

Geometric Mean Maximization: An Overlooked Portfolio Approach?

Javier Estrada *

IESE Business School, Avenida Pearson 21, 08034 Barcelona, Spain
Tel: +34 93 253 4200, Fax: +34 93 253 4343, Email: jestrada@iese.edu

Abstract

Academics and practitioners usually optimize portfolios on the basis of mean and variance. They set the goal of maximizing risk-adjusted returns measured by the Sharpe ratio and thus determine their optimal exposures to the assets considered. However, there is an alternative criterion that has an equally plausible underlying idea; geometric mean maximization aims to maximize the growth of the capital invested, thus seeking to maximize terminal wealth. This criterion has several attractive properties and is easy to implement, and yet it seems to have taken a back seat to the maximization of risk-adjusted returns. The ultimate goal of this article is to compare both criteria from an empirical perspective. The results reported and discussed leave the question posed in the title largely intact: Are academics and practitioners overlooking a useful portfolio approach?

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1. Introduction

Academics have long advocated to optimize portfolios on the basis of mean and variance. Practitioners followed this recommendation and adopted the maximization of risk-adjusted returns, measured by the Sharpe ratio, as their basic criterion for portfolio selection. Obviously, there is an inherent plausibility in selecting the portfolio that provides the highest (excess) return per unit of volatility risk.

However, there is an alternative criterion that seems to be less popular with academics and practitioners that consists of maximizing the growth of the capital invested, thus maximizing terminal wealth. This criterion, which amounts to maximizing a portfolio's geometric mean return (or mean compound return) in principle appears to be at least just as plausible as the maximization of risk-adjusted returns. One of the main goals of this article, then, is to compare both criteria from an empirical perspective.

Markowitz (1952, 1959) was the first to advocate the focus on mean and variance and the selection of portfolios with the lowest risk (volatility) for a target level of return, or the highest

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return for a target level of risk. Sharpe (1964), Lintner (1965), and Mossin (1966) complemented this insight by arguing that, given a risk-free rate, the optimal combination of risky assets is given by the market (or tangency) portfolio, which is the one that maximizes returns in excess of the risk-free rate per unit of volatility risk.¹ Selecting the portfolio of risky assets that maximizes the Sharpe ratio has been the standard criterion for academics and practitioners ever since.

Investors, however, find risk-adjusted returns more difficult to digest. Few (if any) of them, upon receiving their periodic financial statements, hasten to look at the Sharpe ratio of their investments; rather, they tend to focus on whether or not their invested capital grows and the rate at which it does. Furthermore, fund management companies tend to summarize performance with the mean compound return of their funds. For both reasons, then, a potential plausible goal for portfolio managers to adopt would be to grow the capital entrusted to them at the fastest possible rate; that is, to maximize the geometric mean return of their portfolios.

At least two questions arise naturally from this discussion. First, is the portfolio that grows at the fastest rate the one that yields the highest risk-adjusted returns? As discussed in more detail below, in general, that is not the case. Second, given that the portfolio that maximizes the geometric mean return and the one that maximizes the Sharpe ratio are in general different, which one of the two is more attractive? Providing an empirical perspective on this question is one of the main goals of this article.

Needless to say, theory has a role to play in this debate; however, it does not unambiguously point out to the superiority of either criterion. As discussed in more detail below, in general, neither maximizing the geometric mean return nor maximizing the Sharpe ratio are consistent with expected utility maximization. This fact raises a third interesting question: Which of the conditions that make each criterion consistent with expected utility maximization are more plausible? This issue is also addressed below.

In a nutshell, the main arguments and results in this article can be summarized as follows. First, from a theoretical perspective, it is far from clear that the standard criterion of maximizing the Sharpe ratio is superior to the alternative criterion of maximizing the geometric mean return; in fact, the opposite may actually be the case. Second, the analysis of in-sample optimizations shows that portfolios built with the goal of maximizing the growth of the capital invested are less diversified, have a higher expected return, and higher volatility than those built with the goal of maximizing risk-adjusted returns; this is the case for developed markets, emerging markets, and asset classes.

¹ Credit for this insight is also usually given to Treynor (1961), who never published his article.

Third, the analysis of out-of-sample performance largely confirms the in-sample results but also provides some additional interesting insights; in particular, although portfolios built to maximize the growth of the capital invested tend to achieve their goal, those built to maximize risk-adjusted returns often do not. All in all, the results reported and discussed here suggest that academics and practitioners may be overlooking a compelling portfolio optimization criterion.

The criterion at the heart of this article has been variously referred to in the literature as the Kelly criterion, the growth optimal portfolio, the capital growth theory of investment, the geometric mean strategy, investment for the long run, maximum expected log, and here as *geometric mean maximization* (GMM). The standard criterion accepted by academics and practitioners will be referred to here as Sharpe ratio maximization (SRM). Furthermore, the optimal portfolios that result from the GMM and the SRM criteria will be respectively referred to as the *G* portfolio and the *S* portfolio.

The rest of this article is organized as follows. Section 2 briefly discusses the origins and evolution of the GMM criterion, highlights its most relevant characteristics, and provides a brief review of the relevant literature. Section 3 discusses the implementation of the two optimization criteria considered in this article. Section 4 discusses the evidence, focusing on the characteristics of the in-sample and out-of-sample portfolios generated by both optimization criteria. Section 5 rounds up by discussing the circumstances under which the GMM criterion may be particularly appealing. Finally, section 6 provides an assessment. An appendix at the end of the article contains an example of the Kelly criterion; the derivation of the expression to be maximized when implementing the GMM criterion; a description of the data; an evaluation of different approximations to the geometric mean; and figures depicting the performance of out-of-sample portfolios.

2. The Issue

This section introduces the GMM criterion, highlights its most relevant characteristics, and provides a very brief overview of the literature. An exhaustive literature review is not attempted here; both Christensen (2005) and Poundstone (2005) provide thorough accounts of the origins and evolution of this criterion, the former from a theoretical perspective and the latter through a fascinating and entertaining story.²

² Estrada (2007) provides a review of Poundstone's (2005) book.

2.1. The GMM Criterion in Gambling – Kelly

The origins of the GMM criterion are inevitably intertwined with those of the gambling strategy developed by John Kelly based on the insights of his Bell Labs colleague Claude Shannon.³ Kelly (1956) analyzed the optimal betting strategy of a gambler with private information. He considered a gambler having noisy (rather than certain) private information; making a large number of bets with cumulative effects (reinvesting gains and losses); and betting a fixed (constant) proportion of his capital on each round. In this setting, Kelly asked how much should this gambler bet if he aimed to maximize his expected terminal wealth.⁴

Note that because the gambler's private information is not certain, betting too much would eventually lead to ruin; on the other hand, because his private information is valuable, betting too little would not fully exploit his informational advantage. Hence, there exists an optimal proportion of capital the gambler should risk on each round. This proportion, usually referred to as the Kelly criterion (K), is often expressed as $K = E/O$, where E denotes the 'edge' or expected value of the gamble, and O the 'odds' or potential payoff per \$1 bet. Three examples on Exhibit A1, in section A1 of the appendix, illustrate that wealth at the end of a large number of cumulative bets is maximized (that is, capital grows at the fastest possible rate) when the fraction of capital bet on each round is that derived by Kelly.

There is a vast gambling literature on the Kelly criterion. Although largely irrelevant for the purpose of this article, this literature does highlight some properties of Kelly's betting scheme that are shared by the investing version of the GMM criterion. These properties include that the gambler never risks ruin (his capital can never be totally lost);⁵ terminal wealth is very likely (though not certain) to be higher than with any other strategy; the bets may be very aggressive (when the opportunities are very attractive); the ride may be very bumpy (volatile); and betting more (less) than K increases (decreases) risk and decreases the growth of capital.⁶

³ Shannon is widely acknowledged as having single-handedly created the field of information theory, which deals with the transmission of information over a noisy channel. Shannon's original insight has far-reaching applications, ranging from cryptography to neurobiology, and from DNA sequencing to all forms of communication. Trying to put Shannon's genius into perspective, Poundstone (2005) mentions that many at Bell Labs and MIT compared Shannon to Einstein, and some thought that the comparison was unfair ... *to Shannon*.

⁴ Formally, Kelly stated the goal of maximizing the expected value of the log of terminal wealth, which amounts to maximizing the geometric mean return of the capital invested.

⁵ This property is shared by any strategy that invests a fixed proportion of capital and it implicitly assumes the infinite divisibility of money.

⁶ The aggressiveness and high volatility of Kelly's method are well known and have led to the development of fractional Kelly strategies, such as half-Kelly, that limit the volatility (and growth) of the capital at risk.

2.2. The GMM Criterion in Investing – Latane

At the same time that John Kelly was applying information theory to develop a gambler's optimal betting strategy, Henry Latane was independently considering the problem of an individual making rational choices under uncertainty. More precisely, he was interested in the optimal decision of an individual facing a large number of uncertain and cumulative choices and aiming to maximize his expected terminal wealth.

Latane (1959) argued that Markowitz (1952) had developed a method for determining efficient portfolios but not a way to select one from among them. In fact, in the framework proposed by Markowitz, the choice of a specific portfolio from those in the efficient frontier is subjective. However, Latane was looking for an *objective* criterion.⁷ Therefore, he argued that the optimal choice of a rational individual aiming to maximize his expected wealth at the end of a large number of uncertain and cumulative choices was the strategy with the highest probability of leading to more terminal wealth than any other strategy. He also argued that such strategy would be the one with the highest geometric mean return. Thus, just as Kelly (1956) introduced the GMM criterion to the world of gambling, Latane (1959) introduced it to the world of investing.

