Fragment Abstraction for Concurrent Shape Analysis

No Institute Given

1 Introduction

Concurrent algorithms with an unbounded number of threads that concurrently access a dynamically allocated shared state are of central importance in a large number of software systems. They provide efficient concurrent realizations of common interface abstractions, and are widely used in libraries, such as the Intel Threading Building Blocks or the `java.util.concurrent` package. They are notoriously difficult to get correct and verify, since they often employ fine-grained synchronization and avoid locking when possible. A number of bugs in published algorithms have been reported [9, 27]. Consequently, significant research efforts have been directed towards developing techniques to verify correctness of such algorithms, in particular to verify that concurrent algorithms that implement standard data structure interfaces are *linearizable*, meaning that each method invocation can be considered to occur atomically at some point between its call and return. Many such techniques require significant *manual* effort for constructing a proof of correctness (e.g., [25, 39]), in some cases with the support of an interactive theorem prover (e.g., [37, 6, 7, 32, 31]). Development of automated verification techniques is a difficult challenge.

A major difficulty is that a successful verification technique must be able to reason about fine-grained concurrent algorithms that are infinite-state in many dimensions: they consist of an unbounded number of concurrent threads, which operate on an unbounded domain of data values, and use unbounded dynamically allocated memory. Perhaps the hardest of these is the problem of handling dynamically allocated memory. Consequently, all existing techniques for automatically proving correctness of such concurrent algorithms restrict attention to the case where the heap represents shared data byp singly-linked lists [1, 17, 2, 33, 40]. Many of these techniques impose additional restrictions on the considered problem, such as bounding the number of accessing threads [3, 44, 42]. [Add other restrictions] However, many concurrent data structure implementations employ more sophisticated structures, such as skip lists [13, 22, 35], trees, and arrays of of singly-linked lists [8]. There are no techniques that have been applied to automatically verify concurrent algorithms that operate on such data structures.

The last list and its description must be improved. We actually should be more precise about what previous work has achieved, in terms of which combinations of challenges have been overcome

Contributions In this paper, we present a technique for automatic verification of concurrent data structure implementations that operate on dynamically allocated heap structures which are more complex than just singly-linked lists. Our approach is the first

framework that can automatically verify concurrent data structure implementations that employ skip lists, singly linked lists, as well as arrays of singly linked lists, at the same time as handling an unbounded number of concurrent threads, an unbounded domain of data values (including timestamps), and an unbounded shared heap.

Our technique is based on a novel shape abstraction, called *fragment abstraction*, which in a simple way is able to represent different kinds of unbounded heap structures, such as arrays of singly linked lists and skip lists. Its main idea is to represent a set of heap states by a set of *fragments*. Each fragment is simply a pair of node types (called *tags*) that are connected by a pointer. A tag can be seen as a finitary abstraction of a heap node, which summarizes both local information about values of its data fields as well as global information about its position in the heap, including how it can reach to and be reached from (by following chains of pointers) other heap cells that are pointed to by global variables. A set of fragments represents the set of heap structures in which each pair of pointer-connected nodes are represented by some fragment in the set. Inuitively, a set of fragments describes the set of heaps that can be formed by "pieced together" fragments in the set. This "piecing together" must be both locally consistent (appending only fragments that agree on their common node), and globally consistent (respecting the global reachability information). By construction, our fragment abstraction is finitary, since there is a bounded set of tags.

Fragment abstraction can, in a natural way, be combined with other abstractions for handling unbounded data domains and for handling an unbounded number of threads. Our fragment abstraction technique copes with an unbounded data domain by letting the definition of tags incorporate a suitable data abstraction to the data fields in heap nodes. We cope with the challenge of an unbounded number of threads by incorporating the successful thread-modular approach [4]; this is done simply by letting each set of fragments represent only the heap cells that are accessible to an arbitrary single thread.

We have implemented our approach and applied it to automatically verify correctness, in the sense of linearizability, of a large number of concurrent data structure algorithms. More specifically, we have automatically verified linearizability of most linearizable concurrent implementations of sets, stacks, and queues, which emply singly-linked lists, skip lists, or arrays of timestamped singly-linked lists, which are known to us in the literature on concurrent data structures.

For this verification, we we specify linearizability using the simple and powerful technique of *observers* [1], which can be seen as monitors that report violations of the linearizability criterion. Observers synchronize with the monitored concurrent programs at designated actions. This can be done in two ways. (1) For concurrent implementations of stacks and queues, linearizability can be precisely specified by observers that synchronize on call and return actions of methods, as shown by [5, 20]; this is done without any user annotation. (2) For sets, the verification requires the user to annotate how linearization points are placed in each method; in most cases this is a small burden for the verifier. The observer then synchronizeson these linearization points. Our implementation then automatically checks, using our novel technique based on fragment abstraction, that a supplied C-like description of a concurrent data structure is a correct linearizable implementation of a stack, queue, or set.

The fact that our fragment abstraction has been able to automatically verify all supplied concurrent algorithms, also those that employ skiplists or arrays of SLLs, indicates that the fragment abstraction is a simple mechanism for capturing both the local and global information about heap cells that is necessary for verifying correctness, in particular for concurrent algorithms where an unbounded number of threads interact via a shared heap.

Here goes the outline

Related Work [14] presents a thread-modular shape analysis for multi-threaded programs in which locks protect portions of the heap, and threads have to acquire a lock before accessing the corresponding portion of the heap. Fine-grained concurrent algorithms do not follow this pattern.

Thread-modular approaches for verifying fine-grained concurrent algorithms have been presented in several works. [1] introduce observers to specify the semantics of data structures. It verifies program using the thread-modular approach, where heaps are specified by reachability constraints between cells pointed to by program variables. The work is applied to concurrent stack and queue implementaitons based on singly-linked lists. [23] present a more efficient way to handle the expensive interference steps for common programming idioms in lock-free data structures. [2] extend this approach to be able to verify linearizability for a large class of concurrent data structures based on singly-linked lists. It used a specifically designed finitary abstraction of unbounded singly-linked list segments.

[We should say something about to CAVE]

[We should have a paragraph on verification of skiplists, and on verification of the TS queue/stack]

Below is related work from the SAS paper

Much previous work has been devoted to the *manual* verification of linearizability for concurrent programs. Examples include [25, 37]. In [30], O'Hearn *et al.* define a *hindsight lemma* that provides a non-constructive evidence for linearizability. The lemma is used to prove linearizability of an optimistic variant of the lazy set algorithm. Vafeiadis [39] uses forward and backward simulation relations together with history or prophecy variables to prove linearizability. These approaches are manual, and without tool implementations. *Mechanical* proofs of linearizability, using interactive theorem provers, have been reported in [6, 7, 32, 31]. For instance, Colvin *et al.* [6] verify the lazy set algorithm in PVS, using a combination of forward and backward simulations.

Several techniques for verifying linearizability are based on establishing commutation properties between atomic actions inside each method. Such methods can be partly automated [12, 34] or part of automated program analysis [24].

There are several works on *automatic* verification of linearizability. In [40], Vafeiadis develops an automatic tool for proving linearizability that employs instrumentation to verify logically pure executions. However, this work can handle non-fixed LPs only for read-only methods, i.e, methods that do not modify the heap. This means that the method cannot handle algorithms like the *Elimination* queue [29], *HSY* stack [19], *CCAS* [16], *RDCSS* [16] and *HM* set [22] that we consider in this paper. In addition,

their shape abstraction is not powerful enough to handle algorithms like *Harris* set [15] and *Michael* set [26] that are also handled by our method. Chakraborty *et al.* [20] describe an "aspect-oriented" method for modular verification of concurrent queues that they use to prove linearizability of the Herlihy/Wing queue. Bouajjani et al. [5] extended this work to show that verifying linearizability for certain fixed abstract data types, including queues and stacks, is reducible to control-state reachability. We can incorporate this technique into our framework by a suitable construction of observers. The method can not be applied to sets. The most recent work of Zhu *et al.* [46] describe a tool that is applied for specific set, queue, and stack algorithms. For queue algorithms, their technique can handle queues with helping mechanism except for *HW* queue [21] which is handled by our paper. For set algorithms, the authors can only handle those that perform an optimistic contains (or lookup) operation by applying the *hindsight lemma* from [30]. Hindsight-based proofs provide only *non-constructive* evidence of linearizability. Furthermore, some algorithms (e.g., the unordered list algorithm considered in Sec. 8 of this paper) do not contain the code patterns required by the hindsight method. Algorithms with non-optimistic contains (or lookup) operation like *HM* [22], *Harris* [15] and *Michael* [26] sets cannot be verified by their technique. Vechev *et al.* [44] check linearizability with user-specified non-fixed LPs, using a tool for finite-state verification. Their method assumes a bounded number of threads, and they report state space explosion when having more than two threads. Dragoi *et al.* [11] describe a method for proving linearizability that is applicable to algorithms with non-fixed LPs. However, their method needs to rewrite the implementation so that all operations have linearization points within the rewritten code. Černý *et al* [42] show decidability of a class of programs with a bounded number of threads operating on concurrent data structures. Finally, the works [1, 4, 38] all require fixed linearization points.

We have not found any report in the literature of a verification method that is sufficiently powerful to automatically verify the class of concurrent set implementations based on sorted and non-sorted singly-linked lists having non-optimistic contains (or lookup) operations we consider. For instance the lock-free sets of *HM* [22], *Harris* [15], or *Michael* [26], or unordered set of [45],

2 Overview

This is derived from the previous overview section, which described TS-stack

In this section, we illustrate our technique by using it to prove correctness, in the sense of linearizability, of a concurrent data structure implementation which uses skiplists. We consider an implementation of a set data structure, which operates on a shared heap which represents a skiplist. Here, we consider the lock-free implementation in [22]. To the best of our knowledge, no existing automated verification technique has succeeded in verifying functional correctness of concurrent skiplist algorithms. We have verified both lock-based and lock-free algorithms automatically using fragment abstraction, as reported in Section 8.

2.1 The Skiplist Algorithm

In this section, we describe a concurrent lock-free skiplist algorithm [22]. Firstly, let us describe the definition of a skiplist structure. A skiplist is a list consisting of a number of sorted linked lists, each of which is located at a layer. Each skiplist node has a key value, one successor at each layer in which it participates, and is assigned a height. The lowest-level list is an ordered list of all stored elements. Higher-level lists serve as shortcuts into lower-level lists. A skiplist has head and tail nodes with the maximum heights at the head and tail positions respectively. The heads key value is assigned $-\infty$, and the tails key value is $+\infty$. Now, let us describe an important method `add` of the algorithm. The method is to add a new node into the list. Figure 1 shows the codes of the method.

```
struct Node { int data; int topLayer; Node *next[]; bool marked;}
```

```
boolean find(int x,Node* preds[],Node* succs[])
 1  int mLevel = 0;
 2  boolean marked[MAXLEVEL] = false;
 3  boolean snip;
 4  Node* pred = null,curr = null,succ = null;
 5  int k;
 6  for(int i=0; i<maxThreads; i++)
 7    Node* pred = null,curr = null,succ = null;
 8    retry:
 9    while (true)
10      pred = head;
11      for (int i = MAXLEVEL; i >= mLevel; i--)
12      curr = pred.next[i];
13      while (true)
14       succ = curr.next[i].get(marked);
15       while (marked[0])
16         s= CAS(pred.next[i],curr,succ,false
          false);
17         if (!s) continue retry;
18         curr = pred.next[i];
19         succ = curr.next[i].get(marked);
20       if (curr.key < x)
21         pred = curr;
22         curr = succ;
23       else break;
24    preds[i] = pred;
25    succs[i] = curr;
26  return (curr.key == x);
```

```
boolean add (int x):
 1  int topLevel = randomLevel; mLevel = 0;
 3  Node* preds[MAXLEVEL+1];
 4  Node* succs[MAXLEVEL+1];
 5  while (true);
 6    boolean found = find(x,preds,succs);
 7    if (found);
 8      return false;
 9    else
10      Node* new = new Node(x, topLevel);
11      for(int i= mLevel;level<=topLevel;i++)
12       Node* succ = succs[i];
13       new.next[i].set(succ, false);
14      Node* pred = preds[mLevel];
15      Node* succ = succs[mLevel];
16      new.next[mLevel].set(succ, false);
        if (!CAS(pred.next[mLevel],succ,new;
        ,false,false));
17       continue;
18      for (int i=mLevel+1;i<=topLevel;i++)
19       while (true);
20         pred = preds[i];
21         succ = succs[i];
22         if (CAS(pred.next[i],succ,new
22         ,false,false));
23           break;
24         find(x,preds,succs);
25  return true;
```

Fig. 1: Description of the find function of the Skip-list algorithm

The `add` method uses `find` method to check whether a node with key `x` is already in the list. If an unmarked node with key `k` is found in the bottom-level list, the `add` method returns `false`. If no node is found, then the next step is to try to add the new

node into the list. It is performed by linking it into the bottom-level list between the `preds[0]` and `succs[0]` nodes returned by `find`. They use `compareAndSet` to set the reference while validating that these nodes still refer one to the other and have not been removed from the list (Line 16). If `compareAndSet` fails, the method call will restart. Otherwise, the node is added into the list. The method then links the node in at higher levels. The `find` method traverses the SkipList using two pointer variables `pred` and `curr` which start from the head and at the highest level. It then proceeds in each level down the list, filling in `preds` and `succs` nodes that are repeatedly advanced until `pred` points to a node with the largest value on that level that is strictly less than the `x`. The method repeatedly remove marked nodes from the given level as they are encountered (Lines 15 to 19) using a `compareAndSet` statement. Once an unmarked `curr` is found (Line 20), it is tested to see if its key is less than the target key. If so, `pred` is advanced to `curr`. Otherwise, `curr`'s key is greater than or equal to `x`, so the current value of `pred` is the target nodes immediate predecessor. The find method breaks out of the current level search loop, saving the current values of `pred` and `curr`. It proceeds by this way until reaching the bottom level. The `add` method for adding a node with the key `k` works as follows:

2.2 Specifying the Correctness Criterion of Linearizability

In our verification, we establish that the skiplist algorithm of Figure 1 is correct in the sense that it is a linearizable implementation of a set data structure. Linearizability intuitively states that each operation on the data structure can be considered as being performed atomically at some point (called the *linearization point (LP)*) between its invocation and return [21].

We specify linearizability by extending of the technique of *observers* [1], which can be extended in different ways [5, 20, 2]. In the case of the skiplist algorithm, LPs can be associated to fixed statements in the code. The user then instruments these statements so that they also announce the corresponding operation on the data structure. For instance, The linearization point of the successful add is line 16 of the `add` method, whereas the linearization point of the unsuccessful add is line 7 of the `add` method. Having instrumented methods at LPs, we then consider an arbitrary concurrent program consisting of an arbitrary collection of threads, each of which executes some method call. We must check that the concurrent execution of such a program generates (through its instrumentation) a sequence of operations which satisfies the semantics of the set data structure. This check is performed by an *observer*, which monitors the sequence of operations that is announced by the instrumentation, and report when it violates the semantics of the set data structure.

Observers are finite automata extended with a finite set of *registers* that assume values in $\mathbb{Z}$, which are nondeterministically initialized with arbitrary values, which never change during a run of the observer. The observer accepts a trace if, for *some* initial values of the registers, the trace can be processed in such a way that an accepting state is reached. In other words, the observer is defined in such a way that it accepts precisely those traces that do *not* belong to the behavior of the data structure. Fig. 2 depicts an observer that accepts the sequences of operations that do *not* conform to the semantics of a set data structure.

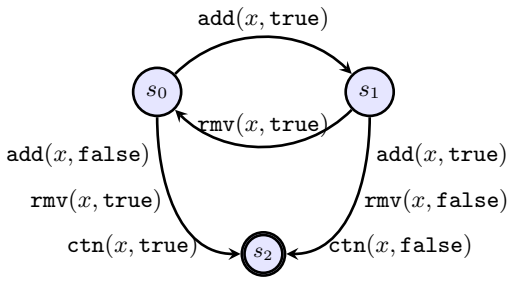

Fig. 2: Set observer.

To verify that no execution of the program may cause the observer to accept, we form as in the automata-theoretic approach [41] (adapted in [1]), the cross-product of the program and the corresponding observer, where the observer synchronizes with the program on the operations that are announced at LPs. This reduces the problem of checking linearizability to the problem of checking that, in the cross-product, the observer cannot reach an accepting state.

We note that linearizability of stacks and queues can be verified without any user-supplied instrumentation. For stacks and queues, linearizability can be precisely specified by observers that process the sequence of call and return actions of methods, instead of the user-supplied LPs [5, 20]. We use this technique for sets and stacks in Section XXX.

