An Introduction to Combinatorial Game Theory

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Combinatorial Games

A two-player game is described as combinatorial if it satisfies the following conditions:

- The game is deterministic, with no elements of randomness.
- The game's rules give no inherent advantage to either player.
- Both players have perfect information, meaning both have complete understanding of the state of the game at all times, and nothing is hidden.

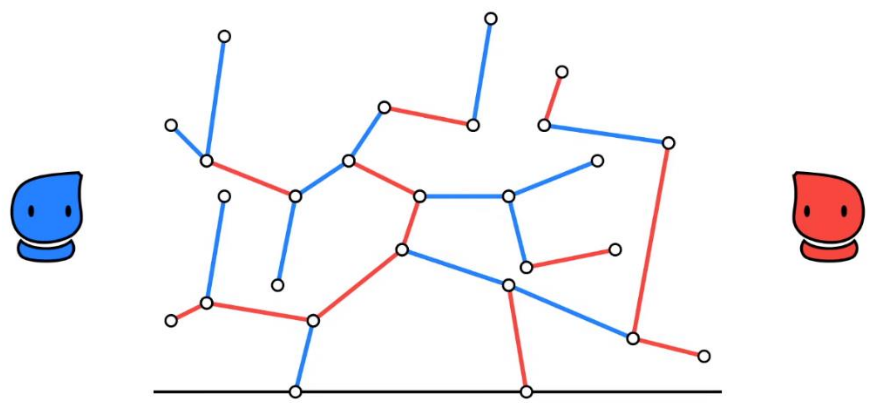
Combinatorial games can be played by anyone, regardless of their level of understanding, but for our analysis of such games we will assume that each player makes perfect moves, always taking their best option and ignoring all others.

The main game we will focus on in this introduction is one called Hackenbush, though there are many other combinatorial games that can be analyzed in the same way, all yielding fascinating and varied results. For Hackenbush, the visuals used herein are taken or derived from those used in the video, "HACKENBUSH: a window to a new world of math", by Own Maitzen.

Hackenbush

Hackenbush is a game where blue and red line segments are joined together into a graph called a “tree” which is attached to the “ground”. The Blue and Red players figure out who goes first, and they take turns cutting segments of their own color. If any “branches” of the tree are left disconnected from the ground, they “fall” to the ground and are also deleted from play. Whoever is the first to have no valid move is the loser. Without loss of generality, it is traditional to consider this game from the perspective of Blue, such that things that favor Blue are considered positive while things that favor Red are considered negative. Reversing the convention would make no change in the system since the game has no inherent bias toward any player, so we will stick with the standard of calling Blue positive.

Here is an example of a state in Hackenbush (also called a “position”, or simply a “game”):

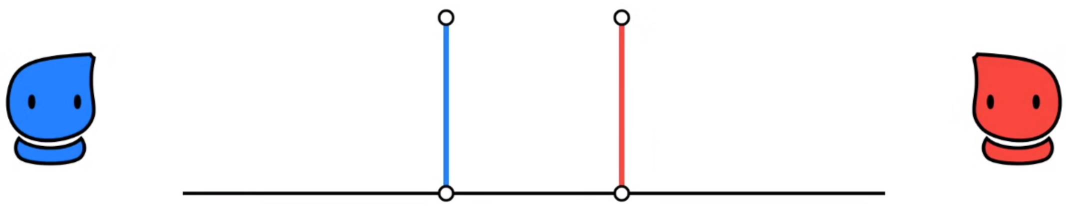


Also, note that Blue is often represented as male while Red is represented as female. This is convenient because it allows us to use gendered pronouns as shorthand for the players. We will use that convention in this document.

The first situation we will consider is the empty game, in which there are no line segments present of any color. This is a loss for whoever goes first, since they will have no valid move, and the second player wins by default. We will call the value of this position zero, since it does not inherently favor either player – who wins depends on who goes first.

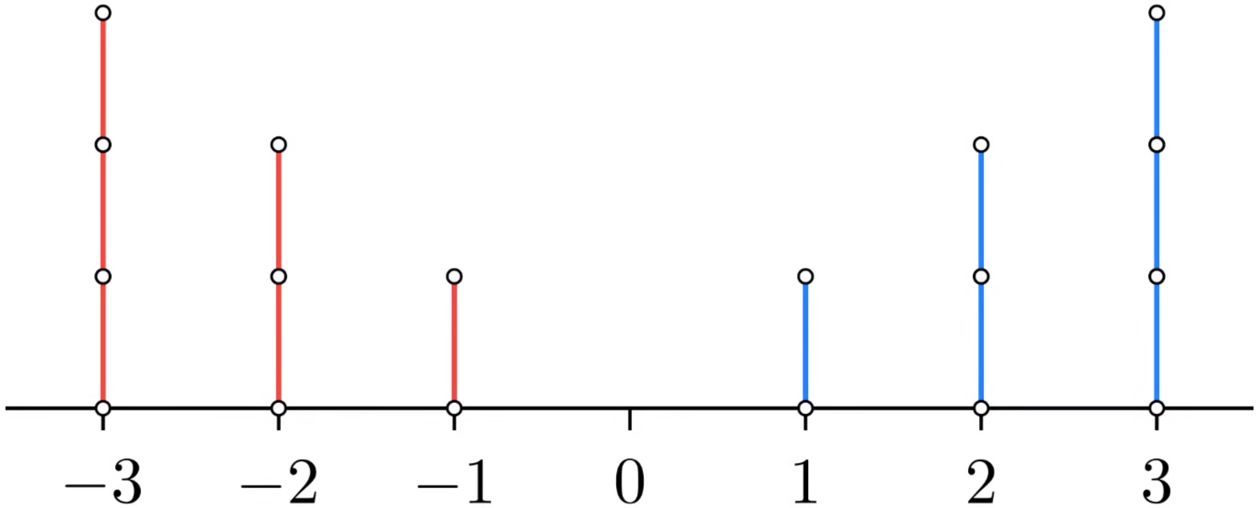
Now, because of our assumption that Blue is positive and Red is negative, a single blue line segment has a value of 1 while a single red segment has value -1, for simplicity. Choosing another number to be that basic value would simply scale all the other values, so 1 and -1 are convenient. Clearly, in a game where there is only one line segment, whoever owns that segment wins the game because the other player has no valid moves, so no matter who moves first that player who owns the line segment will win. It would be profitable for you to practice some simple situations to get a feel for how games play out.

Placing two Hackenbush trees on the same playing field is equivalent to adding them, and their values sum like real numbers do. We will explore later the fact that the values of such Hackenbush trees form a field. Because of this simple form of addition, we see that putting one blue segment beside one red segment, both connected to the ground, together they have value $1 + (-1) = 0$. We can prove that these two lines together (depicted below) make zero by looking at the possible outcomes of the game. Suppose Blue goes first. He takes his segment, she takes hers, and he loses because he has no more valid moves. So, Blue going first means Blue loses. Now suppose Red goes first. She takes her segment, he takes his, and she loses for lack of a valid move. Thus we see that the position $1 + (-1)$ is in fact a zero game, as desired.