Importantly, note that the SRM criterion, based on the static model of Markowitz (1952, 1959), Sharpe (1964), Lintner (1965), and Mossin (1966), is a *one-period* framework. In contrast, the GMM criterion introduced by Kelly (1956) and Latane (1959) is a *multiperiod* framework with *cumulative* results, which is consistent with the way most investors think about their portfolios. This distinction is critical because optimal decisions for a single period may be suboptimal in a multiperiod framework.⁸ Similarly, the relevant variables in a cumulative framework are different from those relevant when gains and losses are not reinvested; the geometric mean is relevant in the first case, and the arithmetic mean in the second.⁹

Finally, note that Latane (1959) admitted that his proposed criterion is less general than maximizing expected utility (but at the same time more operational); that it is not the only rational criterion (but a useful one to deal with a broad range of problems); and, as mentioned above, that it does not apply to either one-time choices or repeated but noncumulative choices.

⁷ In fact, he agreed with Roy (1952), who argued that in “calling a utility function to our aid, an appearance of generality is achieved at the cost of a loss of practical significance and applicability in our results. A man who seeks advice about his actions will not be grateful for the suggestion that he maximize expected utility.”

⁸ As an example consider two investments, one with a 5% certain return, and another with a 50-50 chance of a 200% gain or a 100% loss. Although this second alternative (with an expected value of 50%) may be, at least to some investors, more attractive than the first when making a one-time choice, it is a bad choice for *all* investors in a (long-term) multiperiod framework with reinvestment of gains and losses. This is the case because sooner or later the 100% loss will occur and wipe out all the capital accumulated.

⁹ This had been recognized before Latane (1959). Williams (1936) had previously argued that a speculator who repeatedly risks capital plus profits should focus on the geometric (rather than on the arithmetic) mean. Furthermore, Kelly (1956) had previously argued that a gambler restricted to bet the same absolute amount repeatedly (thus not reinvesting his proceeds) should focus on the arithmetic (rather than on the geometric) mean.

Latane and Tuttle (1967) elaborated further and provided additional support for the GMM criterion.

2.3. Early Supporters – Breiman, Hakansson, and Markowitz

Breiman (1961), citing Kelly (1956) but not Latane (1959), argued that when considering how much to bet on a long sequence of favorable gambles two criteria seemed reasonable: First, given a target level of wealth, to ask what strategy would minimize the time to reach it; and second, given a target holding period, to ask what strategy would maximize expected wealth at the end of that period. He showed that the optimal strategy to attain *both* goals is to maximize the expected value of the log of terminal wealth which, as Kelly (1956) and Latane (1959) had also established, is the same as maximizing the rate at which capital grows, or the geometric mean return.

Hakansson (1971a) emphasized that the only type of preferences that make mean-variance optimization consistent with expected utility maximization are those represented by a quadratic utility function,¹⁰ which has the implausible property of displaying increasing absolute risk aversion. On the contrary, Hakansson argued, the GMM criterion is consistent with expected utility maximization when the underlying utility function is logarithmic, which has the plausible property of displaying decreasing absolute risk aversion; has a longer history than any other utility function; and was derived by Bernoulli (1954) purely on intuitive grounds.¹¹ He also emphasized two desirable properties of the GMM criterion, namely, myopia and solvency. A strategy is myopic if each period can be treated as the last one, thus requiring to look only one period ahead even in a multiperiod horizon;¹² solvency, on the other hand, implies that the strategy never risks ruin. Hakansson (1971b) elaborated further and provided additional support for the GMM criterion.

Although GMM was proposed as an alternative to mean-variance optimization, curiously, one of the strongest early supporters of this alternative criterion was Harry Markowitz. In fact, not only did he allocate the entire chapter VI of his pioneering book (Markowitz, 1959) to ‘Return in the Long Run’ but he also added a ‘Note on Chapter VI’ on a later edition. (More on it below.) Markowitz (1976) subsequently reaffirmed his support for the GMM criterion.

¹⁰ This was originally shown by Tobin (1958).

¹¹ Bernoulli (1954) is the English translation from the original article written in Latin in 1738. This article clearly states that the value of a risky proposition should be evaluated with the geometric mean, thus making Bernoulli the earliest proponent of both the logarithmic utility function and the use of the geometric mean to evaluate uncertain outcomes.

¹² Mossin (1968) was the first to formally show that GMM is a myopic strategy when the underlying utility function is logarithmic.

2.4. Main Challenger – Samuelson

Without a doubt, the most adamant challenger of GMM has been Paul Samuelson. Latane (1959) argued that the strategy with the highest probability of leading to more wealth at the end of a large number of uncertain and cumulative decisions was a plausible choice; he also argued that this would not apply to decisions for a single period.

Samuelson (1963), without citing or referring to Latane (1959) but aiming to “dispel one fallacy of wide currency,” disagreed. He argued that if a rational individual rejected a single favorable bet, he would also reject a large number of such bets.¹³ However, Ross (1999) has shown that rejecting a single favorable bet but accepting a large number of such bets is both common and consistent with expected utility.¹⁴ Importantly, Samuelson’s argument defines rationality as the maximization of expected utility. However, as mentioned above, Latane (1959) did admit that his criterion was less general than maximizing expected utility and that it was not the only rational criterion;¹⁵ he actually proposed it as an *alternative* criterion.

Samuelson (1971) admits as an obvious truth that aiming to maximize the geometric mean return would almost certainly lead to the maximization of terminal wealth and utility if the period considered is sufficiently long. But he warned against believing in a false corollary, namely, that such a strategy would maximize expected utility unless the underlying utility function were logarithmic. This argument is mathematically irrefutable and universally accepted; those who disagree with Samuelson do it on different grounds. Perhaps surprisingly given his strong opposition to the criterion, Samuelson (1971) concludes by admitting that GMM “still avoids some of the even greater arbitrariness of conventional mean-variance analysis.”¹⁶

In ‘Note on Chapter VI,’ on a later edition of his 1959 pioneering book, Markowitz tried to reconcile the arguments of Kelly (1956), Latane (1959), Breiman (1961), and his own with those of Samuelson (1971). He concluded that these two seemingly opposing positions can be true at the same time: Samuelson is correct in pointing out that maximizing the geometric mean return is not necessarily consistent with maximizing expected utility; and Kelly-Latane-Breiman-

¹³ As an example, he tells the story of a colleague who rejected a favorable bet (a 50-50 chance of winning \$200 or losing \$100) but would have accepted 100 of such bets. Samuelson wrote his paper to prove that such behavior is irrational.

¹⁴ Samuelson (1984) himself constructed an example of a utility function for which it is rational to reject a favorable bet and accept a sequence of such bets. Vivian (2003) discusses other reasons why it may be plausible to behave this way. See, also, Hakansson (1974) and De Brouwer and Van den Spiegel (2001).

¹⁵ In fact, he even admitted that for some utility functions no amount of repetition justifies choosing the strategy most likely to maximize expected terminal wealth; see Latane (1959), footnote 3.

¹⁶ Latane and Samuelson kept disagreeing on the GMM criterion in several other articles; see, for example, Merton and Samuelson (1974), Latane (1978, 1979), and Samuelson (1979). See, also, Ophir (1978, 1979).

Markowitz are correct in pointing out that, despite Samuelson's irrefutable argument, GMM may still be a plausible and useful criterion.¹⁷

2.5. Empirical Literature

By the mid 1970s the main theoretical arguments in favor of and against the GMM criterion had been largely established. Research on the topic from that point on, some of which has been cited above and some that is cited below, largely refined or elaborated on previous arguments. Table 1 in MacLean, Ziembra, and Blazenko (1992) summarizes many desirable and undesirable analytical properties of the GMM criterion and provides references for each of those properties. McEnally (1986) provides a good overview of this criterion and some of the controversies surrounding it.

Empirical research on the GMM criterion has been rather scarce, which is precisely one of the voids this article attempts to fill. The earliest empirical contribution appears to be that of Roll (1973), who derives from theory some testable implications of this criterion; tests them using a sample of NYSE and AMEX stocks; and finds that the G portfolio is statistically indistinguishable from the market portfolio.

Similarly, Fama and McBeth (1974), using a sample of NYSE stocks, cannot reject the hypothesis that the market portfolio is growth optimal. For several time periods they estimate the G portfolio, compare it to the market portfolio, and conclude that although the G portfolios are *statistically* indistinguishable from the market portfolios, the *economic* differences between them are substantial. In particular, they find that the G portfolios have much higher (geometric mean) return and (beta) risk than the market portfolios.

Grauer (1981) focuses on comparing the asset mix selected by the GMM and SRM criteria. Using a sample of 20 Dow stocks and 20 NYSE portfolios, he runs over 200 optimizations and finds that these two criteria rarely select the same asset mix; that neither criterion produces highly diversified portfolios; that G portfolios are less diversified than S portfolios; and that G portfolios have higher expected return and volatility than S portfolios. In contrast, using 10 samples of 25 US stocks, Pulley (1983) finds that portfolios selected by the GMM and SRM criteria are rather similar.

¹⁷ He stresses that the recommendation of his book is not that investors should maximize the geometric mean return. Rather, he recommends not to select a portfolio with higher arithmetic mean and variance than the combination that maximizes the geometric mean (although he admits that some investors may prefer portfolios with a lower arithmetic mean, thus sacrificing long-term growth in exchange for short-term stability). Markowitz (1976) reaffirms this recommendation.

Rotando and Thorp (1992) focus on selecting the strategy that maximizes real growth by investing in the S&P 500. They find that such strategy is achieved with a position 117% long the index and 17% short the risk-free rate.

Finally, Hunt (2005a, 2005b) using a sample of 25 Australian stocks in the first case and the 30 stocks in the Dow in the second case finds that unconstrained G portfolios are largely unattainable, implying short positions in excess of 1000% in some cases. He also finds that long-only G portfolios are less diversified, have over twice the geometric mean return, and almost twice the volatility than his three benchmark portfolios (the equally-weighted portfolio, the minimum variance portfolio, and the portfolio with minimum risk for a target return of 15%).

