17 Nov 2021

Max Flow / Min Cut

$\boxed{\text{input:}}$ weighted (di)graph $G = (V, E, c)$

source $s \in V$

sink $t \in V$

Def$^{\underline{n}}$ A <u>feasible (s,t)-flow</u> (or simply, a flow) on G is a fcn $f : E \to \mathbb{R}$ such that

1. $\forall e \in E \quad 0 \leq f(e) \leq c(e)$
2. $\forall v \in V / \{s,t\}$, flow into v = flow out of v

$$\sum_{e=(w,v)\in E} f(e) = \sum_{(v,u)\in E} f((v,u))$$

<u>max flow problem</u>: asks for the <u>value</u> of the maximum flow on G, where the value of f is:

$$|f| = \sum_{\underline{(s\to x)\in E}} f(s\to x) - \sum_{(x\to s)\in E} f(x\to s)$$

$$= \sum_{x\in V} f(s\to x) - \sum_{x\in V} f(x\to s)$$

here, $f(x\to y) = 0$ if $x \to y \notin E$

Def'n An (s,t)-cut in G is a partition of the vertices $V = S \sqcup T$ such that $s \in S$ and $t \in T$.

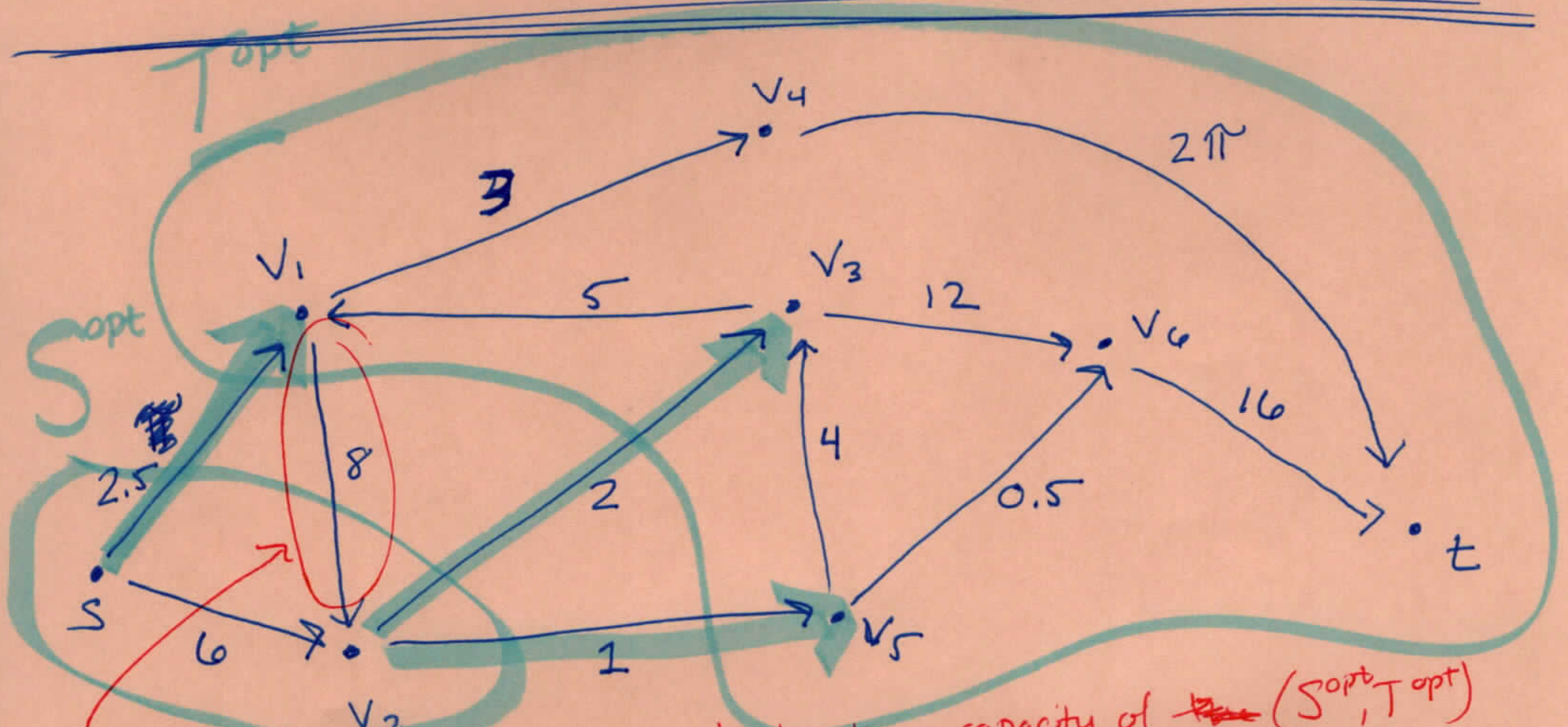
disjoint union! $S \cap T = \emptyset$

The capacity of a cut (S,T) is "the amount that can leave S"

