Correctness of Local Probability Propagation in Graphical Models with Loops

by

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Ben Eyal

November 16, 2016

Outline

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1. Introduction
   - Definitions
   - The Problem

2. Belief Propagation on PGMs with a Single Loop
   - Correctness

3. Conclusion
   - What Have We Achieved?
   - The End

Breaking Down the Title - Part I

Correctness of Local Probability Propagation in Graphical Models with Loops

Breaking Down the Title - Part I

Correctness of Local Probability Propagation in Graphical Models with Loops

Probabilistic Graphical Models

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Definition

A Probabilistic Graphical Model (PGM) is a graph, either directed or undirected, in which the nodes correspond to random variables, and the edges correspond to direct probabilistic interactions between them.

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A **Bayes network** is a directed acyclic PGM whose edges can be seen as "cause" and "effect", e.g. an edge $X \to Y$ means that X directly influences Y, or "X is the cause of Y".

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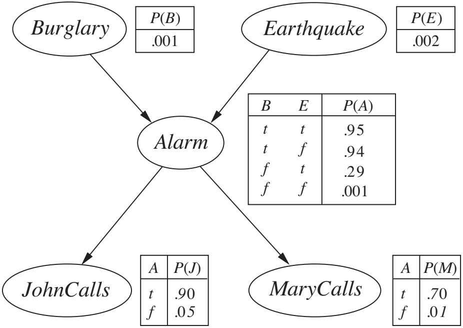

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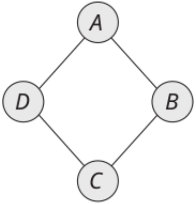

$\phi_1(A,B)$

a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\phi_2(B,C)$

b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

$\phi_3(C,D)$

c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

$\phi_4(D,A)$

d^0	a^0	100
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Reminder

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Bayes' Theorem

Bayes' theorem gives us the relation between the posterior probability, $\Pr(A \mid B)$, and the prior probability, $\Pr(A)$.

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}$$

Reminder

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Marginal Probability

Given a joint probability distribution $\Pr(X, Y)$,

$$\begin{aligned}\Pr(X = x) &= \sum_y \Pr(X = x, Y = y) \\ &= \sum_y \Pr(X = x \mid Y = y) \Pr(Y = y)\end{aligned}$$

is the marginal distribution of X.

Belief Propagation

The Task of Inference

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- *Maximum Marginal (MM) assignment*: Finding assignments to the unobserved variables that maximize the marginal probability of the assignment – e.g. finding (b, c, d, e) such that $\Pr(B = b \mid \mathcal{O})$, $\Pr(C = c \mid \mathcal{O})$, $\Pr(D = d \mid \mathcal{O})$, $\Pr(E = e \mid \mathcal{O})$ are all maximized.

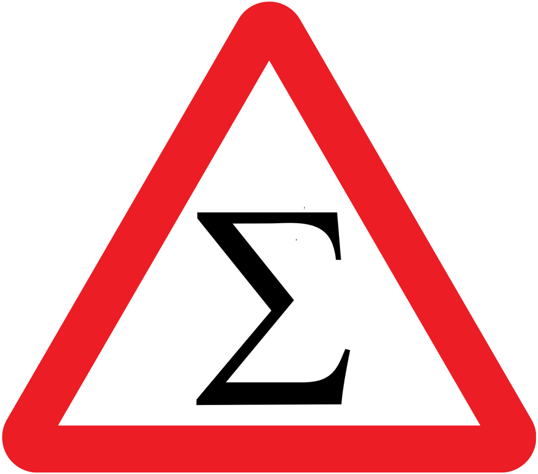
Belief Propagation

Definition

Belief Propagation (Pearl, 1982) is an iterative message-passing algorithm used to perform inference on PGMs.
The algorithm gives exact marginals on polytrees (a directed graph with only a single path between any two nodes).

Example

Example



Example

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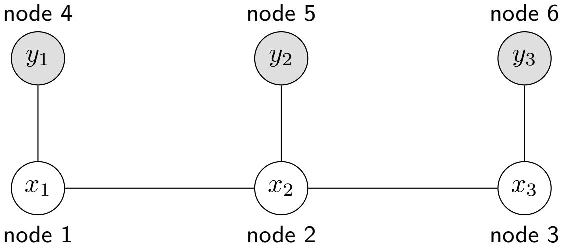
We would like to compute the maximum marginal probability $\Pr(X = x_1 \mid Y)$ given the following Markov network:



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$$\Pr(X = x_1 \mid Y) = \sum_{x_2} \sum_{x_3} \Pr(x_1, x_2, x_3 \mid Y)$$

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As in $\mathcal{NP}$-hard?

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$$\Pr(x_1|Y)=\frac{1}{\Pr(Y)}\sum_{x_2}\sum_{x_3}\phi_{12}(x_1,x_2)\phi_{23}(x_2,x_3)\psi_1(y_1,x_1)\psi_2(y_2,x_2)\psi_3(y_3,x_3)$$

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Do not despair! This is going to be very clear – very quick.

