ECON42720 Causal Inference and Policy Evaluation

2 Regression Recap

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About this Lecture

Linear regression is by far the most important **estimation technique in causal inference**

Yes, I know, machine learning is all the rage these days

- But a linear approximation is often the best we can do
- It's already hard enough to get the linear approximation right
- Fancy techniques are not always better

Resources

The material behind these slides can be found in any good econometrics textbook. For introductory econometrics, I recommend

- Wooldridge, J. Introductory Econometrics: A Modern Approach. 7th Edition. South-Western College Publishing, 2019.
- Stock, J. and M. Watson. Introduction to Econometrics. 3rd Edition. Pearson, 2017.

Regression recap

Why do we use linear regression?

- What we want to approximate: the conditional expectation function (CEF)

How does regression work?

- Interpretation of coefficients with and without controls
- Estimation with OLS and Inference

Undergrad Recap: Goal of Linear Regression

Quantify the **expected effect** of a **one unit change in X on Y**

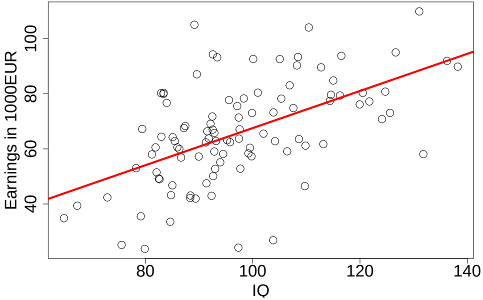
- If X goes up by one unit, by how many units does Y go up or down?
- **Causal interpretation:** If we/nature/an experimenter changes X by one unit, what is the expected effect on Y?

This **effect** is equivalent to the **slope** coefficient β_1 in a **linear regression model**

$$Y = \beta_0 + \beta_1 X + u$$

Linear Regression

Regression analysis means that we **fit a straight line** through (X, Y) data points



In this example, the **regression line** is Earnings $= -3000 + 700$ IQ

- An increase in the IQ by one point increases earnings on average by 700 EUR

What we are looking for: the conditional expectation function

In undergraduate econometrics, you probably learned about a **population regression model**

- the idea is that there is a **true relationship** between X and Y that we want to estimate
- we have a **sample** of n observations from this population and estimate $\widehat{\beta}_0, \widehat{\beta}_1$ using OLS

But is the population regression model (PRM) really what we are looking for?

- Yes and no. The PRM is an approximation of our object of interest
- This object is called the **conditional expectation function** (CEF)

Ingredients: Random Variables

Random variables are variables that take on different values with a certain probability

- x is a random variable that takes on values $x_1, x_2, \ldots, x_n$ with probabilities $f(x_1), f(x_2), \ldots, f(x_n)$

Expected value: the average value of a random variable

$$
\begin{aligned}
E(x) &= x_1 f(x_1) + x_2 f(x_2) + \cdots + x_k f(x_n) \\
&= \sum_{j=1}^{n} x_j f(x_j)
\end{aligned}
$$

Expected Value: Example

$x \in \{-1, 0, 2\}$ with probabilities $f(-1) = 0.3, f(0) = 0.3, f(2) = 0.4$. The expected value of X is

$$\begin{aligned} E(x) &= (-1)(0.3) + (0)(0.3) + (2)(0.4) \\ &= 0.5 \end{aligned}$$

Notation

We denote **random variables with lower case letters** x, y.

Typically, we **do not use indices when we talk about the population**. For example, the linear relationship between x and y in the population is

$$y = \beta_0 + \beta_1 x + u$$

Realisations of a random variable are denoted with **lower case letters with indices** x_i with $i = 1, \ldots, n$. We also use indices when

- Talking about **relationships in the sample**
- Talking about particular realisations of a random variable: $x_i = x$, for example $x_i = \textit{female}$ or $x_i = 5$

This can be confusing at the start but you'll get used to it!

The Conditional Expectation Function

We are interested in **explaining the relationship between x and y** in the population

A useful concept in this regard is the **Conditional Expecation Function (CEF)**: $E(y_i|x_i)$

- what is the **population average of y_i for a given value of x_i**
- i.e. what if x_i takes value x?