$$\|S,T\| := \sum_{v \in S} \sum_{w \in T} c(v \to w)$$

} sum capacities of edges from S to T, ignoring edges from T to S & edges w/in S or w/in T.

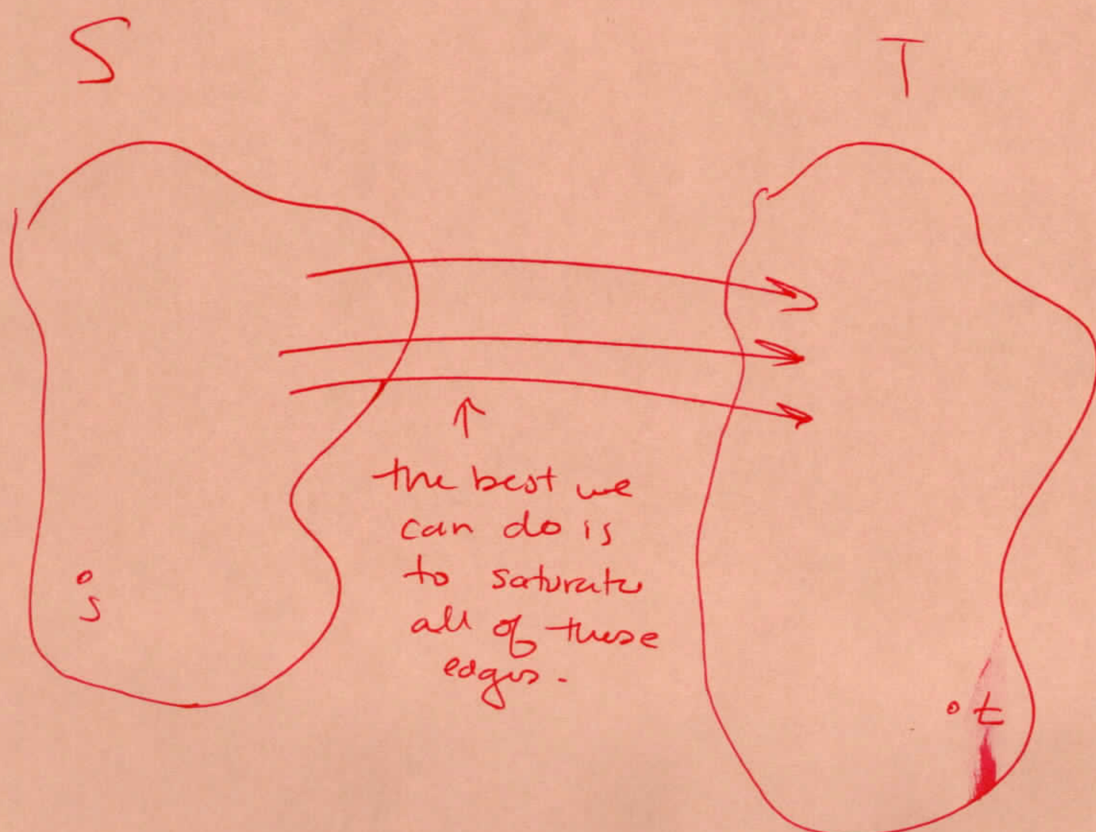
min cut problem: find the capacity of the minimum cut.



note: this edge does not count to the capacity of (S^{opt}, T^{opt})

Groups: ① Find 3 cuts + give their capacities
② Find a flow that realizes the max flow.
③ Find a cut that realizes the min cut.

$\|(S^{opt}, T^{opt})\| = 5.5 = 2.5 + 2 + 1$


S
T
s
t
the best we can do is to saturate all of these edges.

Lemma: Let $G=(V,E,c)$ be a weighted graph.
Let $s,t \in V$
Let $f: E \to \mathbb{R}$ be <u>any</u> (s,t)-feasible flow.
Let (S,T) be <u>any</u> (s,t) cut.

