BIOS 617 - Lecture 20

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3/25/2020

JITT: “A fundamental identity in statistics” - X.L. Meng

$$
\begin{aligned}
\bar{y}_n - \bar{Y} &= \frac{\mathrm{E}_J(I_J \cdot Y_J)}{E_J(I_J)} - E_J(Y_J) \\
&= \frac{\mathrm{E}_J(I_J \cdot Y_J) - E_J(Y_J)E_J(I_J)}{E_J(I_J)} \\
&= \frac{\mathrm{Cov}_J(I_J, Y_J)}{E_J(Y_J)} \\
&= \frac{\mathrm{Cov}_J(I_J, Y_J)}{\sqrt{V_J(I_J)V_J(Y_J)}} \frac{\sqrt{V_J(I_J)}}{E_J(I_J)} V_J(Y_J) \\
&= \rho_{I,Y} \times \sqrt{\frac{1-f}{f}} \times \sigma_Y
\end{aligned}
$$

What do these factors represent

$$\underbrace{\rho_{I,Y}}_{\text{Data Quality}} \times \underbrace{\sqrt{\frac{1-f}{f}}}_{\text{Data Quantity}} \times \underbrace{\sigma_Y}_{\text{Problem Difficulty}}$$

- *Data quantity* : how much of the population have we observed
- *Problem difficulty*: If Y is a constant across population then $\sigma_Y = 0$ and the problem is super simple. The overall variation in the population tells you how much data is needed
- *Data quality*: Most challenging to assess yet most critical!
  - Measures sign and degree of selection bias caused by the selection mechanism
  - Absence of selection bias (e.g., probabilistic sampling) then $\rho_{I,Y} \approx 0$
  - If larger values of Y have higher/lower chances to be recorded, then $\bar{y}_n$ overestimates/underestimates $\bar{Y}$. Sign and degree of $\rho_{I,Y}$ measure direction and magnitude of these effects respectively.

Applied to SRS

$$\text{MSE}_{\mathbf{I}}(\bar{y}_n) = E_{\mathbf{I}}\left[\rho^2_{I,Y}\right] \times \left(\frac{1-f}{f}\right) \times \sigma^2_G$$

- Under SRS, we had

$$V_{SRS}(\bar{y}_n) = \frac{1-f}{n}\frac{N}{N-1}\sigma^2_Y = \frac{1}{N-1}\frac{1-f}{f}\sigma^2_Y$$

- For SRS, since there's no bias, we have MSE is the variance and so

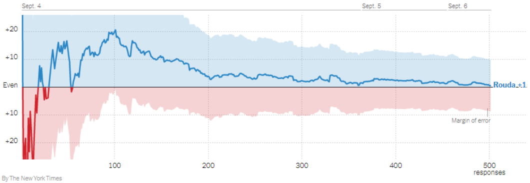
$$E_{SRS}\left[\rho^2_{I,Y}\right] = \frac{1}{N-1}$$

JITT: For Monday's Diary

- What's the implicit assumption in the following from the NYT:
  - *As we reach more people, our poll will become more stable and the margin of sampling error will shrink*

How our poll result changed

As we reach more people, our poll will become more stable and the margin of sampling error will shrink. The changes in the timeline below reflect that sampling error, not real changes in the race.



The margin of sampling error on the overall lead is nine points, roughly twice as large as the margin for a single candidate's vote share. One reason we're doing these surveys live is so you can see the uncertainty for yourself.

Replication methods for variance estimation

An alternative to linearization is replication or resampling: elements of the sample are dropped, a new estimator is computed using the remaining elements of the sample, and the resulting estimates resulting from repeated applications of this process are used to compute a variance estimator.

This approach is an extension of variance estimation under interpenetrated subsamples. In this setting, a sample is drawn and randomly divided into K subsamples reflecting the original sample design. The resulting K estimators $\hat{\theta}_k$ can then be viewed as an SRS from all possible samples under the design the mean $\hat{\theta} = K^{-1} \sum_{k=1}^{K} \hat{\theta}_k$ and estimated variance $v(\hat{\theta}) = K^{-1}(K-1)^{-1} \sum_{k=1}^{K}(\hat{\theta}_k - \hat{\theta})^2$

Balanced repeasted replication (BRR)