We may have to say that we check the trivial conditions that were checked by the monitor in SAS 16

2.3 Verification by Fragment Abstraction

In the actual verification, we must compute a symbolic representation of an invariant that is satisfied by all reachable configurations of the cross-product of the program and an observer. The verification must address the challenges of an unbounded domain of data values, an unbounded number of concurrently executing threads, and an unbounded heap. For this, we have developed a novel shape representation, called *fragment abstraction*, which can also be combined with data abstraction and thread abstraction. Let us illustrate how fragment abstraction applies to the skiplist algorithm.

Figure 3 shows an example state of the heap of the skiplist algorithm with three levels. Each heap cell is shown with the values of its fields. In addition, each cell is labeled by the pointer variables that point to it. We use $\texttt{lvar(i)}$ to denote the local variable $\texttt{lvar}$ of thread $\texttt{th}_i$. In the heap state, thread $\texttt{th}_1$ is trying to add a new node with key 9. It has done the first iteration of the for loop in it's $\texttt{find}$ function. Thread $\texttt{th}_2$ is also doing the same thing as $\texttt{th}_1$ and it has done the second iteration. Thread $\texttt{th}_2$ is trying remove a node with key 9 and it has done the third iteration. However, before $\texttt{th}_3$ removes the node with key 9, the node is removed by another thread and a new node with key 10 is also added to the list. One of the main properties of the algorithm

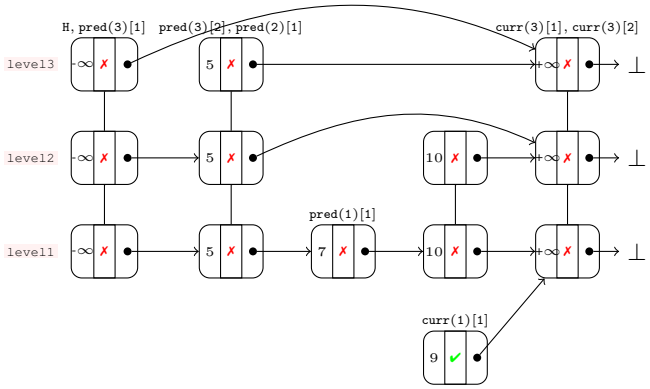

Fig. 3: A concrete shape of 3-level skiplist with three threads

is that the `pred`, in any level, never refers to an unmarked node whose key is greater than or equal to the target key. The `pred` variable arrives at the bottom-level list at a node before, and never after, the target node. If the node is physically removed before the `find` starts, then it will not be found.

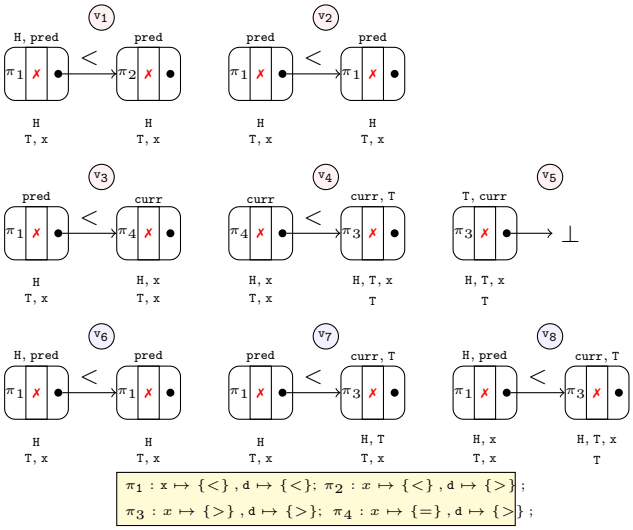

Fig. 4: Fragment abstraction of the skiplist algorithm

Our verification technique is based on a combination of a thread abstraction, a data abstraction, and a shape abstraction.

– Our thread abstraction adapts the thread-modular approach by representing only the view of single, but arbitrary, thread `th`. Such a view consists of (i) the local state of thread `th`, (ii) the part of the heap that is accessible to thread `th` via its local pointer variables or via global pointer variables, and (iii) the state of the observer.
– A natural data abstraction is applied to the non-pointer local variables of the thread, the non-pointer fields of each fragment, as well as the observer registers. For our skiplist algorithm, fields that range over small finite domains are represented with their concrete values, whereas the fields that range over the domain of data values, as well as the observer registers, are represented by constraints over their relative ordering (wrp. to the order $<$).
– The part of the heap that is accessible to a thread `th` is represented by a set of *fragments*. Each fragment is an abstraction of a pair of nodes in the heap that are connected by a pointer. The abstraction represents
  - the contents of the data fields in the pair of nodes (under the empoyed data abstraction),
  - which pointer variables (either global or local to `th`) point to the nodes, and
  - global reachability information, which expresses how each node in the pair can reach to and be reached from (by following a chain of pointers) a finite set of *globally significant* heap cells. A cell is globally significant if it is pointed to by a global pointer variable or if it triggers some significant behavior in a thread that reaches it: in the [WHICH?] skiplist algorithm, this happens for nodes whose `key` field has the same value as the register of the set observer.

  In our fragment abstraction, a set of fragments represents the set of heap structures in which each pair of pointer-connected nodes is represented by some fragment in the set. Inuitively, a set of fragments describes the set of heaps that can be formed by "piecing together" fragments in the set. This "piecing together" must be both locally consistent (appending only fragments that agree on their common node), and globally consistent (respecting the global reachability information). By construction, our fragment abstraction is finitary, since there is a bounded set of tags.

Let a *local symbolic configuration* be an abstraction of the program counter and local data variables of an arbitrary thread `th`. Our symbolic representation of a set of program configurations consists of a mapping from a set of local symbolic configurations, which maps each local symbolic configuration in its domain to a set of fragments. A global configuration satisfies a symbolic representation Ψ if the view of each thread `th` satisfies some local symbolic configuration in the domain of Ψ, which is mapped to a set of fragments, which represents the heap that is accessible to `th`.

Quy: After this, you must provide a nice example

Figure 5 shows a set of fragments that is satisfied wrp. to $\texttt{th}_2$ by the configuration in Figure **??**. There are 7 fragments, named $\texttt{v}_1, \ldots, \texttt{v}_7$. Two of these ($\texttt{v}_3$ and $\texttt{v}_7$) consist of a tag that points to $\bot$, and the other consist of a pair of pointer-connected tags. Consider the tag which occurs in fragment $\texttt{v}_7$; it is the bottom-rightmost tag. This tag is an abstraction of the bottom-rightmost heap cell in Figure **??**, using the same layout for fields as Figure **??**. The different non-pointer fields are represented as follows.

– The `data` field of the tag (to the left) abstracts the data value 2 to the set of observer registers with that value: in this case $\texttt{x}_2$.

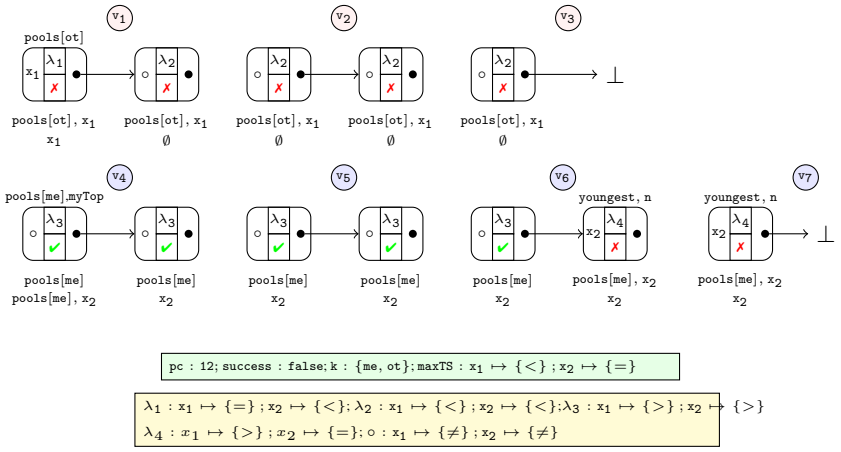

Fig. 5: Fragment abstraction

- The `ts` field (at the top) abstracts the timer value 15 to the possible relations with `ts`-fields of heap cells with the same data value as each observer registers. Recall that observer registers x_1 and x_2 have values 4 and 2, respectively. There are three heap cells with `data` field value 4, all with a `ts` value less than 15. There is one heap cell with `data` field value 2, having `ts` value 15. Consequently, the abstraction of the `ts` field maps x_1 to $\{>\}$ and x_2 to $\{=\}$: this is shown as the mapping λ_4 in Figure 5.
- The `mark` field assumes values from a small finite domain and is represented precisely as in concrete heap cells

Above the top, the tag contains the thread-local and global pointer variables that point to the cell, in this case `youngest` and `n`. At the bottom of the tag, the first row contains the global variables pointing to cells from which the cell can be reached, in this case `pools`[3], as well as observer registers whose value is equal to the `data` field of a cell from which the cell can be reached, in this case x_2 (since the cell itself has the same data value as x_2). The second row contains dual information: now for cells that can be reached from the cell itself (this is again x_2).

Fix the issue with `pools`[3]

Each cell in the heap state of Figure **??** now satisfies some tag in Figure **??**. Moreover, each pair of pointer-connected cells (where the pointed-to "cell" can also be $\perp$) satisfies some fragment in Figure **??** in the obvious way. Conversely, the set of fragments in Figure **??** represents the set of heaps in which each pair of pointer-connected cells satisfies one of its fragments. For instance, the list pointed to by `pools`[3] is represented by the sequence of fragments $v_4v_5v_6v_7$.

In order to obtain a complete representation of reachable program configurations, we must also represent the local states of a thread. This is done in a standard manner, by applying the same data abstraction as for heap cells. For instance, the local state of thread th_2 corresponding to Figure **??** is represented by a *local symbolic configuration*

that contains the values of the program counter and variable success, abstracts the value of k into the set $\{\texttt{me}, \texttt{ot}\}$ and applies the timestamp abstraction to maxTS.

In the verification, we must compute a symbolic representation that is satisfied by all reachable program configurations (recall that program configurations include the state of the observer). This invariant is obtained by an abstract-interpretation-based fixpoint procedure, which starts from a representation of the set of initial configurations, and thereafter repeatedly performs postcondition computations that extend the symbolic representation by the effect of any execution step of the program, until convergence. This procedure is presented in Section 4.1.

3 Concurrent Data Structure Implementations

In this section, we introduce our representation of concurrent data structure implementations, we define the correctness criterion of linearizability, we introduce observers and how to use them for specifying linearizability.

3.1 Concurrent Data Structure Implementations

We begin by introducing (sequential) data structures. A *data structure* DS is a pair $\langle \mathbb{D}, \mathbb{M} \rangle$, where $\mathbb{D}$ is a (possibly infinite) *data domain* and $\mathbb{M}$ is an alphabet of *method names*. An *operation* op is of the form $\texttt{m}(d^{in}, d^{out})$ where $\texttt{m} \in \mathbb{M}$ is a method name and d^{in} are the *input* resp. *output* values, each of which is either in $\mathbb{D}$ or in some fixed finite domain (such as the booleans). For some method names, the input or output value is absent from the operation. A *trace* of DS is a sequence of operations. The (sequential) semantics of a data structure DS is given by a set $[\![\texttt{DS}]\!]$ of allowed traces, called the *legal* traces of DS.

Quy: replace the below by an example of a set trace.

For example, for the Stack data structure, the method names are push and pop. The operation push(3), where 3 is an input value, pushes the value 3, whereas pop(4), where 4 is an output value, pops the value 4.

A *concurrent data structure implementation* operates on a shared state consisting of shared global variables and a shared heap. It assigns, to each method name, a method which performs operations on the shared state. Each data structure implementation also comes with an initialization method, named init, which initializes its shared state.

We assume that all global variables are pointer variables. Heap cells have a fixed set $\mathcal{F}$ of fields, namely data fields that assume values in $\mathbb{D}$ or $\mathbb{F}$, and possibly lock fields. We use the term $\mathbb{D}$-field for a data field that assumes values in $\mathbb{D}$, and the terms $\mathbb{F}$-field and lock field with analogous meaning. Furthermore, each cell has one or several named pointer fields. For instance, in data structure implementations based on singly linked lists each heap cell has a pointer field named next; in implementations based on skiplists there is an array of pointer fields named next[k] where k ranges from 1 to the maximum level of the skiplist.

Each method declares local variables and a method body. The set of local variables include the input parameter of the method and the program counter pc. The global variables can be accessed by all threads, whereas local variables can be accessed only

by the thread which is invoking the corresponding method. Variables are either pointer variables (to heap cells) or data variables, assuming values from $\mathbb{D}$ or from some finite set $\mathbb{F}$ that includes the boolean values. The body is built in the standard way from atomic commands, using standard control flow constructs (sequential composition, selection, and loop constructs). Method execution is terminated by executing a `return` command, which may return a value. Atomic commands include assignments between data variables, pointer variables, or fields of cells pointed to by a pointer variable. The command `new Node()` allocates a new structure of type `Node` on the heap, and returns a reference to it. The compare-and-swap command `CAS(&a,b,c)` atomically compares the values of `a` and `b`. If equal, it assigns the value of `c` to `a` and returns `true`, otherwise, it leaves `a` unchanged and returns `false`. We assume a memory management mechanism, which automatically collects garbage, and ensures that a new cell is fresh, i.e., has not been used before; this avoids the so-called ABA problem (e.g., [28]).

We define a *program* (over a concurrent data structure) to consist of an arbitrary number of concurrently executing threads, each of which executes a method that performs an operation on the data structure. We assume concurrent execution according to sequentially consistent memory model. We assume that the data structure has been initialized by the `init` method prior to the start of program execution.

3.2 Linearizability

In a concurrent data structure implementation, we represent the calling of a method by a *call action* $\mathtt{call}_{\mathtt{o}}\ \mathtt{m}\left(d^{in}\right)$, and the return of a method by a *return action* $\mathtt{ret}_{\mathtt{o}}\ \mathtt{m}\left(d^{out}\right)$, where $\mathtt{o} \in \mathbb{N}$ is an *action identifier*, which links the call and return of each method invocation. A *history* h is a sequence of actions such that (i) different occurrences of return actions have different action identifiers, and (ii) for each return action a_2 in h there is a unique *matching* call action a_1 with the same action identifier and method name, which occurs before a_2 in h. A call action which does not match any return action in h is said to be *pending*. A history without pending call actions is said to be *complete*. A *completed extension* of h is a complete history h' obtained from h by appending (at the end) zero or more return actions that are matched by pending call actions in h, and thereafter removing the call actions that are still pending. For action identifiers $\mathtt{o}_1, \mathtt{o}_2$, we write $\mathtt{o}_1 \preceq_{\mathtt{h}} \mathtt{o}_2$ to denote that the return action with identifier $\mathtt{o}_1$ occurs before the call action with identifier $\mathtt{o}_2$ in h. A complete history is *sequential* if it is of the form $a_1 a_1' a_2 a_2' \cdots a_n a_n'$ where a_i' is the matching action of a_i for all $i : 1 \leq i \leq n$, i.e., each call action is immediately followed by the matching return action. We identify a sequential history of the above form with the corresponding trace $op_1\, op_2 \cdots op_n$ where $op_i = \mathtt{m}(d_i^{in}, d_i^{out})$, $a_i = \mathtt{call}_{\mathtt{o}_i}\ \mathtt{m}\left(d_i^{in}\right)$, and $a_i = \mathtt{ret}_{\mathtt{o}_i}\ \mathtt{m}\left(d_i^{out}\right)$, i.e., we merge each call action together with the matching return action into one operation. A complete history h' is a *linearization* of h if (i) h' is a permutation of h, (ii) h' is sequential, and (iii) $\mathtt{o}_1 \preceq_{\mathtt{h}'} \mathtt{o}_2$ if $\mathtt{o}_1 \preceq_{\mathtt{h}} \mathtt{o}_2$ for each pair of action identifiers $\mathtt{o}_1$ and $\mathtt{o}_2$. A sequential history h' is *valid* wrt. `DS` if the corresponding trace is in $[\![\mathtt{DS}]\!]$. We say that h is *linearizable* wrt. `DS` if there is a completed extension of h, which has a linearization that is valid wrt. `DS`. We say that a program $\mathcal{P}$ is linearizable wrt. `DS` if, in each possible execution, the sequence of call and return actions is *linearizable* wrt. `DS`.

