BPC

Problem Set 1

Written Component

Secret Code: 20

B1:

Binomial Distribution $\longrightarrow$ $P(N_A ; N) = \frac{(N_{tot})!}{(N_A)!(N_{tot}-N_A)!} \times P_a^{N_A} \times (1-P_a)^{N_{tot}-N_a}$ $\Rightarrow$ let $N_{tot} = N$ and know $1-P_a = P_b$ $\Rightarrow$ $\frac{N!}{N_A!(N-N_A)!} \times P_a^{N_A} \times P_b^{N-N_A}$

for binomial distribution to be normalized, must show that: $\sum_{i=0}^{N} P(i;N) = 1$ (where $\sum P(i;N)$ is sum of all probabilities for N_a from zero $\rightarrow N$

If we sum binomial distribution over this range: $\sum_{i=0}^{N} P(i;N) = \sum_{i=0}^{N} \frac{N!}{i!(N-i)!} P_a^i P_b^{N-i}$

given Binomial Theorem: $(a+b)^N = \sum_{i=0}^{N} \binom{N}{i} a^i b^{N-i} = \sum_{i=0}^{N} \frac{N!}{i!(N-i)!} a^i b^{N-i}$

$\hookrightarrow$ Substitute $a = P_a$, $b = P_b$ $\Rightarrow$ $(P_a + P_b)^N = \sum_{i=0}^{N} \frac{N!}{i!(N-i)!} P_a^i P_b^{N-i}$

these expressions are equivalent therefore

$\therefore = \sum_{i=0}^{N} P(i;N)$

$\hookrightarrow (P_a + P_b)^N = \sum_{i=0}^{N} P(i;N)$

given a and b collectively exhaustive and mutually exclusive, by normalization axiom of probability $P_a + P_b = 1$ as $P_a + P_b = \Omega$ and $P(\Omega) = 1$

$$\boxed{\therefore \sum_{i=0}^{N} P(i;N) = (P_a+P_b)^N = (1)^N = 1 \quad \text{(for any N)}}$$

B2:

- See Jupyter notebook for solution.

B3:

- See Jupyter notebook for solution.

B4:

$P(K) = \frac{2{,}000}{300{,}000{,}000}$ and $P(F) = \frac{200}{300{,}000{,}000}$

Want: $P(F \cap K) = P(F) \cdot P(K_A) = \frac{2000}{300{,}000{,}000} \times \frac{200}{300{,}000{,}000}$

(where F is outcome that any given person in country is on flight and K is outcome that you know any given person in country

$$P(K_A) = \sum_{i=1}^{200} P(i;200) = \sum_{i=1}^{200} \binom{200}{i} \times P_{F\cap K}^{\,i} \times (1 - P_{F\cap K})^{200-i} \approx 8.88 \times 10^{-10}$$

Probability of knowing at least 1 person on flight given $P_{F\cap K}$

  - See Jupyter notebook for confirmation of solution.

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C1:

C1:

a) $f(x) = 3x^4 + 2x + 5$

$$\boxed{f'(x) = 12x^3 + 2}$$

b) $f(x) = \dfrac{Kx}{1 + Kx}$

$$f'(x) = \frac{(1 + Kx)(K) - (Kx)(K)}{(1 + Kx)^2} = \frac{K + K^2x - K^2x}{(1 + Kx)^2} = \boxed{\frac{K}{(1 + Kx)^2}}$$

c) $f(x) = \ln(x) + e^{-Kx}$

$$f'(x) = \frac{1}{x} + e^{-Kx}(-K)$$

$$= \boxed{\frac{1}{x} - Ke^{-Kx}}$$

C2:

C2:

a) $f(x,y) = xy^3 - x^3y + 3$

$$\boxed{\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= y^3 - 3x^2y \\ \frac{\partial f(x,y)}{\partial y} &= 3y^2x - x^3 \end{aligned}}$$

b) $f(x,y) = \dfrac{e^{-ax}}{1 + e^{-ax} + e^{-by} + e^{-ax}e^{-by}}$

$$f(x,y) = \frac{1}{e^{ax}(1 + e^{-ax} + e^{-by} + e^{-ax}e^{-by})}$$

$$= \frac{1}{\cancel{e^{ax}}(e^{ax}\cdot 1 - \frac{1}{\cancel{e^{ax}}} + \frac{1\cdot e^{ax}}{e^{by}} + \frac{1}{\cancel{e^{ax}}}e^{-by})}$$

$$= (e^{ax-by} + e^{ax} + e^{-by} + 1)^{-1}$$

$$\frac{\partial f(x,y)}{\partial x} = -1\,(e^{ax-by} + e^{ax} + e^{-by} + 1)^{-2} \cdot (ae^{ax-by} + ae^{ax})$$

$$= \boxed{\frac{-ae^{ax}(e^{-by} + 1)}{(e^{ax-by} + e^{ax} + e^{-by} + 1)^2}}$$

$$\frac{\partial f(x,y)}{\partial y} = -1\,(e^{ax-by} + e^{ax} + e^{-by} + 1)^{-2} \cdot (-be^{ax-by} + -be^{-by})$$

$$\boxed{= \frac{be^{-by}(e^{ax} + 1)}{(e^{ax-by} + e^{ax} + e^{-by} + 1)^2}}$$

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C3:

C3:

a) $f(x,y) = x^3 e^{-2y}$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\boxed{df = 3x^2 e^{-2y} dx + -2e^{-2y} x^3 dy}$$

b) $f(x,y) = 2x^{-1} + \cos(3y)$

$$\boxed{df = -2x^{-2} dx + 3\cos(3y) dy}$$

C4:

C4: dQ exact if and only if

for: $dQ = A(x\ y)dx + B(x,y)dy$

$$\left(\frac{\partial A}{\partial y}\right)_x = \left(\frac{\partial B}{\partial x}\right)_y$$

a) $df(x,y) = 4x^3 y\, dx + x^4 dy$

$$\left(\frac{\partial A}{\partial y}\right)_x = 4x^3$$

$$\left(\frac{\partial B}{\partial x}\right)_y = 4x^3$$

$$\boxed{\therefore \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x} \text{ and } df \text{ is exact and } f(x,y) = x^4 y + C}$$

b) $df(x,y) = x y^{-1} dx + y x^{-1} dy$

$$\left(\frac{\partial A}{\partial y}\right)_x = -1 x y^{-2}$$

$$\left(\frac{\partial B}{\partial x}\right)_y = -1 y x^{-2}$$

$$\boxed{\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x} \text{ and } df \text{ is not exact}}$$

C5:

C5: a) $\int 3x\, dx = \boxed{\frac{3}{2}x^2 + C}$

b) $$\int_1^4 (x^3 + 2)dx = \frac{x^4}{4} + 2x \Big|_1^4 = \left[\frac{(4)^4}{(4)} + 2(4)\right] - \left[\frac{(1)^4}{4} + 2(1)\right]$$

$$= [\ 72\ ] - \left[\ \frac{9}{4}\ \right]$$

$$= 72 - \frac{9}{4} = \boxed{69.75}$$

c) $\int y^2 e^{-2x} dx = y^2 \int e^{-2x} dx$

$u = -2x$
$du = -2dx$

$$= y^2 \cdot \frac{1}{-2} \int -2e^{-2x} dx = -y^2 \frac{1}{2} \int e^u du$$

$$= -\frac{y^2}{2}\left[e^u + C\right]$$

$$= \frac{-y^2}{2}\left[e^{-2x} + C\right]$$

$$\boxed{= \frac{-y^2 e^{-2x}}{2} + B}$$

$B = \frac{-y^2 C}{2}$