1 Sept. 2021

Suppose we have a fcn

$$T : \mathbb{N} \to \mathbb{N}$$

defined by

$$T(n) = \begin{cases} 1, & n = 1 \\ T(n-1) + n, & n > 1 \end{cases}$$

{Note: Often w/ writing recurrences, we take a shortcut and write simply

$$T(n) = T(n-1) + n$$
}

recursive formulation: $T(n) = T(n-1) + n$

closed form: $\underline{T(n) = \frac{n(n+1)}{2}}$

asymptotic form: $\underline{T(n) \text{ is } \Theta(n^2)}$

In groups: 1 find the closed form using top-down and/or bottom-up

recursion tree — start w/ small ex.

2 what is the asymp. form? Can you prove it using 1 the definitions?

$$T(n) = n + (n-1) + \dots + 2 + 1$$

$$= \sum_{i=1}^{n} i$$

Why? Induction!

$$= \frac{n(n+1)}{2}$$

Suppose $T(n) = \frac{1}{2}(n)(n+1) = \frac{1}{2}n^2 + \frac{1}{2}n$

Why is $T(n) \in \Theta(n^2)$?

- Recall the definition.

  $T(n)$ is $\Theta(n^2)$ iff

  $\exists N \in \mathbb{N}$ and $a, b \in \mathbb{R}$ such that

  $\forall n \geq N$

  $$bn^2 \leq T(n) \leq an^2.$$

- Recall from 246

  Claim: $\exists N, a, b$ such that $\boxed{\text{stmt}}$

  Proof: Let $N = \square$, $a = \square$, $b = \square$.

  Then, ...

want

$$b\,n^2 \le T(n) = \tfrac{1}{2}n^2 + \tfrac{1}{2}n \le a \cdot n^2$$

$b = \frac{1}{2}$ works as long as $n \ge 0$.

What a will work? $a = 1$ is ok. $(\forall n \in \mathbb{Z})$

to find N, choose the max needed by either side. or can be larger.

★ The choice of a, b, N does not need to be the tightest choice possible!!

Claim: $T(n)$ is $\Theta(n^2)$.

Proof: Let $N = 5$,
$a = 1$, and
$b = 1/2$.

Note: In LaTex, your proof should look like a paragraph or textbook-style proof. No forced line breaks!

Let $n \geq N$. ← introduce variables before they are used!

Then,

$$bn^2 \underbrace{=}_{\text{substitution}} \frac{1}{2}n^2 \underbrace{\leq}_{\text{adding a pos. \#}} \frac{1}{2}n^2 + \frac{1}{2}n.$$

even math is in a sent.!

In addition, we have, since $n \geq N$ and $\frac{1}{2}n \leq \frac{1}{2}n^2$,

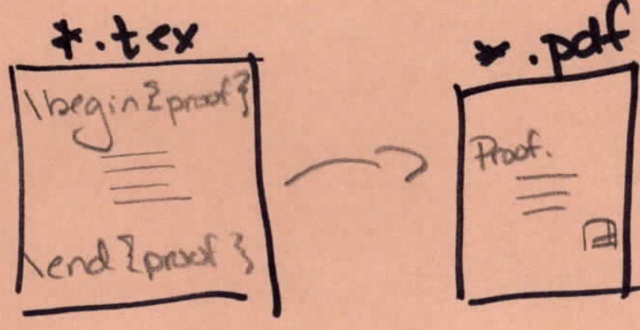
$$\frac{1}{2}n^2 + \frac{1}{2}n \leq \frac{1}{2}n^2 + \frac{1}{2}n^2 = 1 \cdot n^2 = an^2.$$

Therefore, we have $\forall n \geq N$

$$\frac{1}{2}n^2 \leq \frac{1}{2}n^2 + \frac{1}{2}n \leq n^2,$$

which gives us that $T(n)$ is $\Theta(n^2)$. $\square$

We always conclude our proofs!



Recall Hanoi (from the last class).

Q-1: How can we use the <u>concept</u> of induction to show the correctness of the algorithm?

← Dominoes!

Q-2: Why is it important to show that your algorithms are correct?

→ so we don't waste company resources on irrelevant computations

→ to be correct, the algorithm must terminate & do that it is supposed to do.

→ we show this through loop / recursion invariants

→ avoid disasters!

1960s Mariner I (used Radius instead of avg. radius)

1980s radiation errors due to race conditions

imperial -vs- metric