- In practice actually creating interpenetrated samples can be practically onerous, especially if stratification and clustering is involved. Hence the practical methods we will discuss do not rely on actual interpenetrated sample.
- BRR is a method that assumes 2 PSUs per stratum (paired selection model).
- In practice this might not be the case, so approximations are made by collapsing/combining strata
  - Ultimate cluster sampling (ignore lower levels of clustering)
  - With-replacement approximations
  - Creating of Sampling Error Computation Units (SECUs)

Example: National Health and Nutrition Examination Survey I (1971-1974)

- Divide the US into 1900 PSUs (counties, groups of counties), combined into 65 strata:
  - 15 strata selected with certainty
  - 50 PSUs sampled from each of the non-certainty strata
  - Problem: have one PSU per stratum
  - For the certainty strata, 1213 second-stage neighborhoods were selected: these are collapsed 20 SECUs, 2 per 10 (new) strata
  - For the non-certainty strata, they are paired to create 50 SECUs in 25 (new) strata
- 1263 PSUs in 65 strata combined and collapsed to 35 strata with 2 SECUs each.

Paired selection design

- Consider estimating a total Y in a paired selection design
- Assume we have appropriate sample selection probabilities/weights so that $y_h = y_{h1} + y_{h2}$ is an unbiased estimator of the populatoin total Y_h in the hth stratum,

$$E(y) = E\left(\sum_{h=1}^{H}(y_{h1} + y_{h2})\right) = Y$$

- Drawing one PSU per stratum α' at random to form a half-sample
- Letting the remaining PSUs α'' to form the other half-sample, yields

$$E(y') = 2E\left(\sum_{h=1}^{H} y_{h\alpha'}\right) = E(y'') = 2E\left(\sum_{h=1}^{H} y_{h\alpha''}\right) = Y$$

Alternative view

This can be viewed as case of interpenetrated subsamples with $K = 2$:

$$
\begin{aligned}
v(y) &= \frac{\sum_{k=1}^{2}(y_k - y)^2}{2 \times 1} \\
&= \frac{(y' - y)^2 + (y'' - y)^2}{2} \\
&= (y' - y)^2 = (y'' - y)^2
\end{aligned}
$$

Alternative view

- This is a very unstable estimator, as it only has a single degree of freedom based on that random split between SECUs within each stratum.
- So let's consider this estimator as a function of the underlying strata.

$$
\begin{aligned}
(y' - y)^2 &= \left(2 \sum_{h=1}^{H} y_{h\alpha'} - \sum_{h=1}^{H} (y_{h\alpha'} + y_{h\alpha''}) \right)^2 = \left(\sum_{h=1}^{H} (y_{h\alpha'} - y_{h\alpha''}) \right)^2 \\
&= \sum_{h=1}^{H} (y_{h\alpha'} - y_{h\alpha''})^2 + 2 \sum_{h=k+1}^{H} \sum_{k=1}^{H} (y_{h\alpha'} - y_{h\alpha''})(y_{k\alpha'} - y_{k\alpha''}) \\
&= \sum_{h=1}^{H} (y_{h1} - y_{h2})^2 + 2 \sum_{h=k+1}^{H} \sum_{k=1}^{H} \xi_h \xi_k (y_{h1} - y_{h2})(y_{k1} - y_{k2})
\end{aligned}
$$

Alternative view

- So let's consider this estimator as a function of the underlying strata.

$$
\begin{aligned}
(y' - y)^2 &= \sum_{h=1}^{H} (y_{h1} - y_{h2})^2 + 2 \sum_{h=k+1}^{H} \sum_{k=1}^{H} \xi_h \xi_k (y_{h1} - y_{h2})(y_{k1} - y_{k2}) \\
&= \sum_{h=1}^{H} d_h^2 + 2 \sum_{h=k+1}^{H} \sum_{k=1}^{H} \xi_h \xi_k d_h d_k
\end{aligned}
$$

where $\xi_h = 1$ if $\alpha' = 1$ and $\xi_h = -1$ if $\alpha' = 2$ in stratum h, and $y_{(h1)} - y_{(h2)} = d_h$.