3.3 Specification by Observers

To verify correctness of a data structure implementation, we must verify that any history, i.e., sequence of call and return actions, of any program execution satisfies the linearizability criterion. It can be shown [IN WHICH CITATION?] that this is equivalent to requiring that each operation on the data structure can be considered as being performed atomically at some point (called the *linearization point (LP)*) between its invocation and return. To check linearizability of set implementations, the user first instruments each method so that it generates a corresponding operation precisely when the method executes its LP, using the technique of linearization policies [2]. In the case of the skiplist algorithm, this instrumentation simply consists of associating LPs to fixed statements in the code. [Quy: Please illustrate here how this is done for the skiplist] Thereafter, it should be checked that the concurrent execution of any program generates (through its instrumentation) a sequence of operations which satisfies the semantics of the data structure. This check is performed by an *observer*, which monitors the sequence of operations that is announced by the instrumentation, and report when it violates the semantics of the set data structure.

Formally, an observer $\mathcal{O}$ is a tuple $\langle S^{\mathcal{O}}, s^{\mathcal{O}}_{\mathtt{init}}, \mathtt{X}^{\mathcal{O}}, \Delta^{\mathcal{O}}, s^{\mathcal{O}}_{\mathtt{acc}} \rangle$ where $S^{\mathcal{O}}$ is a finite set of *observer states* including the *initial state* $s^{\mathcal{O}}_{\mathtt{init}}$ and the *accepting state* $s^{\mathcal{O}}_{\mathtt{acc}}$, a finite set $\mathtt{X}^{\mathcal{O}}$ of *registers*, and $\Delta^{\mathcal{O}}$ is a finite set of *transitions*. Transitions are of the form $\langle s_1, \mathtt{m}(d^{in}, d^{out}), s_2 \rangle$, $\mathtt{m} \in \mathbb{M}$ is a method name and x^{in} and x^{out} are either registers or constants, i.e., transitions are labeled by operations whose input or output data may be parameterized on registers. The observer processes a sequence of operations one operation at a time. If there is a transition, whose label (after replacing registers by their values) matches the operation, such a transition is performed. If there is no such transition, the observer remains in its current state. The observer accepts a sequence if it can be processed in such a way that an accepting state is reached. The observer is defined in such a way that it accepts precisely those sequences that are *not* in $[\![\mathtt{DS}]\!]$. Fig. 2 depicts an observer for the set data structure.

For stacks and queues, it was recently established [5, 20] that the set of linearizable histories (i.e., sequences of call and return actions) can be exactly specified by an observer. Thus, we check linearizability for queue and stack implementations without any user-supplied instrumentation, by using an observer which accepts precisely the set of linearizable histories. We illustrate this in Section XXX.

It remains to take care of the criterion that the call of pop already has a return value

3.4 Formal Semantics

For preciseness, we outline how we formalize the semantics of programs and observers. We assume a program $\mathcal{P}$ and a specification given by an observer $\mathcal{O}$. We assume that each thread `th` executes one method denoted `Method(th)`.

For a function $f : A \mapsto B$ from a set A to a set B, we use $f[a_1 \leftarrow b_1, \ldots, a_n \leftarrow b_n]$ to denote the function f' such that $f'(a_i) = b_i$ and $f'(a) = f(a)$ if $a \notin \{a_1, \ldots, a_n\}$.

Heaps. A *heap (state)* is a tuple $\mathcal{H} = \langle \mathbb{C}, \mathtt{ptr}_1, \ldots, \mathtt{ptr}_k, \mathtt{Val}^{\mathtt{gl}}, \mathtt{Val}^{\mathbb{C}} \rangle$, where (i) $\mathbb{C}$ is a finite set of cells, including the two special cells `null` and $\perp$ (dangling); we define $\mathbb{C}^- = \mathbb{C} \setminus \{\mathtt{null}, \perp\}$, (ii) for each pointer field $\mathtt{ptr}_i$ there is a total function $\mathtt{ptr}_i : \mathbb{C}^- \to \mathbb{C}$ that defines where that pointer field points, (iii) $\mathtt{Val}^{\mathtt{gl}} : \mathtt{X}^{\mathtt{gl}} \to \mathbb{C}$ maps the global (pointer) variables $\mathtt{X}^{\mathtt{gl}}$ to their values, and (iv) $\mathtt{Val}^{\mathbb{C}} : \mathbb{C} \times \mathcal{F} \to \mathbb{F} \cup \mathbb{D}$ maps data and lock fields of each cell to their values. We let $\mathcal{H}_{\mathtt{init}}$ denote the initial heap produced by the `init` method.

Threads. A *local state* `loc` of a thread `th` wrt. a heap $\mathcal{H}$ defines the values of its local variables, including the program counter `pc` and the input parameter for the method executed by `th`. In addition, there is the special initial state `idle`, and terminated state `term`. We formalize the behavior of a thread `th` by a labeled transition relation $\rightarrow_{\mathtt{th}}$ on pairs $\langle \mathtt{loc}, \mathcal{H} \rangle$ consisting of a local state `loc` and a heap $\mathcal{H}$, with transitions of the form: $\langle \mathtt{loc}, \mathcal{H} \rangle \xrightarrow{\lambda}_{\mathtt{th}} \langle \mathtt{loc}', \mathcal{H}' \rangle$, which denote execution of method statements, labeled by λ. The label λ is empty, except when the executed statement is annotated as a linearization point, in which case it is labeled by the corresponding operation (of form $\mathtt{m}(d^{in}, d^{out})$).

Programs. A *configuration* of a program $\mathcal{P}$ is a tuple $\langle \mathtt{T}, \mathtt{LOC}, \mathcal{H} \rangle$ where `T` is a set of threads, $\mathcal{H}$ is a heap, and `LOC` maps each thread $\mathtt{th} \in \mathtt{T}$ to its local state $\mathtt{LOC}(\mathtt{th})$ wrt. $\mathcal{H}$. The initial configuration $c^{\mathcal{P}}_{\mathtt{init}}$ is the pair $\langle \mathtt{LOC}_{\mathtt{init}}, \mathcal{H}_{\mathtt{init}} \rangle$, where $\mathtt{LOC}_{\mathtt{init}}(\mathtt{th}) = \mathtt{idle}$ for each $\mathtt{th} \in \mathtt{T}$, A program $\mathcal{P}$ induces a transition relation $\rightarrow_{\mathcal{P}}$ where each step corresponds to one move of a single thread. I.e., there is a transition of form $\langle \mathtt{T}, \mathtt{LOC}, \mathcal{H} \rangle \xrightarrow{\lambda}_{\mathcal{P}} \langle \mathtt{T}, \mathtt{LOC}[\mathtt{th} \leftarrow \mathtt{loc}'], \mathcal{H}' \rangle$ whenever the transition relation $\rightarrow_{\mathtt{th}}$ has a transition $\langle \mathtt{loc}, \mathcal{H} \rangle \xrightarrow{\lambda}_{\mathtt{th}} \langle \mathtt{loc}', \mathcal{H}' \rangle$, where the label λ is either a call or return action or the empty label.

Cross-Product of Program and Observer We use $\mathcal{S} = \mathcal{P} \otimes \mathcal{O}$ to denote the cross-product obtained by running $\mathcal{P}$ and $\mathcal{O}$ together. The initial configuration of $\mathcal{S}$ is $\langle c^{\mathcal{P}}_{\mathtt{init}}, s^{\mathcal{O}}_{\mathtt{init}} \rangle$. Transitions of $\mathcal{S}$ are of the form $\langle c^{\mathcal{P}}, s \rangle, \rightarrow_{\mathcal{S}}, \left\langle c^{\mathcal{P}\prime}, s' \right\rangle$, obtained from a transition $c^{\mathcal{P}} \xrightarrow{\lambda}_{\mathcal{P}} c^{\mathcal{P}\prime}$ of the program with some (possibly empty) label λ, where the observer makes a transition if it can perform a matching transition $\langle s, \mathtt{ret}\, \mathtt{m}(l), s' \rangle$. The verification problem is now to check that $\mathcal{S}$ cannot reach a configuration $\langle c^{\mathcal{P}}, s^{\mathcal{O}}_{\mathtt{acc}} \rangle$ with an accepting observer state.

Here, we need to state a the theorem that we have reduced linearizability checking to reachability

4 Fragment Abstraction: Singly Linked Lists

In the previous section, we reduced the problem of verifying linearizability to the problem of verifying that, in any execution of the cross-product of a program and an observer, the observer cannot reach an accepting state. In this section, we describe our

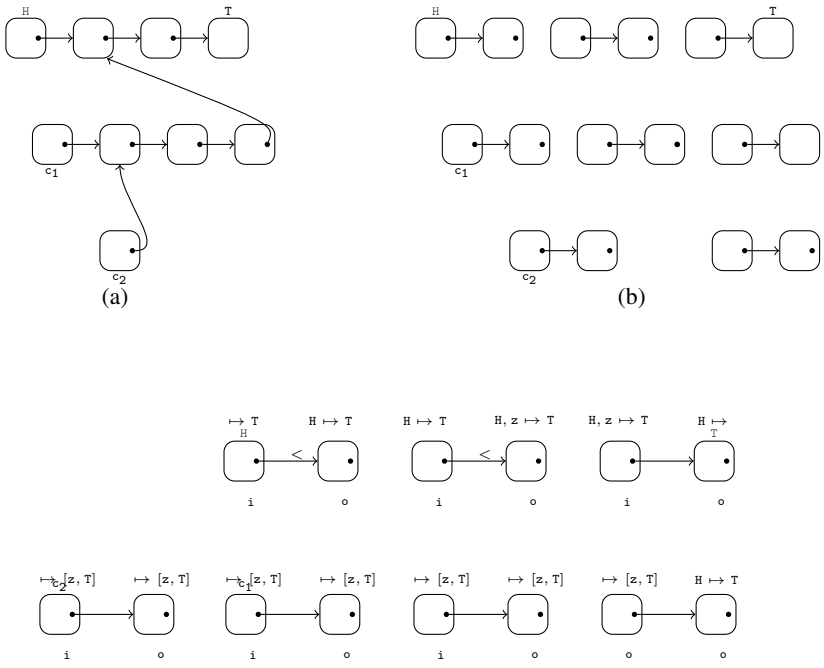

Fig. 6: Example of Fragment Abstraction

technique for performing this verification. The main novel contribution is our fragment abstraction for representing the possible heap configurations of a concurrent program. In the following subsection, we describe our symbolic representation and verification technique, using fragment abstraction, for programs that operate on singly-linked lists (SLLs). This representation is also the basis for our representation for programs operating on skiplists, described in Subsection **??** and programs operating on arrays of SLLs, in Subsection **??**.

[Where to put this paragraph?] We consider a program with global variables $\mathtt{x}^{\mathtt{gl}}$ and thread-local variables $\mathtt{x}^{\mathtt{loc}}$. We assume that all global variables are pointer variables.

4.1 Fragment Abstraction for Singly-Linked List-Based Programs

[Say that for now, we ignore timers?] [Introduce notation for the set of local $\mathbb{D}$-variables?. Maybe define which threads we talk about? For now, we skip timestamps.]

In this subsection, we present our fragment abstraction for programs that operate on singly-linked lists. This means that each cell has exactly one pointer field, named `next`. In order to simplify the presentation in this section, we assume that each heap cell has at most one $\mathbb{D}$-field, named `data`. Our verification technique is based on a combination of a thread abstraction, a shape abstraction, and a data abstraction.

Assume a configuration $c^{\mathcal{S}}$ of $\mathcal{S} = \mathcal{P} \otimes \mathcal{O}$ and thread `th`. We say that a heap cell $\mathbb{c}$ is *accessible* to a thread `th` in $c^{\mathcal{S}}$, if $\mathbb{c}$ is reachable (directly or via sequence of `next`-pointers) from a global pointer variable or local pointer variable of `th`. For a pointer variable p, let a p-cell be the cell to which p points. For an observer register $\mathtt{x}_i$, define a $\mathtt{x}_i$*-cell* to be a heap cell whose `data` field has the same value as $\mathtt{x}_i$. [Maybe skip the following] A heap cell is *globally significant* if it is either a p-cell for a global pointer variable p, or a $\mathtt{x}_i$-cell for some observer register $\mathtt{x}_i$.

- Our *thread abstraction* adapts the thread-modular approach by representing only the view of a single, but arbitrary, thread `th`. Such a view consists of (i) the local state of thread `th`, (ii) the state of the observer, (iii) the part of the heap (including global variables) that is accessible to thread `th`, and (iv) a predicate, which for each heap cell accessible to `th` says whether it is *private* to `th`, i.e., not accessible to any other thread.
- We use a novel *shape abstraction*, which represents the part of the heap that is accessible to `th` by a set of *fragments*. A fragment consists of a pair of heap cells that are connected by a pointer. For each of these cells, the fragment represents (i) the values of its data fields, (ii) the pointer variables (either local to `th` or global) that point to it, and (iii) global pointer variables p and observer registers $\mathtt{x}_i$ such that the cell can either reach to or be reached from (by a chain of `next`-pointers) a p-cell or a $\mathtt{x}_i$-cell, respectively. A set of fragments represents the set of heaps in which each pair of pointer-connected nodes is represented by some fragment in the set. Thus, a set of fragments describes the set of heaps that can be formed by "piecing together" fragments, in a way that is locally consistent (connecting only fragments that agree on their common node) and globally consistent (respecting the reachability information wrp. to global pointer variables and observer registers).

– We apply a natural *data abstraction* to the local state of a thread, the state of the observer, and to each fragment. This abstraction represents small finite domains exactly, and abstracts a set of data values in $\mathbb{D}$ by their relative ordering.

Let us now describe our symbolic representation in concrete detail. Our data abstraction is conveniently represented by defining an abstract domain for each concrete domain of data values.

– For small concrete domains (including that of the program counter, and of the observer location), the abstract domain is the same as the concrete one.
– For locks, the abstract domain is $\{me, other, free\}$, mening that the lock is held by $\mathtt{th}$, held by some other thread, or is free, respectively.
– For the concrete domain $\mathbb{D}$ of data values, the abstract domain is the set of mappings from observer registers and local variables ranging over $\mathbb{D}$ to subsets of $\{<, =, >\}$. An element in this abstract domain represents a concrete data value $\mathtt{d}$ if it maps each local variable and observer register with a value $\mathtt{d}' \in \mathbb{D}$ to a set which includes a relation $\sim$ such that $\mathtt{d} \sim \mathtt{d}'$.

In our shape abstration, we represent each fragment by a pair of *tags*. A *tag* is a tuple $\mathtt{tag} = \langle \mathtt{dabs}, \mathtt{pvars}, \mathtt{reachfrom}, \mathtt{reachto}, \mathtt{private} \rangle$, where

– $\mathtt{dabs}$ is a mapping from non-pointer fields to their corresponding abstract domains,
– $\mathtt{pvars}$ is a set of (global or local) pointer variables,
– $\mathtt{reachfrom}$ and $\mathtt{reachto}$ are sets of global pointer variables and observer registers,
– $\mathtt{private}$ is a boolean value.

For a heap cell $\mathbb{c}$ that is accessible to thread $\mathtt{th}$ in a configuration $c^{\mathcal{S}}$, and a tag $\mathtt{tag} = \langle \mathtt{dabs}, \mathtt{pvars}, \mathtt{reachfrom}, \mathtt{reachto}, \mathtt{private} \rangle$, we write $\mathbb{c} \lhd^{c^{\mathcal{S}}}_{\mathtt{th}} \mathtt{tag}$ to denote that

– $\mathtt{dabs}$ is an abstraction of the concrete values of the non-pointer fields of $\mathbb{c}$,
– $\mathtt{pvars}$ is the set of pointer variables (global or local to $\mathtt{th}$) that point to $\mathbb{c}$,
– $\mathtt{reachfrom}$ is the set of (i) global pointer variables from which $\mathbb{c}$ is reachable via a (possibly empty) sequence of $\mathtt{next}$ pointers, and (ii) observer registers $\mathtt{x}_i$ such that $\mathbb{c}$ is reachable from some $\mathtt{x}_i$-cell.
– $\mathtt{reachto}$ is the set of (i) global pointer variables pointing to a cell that is reachable from $\mathbb{c}$, and (ii) observer registers $\mathtt{x}_i$ such that some $\mathtt{x}_i$-cell is reachable from $\mathbb{c}$.
– $\mathtt{private}$ is $\mathtt{true}$ only if $\mathbb{c}$ is private to $\mathtt{th}$ in $c^{\mathcal{S}}$ [Should we add: "is not accessible to any other thread than $\mathtt{th}$"]

Definition 1 (SLL-fragment). *An* SLL-fragment $\mathtt{v}$ *(or just fragment) is a triple of form* $\langle \mathtt{v.i}, \mathtt{v.o}, \mathtt{v}.\phi \rangle$*, of form* $\langle \mathtt{v.i}, \mathtt{null} \rangle$*, or of form* $\langle \mathtt{v.i}, \bot \rangle$*, where* $\mathtt{v.i}$ *and* $\mathtt{v.o}$ *are tags and* $\mathtt{v}.\phi$ *is a subset of* $\{<, =, >\}$.