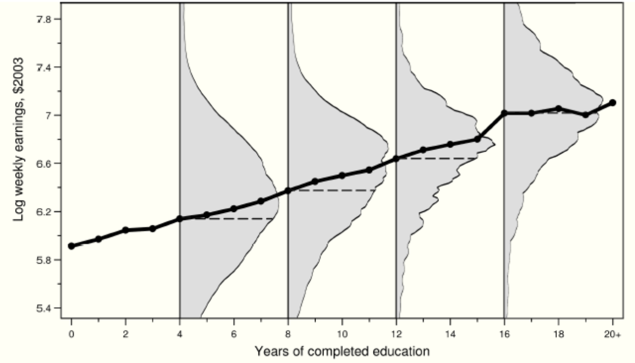
Example: x is a dummy that equals one if a person is female and zero otherwise. y is earnings.

The **CEF can take on two values**:

- Average earnings of women $E(y_i|x_i = 1)$
- Average earnings of men/other $E(y_i|x_i = 0)$

A Continuous CEF

Education vs earnings (from Angrist & Pischke, MHE)



At every level of completed education, we have a different expected value of earnings

- At each value x_i we have a distribution of y_i
- and the CEF comprises the averages of this distribution

Regression is a Linear Approximation of the CEF

The **population regression model** (PRM) is a linear approximation of the CEF

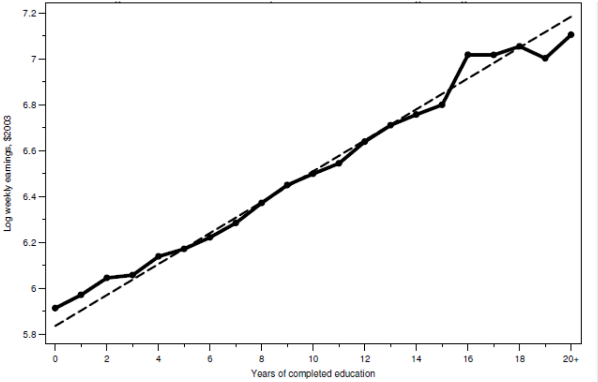
$$y = \beta_0 + \beta_1 x + u$$

- Our **ultimate goal** is to know the **CEF**
- But: with a linear regression, we can estimate the parameters of the PRM
- I.e. we **cannot estimate the CEF directly**

Why a **linear approximation is useful**:

- We typically have small sample sizes, so approximating a non-linear function is difficult
- We are often interested in marginal effects, so a linear approximation is often sufficient

PRM: Linear Approximation of the CEF



Sample is limited to white men, age 40-49. Data is from Census IPUMS 1980, 5% sample.

The dashed line is the Population regression model $y = \beta_0 + \beta_1 x + u$. The solid line is the CEF $E(y|x)$.

It can be shown that the PRM is the **best linear approximation of the CEF** (see Angrist & Pischke, MHE, ch. 3).

CEF and PRM: what's all this about?'

The **CEF is the object of interest** in (most of) econometrics

- The PRM $y = \beta_0 + \beta_1 x + u$ is a **linear approximation** of the CEF.
- But it is an approximation, so it can be wrong.
- With data, we can draw inference on the PRM but not on the CEF

What does this mean for the empirical analysis?

- We need to think about the **relationship between** x **and** y in the population
- Linear approximations are more innocuous when we consider small changes in x

The Sample Regression Function

Now suppose we have a **sample of size n** that was **randomly sampled from the population**, $(y_1, x_1), \ldots, (y_n, x_n)$

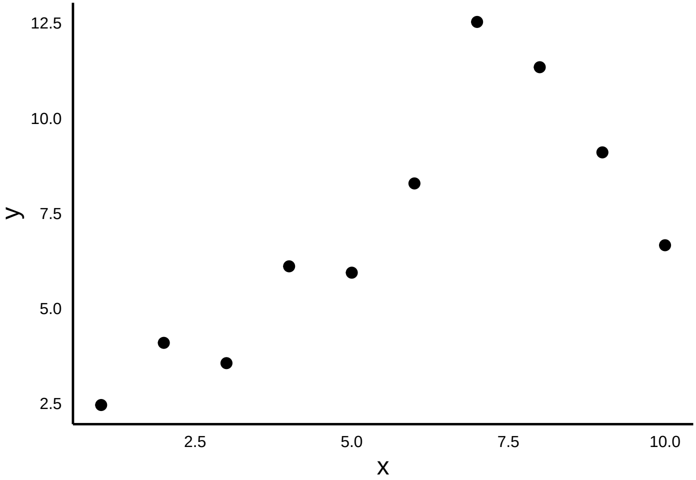
The **sample regression function** is

$$\widehat{y_i} = \widehat{\beta_0} + \widehat{\beta_1} x_i \tag{1}$$

We can estimate the parameters $\widehat{\beta_0}$ and $\widehat{\beta_1}$ with Ordinary Least Squares (OLS)

