

Correctness of Local Probability Propagation in Graphical Models with Loops

by
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November 16, 2016

Outline

- 1 Introduction
 - Definitions
 - The Problem
- 2 Belief Propagation on PGMs with a Single Loop
 - Correctness
- 3 Conclusion
 - What Have We Achieved?
 - The End

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Breaking Down the Title - Part I

Correctness of Local Probability Propagation in Graphical Models with Loops

Breaking Down the Title - Part I

Correctness of Local Probability Propagation in
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Probabilistic Graphical Models

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Definition

A Probabilistic Graphical Model (PGM) is a graph, either directed or undirected, in which the nodes correspond to random variables, and the edges correspond to direct probabilistic interactions between them.

Probabilistic Graphical Models

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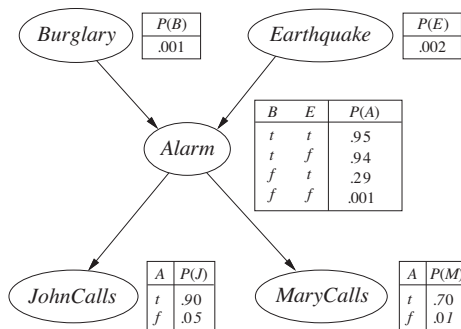
Definition

A **Bayes network** is a directed acyclic PGM whose edges can be seen as “cause” and “effect”, e.g. an edge $X \rightarrow Y$ means that X directly influences Y , or “ X is the cause of Y ”.

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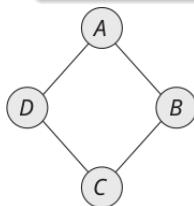
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A Markov Random Field (MRF), or a **Markov network**, is an undirected PGM which is used when the relations between random variables are symmetric, rather than hierarchical, e.g. pixels in an image.

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$\phi_1(A, B)$			$\phi_2(B, C)$			$\phi_3(C, D)$			$\phi_4(D, A)$		
a^0	b^0	30	b^0	c^0	100	c^0	d^0	1	d^0	a^0	100
a^0	b^1	5	b^0	c^1	1	c^0	d^1	100	d^0	a^1	1
a^1	b^0	1	b^1	c^0	1	c^1	d^0	100	d^1	a^0	1
a^1	b^1	10	b^1	c^1	100	c^1	d^1	1	d^1	a^1	100

Breaking Down the Title - Part II

Correctness of Local Probability Propagation in
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Breaking Down the Title - Part II

Correctness of **Local Probability Propagation** in
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Reminder

Conditional Probability

$$\Pr(A \mid B) = \frac{\Pr(A, B)}{\Pr(B)}$$

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Bayes' Theorem

Bayes' theorem gives us the relation between the **posterior probability**, $\Pr(A \mid B)$, and the **prior probability**, $\Pr(A)$.

$$\Pr(A \mid B) = \frac{\Pr(B \mid A) \Pr(A)}{\Pr(B)}$$

Reminder

Reminder

Marginal Probability

Given a joint probability distribution $\Pr(X, Y)$,

$$\begin{aligned}\Pr(X = x) &= \sum_y \Pr(X = x, Y = y) \\ &= \sum_y \Pr(X = x \mid Y = y) \Pr(Y = y)\end{aligned}$$

is the **marginal distribution** of X .

Belief Propagation

The Task of Inference

Inference is divided into three subtasks:

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- *Maximum Marginal (MM) assignment*: Finding assignments to the unobserved variables that maximize the marginal probability of the assignment – e.g. finding (b, c, d, e) such that $\Pr(B = b \mid \mathcal{O})$, $\Pr(C = c \mid \mathcal{O})$, $\Pr(D = d \mid \mathcal{O})$, $\Pr(E = e \mid \mathcal{O})$ are all maximized.

Belief Propagation

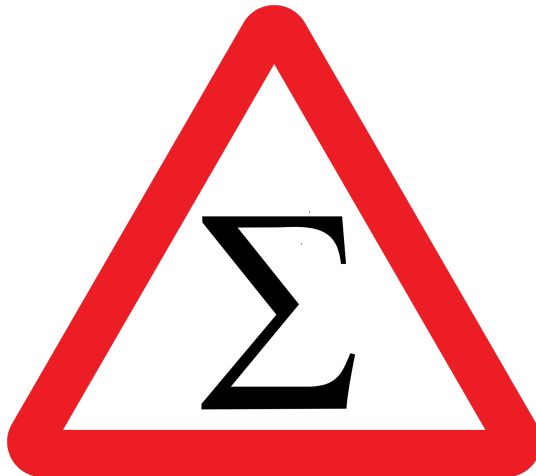
Definition

Belief Propagation (Pearl, 1982) is an iterative message-passing algorithm used to perform inference on PGMs.

The algorithm gives exact marginals on polytrees (a directed graph with only a single path between any two nodes).