Then,

$$|f| \leq ||S,T||.$$

And, if all edges from S to T are saturated and $f(e)=0$ $\forall$ edges from T to S, then $|f| = ||S,T||$.

Proof:

$$|f| = \sum_{v\in V} f(s\to v) - \sum_{w \in V} f(w \to s) \quad \text{by definition}$$

$$= \sum_{x\in S}\left(\sum_{v\in V} f(x\to v) - \sum_{w\in V} f(w\to x)\right) \quad \text{by conservation of flow}$$

(like adding 0's for all $x \neq s$).

$$= \sum_{x\in S}\left(\sum_{v\in T} f(x\to v) - \sum_{w\in T} f(w\to x)\right)$$

(follows from the conservation of flow again → check it out!)

If no edges from T to S used, this is an equality.

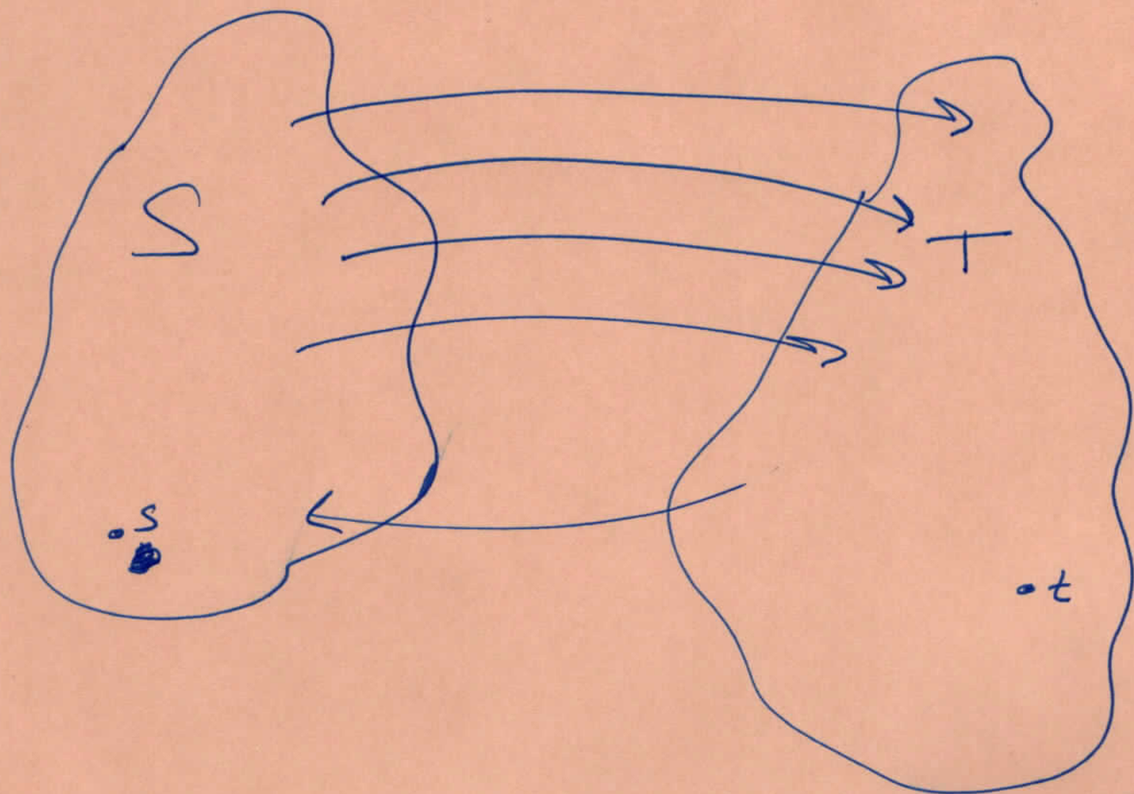
$$\leq \sum_{x\in S}\sum_{v\in T} f(x\to v) \quad \text{by dropping a neg \#}$$

If all edges from S to T saturated, this is an equality.