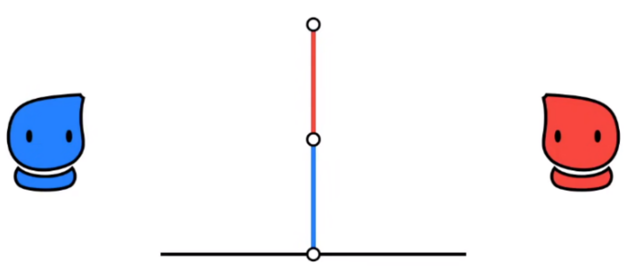
In general, any game that is symmetrical is a zero game. By "symmetrical", I mean that there are two trees that have the same shape but exactly opposite colors – red is changed to blue, and vice versa. Such games always have a value of zero because, no matter who goes first, the second player always can copy the move of the first player on the opposite tree; this process continues until the first player takes their last move, and then the second player takes their last move and the first player loses for lack of a move. Since the first player always loses, the value of the original position is zero, favoring neither player.

Using these units of 1 and -1, we can create any integer by stacking up the lines or setting them side-by-side, as shown below.



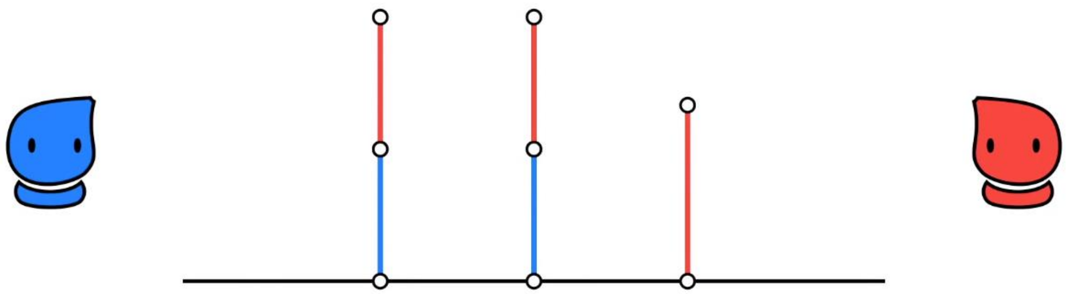
A note about negatives – because of the way these numbers are constructed, the way to negate a number is clear: change each blue line to red and vice versa. This makes any advantage that was Blue's become Red's, and vice versa.

Now we are prepared to talk about fractional values, and from there we will escalate into more complicated values. We will show that the tree depicted below has a value of ½. This can be interpreted as Blue winning by one half of a move, which is not very intuitive, but let's explore it now. First, observe that this tree is positive because Blue always wins, no matter who moves first. If Blue moves first, he chops down the tree and wins; if Red moves first, she cuts her branch, Blue cuts his, and Blue wins. Therefore, this position is strictly in favor of Blue, so we call this a positive game.



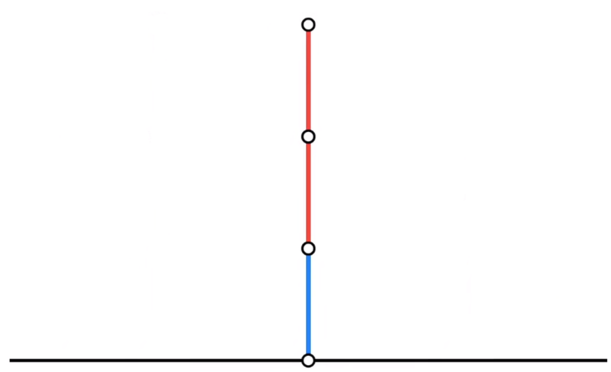
x

Now that we know this is positive, we can figure out exactly how positive it is, or by how much it is in favor of Blue. To do this, we can try to balance multiples of it with other trees to get the modified game to have a value of zero, as defined above. In this case, to show that this tree has a value of ½, we will duplicate this tree and add -1 and show that the new position balances to a zero game.



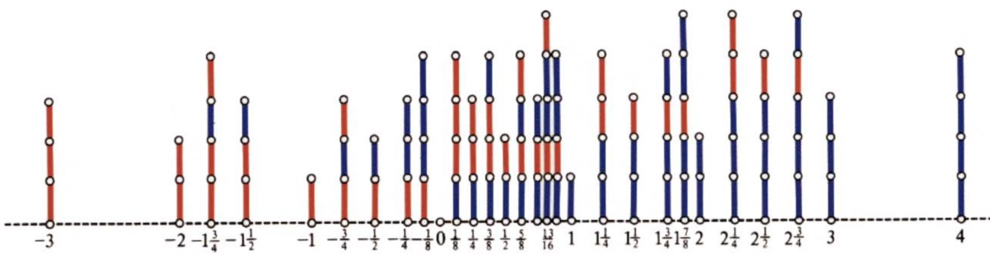
$$x + x - 1$$

If Blue goes first, he takes down one of the two trees, she takes the top off the other, he takes his remaining line, she takes hers, and blue loses. If Red goes first, she takes the top off of one tree, he takes down the entirety of the other tree, she takes her spare move, he takes his, and Red loses. So, no matter who goes first in this game, they lose. This is the definition of a zero game, so we know $x + x - 1 = 0$, and thus $x = \frac{1}{2}$. A similar argument can be given to show that the tree depicted below has a value of ¼.

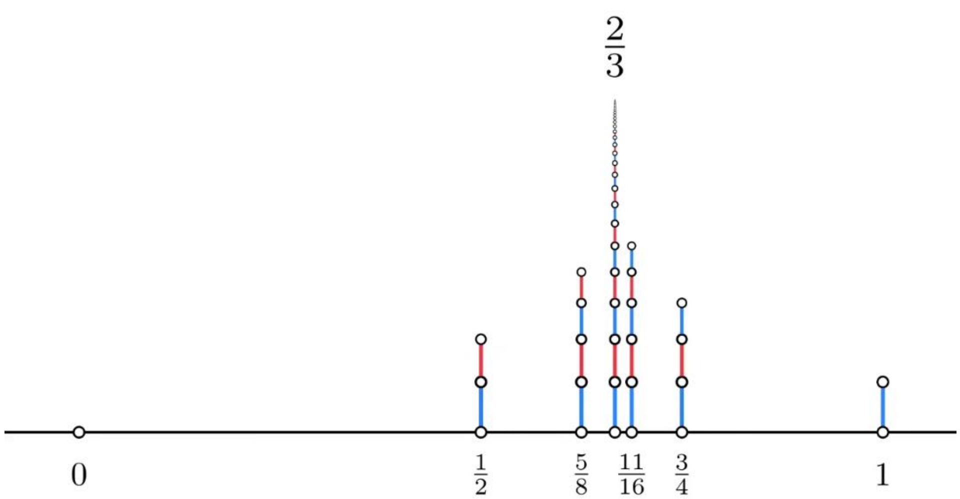


Using these methods, we can construct a Hackenbush tree with the value of any dyadic rational, from which all real numbers arise as the constructing process goes to infinity. To prove the value of any of them, insert multiple of the tree of interest to take care of any fractional denominator, and try to balance that with other trees to become a zero game. This naturally produces an equation that easily shows the value of the original tree.

