Statistical Field Theory 2

2D Classical Ising Model to 1D Quantum Ising Model

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1 Introduction

This will being with a discussion of 2D classical to 1D Quantum Ising Model, forumalting the path integral with an action of Majorana Fermions at the critical temperature which will allow us to derive a CFT for the Ising model at criticality. This disussion will closely follow [Sha17] and [Mol13] borrowing from [Rid13].

Here we will study the continuum limit formulation of the two-dimensional Ising model, starting from the Hamiltonian limit of its transfer matrix. We will first derive the quantum Hamiltonian of the model and then we will see how to diagonalize the quantum Hamiltonian by means of fermionic fields via the Jordan-Wigner Transformation: brining the Hamiltonian to a quadratic form with creation and annihilation operators of the fermions. In the limit where the lattice spacing goes to zero, the Ising model becomes a theory of free Majorana fermions, which satisfy a relativistic dispersion relation and their mass is a direct measurement of the displacement of the temperature from the critical temperature T_c. The fermionic formulation of the two-dimensional Ising model is crucial for understanding many of its physical properties and for the computation of its correlation functions, which we will see when we arrive at Conformal Field Theory.

2 2D Ising Model

On a square lattice with N columns and M rows, we define the model by

$$Z = \sum_{s_i} \exp\left[K \sum_{\langle i,j \rangle} s_i s_j\right], \tag{2.1}$$

where $K = J/kT$ and the symbol $\langle i, j\rangle$ means that sites i and j are nearest neighbours. There are many choices one could make for the edges of the lattice, periodic boundary conditions in one direction which makes a cylinder, or both directions which makes a torus, or open boundary conditions where spins at the edges have no nearest neighbours. For now let us just say that M and N and the number of sites NM are huge and we are nowhere near the ends. There are $2NM$ bonds on a square lattice with NM sites because each site has four bonds enamating from it, but each bond is counted twice, once at each of its endpoints.

Consider quickly the extreme limits. As $K = J/kT \to \infty$ or $T \to 0$, the spins will be all up or down, the system will be magnetised, and $\langle M\rangle$, the average spin per site, will be at its maximum of ± 1. Let us pick $\langle M\rangle = +1$. As $K \to 0$ or $T \to \infty$, the Boltzmann weight will be 1 for all configurations

Here we will now outline how the Ising model can be solved exactly. The plan is to first write down the transfer matrix T, and show how it can be diagonalised by introducing fermionic variables. The actual diagonalisation will be carried out in the τ-continuum limit where $T = e^{-\tau H}$ and H is a simple, readily diagonalised Hamiltonian. Now, as in the $1D$ case we have seen, we

want the transfer matrix corresponding to

$$Z = \sum_{\{s_i\}} \exp\left[\sum_i K_x s_i s_{i+x} + K_\tau (s_i s_{i+\tau} - 1)\right], \tag{2.2}$$

where $i+x$ and $i+\tau$ are neighbours of site i in the x and τ directions, where we have subtracted 1 from the $s_i s_{i+\tau}$ so we can borrow results from the $d=1$ Ising model. We have also set $K_x = \beta J$ and $K_\tau = \beta J'$ for the horizontal and vertical directions respectively. We claim the transfer matrix for a lattice with N columns is

$$\begin{aligned} T &= \frac{\exp\left[\sum_{n=1}^N K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]^N} \cdot \exp\left[\sum_{n=1}^N K_x \sigma_3(n)\sigma_3(n+1)\right] \\ &\equiv V_1 V_3, \end{aligned} \tag{2.3}$$

where $\sigma_1(n)$ and $\sigma_3(n)$ are the usual (Heisenberg) Pauli matrices at sites n. To see why this is the correction operator, recall $Z = \operatorname{Tr} T^N$ from the last talk. Now the transfer matrix is $2^N \times 2^N$ dimensional. So if we sandwich T between $\langle s_1' \cdots s_n' \cdots s_N'|$ and $|s_1 \cdots s_n \cdots s_N\rangle$ where $(s_1', \cdots, s_n', \cdots, s_N')$ and $(s_1, \cdots, s_n, \cdots, s_N)$ are the eigenvalues of $\sigma_3(n)$ for $n = 1, \ldots, N$.