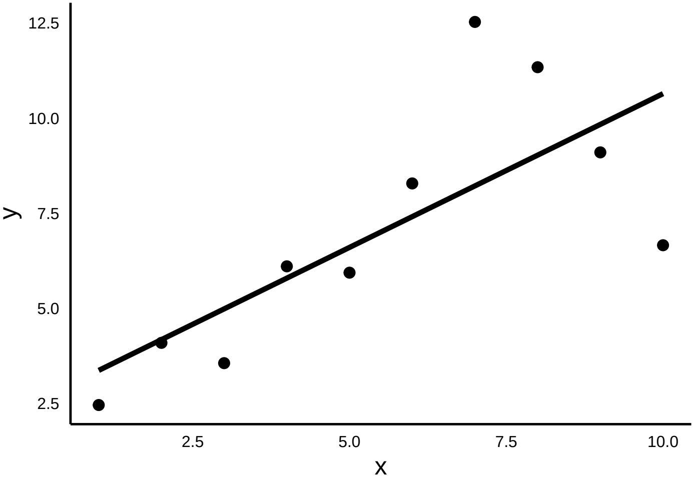
OLS: Intuition

Let's start with some data points



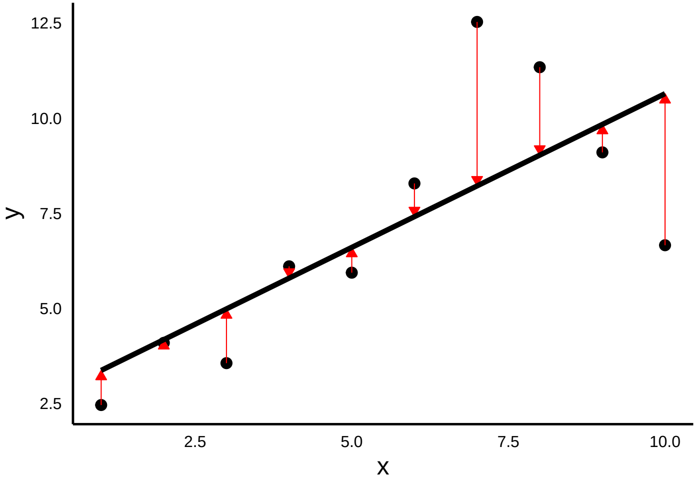
OLS: Intuition

Goal: fit a regression line through those points



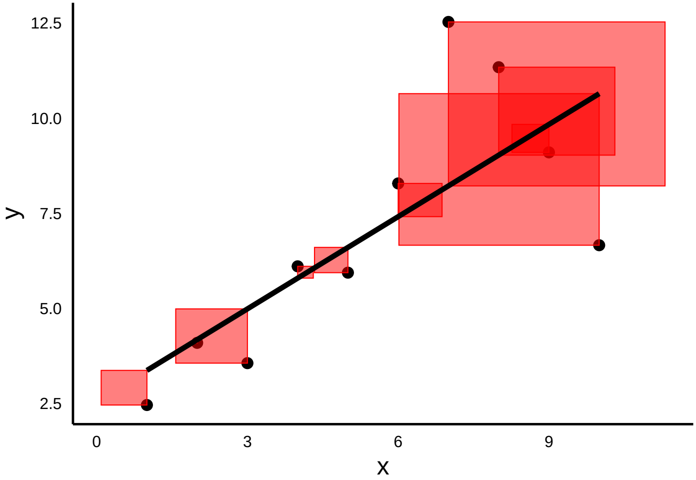
OLS: Intuition

The key ingredient of OLS are the residuals $\widehat{u}_i = y_i - \widehat{b_0} - \widehat{b_1} x_i$