$$\frac{n}{2^k}$$

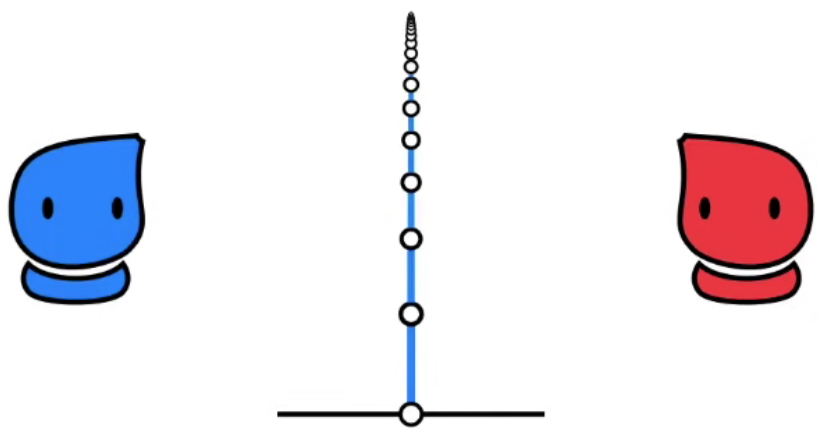


A good case to consider here is the standard Hackenbush tree for the fraction $\frac{2}{3}$, which is the infinite tree depicted below. Obviously $\frac{2}{3}$ is not a dyadic fraction, but it is the limit of a sequence of dyadic fractions, and we will see soon why that is sufficient to fully produce the value $\frac{2}{3}$. One sequence of dyadic fractions that approaches $\frac{2}{3}$ from the left is represented by the trees to the left of $\frac{2}{3}$ depicted below. These are all options that Blue can choose to take, depending only on how high up the tree he cuts – the higher up the tree he chooses to cut, the closer the result is to the value of the original tree. Red has a similar situation, but reversed, with their options forming a sequence of values that approaches $\frac{2}{3}$ from above.



Notice that each successive move that Blue makes has strictly less value for him, since he is essentially "using up" his advantage to advance the game toward an end. This explains why all of Blue's options are below $\frac{2}{3}$ and all Red's options are above it.

Now, for another infinite tree: this is similar to the previous tree, but we have only blue lines. Clearly this is an enormous win for Blue, since there is no move Red can make. Blue can cut the tree arbitrarily high, and doing so will result in some tree with a positive integer value; but it can be any integer at all, which shows that the starting infinite tower must have a value greater than any integer. This is an infinite number! We call this new number ω, and it is our standard "unit" infinite number.



We can even add more to this number in a well-ordered manner, such that $\omega + 1 > \omega$, and in general we can perform any numerical operation on ω and have a valid output, even exponentiation. Likewise, there is a multiplicative inverse of ω, called ε, which is smaller than all dyadic rationals, and is therefore smaller than all reals in the same way that ω is larger than all reals. And still, ε is as well-behaved as ω, and $\omega * \varepsilon = 1$. This number ε is constructed with one blue line topped by infinitely many red lines. This barely scratches the surface of a whole new world of numbers, with some too strange to even represent in Hackenbush.

Surreal Numbers

The collection of all non-loopy numbers in blue-red Hackenbush is called the surreal numbers, which were first described by John H. Conway (1937-2020). (We will explore loopy numbers later.) In the surreals, for any number x we have $x = \{X_L | X_R\}$, where X_L and X_R are sets of surreal numbers. This would have been a circular definition, but these numbers are organized cleverly to avoid that. Suppose there are no surreal numbers, and therefore both X_L and X_R are the empty set, so our new surreal number x is composed of an empty left set and an empty right set. We will call this number zero, so $0 = \{\ |\ \}$. The number zero is considered to be "born" on day zero. If we suppose that Blue owns the left set and Red owns the right set, this definition of zero parallels the standard zero game from Hackenbush, as neither player has any valid move. Thus, the left set can be seen as the set of all valid moves for Left, and the right set can be seen as the set of all valid moves for Right. This, I think, was the greatest breakthrough in my understanding of combinatorial game theory, and understanding this new notation allowed me to be able to write proofs with these symbols and nested sets.

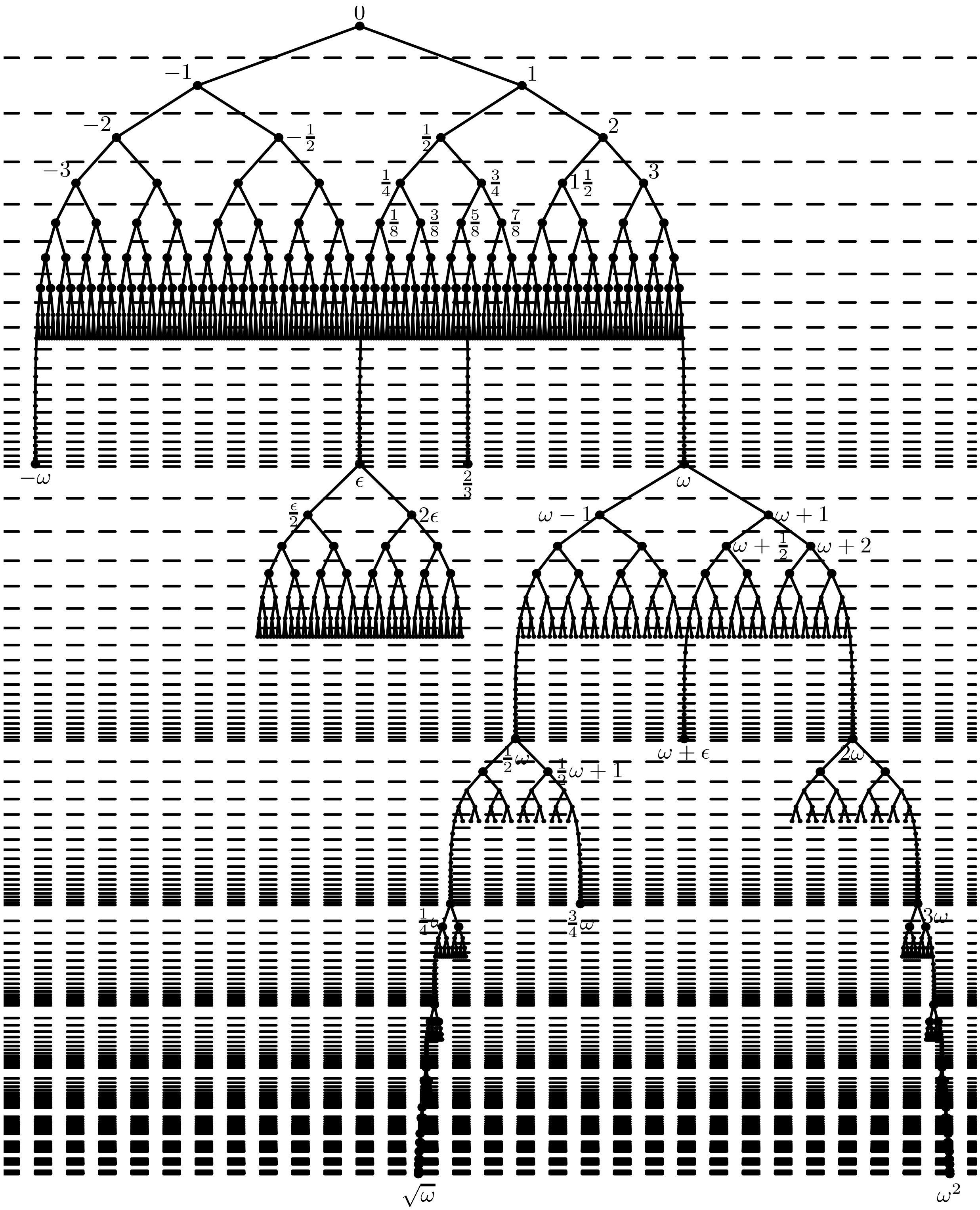
Now that we have the number zero, we can put it into the left and right sides of the braces to create new numbers. This allows us to create the numbers $\{0|\ \}$, $\{\ |0\}$, and $\{0|0\}$. For now, we will institute the restriction that every element of the left set must be strictly less than every element of the right set, which is written $x_L < x_R$ as a shorthand. This restriction means we cannot allow the creation of the number $\{0|0\}$; we will come back to it later. This leaves us with the numbers $\{0|\ \}$ and $\{\ |0\}$ as the only numbers "born" on day 1. The number $\{0|\ \}$ can be seen as a Hackenbush game where Left can move to zero, but Right has no valid move – and this is clearly a win for Left, no matter who moves first. Since this is the first game created after zero and it is positive, we will call it 1. Likewise, the game $\{\ |0\}$ will be called -1. Now that we have these numbers, we can create new combinations such as $\{0|1\}$, which has the value ½. This can be seen from the Hackenbush tree of ½ shown above, since Blue's only move is to go to zero, but Red's only move is to go to 1; and we showed that that tree has value ½.