$$\langle s_1' \cdots s_N'| T |s_1 \cdots s_N\rangle = \langle s_1' \cdots s_N'| \frac{\exp\left[\sum_{n=1}^N K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]^N} \cdot \exp\left[\sum_{n=1}^N K_x \sigma_3(n)\sigma_3(n+1)\right] |s_1 \cdots s_N\rangle \tag{2.4}$$

Acting to the right on their own eigenstates, the $\sigma_3(n)$'s in V_3 will give the spin values $s_n = \pm 1$, yielding the Boltzmann weight in the horizontal bonds of the row containing the s_n's.

$$\begin{aligned} \langle s_1' \cdots s_N'| T |s_1 \cdots s_N\rangle &= \langle s_1' \cdots s_N'| \frac{\exp\left[\sum_{n=1}^N K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]^N} \cdot \exp\left[\sum_{n=1}^N K_x \sigma_3(n)\sigma_3(n+1)\right] |s_1 \cdots s_N\rangle \\ &= \langle s_1' \cdots s_N'| \frac{\exp\left[\sum_{n=1}^N K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]^N} \cdot \exp\left[\sum_{i=1}^N K_x s_i s_{i+x}\right] |s_1 \cdots s_N\rangle \\ &= \exp\left[\sum_{i=1}^N K_x s_i s_{i+x}\right] \langle s_1' \cdots s_N'| \frac{\exp\left[\sum_{n=1}^N K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]^N} |s_1 \cdots s_N\rangle . \end{aligned} \tag{2.5}$$

As for V_1, the matrix that is left inside the inner product, the matrix elements will factorise into a product over the n sites, since each $\sigma_1(n)$ will only act on the n-th eigenstate, giving

$$\begin{aligned} \langle s_1' \cdots s_N'| T |s_1 \cdots s_N\rangle &= \exp\left[\sum_{i=1}^N K_x s_i s_{i+x}\right] \langle s_1' \cdots s_N'| \frac{\exp\left[\sum_{n=1}^N K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]^N} |s_1 \cdots s_N\rangle \\ &= \exp\left[\sum_{i=1}^N K_x s_i s_{i+x}\right] \langle s_1' \cdots s_N'| \prod_{n=1}^N \frac{\exp\left[K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]} |s_1 \cdots s_N\rangle \\ &= \exp\left[\sum_{i=1}^N K_x s_i s_{i+x}\right] \prod_{n=1}^N \langle s_n'| \frac{\exp\left[K_\tau^* \sigma_1(n)\right]}{[\cosh K_\tau^*]} |s_n\rangle . \end{aligned} \tag{2.6}$$

where we can take the sum out of the exponential since the first Pauli matrix will commute with itself. The last step requires some explanation, here we merely mean that if we expand the product, and then act repeatedly with the operator, we will end up with a product of numbers, which the

last step highlights. Now, each matrix element can be simply computed

$$\begin{aligned}
\langle s_n'| \frac{\exp\left[K_\tau^* \sigma_1(n)\right]}{\left[\cosh K_\tau^*\right]} |s_n\rangle &= \frac{1}{\cosh K_\tau^*} \langle s_n'| \cosh K_\tau^* + \sinh\left[K_\tau^* \sigma_1(n)\right] |s_n\rangle \\
&= \frac{1}{\cosh K_\tau^*} \left(\langle s_n'| \cosh K_\tau^* |s_n\rangle + \langle s_n'| \sinh\left[K_\tau^* \sigma_1(n)\right] |s_n\rangle\right) \\
&= \delta_{s_n', s_n} + \langle s_n'| \frac{\sinh K_\tau^*}{\cosh K_\tau^*} |-s_n\rangle \\
&= \delta_{s_n', s_n} + \tanh K_\tau^* \delta_{s_n', -s_n}.
\end{aligned} \tag{2.7}$$