Example

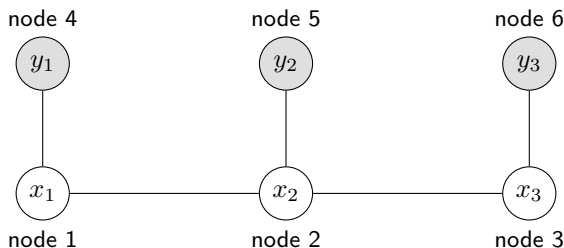
Example



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We would like to compute the maximum marginal probability $\Pr(X = x_1 \mid Y)$ given the following Markov network:



Example

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If we knew nothing about the structure of the joint probability:

$$\Pr(X = x_1 \mid Y) = \sum_{x_2} \sum_{x_3} \Pr(x_1, x_2, x_3 \mid Y)$$

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As in \mathcal{NP} -hard?

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$$\Pr(x_1|Y) = \frac{1}{\Pr(Y)} \sum_{x_2} \sum_{x_3} \phi_{12}(x_1, x_2) \phi_{23}(x_2, x_3) \psi_1(y_1, x_1) \psi_2(y_2, x_2) \psi_3(y_3, x_3)$$

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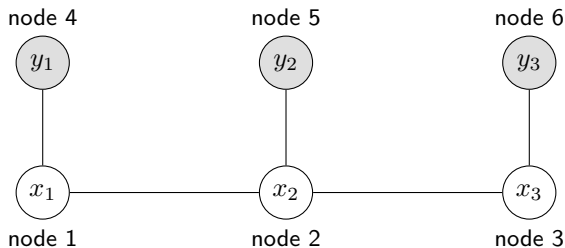
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Do not despair! This is going to be very clear – very quick.

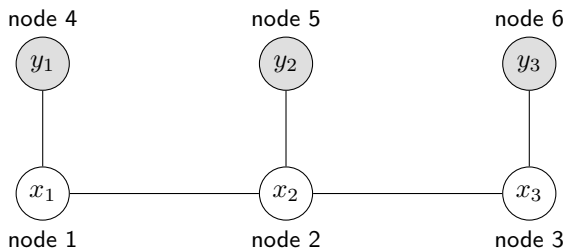
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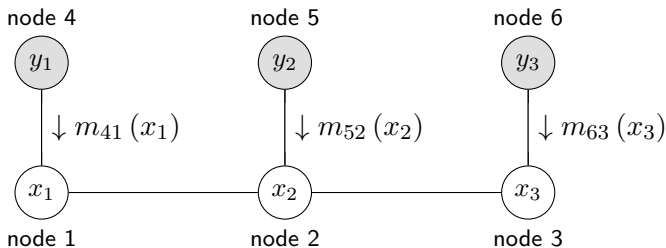
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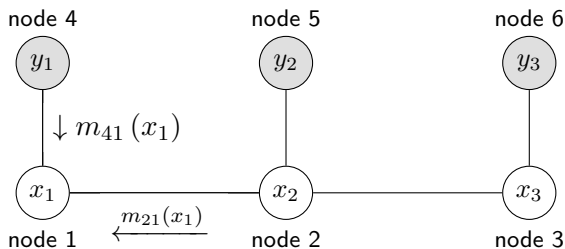
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$$\Pr(x_1|Y) = \frac{1}{\Pr(Y)} m_{41}(x_1) \sum_{x_2} \phi_{12}(x_1, x_2) m_{52}(x_2) \sum_{x_3} \phi_{23}(x_2, x_3) m_{63}(x_3)$$



Example

$$\Pr(x_1|Y) = \frac{1}{\Pr(Y)} m_{41}(x_1) m_{21}(x_1)$$



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Inside Belief Propagation

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- 3 Normalize the product $M_{XY}\vec{v}$ so it sums to 1. The normalized vector is sent to Y .

The belief vector for a node X is obtained by combining all incoming messages to X and normalizing.

Belief Propagation

Maximum A-Posteriori (MAP) Assignment Using Belief Propagation

To find a MAP assignment, we use the algorithm above, with a small change:

Instead of using matrix multiplication (a sum-of-products), we take the argmax, giving us a max-of-products.

This step is called **belief revision (BR)**.

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“Loopy” Belief Propagation

Belief Propagation on General Graphs

As we have already seen, Belief Propagation is guaranteed to converge on the exact marginals or MAP assignments given a singly-connected graph.

What happens when we have loops?

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In practice, “loopy” Belief Propagation is doing surprisingly well!

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Breaking Down the Title - Finale

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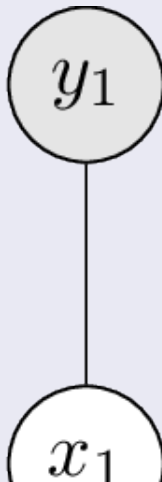
- belief revision in Loopy Belief Propagation will always converge on a correct MAP assignment.
- belief update will converge on a marginal that is not the correct marginal, but is closely related to it.

Motivation

The problem with loops is what Weiss refers to as “double counting”:

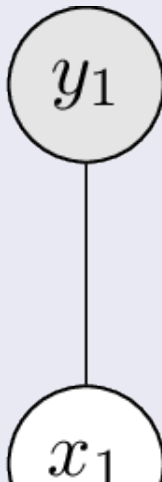
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- We are aware of three problems that may arise in PGMs with loops...
- ...but we also know that loops are not always a problem in practice.
- We know what to expect when running Loopy Belief Propagation on a PGM with a single loop.

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Questions?