For a cell $\mathbb{c}$ which is accessible to thread $\mathtt{th}$, and a fragment $\mathtt{v}$ of form $\langle \mathtt{v.i}, \mathtt{v.o}, \mathtt{v}.\phi \rangle$, let $\mathbb{c} \lhd^{c^{\mathcal{S}}}_{\mathtt{th}} \mathtt{v}$ denote that the $\mathtt{next}$ field of $\mathbb{c}$ points to a cell $\mathbb{c}'$ such that (i) $\mathbb{c} \lhd^{c^{\mathcal{S}}}_{\mathtt{th}} \mathtt{v.i}$, (ii) $\mathbb{c}' \lhd^{c^{\mathcal{S}}}_{\mathtt{th}} \mathtt{v.o}$, and (iii) $\mathbb{c}.\mathtt{data} \sim \mathbb{c}'.\mathtt{data}$ for some $\sim \in \mathtt{v}.\phi$. For a fragment of form $\mathtt{v} = \langle \mathtt{v.i}, \mathtt{null} \rangle$, let $\mathbb{c} \lhd \mathtt{v}$ denote that $\mathbb{c} \lhd \mathtt{v.i}$ and $\mathtt{next}(\mathbb{c}) = \mathtt{null}$. Define $\mathbb{c} \lhd \mathtt{v}$ for $\mathtt{v}$ of form $\langle \mathtt{v.i}, \bot \rangle$ analogously.

Let V be a set of fragments. A global configuration $c^{\mathcal{S}}$ satisfies a set V of fragments wrp. to $\mathtt{th}$, denoted $c^{\mathcal{S}} \models^{heap}_{\mathtt{th}} V$, if for any cell $\mathbb{c}$ that is accessible to $\mathtt{th}$, there is a fragment $\mathtt{v} \in V$ such that $\mathbb{c} \lhd^{c^{\mathcal{S}}}_{\mathtt{th}} \mathtt{v}$.

We are now ready to define our abstract symbolic representation, which is defined as a partial mapping from combinations of abstract observer and local thread states to sets of fragments.

Define a *local symbolic configuration* as a mapping from local variables (including the program counter) to their corresponding abstract domains. We use $c^{\mathcal{S}} \models^{loc}_{\mathtt{th}} \sigma$ to denote that in the global configuration $c^{\mathcal{S}}$, the local configuration of thread $\mathtt{th}$ satisfies the local symbolic configuration σ, defined in the natural way.

Definition 2. *A* symbolic representation Ψ *is a partial mapping from pairs of local symbolic configurations and observer locations to sets of fragments. A system configuration* $c^{\mathcal{S}}$ *satisfies a symbolic representation* Ψ *if for each thread* $\mathtt{th}$*, the domain of* Ψ *contains a pair* $\langle \sigma, s \rangle$ *such that (i)* $c^{\mathcal{S}} \models^{loc}_{\mathtt{th}} \sigma$*, (ii) the observer is in location* s*, and (iii)* $c^{\mathcal{S}} \models^{heap}_{\mathtt{th}} \Psi(\langle \sigma, s \rangle)$.

Skip the following example. Do we need one?

Example: Let us show an example of how a singly linked list (SLL) is split into a set of fragments. Fig. 6(a) shows an example of a concrete shape of a singly linked list. Each cell contains the values of `val`, `mark`, and `lock` from top to bottom, where ✔ denotes `true`, and ✗ denotes `false` (or `free` for `lock`) and the value of `val` is denoted by a pair of the observer register `z` and a subset of $\{<, =, >\}$. There are two threads 1 and 2 with two local variables $\mathtt{c}_1$ and $\mathtt{c}_2$. There are two global variables `H` and `T` pointing to the head and tail of the list. The observer register `z` has value 8. Fig. 6(b) shows the result of splitting the list in Fig. 6(a) into fragments where each heap fragment is a small list of two nodes. Fig. 6(c) shows the abstraction of fragments in Fig. 6(b) where for each cell $\mathbb{c}$, the value of val is abstracted to a subset in $\{< \mathtt{z}, = \mathtt{z}, > \mathtt{z}\}$, the reachability relation between $\mathbb{c}$ and global variables and observer registers is abstracted to the predicate $\mathtt{X} \mapsto \mathtt{Y}$ where `X`, `Y` are sets of global variables and observer registers, `i` or `o` states that $\mathbb{c}$ is an input or output cell.

Computing Postconditions In the verification, we must compute a symbolic representation that is satisfied by all reachable program configurations. This invariant is obtained by an abstract-interpretation-based fixpoint procedure, which starts from a representation of the set of initial configurations, and thereafter repeatedly performs postcondition computations that extend the symbolic representation by the effect of any execution step of the program, until convergence. In this subsection, we describe the symbolic postcondition computation, which is the key step in this procedure.

The symbolic postcondition computation must ensure that the symbolic representation of the reachable configurations of a program is closed under execution of a statement by some thread. More precisely, assume that a global configuration c satisfies a symbolic representation Ψ. Let $\mathtt{th}$ be an arbitrary thread. Assume that there is a local symbolic configuration $\sigma \in Dom(\Psi)$ such that $c \models_{\mathtt{th}} \langle \sigma, \Psi(\sigma) \rangle$. We must ensure

that this property still hols after any execution of a statement by some thread. In the thread-modular approach, we must consider two cases:

- *Local Steps:* The thread th itself executes some statement, which may change its local state and the state of the heap. In this case, we compute a local symbolic configuration σ' and set V' such that the resulting configuration c' satisfies $c' \models_{\mathtt{th}} \langle \sigma', V' \rangle$, and (if necessary) extend Ψ so that $\sigma' \in Dom(\Psi)$ and $V' \in \Psi(\sigma')$.
- *Interference Steps:* Another thread $\mathtt{th}_2$, which satisfies a local symbolic configuration σ_2 in $Dom(\Psi)$ with $c \models_{\mathtt{th}_2} \langle \sigma_2, \Psi(\sigma_2) \rangle$ performs a computation step, which affects the state of the heap in such a way that makes it necessary to extend $\Psi(\sigma)$. We must then compute a set V' of fragments such that the resulting configuration c' satisfies $c' \models^{heap}_{\mathtt{th}} V'$ and make sure that $V' \in \Psi(\sigma)$. To do this, we first combine the the local symbolic configurations σ and σ_2 and the sets of fragments $\Psi(\sigma)$ and $\Psi(\sigma_2)$, using an operation, called *intersection*, into a joint local symbolic configuration of th and $\mathtt{th}_2$ and a set $V_{1,2}$ of fragments that represents the cells accessible to either th or $\mathtt{th}_2$. We thereafter symbolically compute the postcondition of the statement execution of $\mathtt{th}_2$, as in the local case, finally project back the set of fragments onto th in the natural way, to obtain V'.

In the following, we first describe the symbolic postcondition computation for local steps, and thereafter the intersection operation.

Symbolic Postcondition Computation for Local Steps Let th be an arbitrary thread, and assume that $\sigma \in Dom(\Psi)$ with For each statement that th can execute in a configuration c with $c \models_{\mathtt{th}} \langle \sigma, \Psi(\sigma) \rangle$, we must compute a local symbolic configuration σ' and a set V' of fragments such that such that the resulting configuration c' satisfies $c' \models_{\mathtt{th}} \langle \sigma', V' \rangle$. This computation has do be done differently for each statement. For statements that do not affect the heap or pointer variables, this computation is standard, and affects only the local symbolic configuration and data abstraction part of fragments. We therefore here describe how to compute the effect of statements that update pointer variables or the heap, since these are the most interesting cases.

The main difficulty in the postcondition computation is to update the reachability information provided in the fields reachfrom and reachto in each tag of a fragment. For instance, considerthat a statement g := p, which assigns the value of a local pointer variable p to a global pointer variable g. In the postcondition computation, we must for each fragment determine how to update the field reachfrom in its tags, and in particular whether g should be in this set after the statement (the same problem occurs for the set reachto), If v would have been a global variable, this information could be obtained by checking whether v is in the set before the operation. However, since the reachfrom field does not include local variables, we start the postcondition computation by computing a number of transitive-closure-like relations between fragments, which will allow to determine whether g should be in the reachfrom field after the statement with rather good accuracy. Note that if our procedure can not determine whether g should be in a reachfrom field, then it generates fragments for both possibilities.

First, we say that two tags $\mathtt{tag} = \langle \mathtt{dabs}, \mathtt{pvars}, \mathtt{reachfrom}, \mathtt{reachto}, \mathtt{private} \rangle$ and $\mathtt{tag}' = \langle \mathtt{dabs}', \mathtt{pvars}', \mathtt{reachfrom}', \mathtt{reachto}', \mathtt{private}' \rangle$ are *consistent* if if there is some concrete valuation of non-pointer fields represented by both dabs and

$\texttt{dabs}'$, and if $\texttt{pvars} = \texttt{pvars}'$, $\texttt{reachfrom} = \texttt{reachfrom}'$, $\texttt{reachto} = \texttt{reachto}'$, and $\texttt{private} = \texttt{private}'$. Intuitively, $\texttt{tag}$ and $\texttt{tag}'$ are consistent if there can exist a cell $\mathbb{c}$ accessible to $\texttt{th}$ with $\mathbb{c} \lhd^{\mathcal{S}}_{\texttt{th}} \texttt{tag}$ and $\mathbb{c} \lhd^{\mathcal{S}}_{\texttt{th}} \texttt{tag}'$.

To Quy: Can you really require that the data abstrations must be the same?

Let $\texttt{v}_1$ and $\texttt{v}_2$ be two fragments in a set V of fragments.

- Let $\texttt{v}_1 \hookrightarrow_V \texttt{v}_2$ denote that $\texttt{v}_1.\texttt{o}$ and $\texttt{v}_2.\texttt{i}$ are consistent.
- Let $\texttt{v}_1 \leftrightarrow_V \texttt{v}_2$ denote that $\texttt{v}_1.\texttt{o} = \texttt{v}_2.\texttt{o}$ are consistent, and that either $\texttt{v}_1.\texttt{i}.\texttt{pvars} \cap \texttt{v}_2.\texttt{i}.\texttt{pvars} = \emptyset$ or that the global variables in $\texttt{v}_1.\texttt{i}.\texttt{reachfrom}$ are disjoint from those in $\texttt{v}_2.\texttt{i}.\texttt{reachfrom}$.

  Question: Is this correct?

Intuitively, $\texttt{v}_1 \hookrightarrow_V \texttt{v}_2$ denotes that it is possible that $\texttt{next}(\mathbb{c}_1) = \mathbb{c}_2$ for some cells with $\mathbb{c}_1 \lhd \texttt{v}_1$ and $\mathbb{c}_2 \lhd \texttt{v}_2$. Intuitively, $\texttt{v}_1 \leftrightarrow_V \texttt{v}_2$ denotes that it is possible that $\texttt{next}(\mathbb{c}_1) = \texttt{next}(\mathbb{c}_2)$. for different cells $\mathbb{c}_1$ and $\mathbb{c}_2$ with $\mathbb{c}_1 \lhd \texttt{v}_1$ and $\mathbb{c}_2 \lhd \texttt{v}_2$. Note that the above definitions also work for the cases that the output tag is $\texttt{null}$ or $\bot$.

We use the above relations to define several derived relations:

- Let $\overset{+}{\hookrightarrow}_V$ denote the transitive closure of $\hookrightarrow_V$, and $\overset{*}{\hookrightarrow}_V$ the reflexive transitive closure of $\hookrightarrow_V$.
- Let $\texttt{v}_1 \overset{**}{\leftrightarrow}_V \texttt{v}_2$ denote that there are $\texttt{v}'_1, \texttt{v}'_2 \in V$ with $\texttt{v}'_1 \leftrightarrow_V \texttt{v}'_2$ such that $\texttt{v}_1 \overset{*}{\hookrightarrow}_V \texttt{v}'_1$ and $\texttt{v}_2 \overset{*}{\hookrightarrow}_V \texttt{v}'_2$.
- Let $\texttt{v}_1 \overset{*+}{\leftrightarrow}_V \texttt{v}_2$ denote that there are $\texttt{v}'_1, \texttt{v}'_2 \in V$ with $\texttt{v}'_1 \leftrightarrow_V \texttt{v}'_2$ such that $\texttt{v}_1 \overset{*}{\hookrightarrow}_V \texttt{v}'_1$ and $\texttt{v}_2 \overset{+}{\hookrightarrow}_V \texttt{v}'_2$.
- Let $\texttt{v}_1 \overset{*\circ}{\leftrightarrow}_V \texttt{v}_2$ denote that is a $\texttt{v}'_1 \in V$ with $\texttt{v}'_1 \leftrightarrow_V \texttt{v}_2$ such that $\texttt{v}_1 \overset{*}{\hookrightarrow}_V \texttt{v}'_1$.
- Let $\texttt{v}_1 \overset{++}{\leftrightarrow}_V \texttt{v}_2$ denote that there are $\texttt{v}'_1, \texttt{v}'_2 \in V$ with $\texttt{v}'_1 \leftrightarrow_V \texttt{v}'_2$ such that $\texttt{v}_1 \overset{+}{\hookrightarrow}_V \texttt{v}'_1$ and $\texttt{v}_2 \overset{+}{\hookrightarrow}_V \texttt{v}'_2$.
- Let $\texttt{v}_1 \overset{+*}{\leftrightarrow}_V \texttt{v}_2$ denote that there are $\texttt{v}'_1, \texttt{v}'_2 \in V$ with $\texttt{v}'_1 \leftrightarrow_V \texttt{v}'_2$ such that $\texttt{v}_1 \overset{+}{\hookrightarrow}_V \texttt{v}'_1$ and $\texttt{v}_2 \overset{*}{\hookrightarrow}_V \texttt{v}'_2$.
- Let $\texttt{v}_1 \overset{+\circ}{\leftrightarrow}_V \texttt{v}_2$ denote that is a $\texttt{v}'_1 \in V$ with $\texttt{v}'_1 \leftrightarrow_V \texttt{v}_2$ such that $\texttt{v}_1 \overset{+}{\hookrightarrow}_V \texttt{v}'_1$.

We say that $\texttt{v}_1$ and $\texttt{v}_2$ are *compatible* if $\texttt{v}_x \overset{*}{\hookrightarrow} \texttt{v}_y$ or $\texttt{v}_y \overset{*}{\hookrightarrow} \texttt{v}_x$ or $\texttt{v}_x \overset{**}{\leftrightarrow} \texttt{v}_y$. Intuitively, this means that $\texttt{v}_1$ and $\texttt{v}_2$ can be satisfied by two cells in the same heap state. Figure 7 illustrates the above relations for a heap state with 13 heap cells. The figure shows 4 fragments that are satisfied by heap cells, as denoted by green boxes, and how the relationship between heap cells is reflect by relations between the corresponding fragments.

We can now describe how to perform the symbolic postcondition computations for statements that assign to a pointer variable.

Consider a statement of form $\texttt{x} := \texttt{y}$, where $\texttt{x}$ and $\texttt{y}$ are global or local (to thread $\texttt{th}$) pointer variables. We must compute a set V' of fragments which are satisfied by the configuration after the statement. We must ensure that any cell $\mathbb{c}$ which is accessible to $\texttt{th}$ after the statement satisfies some fragement in V'. The cell $\mathbb{c}$ must satisfy some

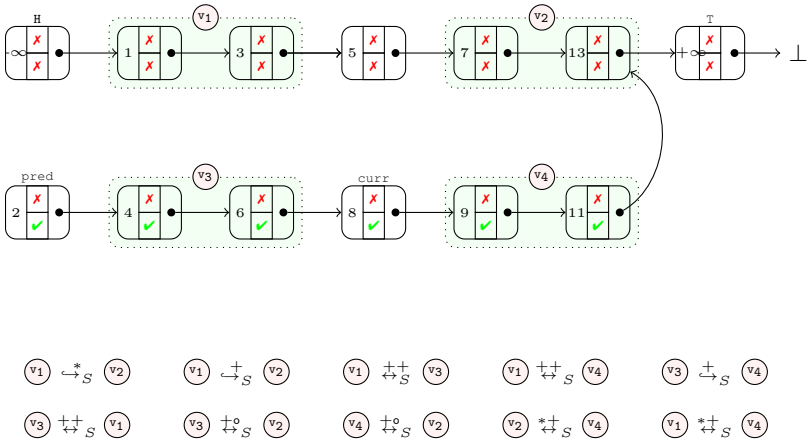

Fig. 7: Illustration of some transitive-closre-like relations between fragments

fragment $\mathtt{v}$ in V, and must be in the same heap state as the cell pointed to by $\mathtt{y}$. This means that we can make a case analysis on the possible relationships between $\mathtt{v}$ and any fragment $\mathtt{v}_y \in V$ such that $\mathtt{y} \in \mathtt{v}_y.\mathtt{i.pvars}$. Thus, for each fragment $\mathtt{v}_y \in V$ such that $\mathtt{y} \in \mathtt{v}_y.\mathtt{i.pvars}$ we let V' contain the fragments obtained by the following transformations on fragments in V.