To see why this is the Boltzmann weight, identify $s_n = s_i$ and $s_n' = s_{i+\tau}$ and observe that when $s_n = s_n' = \pm 1$, the Boltzmann weight in the τ direction $\exp\left(K_\tau(s_i s_{i+\tau} - 1)\right)$ gives 1, and when $s_n = -s_n'$ we get $\exp(-2K_\tau)$. Remembering how K_τ^* is defined, this is simply $\tanh K_\tau^*$. Therefore, we have

$$\delta_{s_n', s_n} + \tanh K_\tau^* \delta_{s_n', -s_n} = \exp\left(K_\tau(s_n s_n' - 1)\right). \tag{2.8}$$

Therefore we have

$$\begin{aligned}
\langle s_1' \cdots s_N'| T |s_1 \cdots s_N\rangle &= \exp\left[\sum_{i=1}^N K_x s_i s_{i+x}\right] \prod_{n=1}^N \delta_{s_n', s_n} + \tanh K_\tau^* \delta_{s_n', -s_n} \\
&= \exp\left[\sum_{i=1}^N K_x s_i s_{i+x}\right] \prod_{n=1}^N \exp\left(K_\tau(s_n s_n' - 1)\right) \\
&= \exp\left[\sum_{i=1}^N K_x s_i s_{i+x} + K_\tau(s_i s_{i+\tau} - 1)\right].
\end{aligned} \tag{2.9}$$

So the operator gives the correct matrix elements. While T above does the job, it is unfortunately not Hermitian, as we promised all operators to be for us to be able to write expectation values $\langle\psi|A|\psi\rangle$ unambiguously. We could trade it for a Hermitian version

$$T = V_3^{1/2} V_1 V_3^{1/2}, \tag{2.10}$$

where now each $V_3^{1/2}$ captures half the energies of the horizontal bonds in the two rows in question, where the other halves come from the next insertion of the transfer matrix in the computation of the partition function Z. We will ignore the Hermicity issue since it will disappear in the τ-continuum limit we will care about and stick with using $T = V_1 V_3$, we will also again drop the $(\cosh K_\tau^*)^N$ in the denominator of V_1 and the additive contribution of $\log\cosh K_\tau^*$ it will make to βf.

Speaking of the partition function, using this tranfer matrix as in the $1D$ case we can express the partition function using the matrix elements as before. To do this, lets make a quick notational change, let the $a-th$ row of the lattice be denoted by

$$\mu_a = \{s_1, \ldots, s_N\}_{a\text{-th row}} \tag{2.11}$$

so that we can define Therefore we can think of a configuration of the system to be specified by the ensemble of rows $\{\mu_1, \ldots, \mu_n\}$. The a-th row interacts with only the nearest neighbour rows, namely μ_{a-1} and μ_{a+1}. Let $E(\mu_a, \mu_{a+1})$ be the interaction energy between two nearest-neighbour rows and let $E(\mu_a)$ be the energy coming from the interactions of spins placed on the a-th row

$$\begin{aligned}
E(\mu, \mu') &= -J' \sum_{k=1}^n \sigma_k \sigma_k', \\
E(\mu) &= -J \sum_{k=1}^n \sigma_k \sigma_{k+1} - B \sum_{k=1}^n \sigma_k,
\end{aligned} \tag{2.12}$$

where J and J' are the coupling constants alng the vertical and horizontal axis respectively. The total energy (Hamiltonian) for the system can therefore we written as

$$H(\mu_1, \ldots, \mu_n) = \sum_{a=1}^n \left[E(\mu_a, \mu_{a+1}) + E(\mu_a)\right]. \tag{2.13}$$