I'll gloss over some details here, but you can continue generating new numbers in this way, and it can be shown that the bracket notation of each given number shows the possible moves of Left and Right from that position. The following are examples of some numbers in this new notation:

$$\begin{aligned}
-4 &= \{\ |-3\}\\
-3 &= \{\ |-2\}\\
-2 &= \{\ |-1\}\\
-1 &= \{\ |0\}\\
0 &= \{\ |\ \}\\
1 &= \{0|\ \}\\
2 &= \{1|\ \}\\
3 &= \{2|\ \}\\
4 &= \{3|\ \}
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2} &= \{0|1\}\\
\tfrac{1}{4} &= \{0|\tfrac{1}{2}\}\\
\tfrac{1}{8} &= \{0|\tfrac{1}{4}\}\\
\tfrac{1}{16} &= \{0|\tfrac{1}{8}\}\\
&\vdots
\end{aligned}$$

These numbers are organized into a tree structure as shown below, where the values are arranged laterally in their order on the number line, while their "birthday" is shown in order vertically. Projecting the entire tree onto a horizontal line would create a standard number line.



One noteworthy aspect of this generating tree is that we can define any given number with a sequence of left and right movements along the branches, traveling from one node to the next. This can be depicted as a sequence of ones and zeroes, so that aspect of this tree is very closely related to the binary representations of numbers, and therefore even Hackenbush trees are connected to binary.

The collection of surreal numbers is denoted No, an abbreviation for "number". Conway described the surreal numbers as containing "all numbers great and small", and it turns out that this collection is a universal ordered field, which means that every other ordered field is a subfield of the surreals – the surreals really do include "all numbers" in that sense. Also, note that the surreals are not a set, but are instead called a class. To prove that they cannot be a set, suppose by way of contradiction that the set S contains all the surreal numbers. Then, according to the construction of all other surreals, we can place the entire set S in the left side of the braces, creating the new number $\{S|\ \}$. Left, therefore, has every surreal number in existence as an option that he can move to. Clearly, he will not move to negative positions because he would lose from there, so we can treat S as though it contains only the positive surreals. This makes clear the fact that this new surreal number we created, $\{S|\ \}$, is a positive number strictly greater than all the elements of S, which contradicts our assumption that S contains all surreals. Therefore, it is instead termed a class. This essentially just kicks the problem down the road, but further analysis of that side of set theory is beyond the scope of this exploration.

Surreal numbers always have a value that is strictly greater than all elements of the left set and less than all elements of the right set, but the value is not always halfway between the values. First, it is profitable to mention now that most surreal numbers do not have singleton sets on their left and right sides, but all finite surreal numbers can be reduced to such a form. One example is the number $\{-1|1\}$, which is zero because neither player's move favors them, so they don't want to take it. Hence we can ignore both moves, which leaves us with $\{\ |\ \}$, which is the "canonical" or simplest form of zero. The same logic dictates that any number of the form $\{-x|y\}$ with $x, y > 0$ is also zero, since whoever moves first obviously loses. Likewise, if we have the number $\{-1,2,5|-20\}$, obviously Left will not pick the option that takes them to -1 since it doesn't benefit him; and he even won't pick 2 because he wants to make the move that benefits him most, which is 5. Therefore, this number can be simplified to $\{5|-20\}$. Notice that this is not a surreal number since the left side is not less than the right side, but we will come back to numbers like these later.

Another aspect of the concept of simplicity is shown in the following example: $\{1|10\}$ has a value of 2, because 2 is the "simplest" number between 1 and 10. In practice, the "simplest" number is the number between the maximum left value and the minimum right value that has the smallest denominator and smallest numerator. This restriction leads to the value being the earliest-born number between the maximum of the left and the minimum of the right. In the tree diagram, this is equivalent to finding the places where the left and right numbers are, and then finding the node above them on the tree where both connect, like their "most recent common ancestor". Because of the fact that we started building the tree of surreals with the empty number, zero, and each number after that is created using only previously constructed numbers, the concept of simplicity is inductive, so there is always a simplest number between any two given numbers.

As a shortcut to the canonical (or, simplest) form of a number, we have the following formulas:

$$n + 1 = \{n|\ \}$$

$$-n - 1 = \{\ |-n\}$$

$$\frac{2p+1}{2^{q+1}} = \left\{\frac{p}{2^q}\middle|\frac{p+1}{2^q}\right\}$$

Operations

Now we will discuss briefly the definitions of operations on these numbers. The definitions of some basic operations are given below.

$$\text{Addition: } x + y = \{x_L + y, y_L + x | x_R + y, y_R + x\}$$

$$\text{Negation: } -x = \{-x_R | -x_L\}$$

$$\text{Multiplication: } xy = \{x_L y + x y_L - x_L y_L, x_R y + x y_R - x_R y_R | x_L y + x y_R - x_L y_R, x_R y + x y_L - x_R y_L\}$$

$$\text{Inverse: } \frac{1}{x} = \left\{0, \frac{1+(x_L-x)\left(\frac{1}{x}\right)_R}{x_L}, \frac{1+(x_R-x)\left(\frac{1}{x}\right)_L}{x_R} \middle| \frac{1+(x_L-x)\left(\frac{1}{x}\right)_L}{x_L}, \frac{1+(x_R-x)\left(\frac{1}{x}\right)_R}{x_R}\right\}$$

Note that x_L ranges over the options in the set X_L, and the case is the same for x_R ranging over the options in X_R; likewise for the options of both sides of y. These formulas must be repeated for every element of X_L and X_R and Y_L and Y_R, all results being treated as options for their respective players. Same as with the concept of simplicity, these operations are inductively defined, considering everything is based on the empty sum, empty product, etc.