1. First, for the fragment $\mathtt{v}_y$ itself, we let V' contain $\mathtt{v}'$, which is the same as $\mathtt{v}_y$, except that
   - $\mathtt{v}'.\mathtt{i.pvars} = \mathtt{v}_y.\mathtt{i.pvars} \cup \{x\}$ and $\mathtt{v}'.\mathtt{o.pvars} = \mathtt{v.o.pvars} \setminus \{x\}$

   and furthermore, if $\mathtt{x}$ is a global variable, then
   - $\mathtt{v}'.\mathtt{i.reachto} = \mathtt{v.i.reachto} \cup \{x\}$ and $\mathtt{v}'.\mathtt{i.reachfrom} = \mathtt{v.i.reachfrom} \cup \{x\}$,
   - $\mathtt{v}'.\mathtt{o.reachfrom} = \mathtt{v.o.reachfrom} \cup \{x\}$ and $\mathtt{v}'.\mathtt{o.reachto} = \mathtt{v.o.reachto} \setminus \{x\}$.
2. for each fragment $\mathtt{v}$ with $\mathtt{v} \hookrightarrow_V \mathtt{v}_y$, let V' contain $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i.pvars} = \mathtt{v.i.pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{o.pvars} = \mathtt{v.o.pvars} \cup \{x\}$,
   - $\mathtt{v}'.\mathtt{i.reachfrom} = \mathtt{v.i.reachfrom} \setminus \{x\}$ if $\mathtt{x}$ is a global variable,
   - $\mathtt{v}'.\mathtt{i.reachto} = \mathtt{v.i.reachto} \cup \{x\}$ if $\mathtt{x}$ is a global variable,
   - $\mathtt{v}'.\mathtt{o.reachfrom} = \mathtt{v.o.reachfrom} \cup \{x\}$ if $\mathtt{x}$ is a global variable,
   - $\mathtt{v}'.\mathtt{o.reachto} = \mathtt{v.o.reachto} \cup \{x\}$ if $\mathtt{x}$ is a global variable,
3. We perform analogous inclusions for fragments $\mathtt{v}$ with $\mathtt{v} \overset{+}{\hookrightarrow}_V \mathtt{v}_y$, $\mathtt{v}_y \overset{*}{\hookrightarrow}_V \mathtt{v}$, $\mathtt{v}_y \overset{*+}{\leftrightarrow}_V \mathtt{v}$, and $\mathtt{v}_y \overset{*\circ}{\leftrightarrow}_V \mathtt{v}$. For space reasons, we show only the case of $\mathtt{v}_y \overset{*+}{\leftrightarrow}_V \mathtt{v}$, in which case we let V' contain $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i.pvars} = \mathtt{v.i.pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{o.pvars} = \mathtt{v.o.pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{i.reachfrom} = \mathtt{v.i.reachfrom} \setminus \{x\}$ if $\mathtt{x}$ is a global variable,

- $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}.\mathtt{i}.\mathtt{reachto} \setminus \{x\}$ if $\mathtt{x}$ is a global variable,
- $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \setminus \{x\}$ if $\mathtt{x}$ is a global variable,
- $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \setminus \{x\}$ if $\mathtt{x}$ is a global variable,

The statement $\mathtt{x} := \mathtt{y}.\mathtt{next}$ is handled rather similarly to the preceding case. Let us therefore describe the computation for statements of the form $\mathtt{x}.\mathtt{next} := \mathtt{y}$. This is the most difficult statement, since it is a destructive update of the heap. The statement affects reachability relations for both $\mathtt{x}$ and $\mathtt{y}$. This means that we can make a case analysis on how a fragment in V is related to some pair of compatible fragments $\mathtt{v}_x$, $\mathtt{v}_y$ in V such that $\mathtt{x} \in \mathtt{v}_x.\mathtt{i}.\mathtt{pvars}$, $\mathtt{y} \in \mathtt{v}_\mathtt{y}.\mathtt{i}.\mathtt{pvars}$. Thus, for each pair of compatible fragments $\mathtt{v}_x$, $\mathtt{v}_y$ in V such that $\mathtt{x} \in \mathtt{v}_x.\mathtt{i}.\mathtt{pvars}$, $\mathtt{y} \in \mathtt{v}_\mathtt{y}.\mathtt{i}.\mathtt{pvars}$, we let V' contain the fragments obtained by the following transformations on fragments in V.

1. First, let V' containt a new fragment $\mathtt{v}_{new}$ of form $\langle \mathtt{v}_{new}.\mathtt{i}, \mathtt{v}_{new}.\mathtt{o}, \mathtt{v}_{new}.\phi \rangle$ $\mathtt{v}_{new}.\mathtt{i}.\mathtt{tag} = \mathtt{v}_x.\mathtt{i}.\mathtt{tag}$ and $\mathtt{v}_{new}.\mathtt{o}.\mathtt{tag} = \mathtt{v}_y.\mathtt{i}.\mathtt{tag}$ except that $\mathtt{v}_{new}.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}_y.\mathtt{i}.\mathtt{reachfrom} \cup \mathtt{v}_x.\mathtt{i}.\mathtt{reachfrom}$, and $\mathtt{v}_{new}.\phi = \{<, =, >\}$. Thereafter, we add all possible fragments that can result from a transformation of some fragment $\mathtt{v}$ which is in $\mathtt{v}$. This is done by an exhaustive case analysis on the possible relationship between $\mathtt{v}$, $\mathtt{v}_x$ and $\mathtt{v}_y$. Let us consider an interesting case, in which $\mathtt{v}_x \overset{*}{\hookrightarrow}_V \mathtt{v}$ and either $\mathtt{v} \overset{+}{\hookrightarrow}_V \mathtt{v}_y$ or $\mathtt{v}_y \overset{*+}{\leftrightarrow} \mathtt{v}$. In this case,
   (a) for each subset $\mathtt{regset}$ of observer registers in $\mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \cap \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom}$, we first create a fragment $\mathtt{v}'$ which is same as $\mathtt{v}$, except that $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = (\mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \setminus \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom}) \cup \mathtt{regset}$.
   (b) Thereafter, for each set $\mathtt{regset}'$ of observer registers in $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} \cap \mathtt{v}_x.\mathtt{i}.\mathtt{reachfrom}$, we let V' contain a fragment $\mathtt{v}''$ which is same as $\mathtt{v}'$, except that $\mathtt{v}''.\mathtt{o}.\mathtt{reachfrom} = (\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} \setminus \mathtt{v}_x.\mathtt{i}.\mathtt{reachfrom}) \cup \mathtt{regset}'$.

We should include an argument why the last case above is correct. Quy, could you produce one?

Symbolic Postcondition Computation for Interference Steps. The key step in this computation is to form the intersection of two sets of fragments V_1 and V_2, such that for the configuration c we have $c \models^{heap}_{\mathtt{th}_i} V_i$ for $i = 1, 2$. In order to distinguish between local variables of $\mathtt{th}_1$ and $\mathtt{th}_2$, we assume that local variable $\mathtt{x}$ of thread $\mathtt{th}_i$ is named as $\mathtt{x}[\mathtt{i}]$. We must compute a set V which for each heap cell accessible to either $\mathtt{th}_1$ or $\mathtt{th}_2$, the set V must contain a fragment $\mathtt{v}$ with $\mathbb{c} \lhd^{c\mathcal{S}}_{1,2} \mathtt{v}$. [This notation to be defined] There are here two possibilities.

- If $\mathbb{c}$ is accessible to both $\mathtt{th}_1$ and $\mathtt{th}_2$, then there are fragments $\mathtt{v}_1 \in V_1$ and $\mathtt{v}_2 \in V_2$ such that $\mathbb{c} \lhd^{c\mathcal{S}}_1 \mathtt{v}_1$ and $\mathbb{c} \lhd^{c\mathcal{S}}_2 \mathtt{v}_2$. We use the notation $\mathtt{v}_1 \sqcap \mathtt{v}_2$ to denote a set of views such that whenever $\mathbb{c} \lhd^{c\mathcal{S}}_1 \mathtt{v}_1$ and $\mathbb{c} \lhd^{c\mathcal{S}}_2 \mathtt{v}_2$ then $\mathbb{c} \lhd^{c\mathcal{S}}_{1,2} \mathtt{v}$ for some $\mathtt{v} \in (\mathtt{v}_1 \sqcap \mathtt{v}_2)$.
- If $\mathbb{c}$ is accessible to only one of $\mathtt{th}_1$ and $\mathtt{th}_2$, say $\mathtt{th}_1$, then V should contain some fragment $\mathtt{v}_1 \in V_1$ with $\mathbb{c} \lhd^{c\mathcal{S}}_{1,2} \mathtt{v}_1$ [Check that we need not change $\mathtt{v}_1$]

For a fragment $\mathtt{v}$, define $\mathtt{v}.\mathtt{i}.\mathtt{greachfrom}$ as the set of global variables in $\mathtt{v}.\mathtt{i}.\mathtt{reachfrom}$. Define $\mathtt{v}.\mathtt{i}.\mathtt{greachto}$, $\mathtt{v}.\mathtt{o}.\mathtt{greachfrom}$, $\mathtt{v}.\mathtt{o}.\mathtt{greachto}$,

v.i.gpvars, and v.o.gpvars analogously. Define v.o.gtag as the tuple $\langle$v.o.gpvars, v.o.dabs, v.o.greachfrom, v.o.greachto, v.o.private$\rangle$.

Question to Quy: You have also used the notation v.i.gdata. Will you need it, and if so what does it mean?

[ANSWER of Quy: Because in the data, I add data constraint between data fields and local data variable. When we do intersection, we do not need to care about this constraint because its local constraint]

Let us now describe how to compute $\mathtt{v}_1 \sqcap \mathtt{v}_2$ for two views $\mathtt{v}_1 \in V_1$ and $\mathtt{v}_2 \in V_2$. Firstly, we consider the case where both $\mathtt{v}_1$ and $\mathtt{v}_2$ have size 2. Let us consider some different cases. They all take into account the observation that if a cell $\mathbb{c}$ satisfies $\mathbb{c} \lhd_1^{\mathcal{C}^S} \mathtt{v}_1$ and $\mathbb{c} \lhd_2^{\mathcal{C}^S} \mathtt{v}_2$, then the information about global variables in $\mathtt{v}_1$ and $\mathtt{v}_2$ must coincide.

- if $\mathtt{v}_1$.i.greachfrom $\neq \emptyset$ and $\mathtt{v}_2$.i.greachfrom $\neq \emptyset$ then the global information in $\mathtt{v}_1$ and $\mathtt{v}_2$ must coincide. We hence obtain:
  - if $\mathtt{v}_1$.i.gtag $= \mathtt{v}_2$.i.gtag and $\mathtt{v}_1$.o.gtag $= \mathtt{v}_2$.o.gtag then $\mathtt{v}_1 \sqcap v_2 = \{\mathtt{v}_{12}\}$ where $\mathtt{v}_{12}$ is identical to $\mathtt{v}_1$ except that
    - $\mathtt{v}_{12}$.i.pvars $= \mathtt{v}_1$.i.pvars $\cup\, \mathtt{v}_2$.i.pvars
    - $\mathtt{v}_{12}$.o.pvars $= \mathtt{v}_1$.o.pvars $\cup\, \mathtt{v}_2$.o.pvars
    - $\mathtt{v}_{12}$.i.reachfrom $= \mathtt{v}_1$.i.reachfrom $\cup\, \mathtt{v}_2$.i.reachfrom
    - $\mathtt{v}_{12}$.o.reachfrom $= \mathtt{v}_1$.o.reachfrom $\cup\, \mathtt{v}_2$.o.reachfrom

    Question to Quy: Why do this union only for pvars and reachfrom, and not for reachto and dabs?

    [ANSWER of Quy: we fixed this in the disscussion]

  Question to Quy: Should we not have an "else" here, with $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \emptyset$?

  [ANSWER of Quy: we fixed this in the disscussion]
- if $\mathtt{v}_1$.i.greachfrom $= \emptyset$, $\mathtt{v}_2$.i.greachfrom $= \emptyset$, $\mathtt{v}_1$.o.greachfrom $\neq \emptyset$ and $\mathtt{v}_2$.o.greachfrom $\neq \emptyset$ then
  - if $\mathtt{v}_1$.o.gtag $= \mathtt{v}_2$.o.gtag, $\mathtt{v}_1$.i.private $=$ false and $\mathtt{v}_2$.i.private $=$ false then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}'_1, \mathtt{v}'_2, \mathtt{v}_{12}\}$ where
    - $\mathtt{v}'_1$ is same as $\mathtt{v}_1$ except that
      - $\mathtt{v}'_1$.o.pvars $= \mathtt{v}_1$.o.pvars $\cup\, \mathtt{v}_2$.o.pvars
      - $\mathtt{v}'_1$.o.reachfrom $= \mathtt{v}_1$.o.reachfrom $\cup\, \mathtt{v}_2$.o.reachfrom
    - $\mathtt{v}'_2$ is same as $\mathtt{v}_2$ except that
      - $\mathtt{v}'_2$.o.pvars $= \mathtt{v}_1$.o.pvars $\cup\, \mathtt{v}_2$.o.pvars
      - $\mathtt{v}'_2$.o.reachfrom $= \mathtt{v}_1$.o.reachfrom $\cup\, \mathtt{v}_2$.o.reachfrom
    - $\mathtt{v}_{12}$ is as in the previous case. [to Quy: I added this, is it correct?]
  - if $\mathtt{v}_1$.o.gtag $= \mathtt{v}_2$.o.gtag and $\mathtt{v}_1$.i.private $=$ true or $\mathtt{v}_2$.i.private $=$ true then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}'_1, \mathtt{v}'_2\}$ where $\mathtt{v}'_1$ and $\mathtt{v}'_2$ are as above.
- if $\mathtt{v}_1$.i.greachfrom $= \emptyset$, $\mathtt{v}_2$.i.greachfrom $= \emptyset$, $\mathtt{v}_1$.o.greachfrom $= \emptyset$ and $\mathtt{v}_2$.o.greachfrom $= \emptyset$ then
  - if gtag($\mathtt{v}_1$, o) $=$ gtag($\mathtt{v}_2$, o), $\mathtt{v}_1$.i.private $= \mathit{false}$, $\mathtt{v}_1$.o.private $= \mathit{false}$, $\mathtt{v}_1$.o.private $= \mathit{false}$ and $\mathtt{v}_2$.o.private $= \mathit{false}$ then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}_1, \mathtt{v}_2, \mathtt{v}'_1, \mathtt{v}'_2, \mathtt{v}_{12}\}$

- if $\texttt{gtag}(\texttt{v}_1, \texttt{o}) = \texttt{gtag}(\texttt{v}_2, \texttt{o})$, ($\texttt{v}_1.\texttt{i}.\texttt{private} = \mathit{true}$ or $\texttt{v}_2.\texttt{i}.\texttt{private} = \mathit{true}$) and $\texttt{v}_1.\texttt{o}.\texttt{private} = \mathit{false}$ and $\texttt{v}_2.\texttt{o}.\texttt{private} = \mathit{false}$ then $\texttt{v}_1 \sqcap \texttt{v}_2 = \{\texttt{v}_1, \texttt{v}_2, \texttt{v}'_1, \texttt{v}'_2\}$
- if $\texttt{gtag}(\texttt{v}_1, \texttt{o}) = \texttt{gtag}(\texttt{v}_2, \texttt{o})$ and ($\texttt{v}_1.\texttt{o}.\texttt{private} = \mathit{true}$ or $\texttt{v}_1.\texttt{o}.\texttt{private} = \mathit{true}$) then $\texttt{v}_1 \sqcap \texttt{v}_2 = \{\texttt{v}_1, \texttt{v}_2\}$
- if $\texttt{gtag}(\texttt{v}_1, \texttt{o}) \neq \texttt{gtag}(\texttt{v}_2, \texttt{o})$ then $\texttt{v}_1 \sqcap \texttt{v}_2 = \{\texttt{v}_1, \texttt{v}_2\}$

5 Fragment Abstraction for Skip-Lists

Here goes a description of the fragment abstraction for skip lists

– In the fragment abstraction, `tag` is define exactly same as `tag` in SLL abstraction where `reachfrom` and `reachto` is defined based on the main level of skip-list. It means that we do not keep the reachability information in higher levels.
– Same as timestamp stacks and queues, in skip-list we keep the main level and abstract all the higher levels. It means that we do not distinguish the differences between high levels. Hence, we have two types of fragments including main level fragments and higher level fragments which are defined same as SLL fragments.