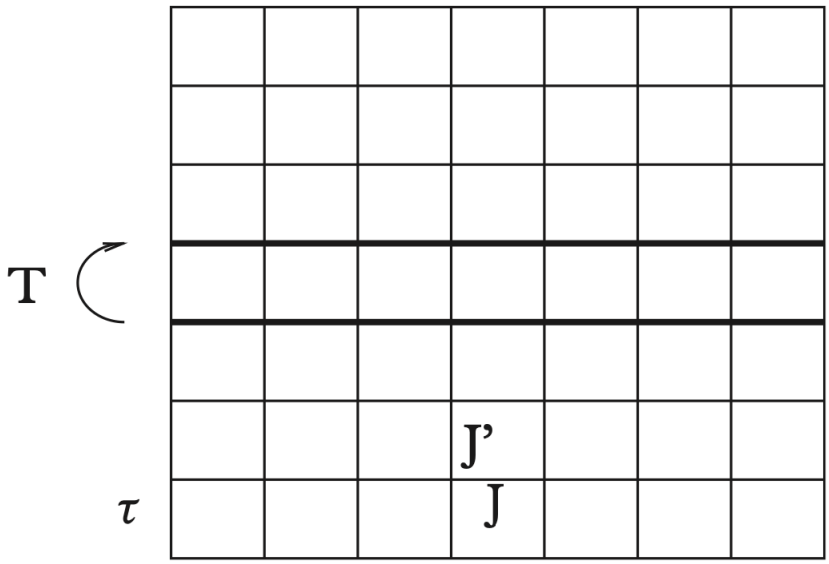

Figure 2.1: Transfer matrix between rows

The partition function will therefore be given by

$$Z = \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \exp\left[-\beta H(\mu_1, \ldots, \mu_n)\right]. \tag{2.14}$$

Therefore we can now write our transfer matrix T in terms of rows to give

$$\langle s_1' \cdots s_N' | T | s_1 \cdots s_N \rangle = \langle \mu | T | \mu' \rangle = \exp\left[-\beta(E(\mu, \mu') + E(\mu))\right] = \exp\left[\sum_{i=1}^{N} K_x s_i s_{i+x} + K_\tau (s_i s_{i+\tau} - 1)\right]. \tag{2.15}$$

We can use this to rewrite our partition function as

$$\begin{aligned}
Z &= \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \exp\left[-\beta \sum_{a=1}^{n} \left[E(\mu_a, \mu_{a+1}) + E(\mu_a)\right]\right] \\
&= \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \prod_{a=1}^{N} \exp\left[-\beta \left[E(\mu_a, \mu_{a+1}) + E(\mu_a)\right]\right] \\
&= \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \prod_{a=1}^{N} \langle \mu_a | T | \mu_{a+1} \rangle \\
&= \sum_{\mu_1} \sum_{\mu_2} \cdots \sum_{\mu_n} \langle \mu_1 | T | \mu_2 \rangle \langle \mu_2 | T | \mu_3 \rangle \cdots \langle \mu_n | T | \mu_1 \rangle \\
&= \sum_{\mu_1} \langle \mu_1 | T \left(\sum_{\mu_2} | \mu_2 \rangle \langle \mu_2 | \right) T \left(\sum_{\mu_3} | \mu_3 \rangle \right) \langle \mu_3 | \cdots \left(\sum_{\mu_n} | \mu_n \rangle \langle \mu_n | \right) T | \mu_1 \rangle \\
&= \sum_{\mu_1} \langle \mu_1 | T^N | \mu_1 \rangle \\
&= \operatorname{Tr} T^N
\end{aligned} \tag{2.16}$$

where we used the complete set of states identity $\sum_a |a\rangle \langle a| = I$. So as before the partition function is the trace of the transfer matrix T.

Let us now look towards obtaining the critical point in terms of the coupling constants (K_x, K_τ) for the Ising model, before taking the limit $\tau \to 0$. Recall from the first talk the relation

$$e^{-2K} = \tanh K^*(K) \tag{2.17}$$

In other words, K^* as a function of K coincides with K as a function of K^*. Consequently, the dual of the dual is the original $(K^*)^* = K$. Now if we assume there is just a single phase transition that occurs at the critical value K_c (and the high/low temperature expansions of the partition function), we have

$$K_c^* = K_c, \text{ or } e^{-2K_c} = \tanh K_c \implies K_c = 0.4407\ldots \tag{2.18}$$