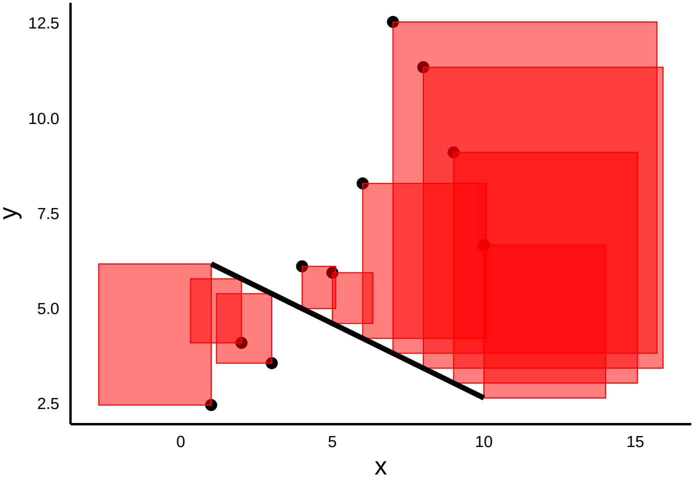
OLS: Intuition

Now consider the square of each residual



OLS: Intuition

Let's consider a different regression line: the squares are much larger!



OLS: Intuition

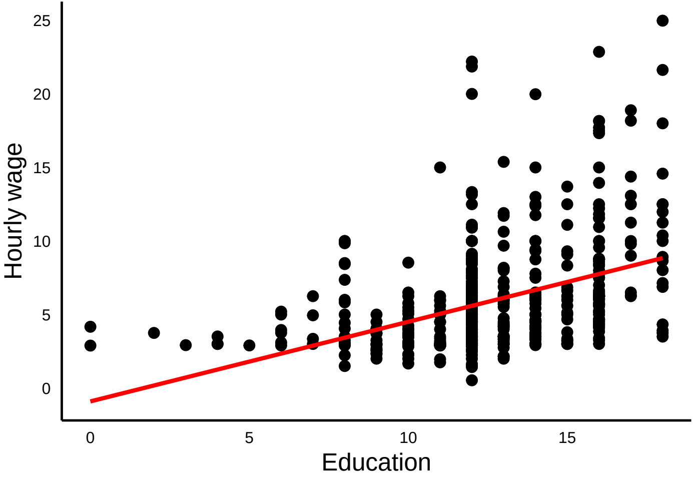
OLS **minimizes the average size of these squares**

It minimizes the sum of squared residuals (SSR)

The result is the **best-fitting line** that describes the relationship between x and y in the sample

OLS: Data Example

Let's look at the relationship between education and wages with data from the U.S.



Regression Output: Interpretation

Table 1: Effect of Education on Wages

	Dependent variable:
	wage
educ	0.541***
	(0.053)
Constant	−0.905
	(0.685)
Observations	526
R^2	0.165
Adjusted R^2	0.163
Residual Std. Error	3.378 (df = 524)
F Statistic	103.363*** (df = 1; 524)
Note:	*p<0.1; **p<0.05; ***p<0.01

A 1-year increase in education is associated with a 0.54 USD increase in hourly wages

OLS: The Math

The Ordinary Least Squares (OLS) estimators $\widehat{\beta_0}$ and $\widehat{\beta_1}$ are derived through the minimization problem

$$(\widehat{\beta_0}, \widehat{\beta_1}) = \underset{\widehat{b_0}, \widehat{b_1}}{\arg\min} \sum_{i=1}^{n} [(y_i - \widehat{b_0} - \widehat{b_1} x_i)^2] \tag{2}$$

The sample means $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ are the sample analogs of the population means $E(y_i)$ and $E(x_i)$

OLS: The Math

The **residuals of the regression** are defined as $\widehat{u_i} = y_i - \widehat{\beta_0} + \widehat{\beta_1} x_i$.