There is a whole field of study called surreal analysis that aims to utilize the surreals in the same way as the reals, though there are some problems that have been encountered in calculus concepts. Considering the surreals have well-defined infinitesimals, differentiation becomes almost trivial, no longer requiring limits; but there are also glaring issues with surreal analysis. One problem is with sequences, in that not every sequence of surreals approaches a surreal number, so the surreals are not closed. One "gap" in the surreals is the space between the finite numbers and ω, and that gap is colloquially named ∞. There are uncountably infinitely many gaps in the surreals, because the islands of infinitesimals around every number create a space between those numbers. It is unintuitive to think that the reals are closed, and yet the surreals, adding uncounted infinities of numbers atop the reals, introduce gaps into the continuum of the reals; but it is what it is. Furthermore, one problem with sums is that they can become trans-finite, which creates a whole new slew of issues, just like infinite sums used to make in the reals. Adding infinitely many numbers together is already hard enough; how can we add more than infinitely many things together? That is why integrals have not been successfully defined on the surreals yet.

One interesting note on surreals is what is called the Conway Normal Form. For any surreal number x, the Conway Normal Form of x is

$$x = \sum_{i \in \mathbf{On}} c_i \omega^i$$

where $\mathbf{On}$ is the class of all ordinal numbers, and thus i ranges through all ordinals; and each c_i is a finite scalar. This sum is finitely nonzero for any number x, so even though it is a trans-finite sum this particular kind of sum is always guaranteed to converge to a surreal number, namely x.

Also, a final note is that, as Conway said, "Just as the real numbers fill in the gaps between the integers, the surreal numbers fill in the gaps between Cantor's ordinal numbers" (*The Book of Numbers*, pg. 283).

Pseudo-numbers

Now we will delve into the even stranger world of what Donald Knuth called pseudo-numbers. These are numbers that do not necessarily obey the requirement that $x_L < x_R$. Numbers with $x_L \geq x_R$ are not surreal numbers, but some of them are smaller or larger than any surreal number. Together, these pseudo-numbers form a commutative ring, with both addition and multiplication (as defined above) being very well behaved. Considering these numbers are so strange, I feel it is important to emphasize that they do not represent quantities, rather they are values of positions, which can have various interpretations. To me, these are purely abstract numbers, though we can still learn from them about the combinatorial games they come from.

Nimbers

We will first return to the number $\{0|0\}$, which we discovered earlier. From this position, if Left moves first, he moves to zero and wins; if Right moves first, she moves to zero and wins. Therefore, this position clearly does not inherently favor either player, so it is neither positive nor negative. But it also is not zero because in a zero position the first player to move always loses, but here the first player always wins. This is a number that does not appear on the surreal number line, so we need a new name for it. We will call it $*$, pronounced “star”. This number motivates the definition of a new relationship that numbers can have, which is a “confused” relationship. This number $*$ is “confused with” zero (written $* \parallel 0$), which simply refers to the fact that the first player wins instead of the second player. According to my understanding, we could reverse our viewpoint and say that a zero position is one where the first player wins, and then $*$ would be what we generally call zero, which would mean the rule changes to be that whoever makes the final move loses – and that would change basically nothing about our number system, so we will stick with the original definitions, without loss of generality.

The number $*$ can be represented in Hackenbush with a modification to the rules: we add green branches which either player can cut. One single green segment connected to the ground has a value of $*$ because either player can move the game to zero and win.

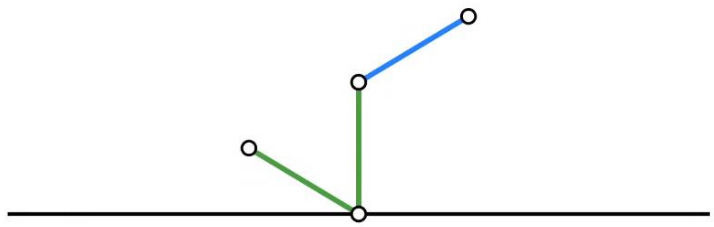
Additionally, we have a whole collection of star numbers of the form

$$* n = \{0, * 1, * 2, * 3, \dots, * (n-1) | 0, * 1, * 2, * 3, \dots, * (n-1)\}$$

where $* 1 = *$. This collection of numbers is called the “nimbers” after the combinatorial game called Nim. Notice that both players have the same options, so these numbers are all neither positive nor negative; also, we have that $* n = - * n$ for any integer n, and therefore $* n + * n = * n - * n = 0$. We also have that $* n \parallel 0$ and $* n \parallel * m$ for any $n, m \in \mathbb{Z}$; and furthermore, for any number x we have that $x + * n \parallel x + * m \parallel x$, and both $x + * n$ and $x + * m$ can be described as being “x-ish”, “fuzzy with x”, or “confused with x”. I prefer the latter, so that is what I will use from now on. I have also learned that the set of nimbers forms an algebraically closed field, and therefore polynomials can even be constructed over the field of nimbers that only have nimber roots, not complex; however, I have not worked with such things, so I have no more to say here about it. Also, adding nimbers together follows the same pattern as binary XOR addition, which is essential knowledge to be able to play the game Nim effectively.

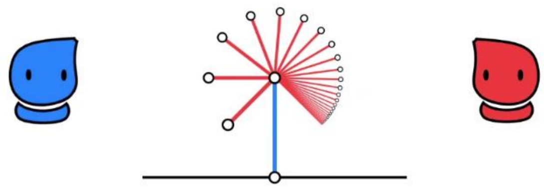
Sub-infinitesimals and Infinity

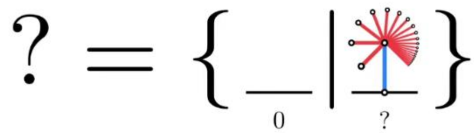
Using the new green segments in our modified Hackenbush, we can construct more strange positions, such as the one depicted below.



A quick test shows that this number is positive, but we can also show that its value is less than all dyadic rationals, and it is also even less than all epsilon numbers, which includes all surreal infinitesimals. This new number is very, very small! Again, we have run out of names to give our numbers, so we will call this number $\uparrow$, pronounced "up". Naturally, switching the blue segment for a red one makes the negative version, called $\downarrow$, and $\uparrow + \downarrow = 0$. We also have that $\uparrow \parallel *$, which is fascinating because it seems to imply that $\uparrow$ is just so small that it is practically zero, even though it is distinctly positive. But even more strangely, it turns out that the number $\uparrow + \uparrow = \Uparrow$ (often called "double up") is not confused with $*$. The "confusion" relationship can be seen as a cloud around a given fuzzy number, and it seems the radius of that cloud only covers $\uparrow$, and specifically does not reach as far as $\Uparrow$. In general, the multiples of $\uparrow$ are denoted $n.\uparrow = \{0|(n-1).\uparrow + *\}$. Another noteworthy number is $\uparrow^2 = \{0| \downarrow + *\}$.

