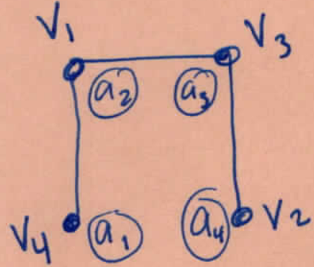
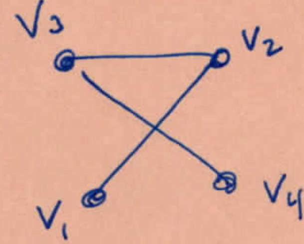
G_1 $\qquad$ G_2 $\qquad$ 20 Oct 2021





Q: Are these the same graph?

→ when we ignore labels, we see 2 copies of the same graph. $G_1 \cong G_2$
"graph isomorphism"

→ G_1 is embedded in $\mathbb{R}^2$ (and also immersed)
but G_2 is immersed but not embedded
↑ allow crossings $\qquad$ ↑ no crossings allowed

→ subgraph isomorphism is NPC
(sub)-tree iso is in P.

Claim: Let G be a plane graph. Then, G has a 6-coloring.

Proof by Induction:

Base case:

For the base case, consider $G = (\{\cdot\}, \emptyset)$. Any color you choose for the vertex is a 6-coloring.

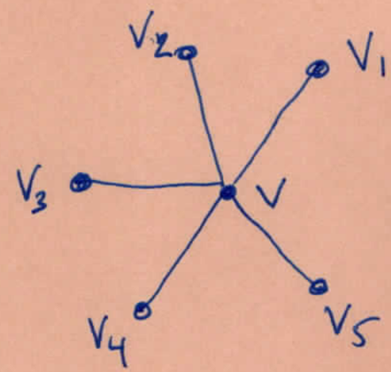
IA Now, let $\boxed{k \geq 1}$ and for all plane graphs ~~Let~~ $G = (V, E)$ ~~be a plane graph~~ such that $|V| = k$, assume that G has a 6-coloring.

allows k to "take on" base case!

Inductive case: Let $H = (V_H, G_H)$ be a plane graph such that $|V_H| = k+1$. By the lemma from last class, $\exists\, v \in V_H$ such that $\deg(v) \leq 5$. Also, $H \setminus \{v\}$ is a plane graph with k vertices + has a 6-coloring by the I.A. Fix one such coloring. Now, since $\deg(v) \leq 5$, we know that there is at least one color available for v. Hence, H has a 6-coloring. $\square$

Claim: Let G be a plane graph.
Then, G has a 5-coloring.

Proof by Ind:
Base + I.A. are nearly identical $(6 \to 5)$.
Let $H = (V_H, G_H)$ be a plane graph s.t. $|V_H| = k+1$. Let $v \in V_H$ be a vertex of degree ≤ 5. Then, $H \setminus \{v\}$ has a 5-coloring. Choose one.



← pick one to be "first neighbor"
+ proceed clockwise numbering them

If $\deg(v) < 5$, choose any remaining color.
If any repeat colors, choose any remaining color.
So, now, we're left w/ the case where v has 5 neighbors, all of the ~~same~~ different colors.
~~Consider~~ Let c_i = the color of v_i.
Consider the subgraph H' of H consisting of all vertices with colors c_1 or c_3 (and their edges, as appropriate)
Are v_1 + v_3 in the same conn. comp?

If no: reverse the colors of the conn. comp. containing v_1

$\Rightarrow v_1$ and v_3 are colored c_3

$\Rightarrow v$ can be colored c_1

If yes: Let's repeat this process w/ v_2 and v_4.

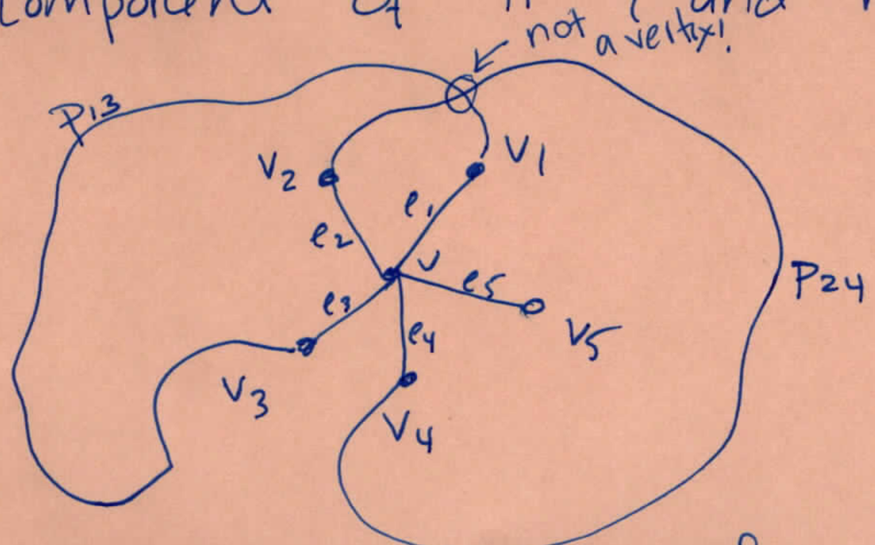
Let H'' be the subgraph restricting to verts of color c_2 and c_4.