When we "run an OLS regression'', we **minimize the sum of squared residuals (SSR)**, $\sum_{i=1}^{n} \widehat{u_i}^2$ and obtain values for $\widehat{\beta_0}$ and $\widehat{\beta_1}$

Solving the minimization problem (2) yields the **estimators**

$$\begin{aligned} \widehat{\beta_1} &= \frac{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\widehat{Cov(y_i, x_i)}}{\widehat{V(x_i)}} \qquad (3) \\ \widehat{\beta_0} &= \bar{y} - \widehat{\beta_1}\bar{x} \end{aligned}$$

The Sampling Distribution of the OLS Estimator

To **draw inference about the population**, we need to know the **sampling distribution of the OLS estimator**

What we want to know:

- When will $\widehat{\beta}_1$ be **unbiased?**
- What is its **variance?**

To answer these questions, we need to make some **assumptions about the sample and population**

1. Population model is linear in parameters
2. Sample is randomly drawn from the population
3. Variation in x
4. **Zero conditional mean assumption** (ZCM)

OLS Assumptions

The four **OLS assumptions must be fulfilled** for the OLS estimator to be **unbiased and consistent**

Unbiasedness: $E(\widehat{\beta}_1) = \beta_1$

- across many random samples, the estimator gets it right on average

Consistency: $\widehat{\beta}_1 \xrightarrow{p} \beta_1$ as $n \to \infty$

- if the sample size increases, the estimator converges to the true value
- this is a consequence of the Law of Large Numbers (LLN)
- As n gets larger, the sample becomes more representative of the population

The Zero Conditional Mean (ZCM) Assumption

The **ZCM assumption is the most important assumption** in this module

- It is **not testable with the data** at hand
- It rarely holds in practice (except in randomised experiments)
- Causal inference techniques exploit scenarios where ZCM holds approximately

Other names for the ZCM assumption:

- **Conditional independence assumption (CIA)**
- **Exogeneity assumption**

The Zero Conditional Mean (ZCM) Assumption

Consider the **population model**

$$y = \beta_0 + \beta_1 x + u$$

The **ZCM assumption** states that the conditional mean of the error term is zero

$$E(u|x) = E(u) = 0$$

What does this mean?

- The error term is **not systematically related** (speak: uncorrelated) with x
- At any level of x, the average value of u is zero

ZCM Assumption: Example

Does **higher education (causally) increase earnings?**

$$wage_i = \beta_0 + \beta_1 education_i + u_i$$

What is the error term u_i here?

- Any determinant of a person's wage that is not education
- E.g., innate ability, motivation, personality, etc.

ZCM Assumption: Example

Say u_i includes ability. According to the ZCM assumption, the following must hold:

$$E(\text{ability} \mid \text{education} = 8) = E(\text{ability} \mid \text{education} = 12) = E(\text{ability} \mid \text{education} = 16)$$

So it **must hold that**:

- the average ability of people with 8 years of education is the same as
- the average ability of people with 12 years of education and
- the average ability of people with 16 years of education

This is hardly plausible $\Rightarrow$ **ZCM assumption is violated**

What if ZCM is Violated?

The OLS estimator of β_1 is **biased and inconsistent**

Bias: $E(\widehat{\beta_1}) \neq \beta_1$

- The expected value of the OLS estimator is not equal to the true value of β_1
- Across many samples, the estimates are systematically too big or too small

Inconsistency: $\widehat{\beta_1}$ does not converge to β_1 as $n \to \infty$

- Even if the sample size is very large, the OLS estimator does not converge to the true value of β_1

How to think about Violations of ZCM

The variable x is **typically a choice**

- People choose how much education to get, how often they go to the gym, how much they save, who they want to date, etc
- Firms choose how much to invest, how many workers to hire, how much to pollute, etc.
- Governments choose how much to spend on education, how much to tax, etc.

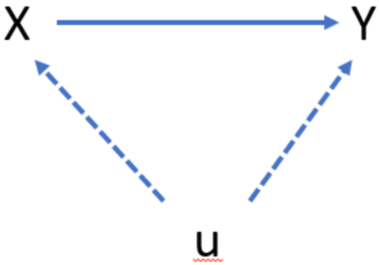
The **choice** of x is typically **influenced by other factors** u

- Individual factors: ability, motivation, preferences, etc.
- Firm factors: technology, market conditions, etc.
- Government factors: ideology, political pressure, etc.