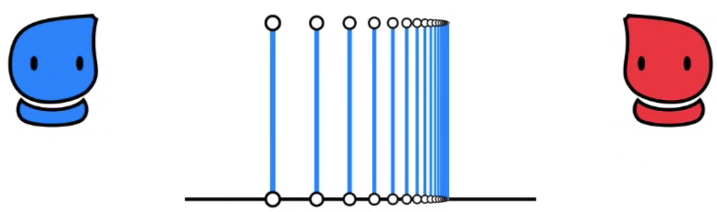
Now we will discuss the following position, with a blue stem and infinitely many red "petals":





This number is also less than every epsilon number, but surprisingly it is greater than all $n.\uparrow$. Once again, we are out of names, so we will call this number $\mathbf{over} = \{0|\mathbf{over}\}$, which is obviously defined circularly. Its negative is called $\mathbf{under} = \{\mathbf{under}|0\}$. Numbers that are circular like this are called “loopy”, and some mathematicians avoid them; but I find them fascinating, and I think they help challenge our preconceptions about numbers, so we will continue to consider them.

Now we consider the following position:



This is yet another new loopy number, which we call $\mathbf{On} = \{\mathbf{On}|\ \}$. This number is greater than any surreal number, and thus it acts like an absolute infinity. Its negative is $\mathbf{Off} = \{\ |\mathbf{Off}\}$. Just like how the form $\infty - \infty$ is indeterminate in the context of reals, the position $\mathbf{On} + \mathbf{Off}$ is also not a number is the same way the previous positions have been. Arbitrarily, this position is given the following name: $\mathbf{On} + \mathbf{Off} = \mathbf{dud} = \{\mathbf{dud}|\mathbf{dud}\}$. The term “$\mathbf{dud}$” serves as an acronym for “deathless universal draw”, which is appropriate since such a game would never end.

Now consider the number $\{0|\{0|-1\}\}$. Clearly this is a win for Left, since he can move to zero if he moves first, or he can move to zero once Right moves. Hence, this is a positive game, but it is also even less in value than $\uparrow$, and is therefore also confused with $*$. This number is called $+_1$, pronounced “tiny one”. There are also infinitely many numbers of this same form, with $+_n = \{0|\{0|-n\}\}$ for any positive ordinal n, and all of them are less than ↑. Furthermore, we have that $+_{n+1} < +_n$, and for each $+_{n+1}$, no number of multiplications or additions can ever get you from $+_{n+1}$ up to $+_n$. Each tiny number is like its own separate island of infinitesimals, completely disconnected from all other numbers. Ultimately, we can define the number $+_{\mathbf{On}} = \mathbf{tiny} = \{0|\{0|\mathbf{Off}\}\}$, which is the smallest of the tinies. As far as I know, this may even be the smallest number in existence, since I cannot find any smaller – all numbers I expected to be smaller led to contradictions when I tried to determine their relative value. See the proofs at the end for more details on that. Also, the negative of any given tiny number is a “miny” number, denoted as $-_n$.

Switches

Earlier we saw the number $\{5|-20\}$, and now we are equipped to talk about such numbers. They are called “switches”, since the left number is greater than the right number. To my knowledge, flipping the convention and making these constitute the surreal numbers and making the normal surreals become switches would not change the ways the respective numbers work, so we will stick with this convention without loss of generality. In a switch like this, both players want to move first, because whoever moves first will win. Therefore, we know that this number $\{5|-20\}$ is confused with zero. It turns out that this number is also confused with every number on the interval $[-20,5]$. In general, for a given switch $\{n|m\}$ with $n \geq m$, the value of the position is confused with every number on the interval $[m, n]$. If we were to add 21 to the position $\{5|-20\}$, the value of the new position would be $\{26|1\}$, and this is not confused with zero but is now strictly positive. But it is still a switch, and its value is still confused with all numbers on the interval $[1,26]$. Surreals have zero temperature because neither player wants to move.

Now we can introduce a concept called the “temperature” of a position, which is the absolute difference between the two sides of the number. The higher the temperature, the more each player “wants” to move in it. For example, the number $\{5|-20\}$ has a temperature of 25, so each player would want very much to make a move in that game, while in a position like $\{1|-1\}$ the players would not be quite so eager to move. The best strategy in a game of sums of switches is to move first in the switch with the highest temperature, and then in the next highest, and so on. There are also “thermographs” of games, which are graphs that track the temperature of a game as it is played, but we will not explore that now.

More Strange Numbers

Here is a miscellaneous collection of other strange numbers I have found in some of the sources I studied, organized into rows of related forms:

- $\mathbf{hot} = \{\mathbf{On}|\mathbf{Off}\}$
- $\mathbf{upon} = \{\mathbf{upon}|*\}$ $\quad$ $\mathbf{downon} = \{*|\mathbf{downon}\}$
- $\mathbf{ono} = \{\mathbf{On}|0\}$ $\quad$ $\mathbf{oof} = \{0|\mathbf{Off}\}$
- $\mathbf{hi} = \{\mathbf{On}|\mathbf{oof}\}$ $\quad$ $\mathbf{lo} = \{\mathbf{ono}|\mathbf{Off}\}$
- $\mathbf{tiny} = \{0|\mathbf{oof}\}$ $\quad$ $\mathbf{miny} = \{\mathbf{ono}|0\}$
- $\mathbf{ace} = \{0|\mathbf{tiny}\}$ $\quad$ $\mathbf{deuce} = \{0|\mathbf{ace}\}$ $\quad$ $\mathbf{trey} = \{0|\mathbf{deuce}\}$
- $\mathbf{joker} = \{0|\{0|\{-\mathbf{ace}|\mathbf{off}\}\}\}$ $\quad$ $♣ = \{\{\mathbf{deuce}|0\}|0\}$ $\quad$ $♦ = \{\mathbf{ace}|\{\mathbf{joker}|0\}\}$
- $2♥ = \{0|\{0|\mathbf{joker}\}\}$ $\quad$ $3♠ = \{0|\{0|\{0|\{0|\{♣|0\}\}\}\}\}$

Notice that **tiny** and **miny** are numbers we have seen before, but here they are written in different terms. Also notice that the number **hot** has the highest temperature possible, having a confusion cloud that covers the entirety of the surreals. These are only a miniscule sample of the wide universe of strange combinatorial numbers.

Returning to the surreals, there are more interesting numbers to examine. They are listed below.

