) Give the perimeter of triangle $\triangle_{PQR}$

   (b) Give the area of triangle $\triangle_{PQR}$

   (c) Can you find the **midpoint** of $\triangle_{PQR}$? (This is the point where the shape would be perfectly balanced if rested on a pin)

Vacation Sheet 8

Ben Eills

December 26, 2011

Submit neat answers to all the questions you attempt (include all the working you wish to be marked, e.g. calculations, diagrams). Partial answers should also be handed in: these may be awarded partial credit. Try to complete the questions at an examination pace; time yourself!

1. Given that $y = 3x^7 - 1 + x^{\frac{1}{2}}$ for $x > 0$
   Find $\dfrac{dy}{dx}$

2. (a) Express $\sqrt{147}$ in the "surd form" $a\sqrt{3}$, where a is an integer.
   (b) Express $(7 - \sqrt{5})^2$ in the "surd form" $a + b\sqrt{3}$, where a and b are integers.
   (c) Express $36 - 16\sqrt{5}$ in the form $(a - b\sqrt{5})^2$, where a and b are integers.

3. Let $f(x) = \dfrac{1}{x}$
   (a) Sketch $4f(x) + 3$, marking all intersections and asymptotes.
   (b) Give the equations of all asymptotes.
   (c) Find both $\dfrac{df}{dx}$ and $\dfrac{d}{dx}(4f(x) + 3)$

4. Consider the two straight lines defined by:

$$3x + 5y = 12$$

$$2x + 10y = 7$$

   (a) Sketch them both and convince yourself that they intersect (i.e. are not parallel)
   (b) Solve the system of equations algebraically.
   (c) What is the area of the triangle bounded by the two lines and the y-axis?

5. The quadratic $3x^2 - 5x + (k + 2)$ has no real roots.

(a) Give the range of possible values for k.

(b) Sketch the graph, giving the value of the y-intercept (in terms of k).

6. Find

$$\int (2 - 7\sqrt{2})^2 dx$$

7. Given that a curve has equation $y = f(x)$, passes through the point $P = (2, 7)$, and satisfies

$$f'(x) = 3x^2 - 4$$

Find and sketch $f(x)$.