Problem: u **affects** y not just through x but also **through other paths** or directly

How to think about Violations of ZCM

The error term u includes **one or more confounders**



Here u includes a confounder y directly and through x

Example: x is education, y is earnings, u is ability

Omitted Variable Bias

Suppose the **true model** is $y = \beta_0 + \beta_1 x + \beta_2 s_1 + e$

However, we **estimate the model** $y = \tilde{\beta}_0 + \tilde{\beta}_1 x + u$

It can be shown that the **OLS estimator is biased**

$$\tilde{\beta}_1 = \beta_1 + \underbrace{\beta_2 \frac{Cov(x, s_1)}{Var(x)}}_{OVB}$$

So when does ZCM hold?

ZCM holds if x is **as good as randomly assigned** to individuals

- This is the case if x is assigned in a **randomised experiment**
- Or if x is assigned in a **quasi-experiment** that mimics random assignment
- Or if we can **control for all confounders** in the analysis

We should **always assume that ZCM is violated**. Researchers need to **think hard about confounders and how to eliminate them.**

Controlling for Confounders: Multivariate Regression

We can include **confounders in the regression model** to control for them

$$y = \beta_0 + \beta_1 x + \boldsymbol{S}\boldsymbol{\gamma} + u$$

Here, $\boldsymbol{S}$ is a vector of covariates $\boldsymbol{S} = (s_1, s_2, \ldots, s_k)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ is a vector of coefficients,[1] i.e.

$$\boldsymbol{S}\boldsymbol{\gamma} = \gamma_1 s_1 + \gamma_2 s_2 + \cdots + \gamma_k s_k$$

We are **only interested in β_1, the causal effect of x on y**

- The other coefficients $\gamma_1, \gamma_2, \ldots, \gamma_k$ are not of interest (nuisance parameters)
- We include the covariates $\boldsymbol{S}$ to control for confounders

[1] Note: each element of $\boldsymbol{S}$ is in itself an $(n \times 1)$ vector, so $\boldsymbol{S}$ is actually an $(n \times k)$ matrix

Interpretation of β_1 in Multivariate Regression

β_1 now has a **ceteris paribus interpretation**

- **Holding all other variables $\boldsymbol{S}$ constant**, a one unit increase in x leads to a β_1 unit increase in y

The **inclusion of $\boldsymbol{S}$** allows for a **like-with-like comparison**

- We **compare units with the same values** of $\boldsymbol{S}$ but different values of x
- But the like-with like comparison is only valid if $\boldsymbol{S}$ contains all confounders

Conditional Mean Independence Assumption

The **Conditional Mean Independence Assumption (CMIA)** is a “light” version of the ZCM assumption.

$$E(u \mid x, \boldsymbol{S}) = E(u \mid S) = 0$$

In plain English: as long as **$\boldsymbol{S}$ is included, the error term u is uncorrelated with x**

- x is exogenous conditional on $\boldsymbol{S}$
- x is as good as random conditional on $\boldsymbol{S}$

Summary: What you need to understand for this module

Logic of linear regression

- Why we use linear regression
- Why and how we use OLS to estimate the parameters of the linear regression model
- How to interpret the OLS estimator $\hat{\beta}_1$

Limitations of linear regression for causal inference

- The ZCM assumption is violated in most applications, leading to OVB
- How control variables can be used to control for confounders

Appendix

Regression with R

```r
# Required packages (install if necessary)
library(tidyverse)
library(wooldridge)
library(stargazer)
```

Regression with R

This code shows how to estimate and present regressions with R

```
data('wage1') # load the data
df <- wage1
reg1 <- lm(wage ~ educ, data = df) # estimate simple regression
reg2 <- lm(wage ~ educ + exper, data = df) # estimate multivariate regress

stargazer(reg1, reg2, type = "text") # print regression results
```

Regression outputs with Stargazer and R Markdown

You can generate nice regression tables with the `stargazer` package and R Markdown/Quarto. Here is a code chunk that gives you a nicely formatted latex table.

````
```{r, results='asis', echo=FALSE}
stargazer(reg1, reg2,
      header=FALSE,
      type='latex',
      title="Effect of Education on Wages")
```
````

Regression with R

We can also generate a scatter with a regression line

```
ggplot(df, aes(x = educ, y = wage)) + # generate scatter plot
  geom_point() + # add points
  geom_smooth(method = "lm", se = FALSE) +  # add regression line
  theme_minimal()
```

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