3. Methodology

This section discusses the two optimization criteria considered in this article.¹⁸ Standard modern portfolio theory establishes that the expected return (μ_p) and variance (σ_p^2) of a portfolio are given by

$$\mu_p = \sum_{i=1}^n x_i \mu_i , \quad (1)$$

$$\sigma_p^2 = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij} , \quad (2)$$

where x_i denotes the proportion of the portfolio invested in asset i ; μ_i the expected return of asset i ; σ_{ij} the covariance between assets i and j ; and n the number of assets in the portfolio.

Maximizing risk-adjusted returns when risk is measured by volatility amounts to maximizing a portfolio's Sharpe ratio (SR_p). This problem is formally given by

$$\text{Max}_{x_1, x_2, \dots, x_n} SR_p = \frac{\mu_p - R_f}{\sigma_p} = \frac{\sum_{i=1}^n x_i \mu_i - R_f}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}}} \quad (3)$$

$$\text{Subject to } \sum_{i=1}^n x_i = 1 \quad \text{and} \quad x_i \geq 0 \text{ for all } i , \quad (4)$$

where R_f denotes the risk-free rate and $x_i \geq 0$ the no short-selling constraint. This is the formal expression of the criterion referred to in this article as SRM. The solution of this problem is well known and available from a wide variety of optimization packages.

¹⁸ GMM and SRM are obviously not the only two portfolio optimization criteria proposed by academics and practitioners. To illustrate, Bekaert et al (1998) propose to consider higher moments in the optimization process; Adler and Kritzman (2006) introduce full-scale optimization, which considers not only some moments but the whole distribution of returns; and Estrada (2008) introduces a simple method to optimize portfolios on the basis of mean and semivariance.

The maximization of a portfolio's geometric mean return can be implemented in more than one way. Ziembra (1972), Elton and Gruber (1974), Weide, Peterson, and Maier (1977), and Bernstein and Wilkinson (1997) all propose different algorithms to solve this problem. The method proposed here is easy to implement numerically and requires the same inputs as those needed to maximize a portfolio's Sharpe ratio. As shown in section A2 of the appendix, maximizing a portfolio's geometric mean return (GM_p) amounts to solving the problem formally given by

$$\begin{aligned} \text{Max}_{x_1, x_2, \dots, x_n} \quad GM_p &\approx \exp \left\{ \ln(1 + \mu_p) - \frac{\sigma_p^2}{2(1 + \mu_p)^2} \right\} - 1 = \\ &\approx \exp \left\{ \ln(1 + \sum_{i=1}^n x_i \mu_i) - \frac{\sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}}{2(1 + \sum_{i=1}^n x_i \mu_i)^2} \right\} - 1 \end{aligned} \quad (5)$$

$$\text{Subject to } \sum_{i=1}^n x_i = 1 \quad \text{and} \quad x_i \geq 0 \quad \text{for all } i. \quad (6)$$

This is the formal expression of the criterion referred to in this article as GMM. Note that maximizing (5) is obviously the same as maximizing the expression inside the brackets. In fact, Markowitz (1959) suggests to approximate the geometric mean of an asset precisely with the expression $\{\ln(1+\mu)-\sigma^2/[2(1+\mu)^2]\}$. The results reported on Exhibit A3, in section A4 of the appendix, show that the approximate geometric mean as defined in (5) is indeed a very good approximation to the exact geometric mean; those results also show that there is virtually no gain from including higher moments in the approximation.

Finally, note that expression (5) highlights an important fact about the role that volatility plays in the GMM framework. In the SRM framework, volatility is undesirable because it is synonymous with risk; in the GMM framework, in turn, volatility is also undesirable but for a different reason, namely, *because it lowers the geometric mean return*. In other words, in the GMM framework volatility is detrimental because it lowers the rate of growth of the capital invested, thus ultimately lowering the expected terminal wealth.

4. Evidence

This section compares the two optimization criteria considered in this article, GMM and SRM, from an empirical perspective. The focus is, first, on comparing the in-sample characteristics of the optimal portfolios determined by each criterion; then, on considering the implications of levering the S portfolio to match some of the characteristics of the G portfolio; and finally, on assessing the out-of-sample performance of the G and S portfolios.

The sample consists of monthly returns for the entire MSCI database of 22 developed markets and 26 emerging markets, as well as monthly returns for five asset classes, namely, US stocks, EAFE stocks, emerging market stocks, US bonds, and US real estate. All returns are in dollars and account for both capital gains and dividends. The sample period varies across assets and goes from the inception of each asset into the MSCI database through Jun/2008 in all cases. Exhibit A2, in section A3 of the appendix, describes the data in detail.

4.1. In-Sample Optimal Portfolios

Perhaps the obvious starting point of an empirical comparison of GMM and SRM is to first to consider the characteristics of the portfolios generated by each criterion. In order to gain a broad perspective, G and S portfolios are obtained for developed markets, emerging markets, and asset classes. Furthermore, in order to avoid drawing conclusions possibly biased by the conditions at a single point in time, G and S portfolios are obtained at three points in time, Jun/2008, Jun/2003, and Jun/1998. Exhibits 1 through 3 report the relevant results. In all cases S portfolios follow from expressions (3)-(4) and G portfolios from expressions (5)-(6); also, in all cases, the inputs of the optimization problems (expected returns, variances, and covariances) are calculated using all the data available at the time of each estimation.

Exhibit 1 focuses on developed markets; panel A shows the composition of all portfolios and panel B summarizes their characteristics. As the exhibit shows, the portfolios generated by the GMM criterion are substantially less diversified than those generated by the SRM criterion; at all three points in time the S portfolios contain at least twice as many assets (developed markets) as the G portfolios. And, as the exhibit also shows, the lower diversification of the G portfolios makes them much more volatile than the S portfolios.

By design, G portfolios have a higher geometric mean return and lower Sharpe ratios than S portfolios, and both characteristics are reflected in the figures displayed on Exhibit 1. The higher expected growth of G portfolios translates into substantially higher levels of expected terminal wealth. The last two lines of the exhibit show the expected terminal value of \$100 invested at the geometric mean return of each portfolio after 10 and 20 years. As the figures clearly show, the differences are substantial; the expected terminal values of G portfolios (relative to S portfolios) are at least 20% higher for a 10-year holding period and at least 40% higher for a 20-year holding period.

Exhibit 1: In-Sample Optimal Portfolios – Developed Markets

This exhibit shows optimal portfolios of developed markets and some of their characteristics. The optimizations are performed in Jun/2008, Jun/2003, and Jun/1998 based on all the data available at each point in time. S portfolios are obtained from expressions (3)-(4) and G portfolios from expressions (5)-(6). Panel A shows the weight of each country in the optimal portfolios and panel B shows some of their characteristics, including the number of markets in each portfolio (n), arithmetic mean return (μ_p), volatility (σ_p), Sharpe ratio (SR_p), geometric mean return (GM_p), and the terminal value of \$100 invested at GM_p after 10 years (TV10) and 20 years (TV20). Mean returns, volatility, and Sharpe ratios in panel B are monthly magnitudes. The monthly risk-free rates used are 0.33% (Jun/2008), 0.29% (Jun/2003), and 0.44% (Jun/1998). The data is described on Exhibit A2 in section A3 of the appendix.

	Jun/2008		Jun/2003		Jun/1998	
	S	G	S	G	S	G
<u>Panel A: Portfolio Weights</u>						
Australia	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Austria	13.8%	0.0%	7.7%	0.0%	0.0%	0.0%
Belgium	10.6%	0.0%	16.7%	2.2%	17.9%	0.0%
Canada	3.6%	0.0%	0.0%	0.0%	0.0%	0.0%
Denmark	33.0%	12.3%	22.3%	0.0%	11.4%	0.0%
Finland	0.6%	9.2%	0.0%	12.4%	2.2%	0.0%
France	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Germany	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Hong Kong	11.8%	56.1%	11.9%	60.7%	7.1%	49.8%
Ireland	0.0%	0.0%	0.0%	0.0%	18.4%	10.8%
Italy	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Japan	3.7%	0.0%	5.3%	0.0%	0.0%	0.0%
Netherlands	3.2%	0.0%	3.9%	0.0%	20.2%	8.3%
New Zealand	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Norway	1.0%	5.5%	0.0%	0.0%	0.0%	0.0%
Portugal	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Singapore	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Spain	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Sweden	12.4%	16.9%	12.8%	24.7%	10.7%	31.1%
Switzerland	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
UK	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
USA	6.2%	0.0%	19.4%	0.0%	12.0%	0.0%
<u>Panel B: Portfolio Characteristics</u>						
n	11	5	8	4	8	4
μ_p	1.3%	1.6%	1.2%	1.6%	1.4%	1.8%
σ_p	4.4%	7.1%	4.3%	8.0%	4.1%	7.0%
SR_p	0.220	0.179	0.208	0.166	0.242	0.187
GM_p	1.2%	1.4%	1.1%	1.3%	1.4%	1.5%
Annualized σ_p	15.1%	24.8%	15.0%	27.7%	14.3%	24.2%
Annualized GM_p	15.2%	17.5%	14.0%	16.8%	17.6%	19.7%
TV10	\$413	\$502	\$370	\$473	\$505	\$604
TV20	\$1,709	\$2,516	\$1,372	\$2,240	\$2,549	\$3,653

The results for developed markets in Exhibit 1 are confirmed and strengthened by those for emerging markets in Exhibit 2. As this exhibit shows, G portfolios (relative to S portfolios) are less diversified, more volatile, and have lower Sharpe ratios; at the same time, they have higher expected (arithmetic and geometric) return and expected terminal values.