- $\omega = \{1,2,3, \dots \mid \ \}$
- $\frac{\omega}{2} = \{1,2,3, \dots \mid \omega, \omega - 1, \omega - 2, \omega - 3, \dots\}$
- $\sqrt{\omega} = \left\{1,2,3, \dots \middle| \omega, \frac{\omega}{2}, \frac{\omega}{4}, \frac{\omega}{8}, \dots\right\}$
- $\varepsilon = \left\{0 \middle| 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$
- $\sqrt{\varepsilon} = \left\{\varepsilon, 2\varepsilon, 3\varepsilon, \dots \middle| 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\}$

And beyond these numbers, all of Cantor's ordinals are in the surreals, as mentioned before. One notable ordinal is $\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} = 1 + \omega + \omega^{\omega} + \omega^{\omega^{\omega}} + \cdots = \epsilon_0$, which is the first ordinal that cannot be created from smaller ones by a finite number of exponentiations. This is also the first solution to Cantor's equation $\omega^{\epsilon} = \epsilon$. The next solution is $\epsilon_1 = (\epsilon_0 + 1) + \omega^{\epsilon_0 + 1} + \omega^{\omega^{\epsilon_0 + 1}} + \cdots$, and then come $\epsilon_2, \dots, \epsilon_{\omega}, \dots, \epsilon_{\omega \times 2}, \dots, \epsilon_{\omega^{\omega}}, \dots, \epsilon_{\epsilon_0}, \dots, \epsilon_{\epsilon_{\epsilon}}, \dots, \epsilon_{\epsilon_{\epsilon_{\ddots}}}$, which is the first solution to the equation $\epsilon_a = a$. And even all those ordinals pale in comparison with the ordinal ω_1, the first uncountable ordinal. Naturally, ω_1 is then followed by $\omega_2, \dots, \omega_{\omega}, \dots, \omega_{\epsilon_0}, \dots, \omega_{\omega_1}, \dots, \omega_{\omega_2}, \dots$ and so on.

Other Topics and Games

In this introduction, we have focused on the values of positions in games, but there are other facets of combinatorial game theory. One such facet is the study of Grundy numbers, which essentially have to do with how far away a player is from a winning move at any given time, which can be very beneficial to understand when playing against a fallible opponent.

We have also been looking at Hackenbush almost solely, but there are infinitely many other games that can be analyzed as we did with Hackenbush. Other examples of combinatorial games include Nim, Domineering, and Amazons, and combinatorial analysis is profitable in many other games, including Go and Dots and Boxes. There is also another version of Hackenbush called “Childish” Hackenbush, in which no line segments are allowed to fall. This can simplify things greatly, but it also opens up new patterns that are worth studying.

Further Reading and Resources

I hope this introduction will pique the interest of some who read this. If you would like to explore this wide new world of numbers and weird math, I recommend the following resources:

- My YouTube playlist of videos on combinatorial game theory and surreal numbers
- *Winning Ways for Your Mathematical Plays*, by John Conway, Elwyn Berlekamp, and Richard Guy
- *The Book of Numbers*, by John Conway and Richard Guy
- *On Numbers and Games*, by John Conway
- *Surreal Numbers*, by Donald Knuth

Also, I intend for this to be freely available to all students, faculty, and employees of BYU-Idaho, so I authorize its use for any nonprofit purpose for that audience.

Proofs

The primary kind of proof that I did in this project is establishing the inequalities between the many infinitesimal numbers. The manner in which this can be accomplished is very straightforward, if not easy. Say we want to prove the inequality $+_2 < +_1$. Instead of delving into the messy direct definitions of what it means for one number to be "less than" another, we can rearrange the terms and prove $+_2 - +_1 < 0$, which means we are looking to prove that the game $+_2 - +_1$ is a win for Right, regardless of who moves first.

If we want to prove that one number is confused with another, we can rearrange the terms similarly. To prove that $x + * \parallel x$, we can instead subtract x from both sides and prove that $* \parallel 0$, which is almost trivially true. For example, to prove that $\uparrow \parallel *$, we can instead prove that $\uparrow - * = \uparrow + \ * \parallel 0$. In practice, this would entail showing that if Left moves first in $\uparrow + \ *$ then he wins, while if Right moves first she wins.

The process of adding these numbers together to evaluate them is usually the most difficult part, since the new set-theoretic definition of addition is unfamiliar, and the results are sometimes extremely complicated. However, because of the fact that the options for Left and Right are always simpler (for finite games) than the original position, by induction we are guaranteed to be able to find a state of the position that is simple enough to evaluate directly.

Because the set of options for each player can get very extensive and crowded while working through a proof, I like to establish another rule which does not change the results of gameplay: neither player may take any move that is distinctly not in their favor. For example, if Right has the option to move to a positive position (which means Left would certainly win from there), then that move is ignored and deleted. If Right only has options with a positive value, then it is the same as if she had no valid moves, because she would still lose if she took any of the positive options. The same goes for Left. Likewise, if taking a certain move obviously makes the other player win from a given position, then the player will not take that option. I found this rule to be extremely helpful as I proved relationships between complicated numbers. In the proofs below, whenever I use this rule to delete an option, I will first highlight that option in red to emphasize that it will be discarded.

Furthermore, another rule that is very helpful in simplifying computations is to ignore the moves of a given player that come right after that player takes a turn. If Left takes a turn, we can discard all their options in the new game state because it is distinctly not his turn anymore; and likewise for Right. In the proofs below, whenever I use this rule to delete an option, I will first highlight that option in yellow to emphasize that it will be discarded.

For simplicity I will refer to the players as L and R.

The following are the relationships between numbers which I have proven or otherwise explored.

$\varepsilon < \dfrac{1}{2^k}$ for all $k \in \mathbb{N}$

We compute

$$\varepsilon - \tfrac{1}{2^k} = \left\{0 \middle| 1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \dots\right\} + \left\{\tfrac{-1}{2^{k-1}} \middle| 0\right\} = \left\{\tfrac{-1}{2^k}, \varepsilon - \tfrac{1}{2^{k-1}} \middle| \varepsilon, 1 - \tfrac{1}{2^k}, \tfrac{1}{2} - \tfrac{1}{2^k}, \tfrac{1}{4} - \tfrac{1}{2^k}, \dots, \tfrac{1}{2^k} - \tfrac{1}{2^k}, \tfrac{1}{2^{k+1}} - \tfrac{1}{2^k}, \dots\right\}.$$

If R moves first, she moves to some position $\frac{1}{2^n} - \frac{1}{2^k}$ with $n > k$, since this is clearly negative; thus, if R moves first then R wins. If L moves first, his only option that is not obviously a losing move is to

$$\varepsilon - \tfrac{1}{2^{k-1}} = \left\{\tfrac{-1}{2^{k-1}}, \varepsilon - \tfrac{1}{2^{k-2}} \middle| \varepsilon, 1 - \tfrac{1}{2^{k-1}}, \tfrac{1}{2} - \tfrac{1}{2^{k-1}}, \tfrac{1}{4} - \tfrac{1}{2^{k-1}}, \dots, \tfrac{1}{2^{k-1}} - \tfrac{1}{2^{k-1}}, \tfrac{1}{2^k} - \tfrac{1}{2^{k-1}}, \dots\right\},$$

from which position R moves to some position $\frac{1}{2^n} - \frac{1}{2^{k-1}}$ with $n > k - 1$, since this is clearly negative; thus, if L moves first then R wins. Because R wins regardless of who moves first, we know that $\varepsilon - \frac{1}{2^k} < 0$ and thus $\varepsilon < \frac{1}{2^k}$.

