Leibniz integral formula for $\frac{\text{Int}}{\text{Diff}}$

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January 31, 2025

1 Mean Value Theorem

Including a previous proof of the MVT, August 2019:
Let $f(x) \in C^{[a,b]}(x)$, where $C^{[a,b]}(x)$ is the set of all non-singular functions of a real variable x that are continuous on an interval $[a, b]$ of $\mathbb{R}$. Then for some $x_0 \in [a, b]$, $f(x_0) = f_{\text{avg}}[a, b]$.
Proof:

First, note that

$$f_{\text{avg}}[a,b] = \frac{\int_a^b f(x)dx}{(b-a)}. \tag{1}$$

Since in one-variable we can interpret a definite integral as the (signed) area bounded between the curve defined by $f(x)$ and the x-axis (Riemann sum), we can put a bound on the integral. Define $\tilde{f}(x) = f(x) - \min f(x) \geq 0$. Then from Eq. (1),

$$0 \leq \tilde{f}_{\text{avg}}[a,b] \leq \max \tilde{f}(x)$$

or equivalently,

$$\min f(x) \leq f_{\text{avg}}[a,b] \leq \max f(x). \tag{2}$$

The coordinates corresponding to the minimum and maximum of $f(x), x_{\min}$ and $x_{\max}$, form a subinterval $[x_{\min}, x_{\max}] \in [a, b]$. Since $f(x) \in C^{[a,b]}(x)$, for $x \in [x_{\min}, x_{\max}]$ $f(x)$ must take on all values in $[\min f(x), \max f(x)]$. Therefore for some (possibly multiple) $x_0 \in [x_{\min}, x_{\max}] \in [a, b]$, $f(x_0) = f_{\text{avg}}[a, b]$.

Continuity of the functions is important, because if the functions are not continuous then this is only true for a restricted set of discontinuous functions. For example, one function that does not satisfy the MVT (for an interval containing the discontinuity) is

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}, \tag{3}$$

for any finite interval $[a, b]$ with $a < 0$ and $b > 0$ since $0 < f_{\text{avg}} = \frac{b}{b-a} < 1$ and $f(x)$ is either 0 or 1. As long as $f(x)$ is continuous (and non-singular) on an interval, then the MVT holds on that interval.

2 Leibniz Formula

What we want here is a formula for the rate of change of the definite integral of a surface $f(x,t)$

$$I[f(x,t);a(x),b(x)] = \int_{a(x)}^{b(x)} f(x,t)dt \tag{4}$$

as we increase x, or dI/dx. Here, $a(x)$ and $b(x)$ are continuous and differentiable paths in the $t-x$ plane, or two definite mappings like $t(x)$. Here it is assumed they do not intersect, and $-\infty < a(x) < b(x) < \infty$, though I believe we can show that the $a < b, \forall x$ condition can be relaxed. Also, we can speak to the limits $|a|, |b| \longrightarrow \infty$ as long as the surface $f(x,t)$ vanishes rapidly enough in these limits, so that the integral remains convergent. We will do this later.

A straightforward approach to obtain the formula, and the only one I have tried so far, is to apply the limit definition of the derivative:

$$\begin{aligned}\frac{dI(x)}{dx} &= \lim_{\Delta x \to 0} \frac{I(x+\Delta x) - I(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\int_{a(x+\Delta x)}^{b(x+\Delta x)} f(x+\Delta x,t)dt - \int_{a(x)}^{b(x)} f(x,t)dt\right) \end{aligned} \tag{5}$$

Now, we are concerned with the limiting behavior of this expression as $\Delta x \longrightarrow 0$, as it becomes a small parameter (as small as we like, but non-zero, yet). So we can perform a Taylor series expansion of both the integrand, and the limits in the first integral in the expression (5). Let us focus on that integral for now:

$$\int_{a(x+\Delta x)}^{b(x+\Delta x)} f(x+\Delta x,t)dt = \int_{a(x)+a'(x)\Delta x....}^{b(x)+b'(x)\Delta x+....} \left(f(x,t) + f'(x,t)\Delta x + \frac{1}{2!}f''(x,t)\Delta x^2 + ...\right) dt \tag{6}$$

