18 Oct 2021

<u>Theorem</u>: All planar graphs are (vertex) 4-colorable.

<u>Weaker Theorem</u>: All planar graphs are (vertex) 5-colorable.

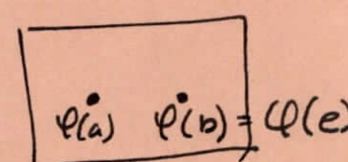
<u>Definition</u>

- G is a <u>plane graph</u> iff G is a (simple) graph, embedded in $\mathbb{R}^2$.
  $G = (V, E, \varphi : G \to \mathbb{R}^2) \Longrightarrow$ edges can't cross!

- A function $\varphi : A \to B$ is an <u>embedding</u> iff φ is continuous + forms a bijection onto its image
  $$\varphi : A \longrightarrow \varphi(A)$$
  is a bijection.

$G = (\{a,b\}, \{\underbrace{\{a,b\}}_{e}\})$

$\mathbb{R}^2$



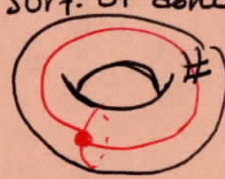
NOT CONTINUOUS

- $G \overset{=(V,E)}{}$ is a <u>planar graph</u> iff $\exists$ a continuous map $m : G \to \mathbb{R}^2$ s.t. (V, E, m) is a plane graph.

- Genus (of a surface)

sphere S^2

torus
surf. of donut, T^2



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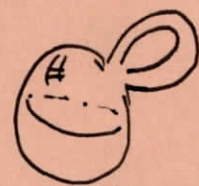


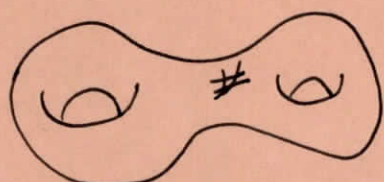
0

1

S^2 + 1 handles

2 hole torus / sphere w/ 2 handles

+ k-handles



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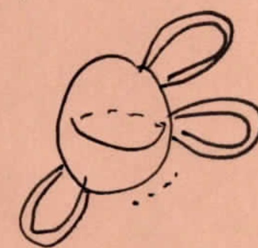
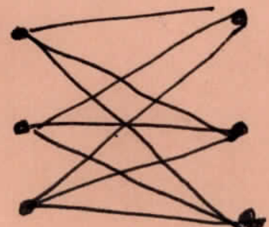


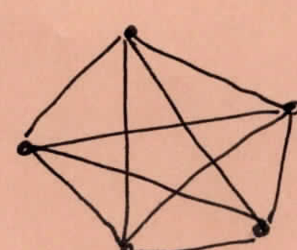
- Let $G=(V,E)$ be a graph. The <u>genus</u> of G is the genus of the simplest surface that the graph can embed into.
- NOTE: $\exists$ an embedding of G into S^2 iff $\exists$ an embedding into $\mathbb{R}^2$
- $K_{3,3}$ and K_5 are two graphs that cannot embed into $\mathbb{R}^2$ (or into S^2). But, they both embed into the torus! Can you draw that?

$K_{3,3}$:



K = Komplete

K_5:



- $\boxed{\text{Lemma}}$ If G is not planar, then $\exists$ a copy of $K_{3,3}$ or K_5 in it!
("a copy" allows us to ignore deg. 2 vertices)

- Euler Characteristic of a plane graph is 2.
$G = (V, E, \varphi)$ is a plane graph.
Let $n = |V|$, $m = |E|$, k = # of (path)-connected components of $\mathbb{R}^2 \setminus \varphi(G)$
= # of faces created by this embedding

$$\boxed{n - m + k = 2}$$

Euler's formula/equation

Q: What is $n - m + k$ when you draw on a torus? (note: faces have to be "nice")

Note: many times, Euler characteristic is denoted by χ.

$$\chi(S^2) = 2$$

Q, reformulated: what is $\chi(\mathbb{T}^2)$?

Lemma Every planar graph has a vertex of degree at most 5.

rewritten:

$\forall$ ~~graphs~~ $G=(V,E) \in$ the set of all planar graphs,

$$\exists v \in V \text{ s.t. } \deg(v) \leq 5.$$

Proof:

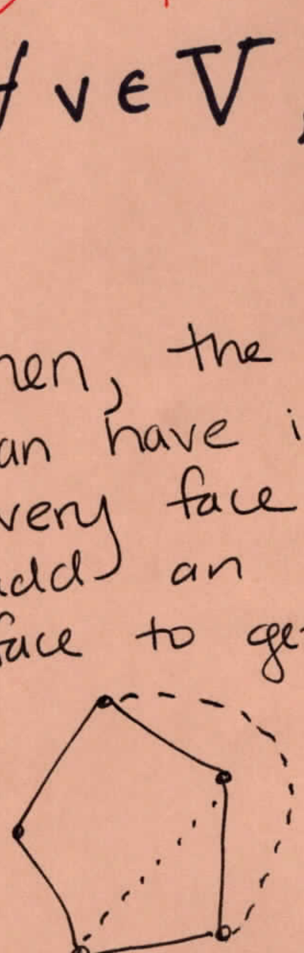
Proof by contradiction. Assume this is false.