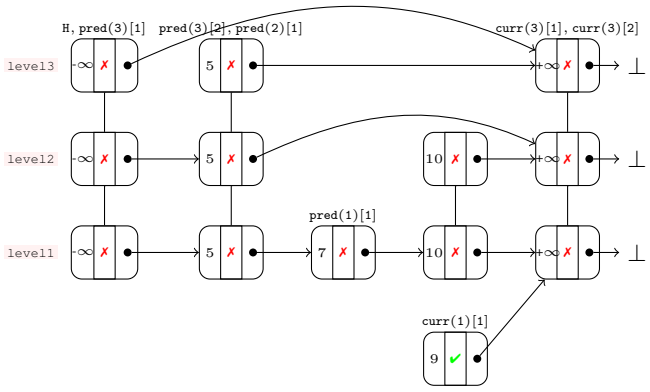

Fig. 8: A concrete shape of 3-level skipl-list with two threads

5.1 Abstract transformers for skip-lists

Let us show how to perform the abstract transformer for skip-list programs on the set of fragments V depending on the particular statement. We consider fragments of size 2 where $\texttt{I}_{\texttt{inp}} = \{\texttt{i}\}$, $\texttt{I}_{\texttt{out}} = \{\texttt{o}\}$, and $\texttt{next}(\texttt{i}) = \texttt{o}$ and fragments of size 1 where $\texttt{I}_{\texttt{inp}} = \{\texttt{i}\}$, $\texttt{I}_{\texttt{out}} = \emptyset$, and $\texttt{next}(\texttt{i}) = \texttt{null}$ or $\texttt{next}(\texttt{i}) = \bot$. For each fragment $\texttt{v}$, let $\texttt{v}.\texttt{level} \in \{1, 2\}$ be the level of $\texttt{v}$.

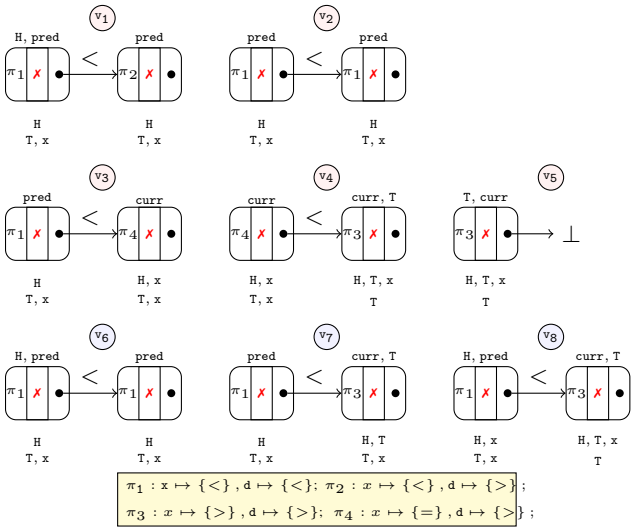

Fig. 9: skipl-list fragments [ANSWER of Quy: i am working with this figure]

Local Abstract Transformers: First, let us show the abstract transformer on the set of fragment V in the fragment of the concurrent thread. Let V_1 be set of fragments of level 1 in V, V_2 be set of fragments of level 2 in V. For each program statement, let V_{post} be the set of fragments after executing the statement. Let V_{post} be initialized as the empty set. Let $\mathtt{R}$ be the set of pairs of fragments. Intuitively, in each element in $\mathtt{R}$, the second fragment is the transformation of the first fragment. Let $\mathtt{R}$ be initialized as the empty set.

- $\mathtt{x} := \mathtt{y}$: The transformer is performed as follows: For each fragment $\mathtt{v_y} \in \mathtt{V_1}$ where $\mathtt{y} \in \mathtt{v_y.i.pvars}$,
  1. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v} \hookrightarrow_{\mathtt{v}} \mathtt{v_y}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
     - $\mathtt{v}'.\mathtt{i.pvars} = \mathtt{v.i.pvars} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{o.pvars} = \mathtt{v.o.pvars} \cup \{x\}$,
     - if
       $gvarof x$ is a global variable
       * $\mathtt{v}'.\mathtt{i.reachfrom} = \mathtt{v.i.reachfrom} \setminus \{x\}$,
       * $\mathtt{v}'.\mathtt{i.reachto} = \mathtt{v.i.reachto} \cup \{x\}$,
       * $\mathtt{v}'.\mathtt{o.reachfrom} = \mathtt{v.o.reachfrom} \cup \{x\}$,
       * $\mathtt{v}'.\mathtt{o.reachto} = \mathtt{v.o.reachto} \cup \{x\}$,

     then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$
  2. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v} \overset{+}{\hookrightarrow}_{\mathtt{v}} \mathtt{v_y}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
     - $\mathtt{v}'.\mathtt{i.pvars} = \mathtt{v.i.pvars} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{o.pvars} = \mathtt{v.o.pvars} \setminus \{x\}$,

- if
  $gvarofx$ is a global variable
  - $\mathtt{v'.i.reachfrom} = \mathtt{v.i.reachfrom} \setminus \{x\}$,
  - $\mathtt{v'.i.reachto} = \mathtt{v.i.reachto} \cup \{x\}$,
  - $\mathtt{v'.o.reachfrom} = \mathtt{v.o.reachfrom} \setminus \{x\}$,
  - $\mathtt{v'.o.reachto} = \mathtt{v.o.reachto} \cup \{x\}$,

  then add $\mathtt{v'}$ to V_{post}, and $(\mathtt{v}, \mathtt{v'})$ to $\mathtt{R}$

3. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v_y} \overset{*}{\hookrightarrow}_{\mathtt{V}} \mathtt{v}$, create $\mathtt{v'}$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v'.i.pvars} = \mathtt{v.i.pvars} \setminus \{x\}$,
   - $\mathtt{v'.o.pvars} = \mathtt{v.o.pvars} \setminus \{x\}$,
   - if
     $gvarofx$ is a global variable
     - $\mathtt{v'.i.reachfrom} = \mathtt{v.i.reachfrom} \cup \{x\}$,
     - $\mathtt{v'.i.reachto} = \mathtt{v.i.reachto} \setminus \{x\}$,
     - $\mathtt{v'.o.reachfrom} = \mathtt{v.o.reachfrom} \cup \{x\}$,
     - $\mathtt{v'.o.reachto} = \mathtt{v.o.reachto} \setminus \{x\}$,

     then add $\mathtt{v'}$ to V_{post}, and $(\mathtt{v}, \mathtt{v'})$ to V'
4. for each fragment $\mathtt{v}$ where $\mathtt{v_y} \overset{*+}{\leftrightarrow}_{\mathtt{V}} \mathtt{v}$, create $\mathtt{v'}$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v'.i.pvars} = \mathtt{v.i.pvars} \setminus \{x\}$,
   - $\mathtt{v'.o.pvars} = \mathtt{v.o.pvars} \setminus \{x\}$,
   - if
     $gvarofx$ is a global variable
     - $\mathtt{v'.i.reachfrom} = \mathtt{v.i.reachfrom} \setminus \{x\}$,
     - $\mathtt{v'.i.reachto} = \mathtt{v.i.reachto} \setminus \{x\}$,
     - $\mathtt{v'.o.reachfrom} = \mathtt{v.o.reachfrom} \setminus \{x\}$,
     - $\mathtt{v'.o.reachto} = \mathtt{v.o.reachto} \setminus \{x\}$,

     then add $\mathtt{v'}$ to V_{post}, and $(\mathtt{v}, \mathtt{v'})$ to V'
5. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v_y} \overset{*\circ}{\leftrightarrow}_{\mathtt{V}} \mathtt{v}$, create $\mathtt{v'}$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v'.i.pvars} = \mathtt{v.i.pvars} \setminus \{x\}$,
   - $\mathtt{v'.o.pvars} = \mathtt{v.o.pvars} \setminus \{x\}$,
   - if
     $gvarofx$ is a global variable
     - $\mathtt{v'.i.reachfrom} = \mathtt{v.i.reachfrom} \setminus \{x\}$,
     - $\mathtt{v'.i.reachto} = \mathtt{v.i.reachto} \setminus \{x\}$,
     - $\mathtt{v'.o.reachfrom} = \mathtt{v.o.reachfrom} \cup \{x\}$,
     - $\mathtt{v'.o.reachto} = \mathtt{v.o.reachto} \setminus \{x\}$,

     then add $\mathtt{v'}$ to V_{post}, and $(\mathtt{v}, \mathtt{v'})$ to V'
6. create $\mathtt{v'}$ which is same as $\mathtt{v_y}$ except that
   - $\mathtt{v'.i.pvars} = \mathtt{v.i.pvars} \cup \{x\}$,
   - $\mathtt{v'.o.pvars} = \mathtt{v.o.pvars} \setminus \{x\}$,
   - if
     $gvarofx$ is a global variable
     - $\mathtt{v'.i.reachto} = \mathtt{v.i.reachto} \cup \{x\}$,

* $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \cup \{x\}$,
* $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \cup \{x\}$,
* $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \setminus \{x\}$,

then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to V'

7. for each fragment $\mathtt{v} \in \mathtt{V_2}$ we do as follows. For each $(\mathtt{v_1}, \mathtt{v}'_1)$, $(\mathtt{v_2}, \mathtt{v}'_2) \in \mathtt{R}$, for each pair of indices $\mathtt{i_1}$, $\mathtt{i_2}$ such that $\mathtt{i_1} \in \{\mathtt{v_1}.\mathtt{i}, \mathtt{v_1}.\mathtt{o}\}$, $\mathtt{i_2} \in \{\mathtt{v_2}.\mathtt{i}, \mathtt{v_2}.\mathtt{o}\}$, $\mathtt{tag}(\mathtt{v}, \mathtt{i}) = \mathtt{tag}(\mathtt{v_1}, \mathtt{i_1})$, and $\mathtt{tag}(\mathtt{v}, \mathtt{o}) = \mathtt{tag}(\mathtt{v_2}, \mathtt{i_2})$. Create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}'_1.i_1.vars$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}'_2.i_2.vars$,
   - $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}'_1.i_1.reachfrom$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}'_2.i_2.reachfrom$,
   - $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}'_1.i_1.reachto$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}'_2.i_2.reachto$,

   then add $\mathtt{v}'$ to V_{post}.

– $\mathtt{x} := \mathtt{y}.\mathtt{next1}$: The local abstract transformer is quite similar to the previous case with slightly differences. For each fragment $\mathtt{v_y} \in \mathtt{S}$ where $\mathtt{y} \in \mathtt{vars}(\mathtt{i})$,

1. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v} \overset{*}{\hookrightarrow}_{\mathtt{V}} \mathtt{v_y}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}.\mathtt{i}.\mathtt{pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}.\mathtt{o}.\mathtt{pvars} \setminus \{x\}$,
   - if $\mathtt{x}$ is a global variable
     * $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \setminus \{x\}$,
     * $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}.\mathtt{i}.\mathtt{reachto} \cup \{x\}$,
     * $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \setminus \{x\}$,
     * $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \cup \{x\}$,

   then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$
2. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v_y} \hookrightarrow_{\mathtt{V}} \mathtt{v}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ then
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}.\mathtt{i}.\mathtt{pvars} \cup \{x\}$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}.\mathtt{o}.\mathtt{pvars} \setminus \{x\}$,
   - if $\mathtt{x}$ is a global variable
     * $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \cup \{x\}$,
     * $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \cup \{x\}$,
     * $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \cup \{x\}$,
     * $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \setminus \{x\}$,

   then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$
3. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v_y} \leftrightarrow_{\mathtt{V}} \mathtt{v}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}.\mathtt{i}.\mathtt{pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}.\mathtt{o}.\mathtt{pvars} \cup \{x\}$,
   - if $\mathtt{x}$ is a global variable
     * $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \setminus \{x\}$,
     * $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}.\mathtt{i}.\mathtt{reachto} \cup \{x\}$,
     * $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \cup \{x\}$,
     * $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \cup \{x\}$,

   then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$

4. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v_y} \overset{+}{\hookrightarrow}_{\mathtt{V}} \mathtt{v}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}.\mathtt{i}.\mathtt{pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}.\mathtt{o}.\mathtt{pvars} \setminus \{x\}$,
   - if $\mathtt{x}$ is a global variable
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \cup \{x\}$,
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}.\mathtt{i}.\mathtt{reachto} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \cup \{x\}$,
     - $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \setminus \{x\}$,

   then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$
5. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v_y} \overset{*\circ}{\hookrightarrow}_{\mathtt{V}} \mathtt{v}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}.\mathtt{i}.\mathtt{pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}.\mathtt{o}.\mathtt{pvars} \setminus \{x\}$,
   - if $\mathtt{x}$ is a global variable
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}.\mathtt{i}.\mathtt{reachto} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \cup \{x\}$,
     - $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \setminus \{x\}$,

   then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$
6. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v_y} \overset{*+}{\hookrightarrow}_{\mathtt{V}} \mathtt{v}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}.\mathtt{i}.\mathtt{pvars} \setminus \{x\{$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}.\mathtt{o}.\mathtt{pvars} \setminus \{x\}$,
   - if $\mathtt{x}$ is a global variable
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}.\mathtt{i}.\mathtt{reachto} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \setminus \{x\}$,

   then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$
7. for each fragment $\mathtt{v} \in \mathtt{V_1}$ where $\mathtt{v} \overset{+\circ}{\hookrightarrow}_{\mathtt{V}} \mathtt{v_y}$, create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}.\mathtt{i}.\mathtt{pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}.\mathtt{o}.\mathtt{pvars} \setminus \{x\}$,
   - if $\mathtt{x}$ is a global variable
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}.\mathtt{i}.\mathtt{reachto} \cup \{x\}$,
     - $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \setminus \{x\}$,
     - $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \cup \{x\}$,

   then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$
8. create $\mathtt{v}'$ which is same as $\mathtt{v_y}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}.\mathtt{i}.\mathtt{pvars} \setminus \{x\}$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}.\mathtt{o}.\mathtt{pvars} \cup \{x\}$,
   - if $\mathtt{x}$ is a global variable
     - $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \setminus \{x\}$,

* $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}.\mathtt{i}.\mathtt{reachto} \cup \{x\}$,
* $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}.\mathtt{o}.\mathtt{reachfrom} \cup \{x\}$,
* $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}.\mathtt{o}.\mathtt{reachto} \cup \{x\}$,

then add $\mathtt{v}'$ to V_{post}, and $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}$

9. for each fragment $\mathtt{v} \in \mathtt{V}_2$ we do as follows. For each $(\mathtt{v}_1, \mathtt{v}'_1)$, $(\mathtt{v}_2, \mathtt{v}'_2) \in \mathtt{R}$, for each pair of indices $\mathtt{i}_1$, $\mathtt{i}_2$ such that $\mathtt{i}_1 \in \{\mathtt{v}_1.\mathtt{i}, \mathtt{v}_1.\mathtt{o}\}$, $\mathtt{i}_2 \in \{\mathtt{v}_2.\mathtt{i}, \mathtt{v}_2.\mathtt{o}\}$, $\mathtt{tag}(\mathtt{v}, \mathtt{i}) = \mathtt{tag}(\mathtt{v}_1, \mathtt{i}_1)$, and $\mathtt{tag}(\mathtt{v}, \mathtt{o}) = \mathtt{tag}(\mathtt{v}_2, \mathtt{i}_2)$. Create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
   - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}'_1.i_1.vars$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}'_2.i_2.vars$,
   - $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}'_1.i_1.reachfrom$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}'_2.i_2.reachfrom$,
   - $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}'_1.i_1.reachto$,
   - $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}'_2.i_2.reachto$,

   then add $\mathtt{v}'$ to V_{post}.