where the primes denote partial differentiation with respect to x, $b'(x) = \partial b(x)/\partial x$. Note that in single-variable calculus (where we suppress the $F(t)dt$),

$$\int_a^{b+r} = \int_a^b + \int_b^{b+r} \tag{7}$$

$$\int_{a+r}^{b} = \int_a^b - \int_a^{a+r}. \tag{8}$$

We can apply these rules to our expanded integral to write Eq. (6) as

$$
\begin{aligned}
&\int_{a(x)+a'(x)\Delta x....}^{b(x)+b'(x)\Delta x+....} \left(f(x,t) + f'(x,t)\Delta x + \frac{1}{2!} f''(x,t)\Delta x^2 + ... \right) dt \\
&= \int_{a(x)}^{b(x)} f(x+\Delta x, t)dt + \int_{b(x)}^{b(x)+b'(x)\Delta x+...} f(x+\Delta x,t)dt - \int_{a(x)}^{a(x)+a'(x)\Delta x+...} f(x+\Delta x,t)dt \\
&= \int_{a(x)}^{b(x)} f(x,t)dt + \Delta x \int_{a(x)}^{b(x)} (f'(x,t) + \frac{1}{2} f''(x,t)\Delta x + ...)dt \\
&\int_{b(x)}^{b(x)+b'(x)\Delta x+...} f(x,t)dt + \Delta x \int_{b(x)}^{b(x)+b'(x)\Delta x+...} (f'(x,t) + \frac{1}{2} f''(x,t)\Delta x + ...)dt \\
&- \int_{a(x)}^{a(x)+a'(x)\Delta x+...} f(x,t)dt - \Delta x \int_{a(x)}^{a(x)+a'(x)\Delta x+...} (f'(x,t) + \frac{1}{2} f''(x,t)\Delta x + ...)dt
\end{aligned} \tag{9}
$$

The first term is canceled by the subtraction in Eq. (5), and when we divide the rest by Δx and take the limit as $\Delta x \longrightarrow 0$, we have

$$
\begin{aligned}
\frac{dI}{dx} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{b(x)}^{b(x)+b'(x)\Delta x+...} & f(x,t)dt - \frac{1}{\Delta x} \int_{a(x)}^{a(x)+a'(x)\Delta x+...} f(x,t)dt \\
&+ \int_{a(x)}^{b(x)} f'(x,t)dt \\
&+ \int_{b(x)}^{b(x)+b'(x)\Delta x+...} f'(x,t)dt - \int_{a(x)}^{a(x)+a'(x)\Delta x+...} f'(x,t)dt
\end{aligned} \tag{10}
$$

Now, the last two terms here both vanish as $\Delta x \to 0$ as the limits of integration become nothing (and we assume f' remains finite). Another way of seeing this is to note that these integrals go as Δx and thus vanish in the limit. Furthermore, using Eq. (1)

$$
\begin{aligned}
\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{b(x)}^{b(x)+b'(x)\Delta x+...} f(x,t)dt &= \lim_{\Delta x \to 0} f_{\text{avg}}(x, [b(x), b(x) + b'(x)\Delta x + ...]) \left(b'(x) + \frac{1}{2} b''(x)\Delta x + ... \right) \\
&= f(x, b(x))b'(x),
\end{aligned} \tag{11}
$$

with a similar result for the $a(x)$ integral. Thus we are left with

$$
\begin{aligned}
&\frac{dI}{dx} = \\
&\quad = \frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt \right) = \left[f(x,b(x)) \frac{\partial b(x)}{\partial x} - f(x,a(x)) \frac{\partial a(x)}{\partial x} \right] + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial x} dt
\end{aligned} \tag{12}
$$

Equation (12) is known as the Leibniz Formula for differentiation under the integral sign.

3 Questions regarding applications of the Leibniz formula

- Use Eq. (12) to derive the Euler-Lagrange equation, by finding an extremum of I with $f(x,t) = L(y(x,t), \dot{y}(x,t); t)$ (dot being derivative w/respect to t).

- Generalize the Euler-Lagrange result to one with prescribed variation of the boundary points $a(x), b(x)$.