25 Oct 2021

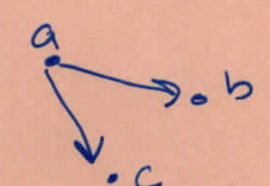
Topological Sort:

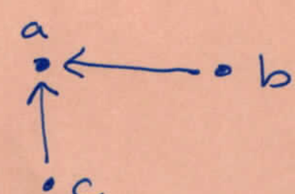
Given a DAG, find a linear ordering that respects all arrows (ie, if $a \to b$ is an edge, then a must appear before b in the linear/total ordering)

(the other day)

Q1: Think about product order on $\mathbb{R}^2$. How can we find a total ordering?

Q2: If G is our dag and $[v_1, v_2, \ldots, v_n]$ is a total ordering that respects all arrows, ~~does~~ is $[v_n, v_{n-1}, \ldots, v_1]$ a total ordering for the graph $\overleftrightarrow{G}$ obtained by reversing all arrows

G: 

$\overleftrightarrow{G}$ 

$[a, b, c]$ $[c, b, a]$

This example works! Do they all?

NO → counterexample? YES → proof?

The product order on $\mathbb{R}^d$:

$a \preceq b$ iff $a_i \leq b_i \ \forall \ i \in \{1, 2, \ldots, d\}$

The dictionary order on $\mathbb{R}^d$:

$a \preceq_D b$ iff $a_1 < b_1$ or $\left\{\begin{matrix} a_1 = b_1 \\ \text{and} \\ a_2 < b_2 \end{matrix}\right\}$ OR $\left\{\begin{matrix} a_1 = b_1 \\ a_2 = b_2 \\ \text{and} \\ a_3 < b_3 \end{matrix}\right\}$ OR ...

Proof:

(A2) Let $G = (V,E)$ be a DAG and let $[v_1, \ldots, v_n]$ be an ordering of V that respects all arrows. (ie., $\forall \ v, v' \in V$ with $v \preceq v'$, v appears before v').

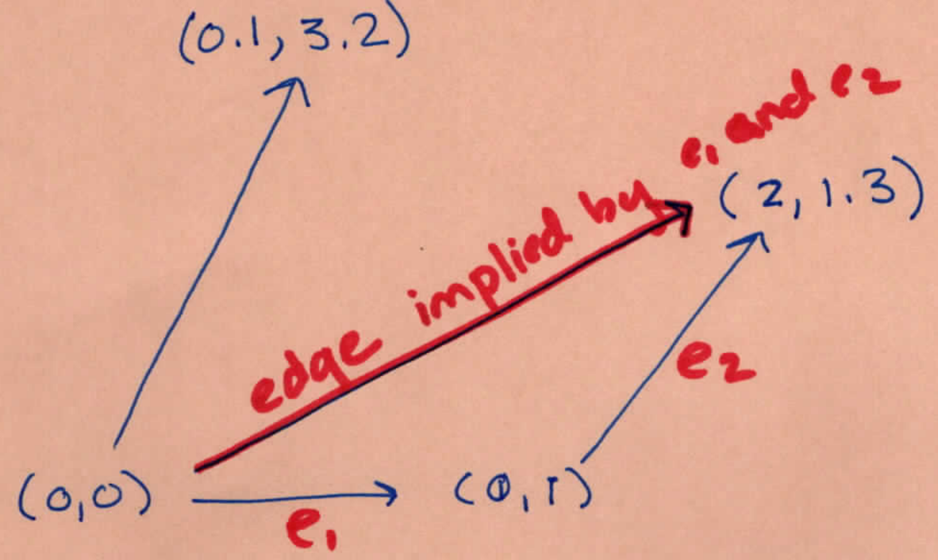
Let $\overleftrightarrow{G} = (V, \overleftrightarrow{E})$ be the graph obtained by reversing all arrows. Assume, by contradiction, that $[v_n, v_{n-1}, \ldots, v_1]$ does not respect all arrows of $\overleftrightarrow{G}$. Thus, (by negating what was in porcn above), $\exists \ v_i, v_j \in V$ such that $v_j \to v_i$ is an edge of $\widetilde{E}$ but $i > j$.

Since $v_j \to v_i \in \widetilde{E}$, we know $v_i \to v_j \in E$, hence, $i < j$, which contradicts $i > j$. $\boxplus$

Challenge: can you prove this constructively instead of by contradiction?

Relation $\preceq$ DAG

$a \preceq b \iff a \rightarrow b \in E$

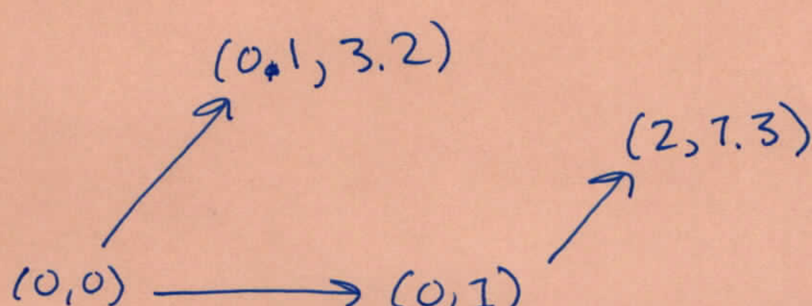


$(0,0) \preceq (0,1) \preceq (2,1.3)$

$\implies (0,0) \preceq (2,1.3)$ by transitivity of $\preceq$.

Note: $\preceq$ is not an equiv. relation since it is not ~~symmetric~~ symmetric $a \preceq b \not\Rightarrow b \preceq a$

Hasse Diagram: the DAG that remains when we remove all implied edges



Note: We often see Hasse dgms where
V = set of sets   E = exists w/ containment

$$
\begin{tikzcd}
 & & \{a,b,c\} & \\
 & \{ab\} \arrow[ur] & & \\
\{a\} \arrow[ur] & \{b\} \arrow[u] & & \{c\} \arrow[uul] \\
 & \emptyset \arrow[ul] \arrow[u] \arrow[urr] & &
\end{tikzcd}
$$

$\boxed{\text{Lemma}}$ If $G=(V,E)$ is a DAG, then there exists a vertex $v \in V$ such that v has no incoming edges.

↑ guiding principle for the Top. Sort algo.