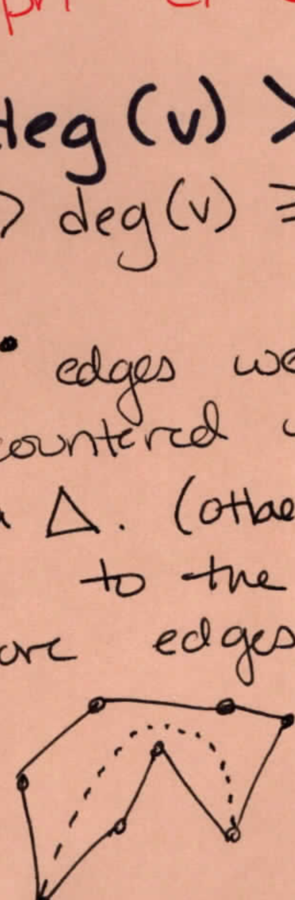
$\exists$ a planar graph $G=(V,E)$ such that

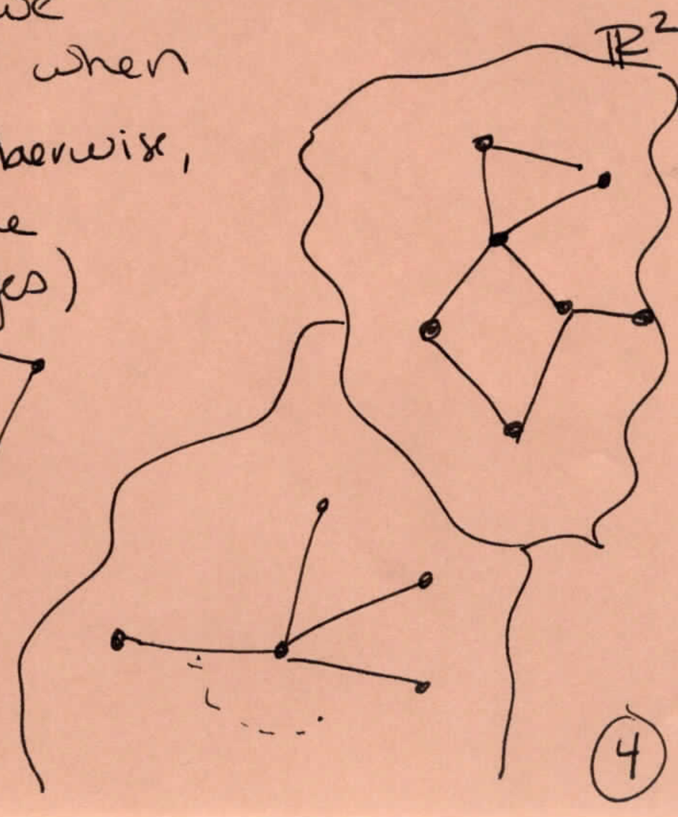
$$\forall v \in V, \quad \deg(v) > 5.$$

$\Rightarrow \deg(v) \geq 6$, since deg must be an integer.

Then, the "most" edges we can have is encountered when every face is a Δ. (otherwise, add an edge to the face to get more edges)







$$|E| \geq \frac{1}{2}(6|V|)$$

$$\boxed{m \geq 3n}$$

Recalling the Euler equation, let $m : G \to \mathbb{R}^2$ be some plane embedding of G. Then,

$$v - m + k = 2 \quad , \text{ where } k = \# \text{ faces.}$$

Again, each face is a Δ so, each face sees 3 edges + each edge sees 2 faces. So,

~~[illegible]~~

$$3k \leq 2m \Rightarrow m \geq \frac{3}{2}k \text{ and } \boxed{k \leq \frac{2}{3}m}$$

Thus $\quad n - m + k = 2 \Rightarrow \boxed{k = 2 + m - n}$

~~[illegible]~~

$$2 + m - n \leq \frac{2}{3}m$$

$$\Rightarrow 6 + 3m - 3n \leq 2m$$

$$\Rightarrow 6 + m - 3n \leq 0$$

$$\Rightarrow \boxed{m \leq 3n - 6}$$

contradicts $m \geq 3n$.

And so, the lemma holds. $\square$