No if we have the case with two different coupling constants K_x and K_τ, once again using the high and low T expansions that we will not consider here one can make the duals as

$$e^{-2K_x} = \tanh K_\tau, \quad e^{-2K_\tau} = \tanh K_x. \tag{2.19}$$

Now we can use the dual equation (2.17) to obtain

$$e^{-2K_x} = e^{-2K_\tau^*}, \quad e^{-2K_\tau} = e^{-2K_x^*} \tag{2.20}$$

Therefore we have the duality maps in the (K_x, K_τ)-plane, with self-dual points obeying

$$(K_x, K_\tau) = (K_\tau^*, K_x^*). \tag{2.21}$$

We can represent the phase space of the coupling constants in Fig. 2.1 and Fig. 3.2 using

$$\begin{aligned}
\sinh 2K &= 2\sinh K \cosh K \\
&= 2\tanh K \cosh^2 K \\
&= \frac{2\tanh K}{1-\tanh^2 K} \\
&= \frac{2e^{-2K^*(K)}}{1-e^{-4K^*(K)}} \\
&= \frac{2}{e^{2K^*(K)} - e^{-2K^*(K)}} \\
&= \frac{1}{\sinh 2K^*(K)}
\end{aligned} \tag{2.22}$$

So we have the relation

$$\sinh 2K \sinh 2K^*(K) = 1, \tag{2.23}$$

which gives us the plot Fig. 2.1. This equation gives the self-dual points in the phase diagram. The system is critical on this whole line, and the isotrophic point like at intersection with $K_\tau = K_x = K_c$ giving

$$\sinh^2 2K_c = 1 \implies K_c = 0.4407. \tag{2.24}$$

3 Analysis in the τ-Continuum Limit

Now, one can show how the Ising model can be solved exactly on a finite periodic lattice by mapping to fermions, we are primarily interested in the case where the vertical lattice spacing goes to zero $\tau \to 0$ to obtain the $1D$ quantum Ising chain. In this limit we have be careful that the physical content of the system does not change.

We will set[1]

$$K_\tau^* = \lambda\tau, \quad K_x = \tau \tag{3.1}$$

where $\lambda \in \mathbb{R}$ is a proportionality factor connecting the disordered and ordered phases of the Ising model. In Fig. 3.2 we have the line showing $\lambda = 1$, where below is $\lambda < 1$ for the disordered phase of the Ising model, and $\lambda > 1$ for the ordered phase of the Ising model. $\lambda = 1$ is the critical point[2].

Now taking (2.3), we would ordinarily need to make use of the Baker-

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([[A,B],B]+[A,[A,B]])+\ldots} \tag{3.2}$$

[1] To see where this comes from, one can keep track of the amount of spins flipping between each row, and then see that $K_x \sim \tau$ and $\exp(-2K_\tau) \sim \tau$, then setting λ proportionality factor, see [Mus20] and [Won05] for this.