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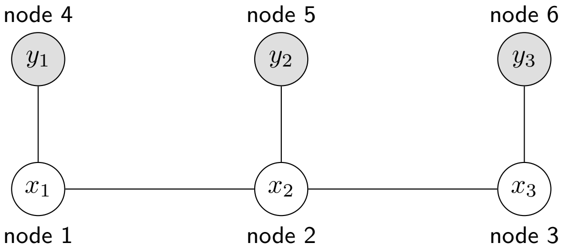

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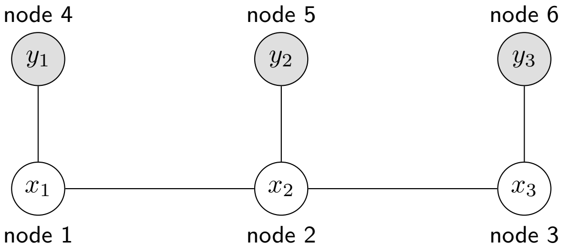
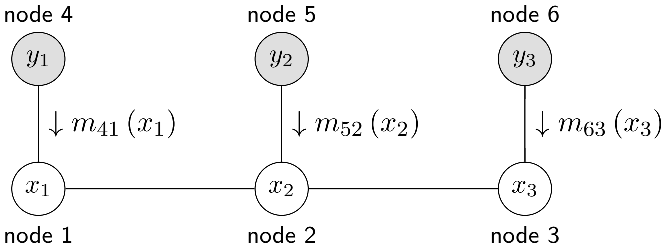


Example

$$\Pr(x_1|Y) = \frac{1}{\Pr(Y)} m_{41}(x_1) \sum_{x_2} \phi_{12}(x_1, x_2) m_{52}(x_2) \sum_{x_3} \phi_{23}(x_2, x_3) m_{63}(x_3)$$



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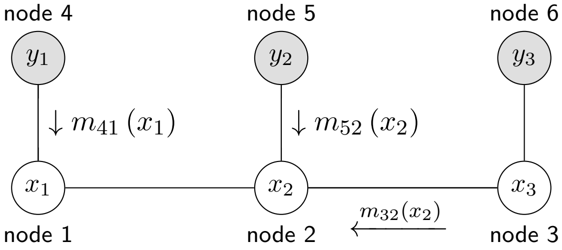

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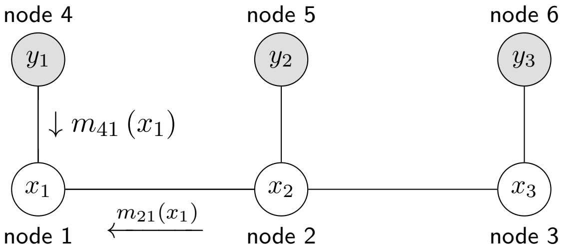
$$\Pr(x_1|Y) = \frac{1}{\Pr(Y)} m_{41}(x_1) m_{21}(x_1)$$



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Inside Belief Propagation

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2. Multiply $\vec{v}$ by the matrix M_{XY} corresponding to the link from X to Y.
3. Normalize the product $M_{XY}\vec{v}$ so it sums to 1. The normalized vector is sent to Y.

The belief vector for a node X is obtained by combining all incoming messages to X and normalizing.

Belief Propagation

Maximum A-Posteriori (MAP) Assignment Using Belief Propagation

To find a MAP assignment, we use the algorithm above, with a small change:
Instead of using matrix multiplication (a sum-of-products), we take the argmax, giving us a max-of-products.
This step is called belief revision (BR).

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"Loopy" Belief Propagation

Belief Propagation on General Graphs

As we have already seen, Belief Propagation is guaranteed to converge on the exact marginals or MAP assignments given a singly-connected graph.

What happens when we have loops?

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In practice, "loopy" Belief Propagation is doing surprisingly well!

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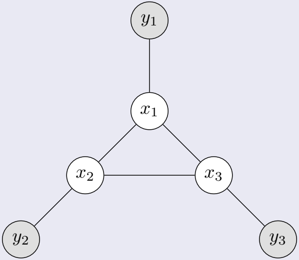

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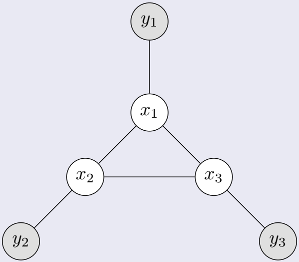

In a graph with a single loop, Weiss shows this “double counting” is guaranteed to occur equally among all nodes in the loop, therefore still reaching convergence.

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What Have We Achieved?

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- We are aware of three problems that may arise in PGMs with loops...
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- We know what to expect when running Loopy Belief Propagation on a PGM with a single loop.

Outline

The End

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Questions?