Exhibit 2: In-Sample Optimal Portfolios – Emerging Markets

This exhibit shows optimal portfolios of emerging markets and some of their characteristics. The optimizations are performed in Jun/2008, Jun/2003, and Jun/1998 based on all the data available at each point in time. S portfolios are obtained from expressions (3)-(4) and G portfolios from expressions (5)-(6). Panel A shows the weight of each country in the optimal portfolios and panel B shows some of their characteristics, including the number of markets in each portfolio (n), arithmetic mean return (μ_p), volatility (σ_p), Sharpe ratio (SR_p), geometric mean return (GM_p), and the terminal value of \$100 invested at GM_p after 10 years (TV10) and 20 years (TV20). Mean returns, volatility, and Sharpe ratios in panel B are monthly magnitudes. The monthly risk-free rates used are 0.33% (Jun/2008), 0.29% (Jun/2003), and 0.44% (Jun/1998). The data is described on Exhibit A2 in section A3 of the appendix.

	Jun/2008		Jun/2003		Jun/1998	
	S	G	S	G	S	G
<u>Panel A: Portfolio Weights</u>						
Argentina	1.9%	18.4%	3.6%	23.0%	2.1%	24.5%
Brazil	3.4%	30.4%	5.1%	19.6%	2.8%	14.3%
Chile	1.7%	0.0%	14.8%	0.0%	4.9%	0.0%
China	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Colombia	2.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Czech Republic	11.1%	0.0%	0.0%	0.0%	0.0%	0.0%
Egypt	11.7%	21.5%	0.0%	0.0%	0.0%	0.0%
Hungary	0.0%	0.0%	0.0%	0.0%	11.4%	34.8%
India	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Indonesia	0.0%	0.0%	1.0%	0.0%	0.0%	0.0%
Israel	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Jordan	8.2%	0.0%	0.0%	0.0%	0.0%	0.0%
Korea	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Malaysia	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Mexico	12.7%	0.0%	21.1%	2.4%	8.4%	0.0%
Morocco	42.3%	0.0%	47.8%	0.0%	69.7%	0.0%
Pakistan	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Peru	2.1%	0.0%	0.0%	0.0%	0.0%	0.0%
Philippines	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Poland	0.1%	0.0%	2.3%	15.1%	0.8%	26.4%
Russia	2.8%	29.7%	1.1%	39.9%	0.0%	0.0%
South Africa	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Sri Lanka	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Taiwan	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Thailand	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Turkey	0.0%	0.0%	3.1%	0.0%	0.0%	0.0%
<u>Panel B: Portfolio Characteristics</u>						
n	12	4	9	5	7	4
μ_p	1.9%	2.9%	1.5%	2.9%	2.7%	3.8%
σ_p	4.0%	9.9%	4.3%	12.6%	3.3%	10.9%
SR_p	0.388	0.263	0.274	0.211	0.681	0.311
GM_p	1.8%	2.5%	1.4%	2.2%	2.7%	3.3%
Annualized σ_p	13.7%	34.3%	15.0%	43.7%	11.5%	37.7%
Annualized GM_p	23.7%	33.8%	17.9%	29.5%	37.0%	46.9%
TV10	\$840	\$1,837	\$520	\$1,331	\$2,333	\$4,673
TV20	\$7,062	\$33,733	\$2,705	\$17,705	\$54,428	\$218,397

The main difference between Exhibits 1 and 2 is simply one of degree; that is, the differences between G portfolios and S portfolios are amplified in the case of emerging markets relative to those discussed before for developed markets. To illustrate, in the case of emerging markets, G portfolios have (relative to S portfolios) at least twice the volatility, at least twice the

expected terminal wealth over 10 years, and at least four times the expected terminal wealth over 20 years. As shown in Exhibit 1, these differences are substantially lower for developed markets.

Finally, Exhibit 3 shows the results for portfolios of asset classes, and again the results confirm and strengthen those of the previous two exhibits. One of the interesting results of Exhibit 3 is the extreme concentration of G portfolios, which contain only one asset class in Jun/2008 and Jun/1998 and two in Jun/2003; S portfolios, in contrast, contain four asset classes in all cases. The rest of the relative characteristics of G and S portfolios in this exhibit are quantitatively different, but qualitatively the same, as those discussed before for developed and emerging markets.

Exhibit 3: In-Sample Optimal Portfolios – Asset Classes

This exhibit shows optimal portfolios of asset classes and some of their characteristics. The optimizations are performed in Jun/2008, Jun/2003, and Jun/1998 based on all the data available at each point in time. S portfolios are obtained from expressions (3)-(4) and G portfolios from expressions (5)-(6). Panel A shows the weight of each asset class in the optimal portfolios and panel B shows some of their characteristics, including the number of asset classes in each portfolio (n), arithmetic mean return (μ_p), volatility (σ_p), Sharpe ratio (SR_p), geometric mean return (GM_p), and the terminal value of \$100 invested at GM_p after 10 years (TV10) and 20 years (TV20). Mean returns, volatility, and Sharpe ratios in panel B are monthly magnitudes. The monthly risk-free rates used are 0.33% (Jun/2008), 0.29% (Jun/2003), and 0.44% (Jun/1998). The data is described on Exhibit A2 in section A3 of the appendix.

	Jun/2008		Jun/2003		Jun/1998	
	S	G	S	G	S	G
<i>Panel A: Portfolio Weights</i>						
US Stocks	4.8%	0.0%	9.3%	77.4%	75.7%	100.0%
EAFE Stocks	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
EM Stocks	16.9%	100.0%	7.7%	22.6%	4.0%	0.0%
US Bonds	64.8%	0.0%	68.7%	0.0%	16.6%	0.0%
US Real Estate	13.5%	0.0%	14.3%	0.0%	3.7%	0.0%
<i>Panel B: Portfolio Characteristics</i>						
n	4	1	4	2	4	1
μ_p	0.8%	1.4%	0.8%	1.1%	1.4%	1.5%
σ_p	1.9%	6.6%	1.8%	4.4%	3.0%	3.5%
SR_p	0.267	0.165	0.299	0.174	0.321	0.318
GM_p	0.8%	1.2%	0.8%	1.0%	1.3%	1.5%
Annualized σ_p	6.6%	22.8%	6.3%	15.2%	10.3%	12.0%
Annualized GM_p	10.3%	15.3%	10.3%	12.1%	17.5%	19.3%
TV10	\$266	\$416	\$266	\$314	\$500	\$586
TV20	\$707	\$1,732	\$705	\$985	\$2,496	\$3,437

The results in Exhibits 1-3 are in general consistent with those previously reported in the few articles that explore the empirical characteristics of G portfolios. In particular, they are consistent with the results in Grauer (1981), who reports that G and S portfolios usually have a substantially different asset mix. They are also consistent with the results in Hunt (2005a, 2005b), who reports that G portfolios are less diversified, have a much higher return, and much higher volatility than S portfolios. And they are useful to highlight some of the advantages and disadvantages of G portfolios relative to those of S portfolios.

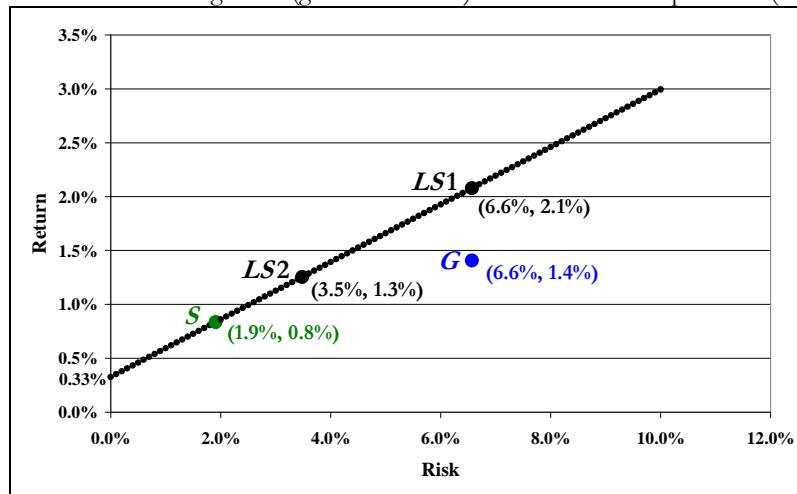
4.2. Leverage

The results in the previous section show that, as expected by design, G portfolios exhibit higher expected growth and terminal wealth than S portfolios. Still, however desirable these two characteristics may be, at least two arguments may be raised against adopting the GMM criterion. First, it may be argued that it is possible to invest in a levered S portfolio with the same level of risk and higher expected return than the G portfolio. Second, it may be argued that it is possible to invest in a levered S portfolio with lower risk and the same expected growth than the G portfolio. Both arguments are considered in this section.

To illustrate both issues consider Exhibit 4, which focuses on asset classes in Jun/2008. The securities market line depicted has an intercept of 0.33% (the risk-free rate) and a slope of 0.267 (the Sharpe ratio). As shown in Exhibit 3, the S portfolio has an expected return of 0.8% and a volatility of 1.9%; the G portfolio, in turn, has an expected return of 1.4% and a volatility of 6.6%. These two portfolios are emphasized in Exhibit 4 and their (arithmetic and geometric) mean return, volatility, and Sharpe ratio are reported again in Exhibit 5.

Exhibit 4: Leverage – Asset Classes – Jun/2008

This figure shows the securities market line for asset classes in Jun/2008; its intercept is given by the risk-free rate (0.33%) and its slope by the Sharpe ratio (0.267). It also shows the risk and return of the portfolios that result from maximizing the Sharpe ratio (S) and the geometric mean return (G), as well as those of a levered portfolio that results from levering S to obtain the same level of risk than that of the G portfolio ($LS1$), and another levered portfolio that results from levering S to obtain the same growth (geometric mean) than that of the G portfolio ($LS2$).



Consider first the $LS1$ portfolio, which is a levered S portfolio designed to match the volatility of the G portfolio. It could be argued that $LS1$ dominates G because, at the same level of risk, it has a higher expected return (which in turn implies, as the figures in Exhibit 5 confirm, that it also has a higher Sharpe ratio and higher expected growth). However, as the last line of Exhibit 5 shows, $LS1$ would require going long 344% the S portfolio (and short 244% the risk-free rate), a level of leverage nearly impossible to obtain for many investors. In other words,

investors that can bear the volatility of the G portfolio will find that $LS1$ is a better choice; but they may also find that this better choice is not attainable.