[2] [Ton17] page 59-60 shows images for each phase of λ

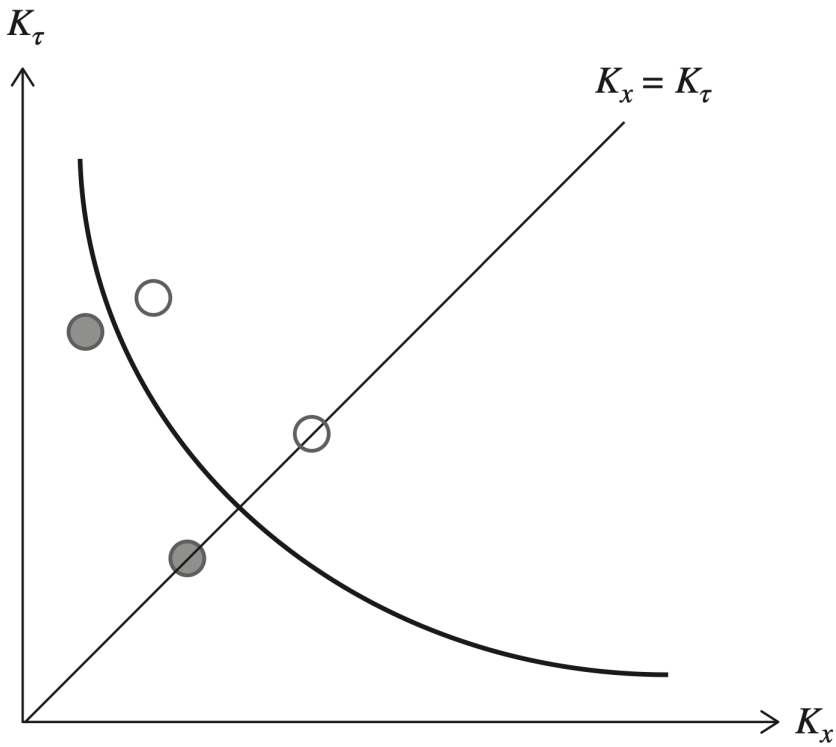

Figure 3.1: Phase diagram for coupling constants in the $K_x - K_\tau$ plane. The critical points lie on the line $\sinh 2K_x \sinh 2K\tau = 1$. Solid circles indicate pairs of points with open cirlces indicating their duals.

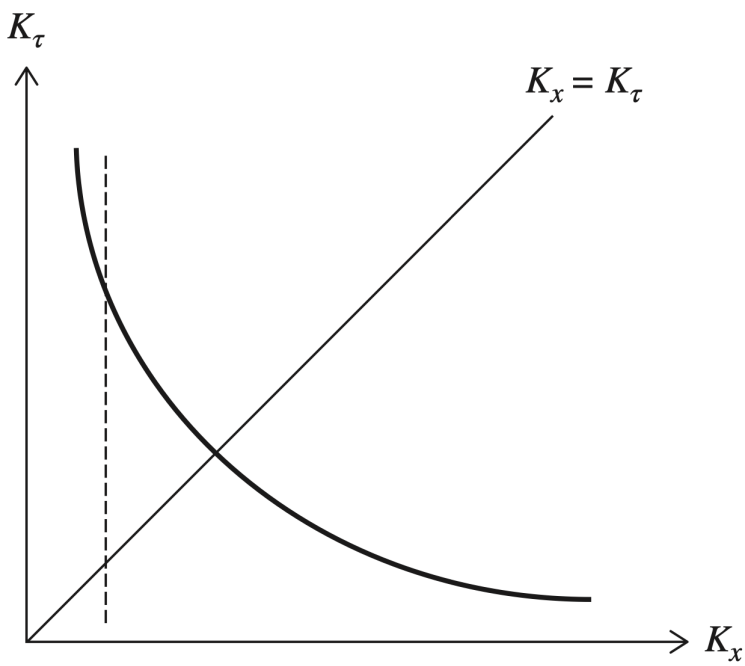

Figure 3.2: The solid curve is $\tanh K_\tau = e^{-2K_x}$ separate's the two phases. The region where both K's are small is disordered, and the other side is ordered. The dotted vertical line at small (eventually infinitesimal) K_x is the range probed by the τ-continuum limit. Moving up and down the dotted line we can see the two phases and the transition.

However, since we are interested in the $\tau \to 0$ case, we can ignore the τ^2 terms and higher, and this allows us to combined the exponentials in (2.3) to give

$$\begin{aligned} T &= \exp\left[\sum_{n=1}^{N} K_\tau^* \sigma_1(n)\right] \cdot \exp\left[\sum_{n=1}^{N} K_x \sigma_3(n)\sigma_3(n+1)\right] \\ &= \exp\left[\sum_{n=1}^{N} \lambda\tau\sigma_1(n)\right] \cdot \exp\left[\sum_{n=1}^{N} \tau\sigma_3(n)\sigma_3(n+1)\right] \\ &= \exp\left[\tau\sum_{n=1}^{N} \lambda\sigma_1(n) + \sigma_3(n)\sigma_3(n+1) + O(\tau^2)\right] \end{aligned} \tag{3.3}$$

where recall we set $K_\tau^* = \lambda\tau$ and $K_x = \tau$ to combine the exponentials. Recall we have defined that $T = e^{-\tau H}$, so we can identify the Hamiltonian as

$$H = -\sum_{n=1}^{N} \lambda\sigma_1(n) + \sigma_3(n)\sigma_3(n+1). \tag{3.4}$$

Now, in the thermodynamic limit $N \to \infty$, this sum is extended to on all sites between $-\infty$ and ∞ in both directions, so the Hamiltonian becomes

$$H = -\sum_{n=-\infty}^{\infty} \left[\lambda\sigma_1(n) + \sigma_3(n)\sigma_3(n+1)\right]. \tag{3.5}$$