– $\mathtt{x}.\mathtt{next1} := \mathtt{y}$: The local abstract transformer is performed by several steps as follows: For each pair of fragments $\mathtt{v}_x$, $\mathtt{v}_\mathtt{y}$ in V_1 where $\mathtt{x} \in \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{pvars}$, $\mathtt{y} \in \mathtt{v}_\mathtt{y}.\mathtt{i}.\mathtt{pvars}$, and $\mathtt{v}_x \overset{*}{\hookrightarrow} \mathtt{v}_y$ or $\mathtt{v}_\mathtt{y} \overset{*}{\hookrightarrow} \mathtt{v}_\mathtt{x}$ or $\mathtt{v}_x \overset{**}{\leftrightarrow} \mathtt{v}_y$, let $\mathtt{R}_1$, $\mathtt{R}_2$, $\mathtt{R}_3$, $\mathtt{R}_4$, $\mathtt{R}_5$, $\mathtt{R}_6$ be initialized as $\mathtt{R}$,
   1. let $\mathtt{v}_\mathtt{new}$ be the fragment of size 2 and of level 1 where $\mathtt{tag}(\mathtt{v}_\mathtt{new}, \mathtt{i}) = \mathtt{tag}(\mathtt{v}_\mathtt{x}, \mathtt{i})$ and $\mathtt{tag}(\mathtt{v}_\mathtt{new}, \mathtt{o}) = \mathtt{tag}(\mathtt{v}_\mathtt{y}, \mathtt{i})$ except that $\mathtt{v}_\mathtt{new}.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}_\mathtt{y}.\mathtt{i}.\mathtt{reachfrom} \cup \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom}$,
   2. for each fragment $\mathtt{v} \in \mathtt{V}_1$ where $\mathtt{v} \overset{*}{\hookrightarrow}_\mathtt{V} \mathtt{v}_\mathtt{x}$, we do as follows: For each subset $\mathtt{regset}$ of observer registers in $\mathtt{v}.\mathtt{i}.\mathtt{reachto} \cap \mathtt{v}_x.\mathtt{i}.\mathtt{reachfrom}$
      - create $\mathtt{v}'$ which is same as $\mathtt{v}$, except that
        * $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = (\mathtt{v}.\mathtt{i}.\mathtt{reachto} \cap \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom}) \cup \mathtt{reachtov}_\mathtt{y}.\mathtt{i} \cup \mathtt{regset}$.
        * $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = (\mathtt{v}.\mathtt{o}.\mathtt{reachto} \cap \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom}) \cup \mathtt{reachtov}_\mathtt{y}.\mathtt{i} \cup \mathtt{regset}$.
      - add $\mathtt{v}'$ to V_{post}
      - add $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}_1$
   3. for each fragment $\mathtt{v} \in \mathtt{V}_1$ where $\mathtt{v}_\mathtt{y} \overset{*}{\hookrightarrow}_\mathtt{V} \mathtt{v}$ or $\mathtt{v}_\mathtt{y} = \mathtt{v}$,
      - create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
        * $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom} \cup \mathtt{v}.\mathtt{i}.\mathtt{reachfrom}$,
        * $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom} \cup \mathtt{v}.\mathtt{o}.\mathtt{reachfrom}$,
      - add $\mathtt{v}'$ to V_{post},
      - add $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}_2$
   4. for each fragment $\mathtt{v} \in \mathtt{V}_1$ where $\mathtt{v}_x \overset{**}{\leftrightarrow}_V v$ and either $\mathtt{v} \hookrightarrow_\mathtt{V} \mathtt{v}_\mathtt{y}$ or $\mathtt{v}_\mathtt{y} \overset{*\mathtt{o}}{\leftrightarrow}_\mathtt{V} \mathtt{v}$,
      - create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom} \cup \mathtt{v}.\mathtt{o}.\mathtt{reachfrom}$,
      - add $\mathtt{v}'$ to V_{post},
      - add $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}_3$
   5. for each fragment $\mathtt{v} \in \mathtt{V}_1$ where $\mathtt{v}_x \overset{*}{\hookrightarrow}_V v$ and either $\mathtt{v} \hookrightarrow_\mathtt{V} \mathtt{v}_\mathtt{y}$ or $\mathtt{v}_\mathtt{y} \overset{*\mathtt{o}}{\leftrightarrow}_\mathtt{V} \mathtt{v}$,
      - create $\mathtt{v}'$ which is same as $\mathtt{v}$ then except that $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom} \cup \mathtt{v}.\mathtt{o}.\mathtt{reachfrom}$,

- for each subset regset of observer registers in $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} \cap \mathtt{v}_x.\mathtt{i}.\mathtt{reachfrom}$
  - create $\mathtt{v}''$ which is same as $\mathtt{v}'$, except that $\mathtt{v}''.\mathtt{i}.\mathtt{reachfrom} = (\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} \setminus \mathtt{v}_x.\mathtt{i}.\mathtt{reachfrom}) \cup \mathtt{regset}$.
  - add $\mathtt{v}''$ to V_{post},
  - add $(\mathtt{v}, \mathtt{v}'')$ to $\mathtt{R}_4$

6. for each fragment $\mathtt{v} \in \mathtt{V}_1$ where $\mathtt{v}_x \overset{**}{\leftrightarrow}_V v$ and either $\mathtt{v} \overset{+}{\hookrightarrow}_\mathtt{V} \mathtt{v}_\mathtt{y}$ or $\mathtt{v} \overset{*+}{\leftrightarrow}_\mathtt{V} \mathtt{v}_\mathtt{y}$,
   - create $\mathtt{v}'$ which is same as $\mathtt{v}$
   - add $\mathtt{v}'$ to V_{post},
   - add $(\mathtt{v}, \mathtt{v}')$ to $\mathtt{R}_5$
7. for each fragment $\mathtt{v} \in \mathtt{V}_1$ where $\mathtt{v}_x \overset{*}{\hookrightarrow}_V v$ and either $\mathtt{v} \overset{+}{\hookrightarrow}_\mathtt{V} \mathtt{v}_\mathtt{y}$ or $\mathtt{v}_\mathtt{y} \overset{*+}{\leftrightarrow} \mathtt{v}$, then for each subset regset of observer registers in $\mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \cap \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom}$,
   - create $\mathtt{v}'$ which is same as $\mathtt{v}$, except that $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = (\mathtt{v}.\mathtt{i}.\mathtt{reachfrom} \setminus \mathtt{v}_\mathtt{x}.\mathtt{i}.\mathtt{reachfrom}) \cup \mathtt{regset}$.
   - for each set $\mathtt{regset}'$ of observer registers in $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} \cap \mathtt{v}_x.\mathtt{i}.\mathtt{reachfrom}$,
     - create $\mathtt{v}''$ which is same as $\mathtt{v}'$, except that $\mathtt{v}''.\mathtt{o}.\mathtt{reachfrom} = (\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} \setminus \mathtt{v}_x.\mathtt{i}.\mathtt{reachfrom}) \cup \mathtt{regset}'$.
     - add $\mathtt{v}''$ to V_{post}
     - add $(\mathtt{v}, \mathtt{v}'')$ to $\mathtt{R}_6$
8. add $\mathtt{v}_\mathtt{new}$ to V_{post}

– for each fragment $\mathtt{v} \in V_2$ then we do as follows. for each $(\mathtt{v}_1, \mathtt{v}_1') \in \mathtt{R}_\mathtt{i}$, $(\mathtt{v}_2, \mathtt{v}_2') \in \mathtt{R}_\mathtt{j}$ where $\mathtt{i} \neq \mathtt{j}$ and $\mathtt{1} \leq \mathtt{i}, \mathtt{j} \leq \mathtt{6}$. For each pair of indices $\mathtt{i}_1$, $\mathtt{i}_2$ such that $\mathtt{i}_1 \in \{\mathtt{v}_1.\mathtt{i}, \mathtt{v}_1.\mathtt{o}\}$, $\mathtt{i}_2 \in \{\mathtt{v}_2.\mathtt{i}, \mathtt{v}_2.\mathtt{o}\}$, $\mathtt{tag}(\mathtt{v}, \mathtt{i}) = \mathtt{tag}(\mathtt{v}_1, \mathtt{i}_1)$, $\mathtt{tag}(\mathtt{v}, \mathtt{o}) = \mathtt{tag}(\mathtt{v}_2, \mathtt{i}_2)$. Create $\mathtt{v}'$ which is same as $\mathtt{v}$ except that
  - $\mathtt{v}'.\mathtt{i}.\mathtt{pvars} = \mathtt{v}_1'.i_1.vars$,
  - $\mathtt{v}'.\mathtt{o}.\mathtt{pvars} = \mathtt{v}_2'.i_2.vars$,
  - $\mathtt{v}'.\mathtt{i}.\mathtt{reachfrom} = \mathtt{v}_1'.i_1.reachfrom$,
  - $\mathtt{v}'.\mathtt{o}.\mathtt{reachfrom} = \mathtt{v}_2'.i_2.reachfrom$,
  - $\mathtt{v}'.\mathtt{i}.\mathtt{reachto} = \mathtt{v}_1'.i_1.reachto$,
  - $\mathtt{v}'.\mathtt{o}.\mathtt{reachto} = \mathtt{v}_2'.i_2.reachto$,

  then add $\mathtt{v}'$ to V_{post}.

Fragment Intersection: Let us describe the intersection of two fragments $\mathtt{v}_1 \in \mathtt{V}_1$ and $\mathtt{v}_2 \in \mathtt{V}_2$ denoted as $\mathtt{v}_1 \sqcap \mathtt{v}_2$. Firstly, we consider the case where both $\mathtt{v}_1$ and $\mathtt{v}_2$ have size 2.

- if $\mathtt{v}_1.\mathtt{greachfrom}(\mathtt{i}, \{1,2\}) \neq \emptyset$ and $\mathtt{v}_2.\mathtt{greachfrom}(\mathtt{i}, \{1,2\}) \neq \emptyset$ then
  - if $\mathtt{gtag}(\mathtt{v}_1, \mathtt{i}) = \mathtt{gtag}(\mathtt{v}_2, \mathtt{i})$ and $\mathtt{gtag}(\mathtt{v}_1, \mathtt{o}) = \mathtt{gtag}(\mathtt{v}_2, \mathtt{o})$ then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}_{12}\}$ where $\mathtt{v}_{12}$ is same as $\mathtt{v}_1$ except that, for all $\mathtt{l} \in \{1, 2\{1,2\}\}$
    - $\mathtt{v}_{12}.\mathtt{vars}(\mathtt{i}) = \mathtt{v}_1.\mathtt{vars}(\mathtt{i}) \cup \mathtt{v}_2.\mathtt{vars}(\mathtt{i})$
    - $\mathtt{v}_{12}.\mathtt{o}.\mathtt{pvars} = \mathtt{v}_1.\mathtt{o}.\mathtt{pvars} \cup \mathtt{v}_2.\mathtt{o}.\mathtt{pvars}$
    - $\mathtt{v}_{12}.\mathtt{reachfrom}(\mathtt{i}, \mathtt{l}) = \mathtt{v}_1.\mathtt{reachfrom}(\mathtt{i}, \mathtt{l}) \cup \mathtt{v}_2.\mathtt{reachfrom}(\mathtt{i}, \mathtt{l})$
    - $\mathtt{v}_{12}.\mathtt{reachfrom}(\mathtt{i}, \mathtt{l}) = \mathtt{v}_1.\mathtt{reachfrom}(\mathtt{o}, \mathtt{l}) \cup \mathtt{v}_2.\mathtt{reachfrom}(\mathtt{o}, \mathtt{l})$
- if $\mathtt{v}_1.\mathtt{greachfrom}(\mathtt{i}, \{1,2\}) = \emptyset$, $\mathtt{v}_2.\mathtt{greachfrom}(\mathtt{i}, \{1,2\}) = \emptyset$, $\mathtt{v}_1.\mathtt{greachfrom}(\mathtt{o}, \{1,2\}) \neq \emptyset$ and $\mathtt{v}_2.\mathtt{greachfrom}(\mathtt{o}, \{1,2\}) \neq \emptyset$ then
  - if $\mathtt{gtag}(\mathtt{v}_1, \mathtt{o}) = \mathtt{gtag}(\mathtt{v}_2, \mathtt{o})$, $\mathtt{v}_1.\mathtt{private}(\mathtt{i}) = \mathtt{false}$ and $\mathtt{v}_2.\mathtt{private}(\mathtt{i}) = \mathtt{false}$ then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}'_1, \mathtt{v}'_2, \mathtt{v}_{12}\}$ where $\mathtt{v}'_1$ is same as $\mathtt{v}_1$ except that, for all $\mathtt{l} \in \{1, 2\{1,2\}\}$
    - $\mathtt{v}'_1.\mathtt{o}.\mathtt{pvars} = \mathtt{v}_1.\mathtt{o}.\mathtt{pvars} \cup \mathtt{v}_2.\mathtt{o}.\mathtt{pvars}$
    - $\mathtt{v}'_1.\mathtt{reachfrom}(\mathtt{o}, \mathtt{l}) = \mathtt{v}_1.\mathtt{reachfrom}(\mathtt{o}, \mathtt{l}) \cup \mathtt{v}_2.\mathtt{reachfrom}(\mathtt{o}, \mathtt{l})$

    and $\mathtt{v}'_2$ is same as $\mathtt{v}_2$ except that
    - $\mathtt{v}'_2.\mathtt{o}.\mathtt{pvars} = \mathtt{v}_1.\mathtt{o}.\mathtt{pvars} \cup \mathtt{v}_2.\mathtt{o}.\mathtt{pvars}$
    - $\mathtt{v}'_2.\mathtt{reachfrom}(\mathtt{o}, \mathtt{l}) = \mathtt{v}_1.\mathtt{reachfrom}(\mathtt{o}, \mathtt{l}) \cup \mathtt{v}_2.\mathtt{reachfrom}(\mathtt{o}, \mathtt{l})$
  - if $\mathtt{gtag}(\mathtt{v}_1, \mathtt{o}) = \mathtt{gtag}(\mathtt{v}_2, \mathtt{o})$ and ($\mathtt{v}_1.\mathtt{private}(\mathtt{i}) = \mathtt{true}$ or $\mathtt{v}_2.\mathtt{private}(\mathtt{i}) = \mathtt{true}$) then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}'_1, \mathtt{v}'_2\}$
- if $\mathtt{v}_1.\mathtt{greachfrom}(\mathtt{i}, \{1,2\}) = \emptyset$, $\mathtt{v}_2.\mathtt{greachfrom}(\mathtt{i}, \{1,2\}) = \emptyset$, $\mathtt{v}_1.\mathtt{greachfrom}(\mathtt{o}, \{1,2\}) = \emptyset$ and $\mathtt{v}_2.\mathtt{greachfrom}(\mathtt{o}, \{1,2\}) = \emptyset$ then
  - if $\mathtt{gtag}(\mathtt{v}_1, \mathtt{o}) = \mathtt{gtag}(\mathtt{v}_2, \mathtt{o})$, $\mathtt{v}_1.\mathtt{private}(\mathtt{i}) = \mathtt{false}$, $\mathtt{v}_1.\mathtt{o}.\mathtt{private} = \mathtt{false}$, $\mathtt{v}_1.\mathtt{o}.\mathtt{private} = \mathtt{false}$ and $\mathtt{v}_2.\mathtt{o}.\mathtt{private} = \mathtt{false}$ then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}_1, \mathtt{v}_2, \mathtt{v}'_1, \mathtt{v}'_2, \mathtt{v}_{12}\}$
  - if $\mathtt{gtag}(\mathtt{v}_1, \mathtt{o}) = \mathtt{gtag}(\mathtt{v}_2, \mathtt{o})$, ($\mathtt{v}_1.\mathtt{private}(\mathtt{i}) = \mathtt{true}$ or $\mathtt{v}_2.\mathtt{private}(\mathtt{i}) = \mathtt{true}$) and $\mathtt{v}_1.\mathtt{o}.\mathtt{private} = \mathtt{false}$ and $\mathtt{v}_2.\mathtt{o}.\mathtt{private} = \mathtt{false}$ then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}_1, \mathtt{v}_2, \mathtt{v}'_1, \mathtt{v}'_2\}$
  - if $\mathtt{gtag}(\mathtt{v}_1, \mathtt{o}) = \mathtt{gtag}(\mathtt{v}_2, \mathtt{o})$ and ($\mathtt{v}_1.\mathtt{o}.\mathtt{private} = \mathtt{true}$ or $\mathtt{v}_1.\mathtt{o}.\mathtt{private} = \mathtt{true}$) then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}_1, \mathtt{v}_2\}$
  - if $\mathtt{gtag}(\mathtt{v}_1, \mathtt{o}) \neq \mathtt{gtag}(\mathtt{v}_2, \mathtt{o})$ then $\mathtt{v}_1 \sqcap \mathtt{v}_2 = \{\mathtt{v}_1, \mathtt{v}_2\}$

6 Timestamp Stack

7 Timestamp Abstraction

7.1 View Abstraction

For timestamp data structures we have to deal with timestamp ordering and unbound number of lists. The solutions are described as follows:

– About timestamp ordering, we add timestamp ordering information for each index of a view. The order of index `i` of view `v` is of the form $\texttt{v.i.ts} \diamond \texttt{x}$ where $\diamond \in \{<, =, >\}$ and `x` is an observer register. Intuitively, $\texttt{v.i.ts} \diamond \texttt{x}$ means that $\diamond$ is the order between the timestamp of `v.i` and timestamp of an index whose data is equal to `x`.
– To deal with the problem of unbounded number of lists. We use two kind of views which are c-views $\texttt{v}_\texttt{c}$ in a current list and o-views $\texttt{v}_\texttt{o}$ in other lists. Note that, current list is the list where the current thread is accessing to.