Consider now the $LS2$ portfolio, which is a levered S portfolio designed to match the growth (geometric mean return) of the G portfolio. On the positive side, $LS2$ has lower volatility than G ; on the negative side, $LS2$ has a lower expected return than G and, as Exhibit 5 shows, also requires a substantial amount of leverage. More precisely, it requires going long 182.3% the S portfolio (and short 82.3% the risk-free rate), a level of leverage much lower than that required to implement $LS1$, but still very high for many investors.

Exhibit 5: Leverage

This exhibit shows the arithmetic mean return (μ_p), geometric mean return (GM_p), volatility (σ_p), and Sharpe ratio (SR_p) of S and G portfolios, all taken from Exhibit 1 for developed markets (DMs); from Exhibit 2 for emerging markets (EMs); and from Exhibit 3 for asset classes (ACs). It also shows the arithmetic mean return, geometric mean return, volatility, and Sharpe ratio of two levered portfolios that result from a short position in the risk-free rate and a long position (x_S) in the S portfolio; one levered portfolio ($LS1$) is designed to match the volatility of the G portfolio, and the other ($LS2$) to match the growth (geometric mean) of the G portfolio. Mean returns, volatility, and Sharpe ratios are monthly magnitudes. The monthly risk-free rates used are 0.33% (Jun/2008), 0.29% (Jun/2003), and 0.44% (Jun/1998). The data is described on Exhibit A2 in section A3 of the appendix.

	Jun/2008				Jun/2003				Jun/1998			
	S	G	$LS1$	$LS2$	S	G	$LS1$	$LS2$	S	G	$LS1$	$LS2$
<i>DMs</i>												
μ_p	1.3%	1.6%	1.9%	1.5%	1.2%	1.6%	2.0%	1.5%	1.4%	1.8%	2.1%	1.6%
σ_p	4.4%	7.1%	7.1%	5.3%	4.3%	8.0%	8.0%	5.6%	4.1%	7.0%	7.0%	4.9%
SR_p	0.220	0.179	0.220	0.220	0.208	0.166	0.208	0.208	0.242	0.187	0.242	0.242
GM_p	1.2%	1.4%	1.6%	1.4%	1.1%	1.3%	1.6%	1.3%	1.4%	1.5%	1.9%	1.5%
x_S		164.0%	121.7%			185.5%	130.0%			168.8%	118.6%	
<i>EMs</i>												
μ_p	1.9%	2.9%	4.2%	2.6%	1.5%	2.9%	3.7%	2.5%	2.7%	3.8%	7.9%	3.3%
σ_p	4.0%	9.9%	9.9%	5.9%	4.3%	12.6%	12.6%	8.0%	3.3%	10.9%	10.9%	4.3%
SR_p	0.388	0.263	0.388	0.388	0.274	0.211	0.274	0.274	0.681	0.311	0.681	0.681
GM_p	1.8%	2.5%	3.7%	2.5%	1.4%	2.2%	3.0%	2.2%	2.7%	3.3%	7.3%	3.3%
x_S		249.3%	149.3%			291.6%	186.1%			326.8%	127.7%	
<i>ACs</i>												
μ_p	0.8%	1.4%	2.1%	1.3%	0.8%	1.1%	1.6%	1.0%	1.4%	1.5%	1.6%	1.5%
σ_p	1.9%	6.6%	6.6%	3.5%	1.8%	4.4%	4.4%	2.3%	3.0%	3.5%	3.5%	3.4%
SR_p	0.267	0.165	0.267	0.267	0.299	0.174	0.299	0.299	0.321	0.318	0.321	0.321
GM_p	0.8%	1.2%	1.9%	1.2%	0.8%	1.0%	1.5%	1.0%	1.3%	1.5%	1.5%	1.5%
x_S		344.0%	182.3%			242.0%	127.8%			116.7%	115.8%	

Exhibit 5 shows for developed markets, emerging markets, and asset classes, as well as for Jun/2008, Jun/2003, and Jun/1998, the (arithmetic and geometric) mean return, volatility, and Sharpe ratio of all S , G , $LS1$, and $LS2$ portfolios. Importantly, it also shows the size of the position that must be taken in S (x_S) to implement the levered portfolios. In some cases the size of this position is moderate (lower than 120–130%) and in some cases very high (over 300%). On average across all assets (developed markets, emerging markets, and asset classes) and all dates (Jun/2008, Jun/2003, and Jun/1998) considered, $LS1$ requires going long 232.1% the S

portfolio (and short 132.1% the risk-free rate); the corresponding number for $LS2$ is 139.9% (short 39.9% the risk-free rate).

These results can be interpreted in a variety of ways. It can be argued that the possibility of leverage renders GMM irrelevant because G portfolios can be dominated in one or more dimensions by levered versions of the S portfolio. Investors that can tolerate the volatility of a G portfolio would prefer to invest in a levered S portfolio with the same volatility but a higher expected return and growth than those of the G portfolio. However, the results discussed show that the leverage required may be unattainable (or undesirable) to many investors.

On the other hand, investors that desire to attain the growth and terminal wealth of a G portfolio may prefer to invest in a levered S portfolio with the same expected growth but lower volatility than the G portfolio. Unlike the previous case, this would require a more moderate amount of leverage. However, this would not render GMM irrelevant; it may still be appealing to investors that cannot or do not want to use leverage and to long-only mutual funds.

4.3. Out-of-Sample Performance

The (arithmetic and geometric) mean return, volatility, Sharpe ratio, and terminal wealth of the portfolios discussed in section 4.1 are all *expected* magnitudes. In other words, they are the characteristics expected from each portfolio given the historical performance of the assets included in them. However, it would be useful to explore the *actual* behavior of the portfolios selected by the GMM and SRM criteria; that is the issue addressed in this section. Exhibit 6 summarizes the relevant results of the analysis and Figures A1 through A6, in section A5 of the appendix, complement this exhibit.

Panel A of Exhibit 6 summarizes the performance of a \$100 investment in the optimal portfolios determined in Jun/1998, passively held through the end of Jun/2008. Panel B, in turn, summarizes the performance of a \$100 investment in the optimal portfolios determined in Jun/1998; passively held through Jun/2003; rebalanced to the optimal portfolios determined at that point in time; and passively held through the end of Jun/2008. This rebalancing half way into the holding period, as the exhibit shows, does not affect the qualitative results substantially.

As expected given the results of the previous in-sample analysis, G portfolios have higher risk than S portfolios regardless of whether risk is measured by the standard deviation, the semideviation, beta, or the minimum monthly return. This result applies to all the assets considered, namely, developed markets, emerging markets, and asset classes.

Exhibit 6: Out-of-Sample Performance

This exhibit shows, for developed markets, emerging markets, and asset classes, the out-of-sample performance of ex-ante optimal portfolios defined as those that maximize the Sharpe ratio (S) according to expressions (3)-(4) or mean compound return (G) according to expressions (5)-(6). Panel A summarizes the performance of \$100 invested in the optimal portfolios formed in Jun/1998, described in exhibits 1-3, and passively held through Jun/2008. Panel B shows the performance of \$100 invested in the optimal portfolios formed in Jun/1998, described in exhibits 1-3; passively held through Jun/2003; rebalanced to the optimal portfolios formed in Jun/2003, described in exhibits 1-3; and passively held through Jun/2008. Performance measures include the arithmetic mean return (μ_p), geometric mean return (GM_p), volatility (σ_p), semideviation with respect to 0 (Σ_p), beta with respect to the world market (β_p), lowest (Min) and highest (Max) return, all expressed in monthly magnitudes, as well as the terminal value of the \$100 investment (TV). The risk-free rate (R_f) used for the calculation of $(\mu_p - R_f)/\sigma_p$ is the monthly average over the Jun/1998-Jun/2008 period (0.39%). The data is described in Exhibit A2 in section A3 of the appendix.

	Developed Markets		Emerging Markets		Asset Classes	
	S	G	S	G	S	G
<u>Panel A: No Rebalancing</u>						
μ_p	0.6%	0.9%	1.3%	1.6%	0.4%	0.3%
GM_p	0.5%	0.8%	1.2%	1.2%	0.4%	0.2%
σ_p	4.9%	6.0%	4.4%	8.2%	3.4%	4.3%
Σ_p	3.4%	3.8%	2.5%	5.5%	2.3%	3.1%
β_p	1.1	1.2	0.6	1.4	0.8	1.0
Min	-13.8%	-15.2%	-11.0%	-35.8%	-11.4%	-13.9%
Max	12.4%	21.2%	16.8%	21.8%	8.4%	10.0%
$(\mu_p - R_f)/\sigma_p$	0.053	0.093	0.199	0.143	0.012	-0.014
μ_p/Σ_p	0.187	0.249	0.503	0.284	0.185	0.107
Annualized σ_p	16.8%	20.7%	15.2%	28.4%	11.7%	15.0%
Annualized GM_p	6.5%	9.6%	14.9%	15.5%	4.6%	2.8%
TV	\$188	\$250	\$402	\$422	\$156	\$132
<u>Panel B: With Rebalancing</u>						
μ_p	0.7%	1.0%	1.3%	1.6%	0.4%	0.6%
GM_p	0.6%	0.8%	1.2%	1.3%	0.4%	0.5%
σ_p	4.9%	6.0%	4.4%	8.3%	3.3%	4.6%
Σ_p	3.4%	3.8%	2.5%	5.5%	2.3%	3.2%
β_p	1.1	1.2	0.6	1.4	0.7	1.0
Min	-13.8%	-15.2%	-12.0%	-35.8%	-11.4%	-13.9%
Max	12.4%	21.2%	15.8%	19.8%	8.4%	10.0%
$(\mu_p - R_f)/\sigma_p$	0.072	0.095	0.198	0.149	0.009	0.044
μ_p/Σ_p	0.216	0.250	0.494	0.296	0.184	0.186
Annualized σ_p	16.8%	20.9%	15.2%	28.6%	11.3%	15.9%
Annualized GM_p	7.7%	9.8%	14.9%	16.3%	4.5%	6.0%
TV	\$209	\$254	\$400	\$452	\$155	\$179