Are c_2 and c_4 in the same con. comp?

If ~~yes~~ no: reverse the colors of the conn. component containing v_2. ✓

Now, $v_1 + v_3$ are in the same conn component of H' (and hence $G \setminus \{v\}$)

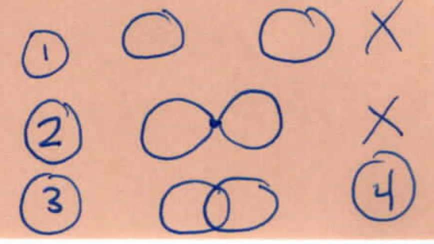
and $v_2 + v_4$ are in the same conn. component of H'' (and hence $G \setminus \{v\}$)

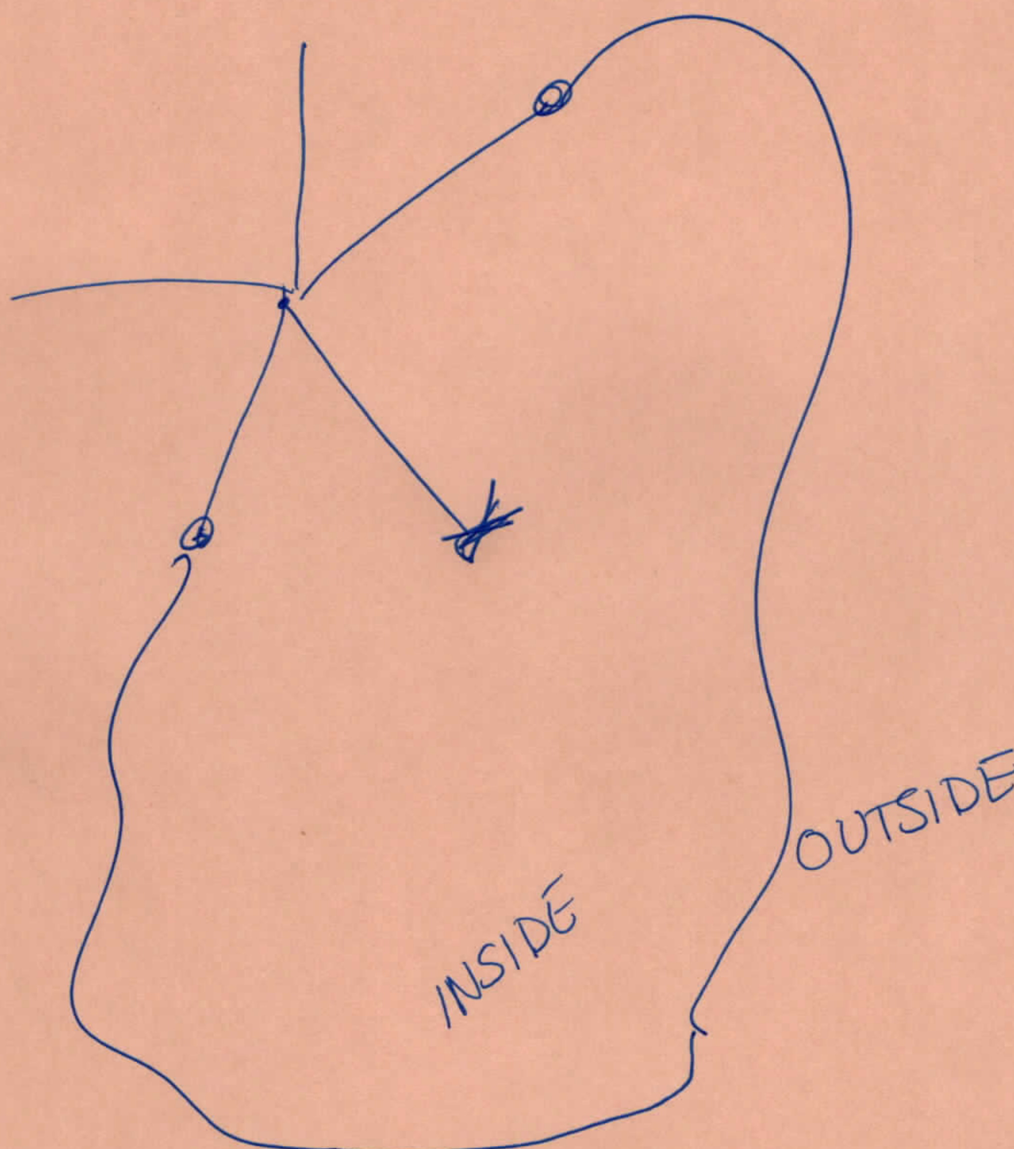


$P_{13} + e_1 + e_3$ is a cycle and

$P_{24} + e_2 + e_4$ is a cycle.

Note: 3 ways 2 (simple) cycles can interact:





Four Colors Suffice ← a great book!