7.2 Post-computation

The post-computation is quite similar to singly-linked lists with several differences as follows: Before computing the post condition of this statement, we change all o-views in the other list to c-views and previous c-views to o-views.

7.3 Intersection

The intersection between two views are computed same as in the case of singly-linked lists with several differences as follows:

– Two views of push methods should not be intersected. The reason for it is that we do not have more concurrent pushes in a same list.
– We can intersect o-views and c-views, o-views and o-views as well as c-views and c-views

8 Experimental Results

I am working here

Based on our framework, we have implemented a tool in OCaml, and used it for verifying concurrent algorithms (both lock-based and lock-free) including timestamps stack and queue, skip-list sets and priority queues as well as singly-linked lists algorithms (stacks, queues, sets). The experiments were performed on a desktop 2.8 GHz processor with 8GB memory. The results are presented in Fig. 10, where running times are given in seconds. All experiments start from the initial heap, and end either when the analysis reaches the fixed point or when a violation of safety properties or linearizability is detected.

Running Times. As can be seen from the table, the running times vary in the different examples. This is due to the types of shapes that are produced during the analysis. For instance, skip-lists algorithm have much longer running times. This is due to the number of pointer variables and their complicated shapes. Whereas, other algorithms produce simple shape patterns and hence they have shorter running times.

Algorithms	Time (s)
TIMESTAMPS	
TS stack [28]	176
TS queue [28]	101
SKIP-LISTS	
Lock-free skip-list [22]	1992
Optimistic skip-list [28]	500
Priority queue skip-list 1 [29]	1320
Priority queue skip-list 2 [29]	599
SINGLY-LINKED LISTS	
Treiber stack [36]	18
MS lock-free queue [28]	21
DGLM queue [10]	16
Vechev-CAS set [43]	86
Vechev-DCAS set [43]	16
Michael lock-free set [26]	178
Pessimistic set [22]	30
Optimistic set [22]	25
Lazy set [18]	34
O′Hearn set [30]	88
HM lock-free set [22]	120

Fig. 10: Experimental results for verifying concurrent programs

Error Detection In addition to establishing correctness of the original versions of the benchmark algorithms, we tested our tool with intentionally inserted bugs. For example, we emitted setting time statement in line 5 of the `push` method in TS stack algorithm. The tool, as expected, successfully detected and reported the bug.

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A Algorithms

In this section, we show several important algorithms including timestamp stack and queue, lazy set, skip-list sets and skip-list priority queue.

```
struct Node {
         int data;
         Timestamp ts;
         Node* next;
         bool mark;
       }

init() :
 Node* pools[maxThreads];
 for(int i=0; i<maxThreads; i++)
   pools[i].next = null;

void push(int d):
1  Node* new := new Node(d,-1,null,false);
   new.next = pools[myID];
   pools[myID] = new;
   Timestamp t = new Timestamp();
   new.ts = t;
   Node* next = new.next;
   while (next.next != next & !next.mark)
     next = next.next;
   new.next = next;
   return new;

Node remove(Node* pt, Node* t, Node* n)
   bool s = CAS(n.mark,false,true);
     if (s)
       CAS(pt, t, n);
       if (t != n);
         t.next = n;
       t.next = n.next;
       Node* next=n.next
       while (next.next != next && next.mark);
         next = next.next;
       n.next = next;
       return s;

int pop(Timestamp ts):
  boolean success = false;
  int maxTS = -1;
  Node* youngest, myTop, n = null;
  Node* empty[maxThreads];
  while (!success)
    int k;
    for(int i=0; i<maxThreads; i++)
      myTop = pools[i]; n = myTop;
      while (n.mark && n.next != n) n = n.next;
      if(n = null)
        empty[i] = pools[i];
        continue;
      if(st < n.ts)
        r = remove(pools[i], top, n);
        return n.data);
      if(maxTS < n.ts)
        maxTS = n.ts;
        youngest = n;
        k = i;
    if (youngest != null)
      success= remove(pools[k],myTop, youngest);
    if (youngest = null)
      for(int i=0; i<maxThreads; i++)
          if (pools[i] != empty[i]);
            return NonEmpty;
      return Empty;
  return youngest.data;
```

Fig. 11: Timestamp Stack.

```
struct Node {
         int data;
         Timestamp ts;
         Node* next;
         bool mark;
       }

init() :
 Node* pools[maxThreads];
 for(int i=0; i<maxThreads; i++)
   pools[i].next = null;

void enq(int d):
 Node* new := new Node(d,-1,null,false);
 new.next = pools[myID];
 pools[myID] = new;
 Timestamp t = new Timestamp();
 new.ts = t;
 Node* next = new.next;
 while (next.next != next & !next.mark)
   next = next.next;
 new.next = next;
 return new;
```

```
int deq(Timestamp ts):
 boolean success = false;
 int maxTS = 10000;
 Node* youngest, myTop, n = null;
 Node* empty[maxThreads];
 while (!success)
   int k;
   for(int i=0; i<maxThreads; i++)
     myTop = pools[i]; n = myTop;
     while (n.mark && n.next != n) n = n.next;
     if(n = null)
       empty[i] = pools[i];
       continue;
     if(maxTS > n.ts)
       maxTS = n.ts;
       youngest = n;
       k = i;
   if (youngest != null)
     success= remove(pools[k],myTop, youngest);
   if (youngest = null)
     for(int i=0; i<maxThreads; i++)
         if (pools[i] != empty[i]);
           return NonEmpty;
     return Empty;
   return youngest.data;
```

```
Node remove(Node* pt, Node* t, Node* n)
  bool s = CAS(n.mark,false,true);
    if (s)
      CAS(pt, t, n);
      if (t != n);
        t.next = n;
      t.next = n.next;
      Node* next=n.next
      while (next.next != next && next.mark);
        next = next.next;
      n.next = next;
      return s;
```

Fig. 12: Timestamp Queue.

```
struct Node(bool lock; int val; Node *next; bool mark);

 <Node, Node> locate(int d):
 local pred, curr
   while (true)
      pred := head;
      curr := pred.next;
      while (curr.val < d)
         pred := curr;
         curr := curr.next
      lock(pred); lock(curr);
      if (! pred.mark &&
           ! curr.mark&&
           pred.next=curr)
         return(pred,curr);
      else
         unlock(pred);
         unlock(curr);

bool ctn(int d):
local curr
  curr := Head;
  while (curr.val < d)
     curr := curr.next
  b := curr.mark
  if (! b && curr.val = d)
     return true;
  else return false;
```

```
bool add(int d):
   local pred, curr, n, r
   1 (pred,curr) := locate(d);
     if (curr.val <> d)
        n :=
        new Node(0,d,curr,false);
        pred.next := n;
        r := true;
     else r := false;
     unlock(pred);
     unlock(curr);
     return r;


   bool rmv(int d):
   local pred, curr, n, r
   1 (pred,curr) = locate(d);
     if (curr.val = d)
        curr.mark = true;
        n = curr.next;
        pred.next = n;
        r = true;
     else r = false;
     unlock(pred);
     unlock(curr);
     return r;
```

Fig. 13: Lazy Set.

```
struct Node(int key; int topLayer; Node *next[]; bool marked; fullylinked;Lock lock);

 locate(int v,Node* preds[], Node* succs[]):
    local lfound, pred, curr;
     pred = H; lfound = -1
     for (int i = maxHeight; i >= 1; i--);
        curr = pred.next[i];
        while (curr.val < v)
           pred = curr;
           curr = curr.next[i]
        if (lfound = 1 && curr.val = v)
           lfound = i
        preds[i] = pred;
        currs[i] = curr
     return lfound

rmv(int v):
 Node* nodeToDelete = null;
 bool isMarked = false;
 int level = -1;
 Node* preds[MaxHeight], succs[MaxHeight];
 while (true)
   int lFound = findNode(v, preds, succs);
   if (isMarked || lFound != -1 &&
     (okToDelete (succs[lFound], lFound)))
     if (!isMarked)
       nodeToDelete = succs[lFound];
       topLayer = nodeToDelete.level;
       lock(nodeToDelete);
       if(nodeToDelete.marked);
         unlock(nodeToDelete);
       return false;
       nodeToDelete.marked = true;
       isMarked = true;
     int highestLocked = -1;
     try
       Node* pred, succ, prevPred = null;
       bool valid = true;
       for (int i = 0;valid &&li<=level; i++)
         pred = preds[i];
         succ = succs[i];
         if (pred != prevPred)
           lock(pred);
         highestLocked = i;
         prevPred = pred;
         valid = !pred.marked && pred.next[i]==succ;
       if (!valid) continue;
       for (int j = level; j >= 0; j--)
         preds[j].nexts[j] = nodeToDelete.nexts[j];
       unlock(nodeToDelete);
       return true;
     finally unlock(preds,highestLocked);
   else return false;

add(int v):
 int level = randomLevel(MaxHeight);
 Node* preds[MaxHeight], succs[MaxHeight];
 while(true)
   int lfound = findnode(v, preds, succs);
   if (lfound != 1)
     Node* nodeFound = succs[lfound];
     if (!nodeFound.marked)
       while (!nodeFound.fullyLinked)
       return false;
     continue;
     int highestLocked = -1;
     try
       Node* pred, succ, prevPred = null;
       bool valid = true;
       for (int i = 0; valid&&
           i<=level;i++)
         pred = preds[i];
         succ = succs[i];
         if (pred != prevPred)
           pred.lock();
           highestLocked = i;
           prevPred = pred;
         valid = !pred.marked && !succ.marked
                && pred.nexts[i]= succ;
       if (!valid ) continue;
       Node* newNode = new Node(v,level);
       for (int j = 0; j <= level; j++)
         newNode.nexts[j] = succs[j];
         preds[j].nexts[layer] = newNode;
       newNode.fullyLinked = true;
       return true;
     finally unlock(preds, highestLocked);


 bool ctn(int v):
 Node* preds[MaxHeight], succs[MaxHeight] ;
   int lFound = findNode ( v,preds,succs);
   return (lFound != -1
            && succs[lFound].fullyLinked
            && !succs[lFound].marked);



 bool okToDelete(Node* candidate,int lFound):
    return (candidate.fullyLinked
          && candidate.topLayer=lFound
          && !candidate.marked);
```

Fig. 14: Optimistic Skiplist.

```
struct Node(int key; int topLayer; Node *next[]; bool marked);

 boolean find(int x, Node* preds[], Node* succs[]):
 int mLevel = 0;
 boolean marked[MAXLEVEL] = false;
 boolean snip;
 Node* pred = null, curr = null, succ = null;
 retry:
   while (true)
     pred = head;
     for (int i = MAXLEVEL; i >= mLevel; i--)
       curr = pred.next[i];
       while (true)
         succ = curr.next[i].get(marked);
         while (marked[0])
         s= CAS(pred.next[i],curr,succ,false,false);
           if (!s) continue retry;
           curr = pred.next[i];
           succ = curr.next[i].get(marked);
         if (curr.key < x)
           pred = curr; curr = succ;
         else
           break;
       preds[i] = pred;
       succs[i] = curr;
     return (curr.key == key);

  bool rmv(int x):
   Node* nodeToDelete = null;
   int mLevel = 0;
   Node* preds[MAXLEVEL+1];
   Node* succs[MAXLEVEL+1];
   Node* succ;
   while (true)
     boolean found = find(x, preds, succs);
     if (!found)
       return false;
     else
       Node* node = succs[mLevel];
       for (int i = node.topLevel; i >= mLevel+1; i--)
         boolean marked[MAXLEVEL+1] = false;
         succ = node.next[i].get(marked);
         while (!marked[0])
           node.next[i].attemptMark(succ, true);
           succ = node.next[i].get(marked);
       boolean marked[MAXLEVEL+1] = false;
       succ = node.next[mLevel].get(marked);
       while (true)
         boolean iMarkedIt =
         CAS(node.next[mLevel],succ, succ, false, true);
         succ = succs[mLevel].next[mLevel].get(marked);
         if (iMarkedIt)
           find(x, preds, succs);
           return true;
         else if (marked[0]) return false;

add(int v):
 int topLevel = randomLevel();
 int mLevel = 0;
 Node* preds[MAXLEVEL+1];
 Node* succs[MAXLEVEL+1];
 while (true)
   boolean found = find(x, preds, succs);
   if (found)
     return false;
   else
     Node* new = new Node(x, topLevel);
     for (int i= mLevel;level<=topLevel;i++)
       Node* succ = succs[i];
       new.next[i].set(succ, false);
     Node* pred = preds[mLevel];
     Node* succ = succs[mLevel];
     new.next[mLevel].set(succ, false);
     if (!CAS(pred.next[mLevel],succ,new,false,false))
       continue;
     for (int i=mLevel+1;i<=topLevel;i++)
       while (true)
         pred = preds[i];
         succ = succs[i];
         if (CAS(pred.next[i],succ,new,false,false))
           break;
         find(x,preds,succs);
     return true;

      bool ctn(int x):
      int bottomLevel = 0;
      bool marked[MAXLEVEL] = false;
      Node* pred = head, curr = null, succ = null;
      for (int i = MAXLEVEL; i >= mLevel; i--)
        curr = pred.next[i];
        while (true)
          succ = curr.next[i].get(marked);
          while (marked[0])
            curr = pred.next[i];
            succ = curr.next[i].get(marked);
          if (curr.key < x)
            pred = curr;
            curr = succ;
          else
            break;
      return (curr.key == v);
```

Fig. 15: Lock-free Skiplist.

```
struct Node(int key; int level; Node *next[]; value value; Timestamp timestamp );

 Node * getLock(node* node1,int key,int level):
 node2 = node1.next[level];
 while (node2.key < key)
   node1 = node2;
   node2 = node1.next[level];
 lock(node1, level);
 node2 = node1.next[level];
 while (node2.key < key);
   unlock(node1, level);
   node1 = node2;
   lock(node1, level);
   node2 = node1.next[level];
 return node1;


  int DeleteMin(value value)):
   time = getTime();
   node1 = head.next[1];
   while (node1 != tail)
     if (node1.timeStamp < time)
       marked = SWAP(node1.deleted, true);
       if (marked = FALSE) break;
       node1 = node1.next[1];
   if (node1 != tail)
     value = node1.value;
     key = node1.key;
     return EMPTY;
   node1 = head;
   for (i = MaxLevel; i > 0; i--)
     node2 = node1.next[i];
     while (node2.key > key)
       node1 = node2;
       node2 = node2.next[i];
     savedNodes[i] = node1;
   node2 = node1;
   while (node2.key != key)
     node2 = node2.next[1];
     lock(node2);
     for (i = node2.level; i > 0; i--)
     node1 = getLock(savedNodes[i], key, i);
     lock(node2, i);
     node1.next[i] = node2.next[i];
     node2.next[i] = node1;
     unlock(node2, i);
   unlock(node1, i);
   unlock(node2);
   return DELETE;

int Insert(key key, value value):
 node1 = head;
 for (int i= MaxLevel; i > 0; i--)
   node2 = node1.next[i];
   while (node2.key > key)
     node1 = node2;
     node2 = node2.next[i];
 savedNodes[i] = node1;
 node1 = getLock(node1, key, 1);
 node2 = node1.next[i];
 if (node2.key = key)
   node2.value = value;
   unlock(node1, 1);
   return UPDATED;
 level = randomLevel();
 newNode = newNode(level, key, value);
 newNode.timeStamp = MAXTIME;
 lock(newNode)
 for (i = 1; i <= level; i++)
   if (i != 1)
     node1 = getLock(savedNodes[i], key, i);
   newNode.next[i] = node1.next[i];
   node1.next[i] = newNode;
   unlock(node1, i);
 unlock(newNode);
 newNode.timeStamp = getTime();
 return INSERTED;
```

Fig. 16: SkiplistBased Concurrent Priority Queues.