The higher volatility of G portfolios, however, is in some cases more than compensated by higher returns. Although S portfolios are designed to produce higher Sharpe ratios than G (and all other) portfolios, this is not achieved out of sample in three of the six cases considered, namely, developed markets with and without rebalancing and asset classes with rebalancing. In these three cases, G portfolios outperform S portfolios in terms of risk-adjusted returns when risk is measured by both the standard deviation and the semideviation. In other words, G

portfolios outperform S portfolios out of sample on the basis of both Sharpe ratios and Sortino ratios.¹⁹

On the other hand, G portfolios, designed to maximize growth, do outperform S portfolios in terms of mean compound return and terminal wealth in all cases but one (asset classes without rebalancing). Interestingly, in the only case in which an S portfolio outperforms a G portfolio in terms of growth, the G portfolio is extremely concentrated and fully invested in US stocks. This result suggests that, when attempting to maximize future growth, the GMM criterion may make a highly concentrated and risky bet that in the end may not pay off. The underperformance of the G portfolio in this specific case may be a useful reminder that the GMM criterion does not *guarantee* outperformance in terms of growth; it merely maximizes the *probability* that growth and terminal wealth will be higher than those obtained from any other strategy.

More generally, note that the shorter the investment horizon, the less certain the outperformance of the G portfolio will be. This is simply because in a short holding period any run of low returns in the G portfolio (or of high returns in the S portfolio) will largely determine the terminal wealth. In other words, in the short term anything can happen because the final outcome may be dominated simply by (good or bad) luck. The longer the holding period, however, the lower the impact of luck, and, therefore, the more likely is the G portfolio to outperform all other portfolios. For this reason, GMM is usually thought of as a *long-term* investment strategy.

5. GMM or SRM?

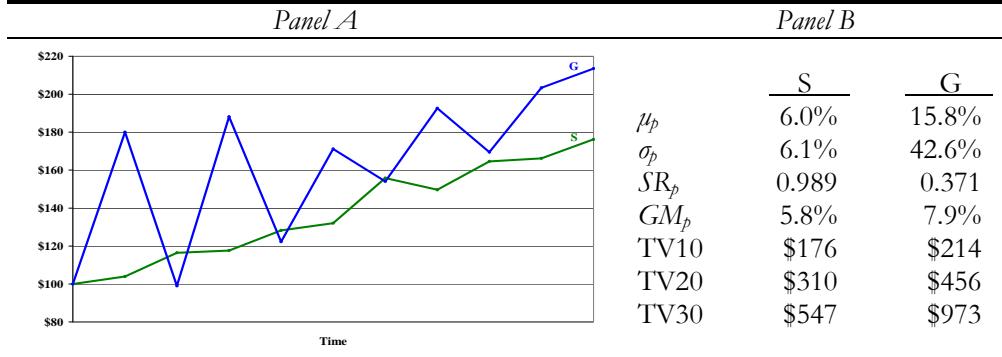
There is little doubt that both GMM and SRM have attractive properties. The question, then, is which one should investors adopt as the standard criterion. Currently, SRM seems to be the preferred choice, but should it be? This section rounds up the discussion above by outlining some of the conditions that would make GMM the more attractive criterion.

Exhibit 7 considers two hypothetical assets, G and S . Panel A depicts their performance and panel B formally summarizes that performance, which can be thought of as representative of the long-term behavior of both assets. In relative terms, S has lower volatility and higher risk-adjusted returns; G , on the other hand, grows at a faster rate. Which of the two assets is more attractive?

¹⁹ Sortino ratios are generally defined as $(R_p - B)/\Sigma_{pB}$, where R_p denotes the return of the portfolio; B a benchmark return chosen by the investor; and Σ_{pB} the semideviation of the portfolio returns with respect to the benchmark B (that is, volatility below B). The benchmark considered in Exhibit 6 is $B=0$. For a practical introduction to downside risk, see Estrada (2006).

Exhibit 7: GMM v. SRM

This exhibit shows, in panel A, the 10-year performance of two hypothetical assets, S and G. Panel B shows the arithmetic mean return (μ_p), volatility (σ_p), Sharpe ratio (SR_p) and geometric mean return (GM_p) of both assets, as well as the terminal value of \$100 invested at GM_p after 10 years (TV10), 20 years (TV20), and 30 years (TV30).



It depends. All investors would prefer the higher terminal value of G, but not all investors would be able to take its very high volatility. This is, precisely, Samuelson's (1971) point: Preferences do play a role and it is not certain that all investors would prefer G just because it grows at a faster rate; G is also far riskier and some investors may avoid it for that reason. This is particularly true given that although an asset (say, a technology stock) may be expected to grow at a faster rate than another (say, a utility stock), it is not certain that it will do so over a given holding period, particularly over a short one.

What conditions would make G the more attractive asset? Besides preferences reflecting a relatively low degree of risk aversion, two seem to stand out: The *longer* and *more certain* the investment horizon, the more attractive G becomes. As mentioned above, in a short holding period luck plays an important role but its impact decreases as the holding period increases; hence, the longer the holding period, the more likely is G to outperform S. In the short term, a utility stock may be preferred over a technology stock, but in the long term the technology stock may be the more plausible choice. For this reason, the longer the investment horizon, the more plausible the choice of G becomes.

Furthermore, how certain an investor is about his holding period also plays an important role. Some investors may have the intention of saving for the long term but may be forced to sell sooner than expected. If an investor's savings are not substantial and are meant to take care of all unforeseen contingencies, the likelihood of having to liquidate the holdings before the end of the expected holding period may be high. In this case, it is not clear that G, given its high volatility, is the better choice. If, on the other hand, the savings can be put away with the certainty that they will not be needed in the short or medium term, then G, given its higher expected terminal value, may be the more attractive choice.

In short, then, SRM may be a more plausible criterion than GMM for relatively more risk-averse investors, those with a short investment horizon, and those that are uncertain about the length of their holding period. GMM, on the other hand, may be a more plausible criterion than SRM for relatively less risk-averse investors, those with a long investment horizon, and those that are likely to stick to their expected (long) holding period.

6. An Assessment

There is little doubt that SRM is the standard criterion of portfolio selection currently chosen by academics and practitioners. The ultimate question posed in this article is whether it *should* be.

The main arguments and results in this article can be summarized as follows. From a theoretical perspective, neither GMM nor SRM are generally consistent with expected utility maximization. However, GMM is consistent with expected utility maximization under a more plausible condition than SRM is, namely, a logarithmic utility function (which exhibits decreasing absolute risk aversion) as opposed to a quadratic one (which exhibits increasing absolute risk aversion).

From an empirical perspective, the in-sample analysis shows that G portfolios are less diversified, have a higher (arithmetic and geometric) mean return, and higher volatility than S portfolios. It also shows that levered versions of S portfolios that aim to match some characteristics of G portfolios may require more leverage than many investors can or are willing to obtain. The out-of-sample analysis, in turn, shows that although GMM tends to achieve its goal of maximizing terminal wealth, SRM often does not achieve its goal of maximizing risk-adjusted returns.

Why then the general preference for SRM over GMM? It is not entirely clear. G portfolios are in fact less diversified and riskier than S portfolios; but at the same time they compound the invested capital faster thus delivering a higher terminal wealth. Is it then the case, as Samuelson (1971) and Markowitz (1976) implied, that many investors (and therefore the practitioners that manage their portfolios) are willing to sacrifice long-term return in exchange for short-term stability? Perhaps that is part of the reason. Mauboussin (2006) suggests that portfolio managers may not be fond of GMM because, more often than not, they are forced to focus on the short term rather than on the long term. He also suggests that investors may find it difficult to deal with the high volatility of the portfolios selected by this criterion. Again, perhaps that is part of the reason.

Nevertheless, it still remains the case that GMM has several desirable characteristics. It is by design equipped to deal with a multiperiod horizon and the reinvestment of capital; it maximizes the probability of ending with more wealth than any other strategy; it minimizes the time to reach any target level of wealth; it empirically does tend to achieve out of sample its goal of maximizing the growth of the capital invested; and it is simple to implement.

And yet GMM does seem to have taken a back seat to SRM. The results reported and discussed in this article question the conventional wisdom and raise two important questions: 1) Are academics and practitioners largely overlooking a compelling portfolio optimization criterion? 2) Perhaps?

Appendix

A1. The Kelly Criterion

Consider a gambler who repeatedly bets a constant proportion of his money; reinvests all his gains and losses; and aims to maximize his expected terminal wealth. Under these conditions, Kelly (1956) established that the optimal proportion of capital (K) a gambler should bet on each round is given by the expression $K = E/O$, where E denotes the ‘edge’ or expected value of the gamble and O the ‘odds’ or potential payoff per \$1 bet.

Exhibit A1 illustrates the Kelly criterion. Panel A describes three gambles and shows (in the last column) the optimal fraction of capital a gambler should bet on each round. Panel B shows the results after 100 rounds of betting different fractions of capital (F) on each gamble; the Kelly fraction is shown in the last column. As this panel shows, the highest geometric mean return and therefore terminal wealth are obtained when the bets are those given by the Kelly criterion.