Now, since we are in the Heisenberg picture of quantum mechanics, where operators depend on time [Vol21]. Let A be a Schrödinger operator so the Heisenberg operator is $A(\tau) = U(\tau)^\dagger A U(\tau)$ for $U(\tau) = e^{-\tau H}$ in Euclidean quantum mechanics with $\hbar = 1$. The equation of motion will be given by

$$\frac{dA(\tau)}{d\tau} = \frac{\partial U^\dagger(\tau)}{\partial\tau} A U(\tau) + U^\dagger(\tau) A \frac{\partial U(\tau)}{\partial\tau} \tag{3.6}$$

Now making use of the Euclidean Schrödinger equation

$$-\frac{\partial U(\tau)}{\partial\tau} = H(\tau)U(\tau) \tag{3.7}$$

We have

$$\frac{\partial U(\tau)}{\partial\tau} = -H(\tau)U(\tau), \quad \frac{\partial U^\dagger(\tau)}{\partial\tau} = U^\dagger(\tau)H(\tau) \tag{3.8}$$

$$\begin{aligned} \frac{dA(\tau)}{d\tau} &= \frac{\partial U^\dagger(\tau)}{\partial\tau} A U(\tau) + U^\dagger(\tau) A \frac{\partial U(\tau)}{\partial\tau} \\ &= U^\dagger(\tau)H(\tau)AU(\tau) - U^\dagger(\tau)AH(\tau)U(\tau) \\ &= U^\dagger(\tau)H(\tau)U(\tau)U^\dagger(\tau)AU(\tau) - U^\dagger(\tau)AU(\tau)U^\dagger(\tau)H(\tau)U(\tau) \\ &= U^\dagger(\tau)H(\tau)U(\tau)A(\tau) - A(\tau)U^\dagger(\tau)H(\tau)U(\tau) \\ &= [U^\dagger(\tau)H(\tau)U(\tau), A(\tau)] \\ &= [H, A(\tau)]. \end{aligned} \tag{3.9}$$

where we inserted $U(\tau)^\dagger U(\tau) = I$ and since H is independent of time in $U(\tau) = e^{-\tau H}$ so we have $U^\dagger(\tau)H(\tau)U(\tau) = H$. Therefore we are left with the Heisenberg equation of motion for Euclidean quantum mechanics

$$\frac{dA(\tau)}{d\tau} = [H, A(\tau)]. \tag{3.10}$$

It is worth mentioning that Heisenberg operators in Euclidean Quantum mechanics are not adjoint at $\tau = 0$ do not evolve into adjoints at later times

$$A^\dagger(\tau) = e^{\tau H} A^\dagger e^{-\tau H} \neq (A(\tau))^\dagger = e^{-\tau H} A^\dagger e^{\tau H}. \tag{3.11}$$

Let's apply the Heisenberg equation of motion (3.10) to our operators in (3.5), we obtain

$$\begin{aligned}
\frac{\partial}{\partial \tau}\sigma_3(r) &= [H, \sigma_3(r)] \\
&= \left[-\sum_{a=-\infty}^{\infty} \lambda\sigma_1(a) + \sigma_3(a)\sigma_3(a+1), \sigma_3(r) \right] \\
&= -\sum_{a=-\infty}^{\infty} \lambda\left[\sigma_1(a), \sigma_3(r)\right] + \left[\sigma_3(a)\sigma_3(a+1), \sigma_3(r)\right] \\
&= -\lambda[\sigma_1(r), \sigma(r)] \\
&= 2\lambda\sigma_2(r) \\
&= -2\lambda\sigma_1(r)\sigma_3(r).
\end{aligned} \tag{3.12}$$

We will now attempt to transform our equations of motion to be those of Majoaran fermions as oppose to Pauli matrices. Since we will be looking towards the conformal field theory, we will set $\lambda = 1$ at the critical point.

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