$$\leq \sum_{x\in S}\sum_{v\in T} c(x\to v) \quad \text{by the capacity constraint}$$

$$= ||S,T|| \qquad \square$$

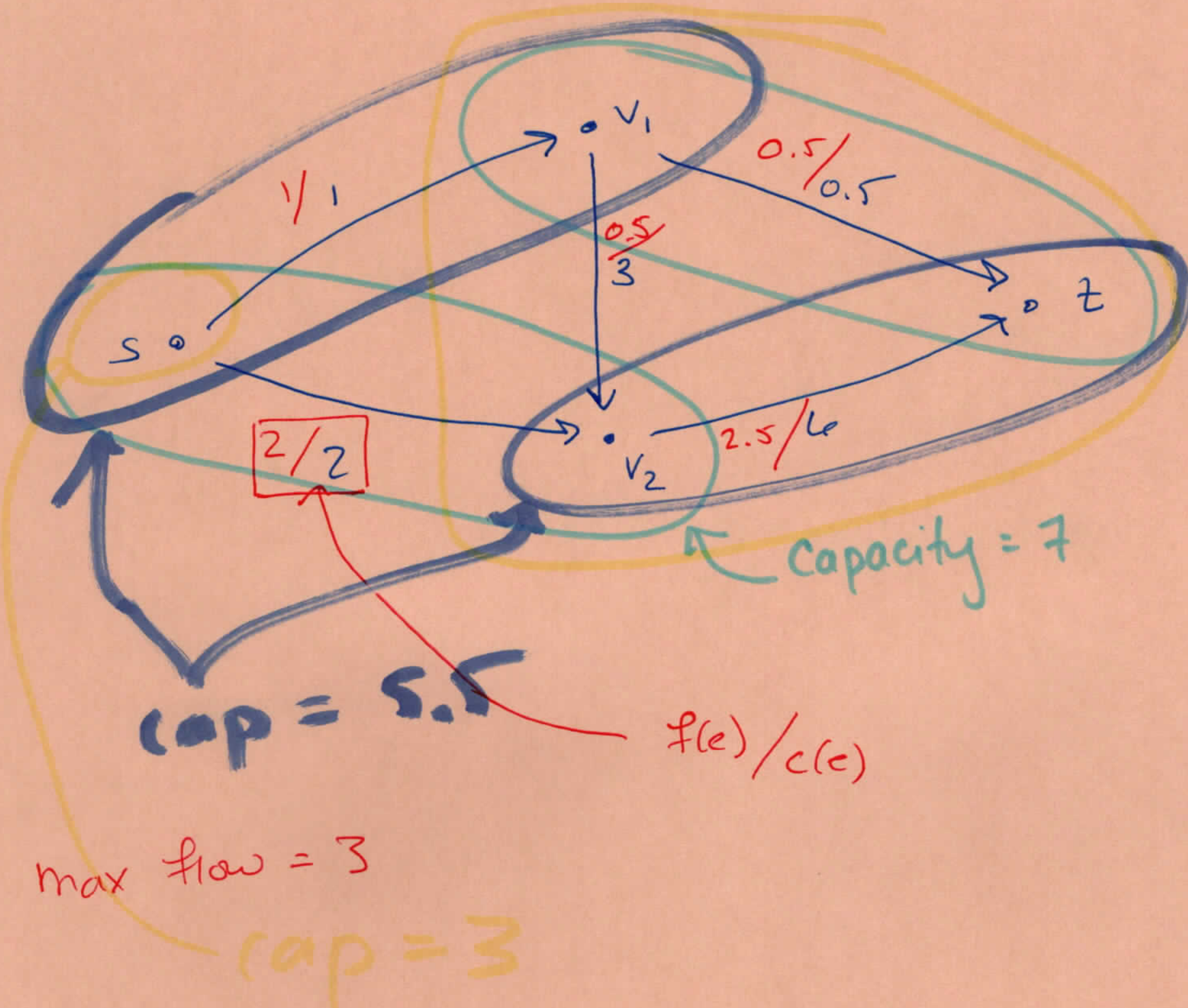
Arbitrary S,T cut:



Any flow will need to move "stuff" from $s \in S$ to $t \in T$. These edges are like bridges. The most I can bring from s to t is upper bounded by the capacity of (S,T) (ie, I would be saturating every edge/bridge)

But, the cut was arbitrary, so if I do this for every cut

$$|f| \leq \min_{(S,T) \text{ cut}} ||S,T||$$


v1
s
t
v2
1/1
0.5/0.5
0.5/3
2/2
2.5/6
capacity = 7
cap = 5.5
f(e)/c(e)
max flow = 3
cap = 3