Exhibit A1: The Kelly Criterion

This exhibit illustrates the Kelly criterion. Panel A describes three gambles, showing for each the probability of the two possible states of the world (P_1 and P_2); the payoff received in each state of the world (W_1 and W_2); the edge (E), defined as the expected value of each gamble; the odds (O), defined as the potential payoff per \$1 bet; and the Kelly criterion (K), defined as $K=E/O$. Panel B shows the terminal value (TV) of \$1 after 100 rounds of betting a fixed proportion of capital (F) in each gamble, reinvesting all gains and losses; it also shows the arithmetic mean return (AM), geometric mean return (GM), and volatility (SD) of the sequence of returns in each gamble.

Panel A	P1	W1	P2	W2	E	O	K				
Gamble 1	0.5	\$1.4	0.5	-\$1.0	\$0.2	1.4/1	14.3%				
Gamble 2	0.5	\$1.8	0.5	-\$1.0	\$0.4	1.8/1	22.2%				
Gamble 3	0.5	\$2.2	0.5	-\$1.0	\$0.6	2.2/1	27.3%				
Panel B											
	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%	K
Gamble 1											
TV (\$)	3.6	3.3	0.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	4.1
AM (%)	2.0	4.0	6.0	8.0	10.0	12.0	14.0	16.0	18.0	20.0	2.9
GM (%)	1.3	1.2	-0.3	-3.3	-7.8	-14.2	-22.9	-34.9	-52.5	-100.0	1.4
SD (%)	12.0	24.0	36.0	48.0	60.0	72.0	84.0	96.0	108.0	120.0	17.1
Gamble 2											
TV (\$)	20.2	67.8	42.8	4.8	0.1	0.0	0.0	0.0	0.0	0.0	70.7
AM (%)	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0	40.0	8.9
GM (%)	3.1	4.3	3.8	1.6	-2.5	-8.8	-17.7	-30.1	-48.8	-100.0	4.4
SD (%)	14.0	28.0	42.0	56.0	70.0	84.0	98.0	112.0	126.0	140.0	31.1
Gamble 3											
TV (\$)	107.2	1,182.0	1,821.0	412.5	11.5	0.0	0.0	0.0	0.0	0.0	1,953.7
AM (%)	6.0	12.0	18.0	24.0	30.0	36.0	42.0	48.0	54.0	60.0	16.4
GM (%)	4.8	7.3	7.8	6.2	2.5	-3.7	-12.7	-25.7	-45.4	-100.0	7.9
SD (%)	16.0	32.0	48.0	64.0	80.0	96.0	112.0	128.0	144.0	160.0	43.6

A2. Geometric Mean Maximization

This section derives the expression to be maximized in order to obtain the optimal weights with the GMM criterion. Let R and r denote simple and continuously compounded returns, such that $\ln(1+R)=r$. Furthermore, let GM denote the geometric mean of R which, by definition, is given by

$$1+GM(R) = \{\prod_t (1+R_t)\}^{1/T}, \quad (\text{A1})$$

where T is the number of returns in the sample. Taking logs on both sides yields

$$\ln\{1+GM(R)\} = (1/T) \cdot \sum_t \ln(1+R_t) = (1/T) \cdot \sum_t r_t = E(r) = \mu, \quad (\text{A2})$$

where μ denotes the arithmetic mean (or expected value) of r . Then,

$$GM(R) = \exp\{E(r)\} - 1 = \exp(\mu) - 1. \quad (\text{A3})$$

Applying the Taylor expansion to approximate $\ln(1+R)=r$ about μ yields

$$\ln(1+R)_\mu \approx \ln(1+\mu) + \frac{R-\mu}{1+\mu} - \frac{(R-\mu)^2}{2 \cdot (1+\mu)^2} + \frac{(R-\mu)^3}{3 \cdot (1+\mu)^3} - \frac{(R-\mu)^4}{4 \cdot (1+\mu)^4} + \dots, \quad (\text{A4})$$

and taking expectations on both sides yields

$$E[\ln(1+R)_\mu] = E(r) \approx \ln(1+\mu) - \frac{E[(R-\mu)^2]}{2 \cdot (1+\mu)^2} + \frac{E[(R-\mu)^3]}{3 \cdot (1+\mu)^3} - \frac{E[(R-\mu)^4]}{4 \cdot (1+\mu)^4} + \dots. \quad (\text{A5})$$

Finally, substituting (A5) into (A3) yields

$$GM(R) \approx \exp \left\{ \ln(1+\mu) - \frac{E[(R-\mu)^2]}{2 \cdot (1+\mu)^2} + \frac{E[(R-\mu)^3]}{3 \cdot (1+\mu)^3} - \frac{E[(R-\mu)^4]}{4 \cdot (1+\mu)^4} + \dots \right\} - 1, \quad (\text{A6})$$

and ignoring all moments of order higher than 2 yields

$$GM(R) \approx \exp \left\{ \ln(1+\mu) - \frac{\sigma^2}{2 \cdot (1+\mu)^2} \right\} - 1, \quad (\text{A7})$$

where $\sigma^2 = E[(R-\mu)^2]$ denotes the variance of R , which is the expression to be maximized.

Section A4 of this appendix evaluates the impact of dropping the third and/or fourth moments of expression (A6) on the accuracy of the approximation. As shown in Exhibit A3, the impact of dropping either one or both moments is negligible, which is consistent with results reported by Young and Trent (1969).

A3. Data and Summary Statistics

Exhibit A2: Data and Summary Statistics

This exhibit shows, for the series of monthly returns, the arithmetic mean (AM) and standard deviation (SD) of all markets and asset classes in the sample, both calculated between the beginning (Start) and the end (Jun/08) of each asset's sample period. The returns of all individual markets, as well as those of EAFE (Europe, Australasia, and the Far East) stocks and EM (Emerging Markets) stocks are summarized by MSCI indices. The returns of US Bonds are summarized by the 10-year government bond total return index (from Global Financial Data) and those of US Real Estate by the FTSE NAREIT (All REITs) total return index. The world market is summarized by the MSCI All Country (AC) World index. All returns are in dollars and account for capital gains and dividends.

Developed	AM	SD	Start	Emerging	AM	SD	Start
Australia	1.1%	6.8%	Dec/69	Argentina	2.8%	16.1%	Dec/87
Austria	1.1%	5.9%	Dec/69	Brazil	3.1%	15.5%	Dec/87
Belgium	1.2%	5.5%	Dec/69	Chile	1.8%	7.0%	Dec/87
Canada	1.1%	5.5%	Dec/69	China	0.5%	11.0%	Dec/92
Denmark	1.3%	5.4%	Dec/69	Colombia	1.7%	9.4%	Dec/92
Finland	1.4%	9.1%	Dec/87	Czech Rep.	1.9%	8.0%	Dec/94
France	1.1%	6.4%	Dec/69	Egypt	2.3%	9.1%	Dec/94
Germany	1.1%	6.1%	Dec/69	Hungary	2.1%	9.9%	Dec/94
Hong Kong	1.8%	10.4%	Dec/69	India	1.2%	8.5%	Dec/92
Ireland	0.9%	5.6%	Dec/87	Indonesia	2.0%	14.9%	Dec/87
Italy	0.9%	7.1%	Dec/69	Israel	1.0%	7.1%	Dec/92
Japan	1.0%	6.3%	Dec/69	Jordan	0.8%	5.1%	Dec/87
Netherlands	1.2%	5.2%	Dec/69	Korea	1.2%	11.1%	Dec/87
New Zealand	0.7%	6.6%	Dec/87	Malaysia	1.0%	8.7%	Dec/87
Norway	1.4%	7.6%	Dec/69	Mexico	2.3%	9.2%	Dec/87
Portugal	0.7%	6.4%	Dec/87	Morocco	1.6%	5.4%	Dec/94
Singapore	1.3%	8.3%	Dec/69	Pakistan	1.3%	11.0%	Dec/92
Spain	1.1%	6.3%	Dec/69	Peru	2.0%	8.8%	Dec/92
Sweden	1.4%	6.8%	Dec/69	Philippines	0.9%	9.4%	Dec/87
Switzerland	1.1%	5.2%	Dec/69	Poland	2.4%	14.5%	Dec/92
UK	1.1%	6.4%	Dec/69	Russia	3.2%	16.9%	Dec/94
USA	0.9%	4.4%	Dec/69	South Africa	1.4%	7.8%	Dec/92
Asset Classes							
US Stocks	1.0%	4.0%	Dec/87	Sri Lanka	0.9%	9.9%	Dec/92
EAFE Stocks	0.7%	4.6%	Dec/87	Taiwan	1.1%	10.9%	Dec/87
EM Stocks	1.4%	6.6%	Dec/87	Thailand	1.2%	11.3%	Dec/87
US Bonds	0.7%	2.0%	Dec/87	Turkey	2.4%	17.3%	Dec/87
US Real Estate	0.9%	3.8%	Dec/87	World Market			
				AC World	0.8%	4.0%	Dec/87

A4. Geometric Mean Approximations

Exhibit A3: Geometric Mean Approximations

This exhibit shows, for the series of monthly returns of all the assets and sample periods in Exhibit A2, the difference between the exact geometric mean and the approximate geometric mean estimated with expression (A6), the latter based on four moments (4M), three moments (3M), and two moments (2M). It also shows, across developed markets, emerging markets, and asset classes, the average difference (Avg), the average of the absolute value of the differences (Avg-Abs), and the maximum difference in absolute value (Max-Abs).