Alternative view

- Conditional on the sampled clustering in each stratum, ξ_h can be considered as a binary variable with probability 0.5 of taking on 1 and 0.5 of taking on -1
  - So $E(\xi_h \mid i \in s) = 0.5 - 0.5 = 0$
- Since sampling across strata are independent,

$$
\begin{aligned}
E\left[(y' - y)^2\right] &= E\left(\sum_{h=1}^{H} d_h^2\right) + 2E\left(\sum_{h=k+1}^{H}\sum_{k=1}^{H} \xi_h \xi_k d_h d_k\right) \\
&= V(y) + 2E\left[E\left[\sum_{h=k+1}^{H}\sum_{k=1}^{H} \xi_h \xi_k d_h d_k \mid i \in s\right]\right] \\
&= V(y) + 2E\left[\sum_{h=k+1}^{H}\sum_{k=1}^{H} E\left[\xi_h \xi_k \mid i \in s\right] d_h d_k\right] \\
&= V(y) + 2E\left[\sum_{h=k+1}^{H}\sum_{k=1}^{H} \underbrace{E\left[\xi_h \mid i \in s\right]}_{=0} \times \underbrace{E\left[\xi_k \mid i \in s\right]}_{=0} d_h d_k\right]
\end{aligned}
$$

More precise estimator

Thus we can obtain a more precise variance estimator by repeating the process of forming half-samples C times, and averaging over the differences between the half-sample estimator y'_k and the full sample estimator y:

$$v_c(y) = \frac{\sum_{c=1}^{C}(y'_c - y)^2}{C}$$

Half-samples without replacement:

For a specific set of draws of half-samples without replacement,

$$
\begin{aligned}
v_c(y) &= C^{-1} \sum_{c=1}^{C} \left[\sum_{h=1}^{H} d_h^2 + 2 \sum_{h=k+1}^{H} \sum_{k=1}^{H} \xi_{ch} \xi_{ck} d_h d_k \right] \\
&= \sum_{h=1}^{H} d_h^2 + 2 \sum_{h=k+1}^{H} \sum_{k=1}^{H} \left(\sum_{c=1}^{C} \frac{\xi_{ch} \xi_{ck}}{C} \right) d_h d_k
\end{aligned}
$$

- Now $\sum_{c=1}^{C} \frac{\xi_{ch}\xi_{ck}}{C} \to 0$ as C gets large, and when $C = 2^H$ (maximum value), $\sum_{c=1}^{C} \xi_{ch}\xi_{ck} = 0$
- However, it is possible to choose samples in a balanced manner to achieve $\sum_{c=1}^{C} \xi_{ch}\xi_{ck} = 0$ for smaller values of C

Example: 3 strata and 4 replicate samples

Replicate c	ξ_{c1}	ξ_{c2}	ξ_{c3}
1	+1	+1	+1
2	+1	-1	-1
3	-1	-1	+1
4	-1	+1	-1

Can verify that cross-products cancel so that

$$(2(y_{11} + y_{21} + y_{31}) - y)^2 + \cdots (2(y_{12} + y_{21} + y_{32}) - y)^2/4$$

yields $v(y) = \sum_{h=1}^{H} (y_{h1} - y_{h2})^2$

Hadamard matrices

- A feature of this matrix is that columns are **orthogonal**: $\sum_c \xi_{ch}\xi_{ck} = 0$ for all h, k
- Matrices are called *Hadamard* matrices, and methods available to generate $C \times C$ matrices of this form for multiples of 4
  - For a 2-SECU design with H strata, use a Hadamard matrix such that $\min(C : C \geq H), \mathrm{mod}(C, 4) = 0)$
  - Drop extra columns (remainder will still be orthogonal)
  - For example, if there are 70 strata, use a 72×72 Hadamard matrix, dropping columns 71 and 72 for the analysis.

Other option

- Of course, we could just compute $v(y)$ directly
- The value of the replication methods is that we compute the variance for a **general statistic** using this method
- Typically this is done using replication weights
- Going back to our simple example, we will generate 4 replication weights w_{ic}, $c = 1, \dots, 4$ from the sampling weights w_i:
  - $w_{i1} = 2w_i \cdot 1[\text{SECU}_i = 1]$
  - $w_{i2} = 2w_i \cdot 1[\text{SECU}_i = 1, h_i = 1 \text{ or } \text{SECU}_i = 2, h_i = 2, 3]$
  - $w_{i3} = 2w_i \cdot 1[\text{SECU}_i = 1, h_i = 3 \text{ or } \text{SECU}_i = 2, h_i = 1, 2]$
  - $w_{i4} = 2w_i \cdot 1[\text{SECU}_i = 1, h_i = 2 \text{ or } \text{SECU}_i = 2, h_i = 1, 3]$

JITT: Derivation continued