Developed	4M	3M	2M	Emerging	4M	3M	2M
Australia	-0.001%	-0.006%	-0.013%	Argentina	0.062%	-0.123%	0.128%
Austria	0.000%	-0.002%	0.001%	Brazil	-0.016%	-0.111%	-0.068%
Belgium	0.000%	-0.001%	-0.001%	Chile	0.000%	-0.003%	-0.003%
Canada	0.000%	-0.001%	-0.004%	China	0.003%	-0.016%	0.013%
Denmark	0.000%	-0.001%	0.000%	Colombia	0.005%	-0.002%	0.000%
Finland	0.009%	0.002%	0.005%	Czech Rep.	0.009%	0.005%	0.002%
France	0.000%	-0.002%	-0.002%	Egypt	0.015%	0.007%	0.031%
Germany	0.000%	-0.002%	-0.003%	Hungary	0.010%	-0.004%	0.000%
Hong Kong	0.009%	-0.031%	0.003%	India	0.002%	-0.001%	-0.002%
Ireland	0.003%	0.002%	0.002%	Indonesia	0.050%	-0.104%	0.091%
Italy	0.000%	-0.002%	0.001%	Israel	0.001%	-0.001%	-0.003%
Japan	0.000%	-0.001%	0.001%	Jordan	0.000%	-0.001%	0.000%
Netherlands	0.000%	-0.001%	-0.003%	Korea	0.008%	-0.026%	0.026%
New Zealand	0.002%	0.000%	0.001%	Malaysia	0.001%	-0.011%	0.002%
Norway	0.000%	-0.003%	-0.006%	Mexico	-0.001%	-0.009%	-0.019%
Portugal	0.002%	0.000%	0.002%	Morocco	0.006%	0.006%	0.008%
Singapore	0.000%	-0.010%	-0.001%	Pakistan	0.002%	-0.014%	-0.002%
Spain	0.000%	-0.002%	-0.003%	Peru	0.006%	-0.001%	-0.002%
Sweden	0.000%	-0.002%	-0.003%	Philippines	0.001%	-0.009%	0.003%
Switzerland	0.000%	-0.001%	-0.001%	Poland	0.110%	-0.125%	0.136%
UK	0.001%	-0.005%	0.007%	Russia	0.011%	-0.080%	-0.050%
<u>USA</u>	0.000%	0.000%	-0.001%	S Africa	0.002%	-0.001%	-0.009%
Avg	0.001%	-0.003%	-0.001%	Sri Lanka	0.004%	-0.012%	0.020%
Avg-Abs	0.001%	0.003%	0.003%	Taiwan	0.001%	-0.015%	0.006%
Max-Abs	0.009%	0.031%	0.013%	Thailand	0.000%	-0.019%	-0.009%
Asset Classes				<u>Turkey</u>	0.016%	-0.083%	0.034%
US Stocks	0.000%	0.000%	-0.001%	Avg	0.012%	-0.029%	0.013%
EAFFE Stocks	0.000%	0.000%	-0.001%	Avg-Abs	0.013%	0.030%	0.026%
EM Stocks	0.000%	-0.002%	-0.008%	Max-Abs	0.110%	0.125%	0.136%
US Bonds	0.000%	0.000%	0.000%	World Market			
<u>US Real Estate</u>	0.000%	0.000%	-0.001%	AC World	0.000%	0.000%	-0.001%
Avg	0.000%	-0.001%	-0.002%				
Avg-Abs	0.000%	0.001%	0.002%				
Max-Abs	0.000%	0.002%	0.008%				

A5. Figures

Figure A1: Developed Markets – No Rebalancing

This figure shows the performance of \$100 invested at the end of Jun/98, and passively held through the end of Jun/08, in two optimal portfolios of developed markets, one selected by maximizing the geometric mean return (G) and the other selected by maximizing risk-adjusted returns (S). See related performance figures on Exhibit 6.



Figure A2: Emerging Markets – No Rebalancing

This figure shows the performance of \$100 invested at the end of Jun/98, and passively held through the end of Jun/08, in two optimal portfolios of emerging markets, one selected by maximizing the geometric mean return (G) and the other selected by maximizing risk-adjusted returns (S). See related performance figures on Exhibit 6.

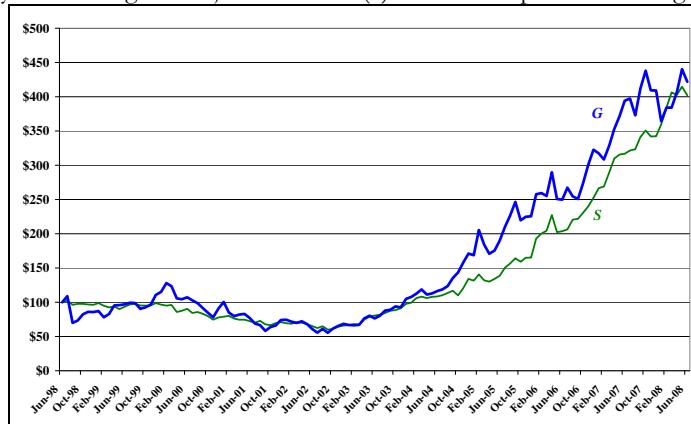


Figure A3: Asset Classes – No Rebalancing

This figure shows the performance of \$100 invested at the end of Jun/98, and passively held through the end of Jun/08, in two optimal portfolios of asset classes, one selected by maximizing the geometric mean return (G) and the other selected by maximizing risk-adjusted returns (S). See related performance figures on Exhibit 6.

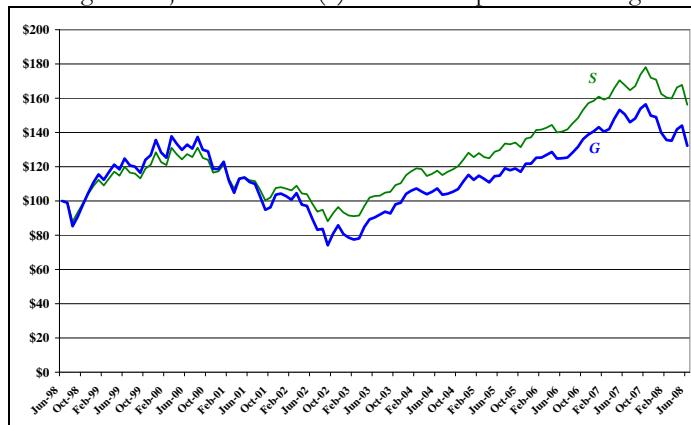


Figure A4: Developed Markets – With Rebalancing

This figure shows the performance of \$100 invested at the end of Jun/98, rebalanced at the end of Jun/03, and held through the end of Jun/08, in two optimal portfolios of developed markets, one selected by maximizing the geometric mean return (G) and the other selected by maximizing risk-adjusted returns (S). See related performance figures on Exhibit 6.

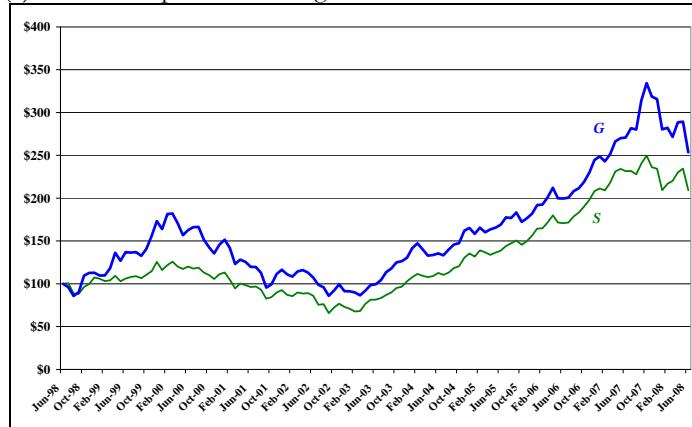


Figure A5: Emerging Markets – With Rebalancing

This figure shows the performance of \$100 invested at the end of Jun/98, rebalanced at the end of Jun/03, and held through the end of Jun/08, in two optimal portfolios of emerging markets, one selected by maximizing the geometric mean return (G) and the other selected by maximizing risk-adjusted returns (S). See related performance figures on Exhibit 6.

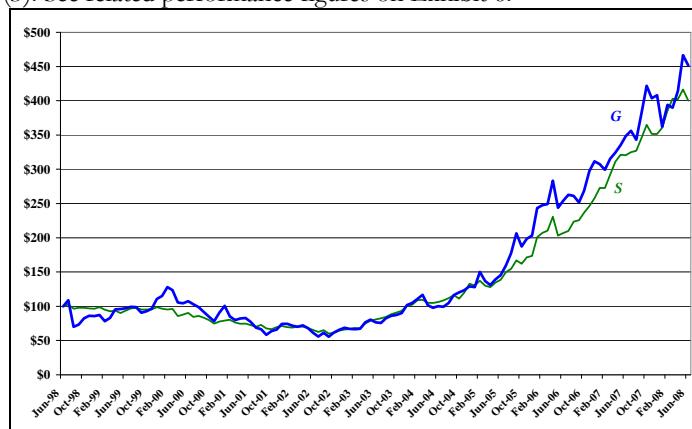
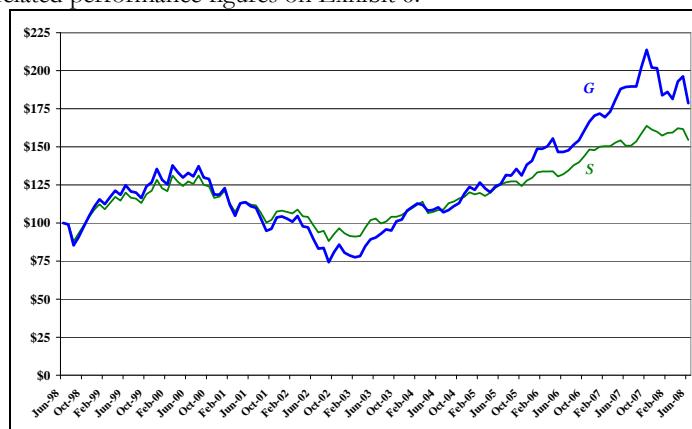


Figure A6: Asset Classes – With Rebalancing

This figure shows the performance of \$100 invested at the end of Jun/98, rebalanced at the end of Jun/03, and held through the end of Jun/08, in two optimal portfolios of asset classes, one selected by maximizing the geometric mean return (G) and the other selected by maximizing risk-adjusted returns (S). See related performance figures on Exhibit 6.



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