$\mathbf{over} < \varepsilon$

For the purposes of this proof, we will use the shorthand $\varepsilon = \left\{0 \middle| \frac{1}{2^n}\right\}$. We compute

$$\mathbf{over} - \varepsilon = \{0|\mathbf{over}\} + \left\{\tfrac{-1}{2^n} \middle| 0\right\} = \left\{-\varepsilon, \mathbf{over} - \tfrac{1}{2^n} \middle| \mathbf{over} - \varepsilon, \mathbf{over}\right\}.$$

If R moves first, she moves back to the starting position, $\mathbf{over} - \varepsilon$; so suppose L moves first. He moves to

$$\mathbf{over} - \tfrac{1}{2^n} = \{0|\mathbf{over}\} + \left\{\tfrac{-1}{2^{n-1}} \middle| 0\right\} = \left\{\mathbf{over} - \tfrac{1}{2^{n-1}}, \tfrac{-1}{2^n} \middle| \mathbf{over}, \mathbf{over} - \tfrac{1}{2^n}\right\},$$

R moves to the same position (effectively skipping her turn again), and L continues to move to positions $\mathbf{over} - \frac{1}{2^k}$ with $\frac{1}{2^k}$ becoming progressively larger until, for example, we reach the position $\mathbf{over} - 2$, which is clearly negative, so R wins. Because R wins regardless of who moves first, we know that $\mathbf{over} - \varepsilon < 0$ and thus $\mathbf{over} < \varepsilon$.

$\uparrow < \mathbf{over}$

We compute

$$\uparrow + \mathbf{under} = \{0| *\} + \{\mathbf{under}|0\} = \{\mathbf{under}, \uparrow + \mathbf{under}|\mathbf{under} + *, \uparrow\}.$$

If L moves first, his best option is to move back to the same position; so, suppose R moves first. She moves to

$$\mathbf{under} + * = \{\mathbf{under}|0\} + \{0|0\} = \{\mathbf{under} + *, \mathbf{under}| *, \mathbf{under}\},$$

L moves back to that same position, and then R moves to $\mathbf{under}$ and wins. Because R wins regardless of who moves first, we know that $\uparrow + \mathbf{under} < 0$ and thus $\uparrow < \mathbf{over}$.

$+_1 < \uparrow$

For more complicated numbers like $+_1$, with nested braces inside them, we will align the dividing lines of the numbers across the steps of computation, for convenience and clarity. We compute

$$+_1 + \downarrow = \{0\big|\{0|-1\}\} + \{*\,|0\}$$

$$= \{\downarrow, * + \{0\big|\{0|-1\}\}\big|\{0|-1\} + \{*\,|0\}, +_1\}$$

$$= \Big\{\Big\{\{0\big|\{0|-1\}\}, *\ \Big|\{0\big|\{0|-1\}\}, \{0|-1\} + *\Big\}\ \Big|\ \downarrow, \Big\{\{0|-1\} + *\ \Big|\{0|-1\}, -1 + *\Big\}\Big\}$$

$$= \Big\{\{\ \ |\{0|-1\} + *\}\Big|\Big\{\{0|-1\} + *\,\big|\ \Big\}\Big\}$$

If L moves first, he moves to $\{\ \ |\{0|-1\} + *\}$, R moves to $\{0|-1\} + * = \{*, \{0|-1\}|\{0|-1\}, -1 + *\}$, L moves to $\{0|-1\}$, and R moves to -1 and wins. If R moves first, she moves to $\{\{0|-1\} + *\,\big|\ \}$, L moves to $\{0|-1\} + * = \{*, \{0|-1\}|\{0|-1\}, -1 + *\}$, and R moves to $-1 + *$ and wins, since $-1 + * \parallel -1 < 0$. Because R wins regardless of who moves first, we know that $+_1 + \downarrow < 0$ and thus $+_1 < \uparrow$.

$+_n < +_m$ for $n > m$, with $n, m > 0$

We compute

$$+_n + -_m = \{0\big|\{0|-n\}\} + \{\{m|0\}\big|0\}$$

$$= \Big\{\{\{m|0\}\big|0\}, \{m|0\} + \{0\big|\{0|-n\}\}\Big|\{0|-n\} + \{\{m|0\}\big|0\}, \{0\big|\{0|-n\}\}\Big\}$$

$$= \Big\{\{m + +_n, \{m|0\}\big|+_n, \{m|0\} + \{0|-n\}\}\Big|+_n, \{-_m, \{m|0\} + \{0|-n\}\big|-_m - n, \{0|-n\}\}\Big\}$$

$$= \Big\{\{\ \ \big|\{m|0\} + \{0|-n\}\}\Big|\{\{m|0\} + \{0|-n\}\big|\ \ \}\Big\}$$

Now we will analyze the position $\{m|0\} + \{0|-n\}$ to gain more context. We compute

$$\{m|0\} + \{0|-n\} = \{m + \{0|-n\}, \{m|0\}\big|\{0|-n\}, -n + \{m|0\}\}$$

$$= \{\{m|m-n\}\big|\{m-n|-n\}\}$$

$$= \{\{\ \ |m-n\}\big|\{m-n|\ \ \}\}$$

If L moves first in this game, he moves to $\{\ \ |m-n\}$, then R moves to $m-n$ which is negative because $n > m$, so R wins. If R moves first, she moves to $\{m-n|\ \ \}$, then L moves to $m-n$, and R wins. Therefore, this position has a negative value, and we will denote it in the other game as $\boldsymbol{negative}$. Thus we have the position

$$+_n + -_m = \{\{\ \ |\boldsymbol{negative}\}\big|\{\boldsymbol{negative}|\ \ \}\}.$$

If L moves first, he moves to $\{\ \ |\boldsymbol{negative}\}$ and R moves to $\boldsymbol{negative}$, winning. If R moves first, she moves to $\{\boldsymbol{negative}|\ \ \}$, L moves to $\boldsymbol{negative}$, and R wins. Because R wins regardless of who moves first, we know that $+_n + -_m < 0$ and thus $+_n < +_m$ for $n > m$, with $n, m > 0$.

$+_{\mathbf{On}} < +_n$ for $n < \mathbf{On}$

We compute

$$+_{\mathbf{On}} + -_n = \{0|\{0|\mathbf{Off}\}\} + \{\{n|0\}|0\}$$

$$= \left\{-_n, \{n|0\} + \{0|\{0|\mathbf{Off}\}\} \,\middle|\, +_{\mathbf{On}}, \{0|\mathbf{Off}\} + \{\{n|0\}|0\}\right\}$$

$$= \left\{\left\{n + \{0|\{0|\mathbf{Off}\}\}, \{n|0\} \,\middle|\, \{0|\mathbf{Off}\} + \{n|0\}, +_{\mathbf{On}}\right\} \,\middle|\, \left\{-_n, \{0|\mathbf{Off}\} + \{n|0\} \,\middle|\, \mathbf{Off} + -_n, \{0|\mathbf{Off}\}\right\}\right\}$$

$$= \left\{\left\{\ \middle|\{0|\mathbf{Off}\} + \{n|0\}\right\} \,\middle|\, \left\{\{0|\mathbf{Off}\} + \{n|0\}\middle|\ \right\}\right\}$$

Now we will analyze the position $\{0|\mathbf{Off}\} + \{n|0\}$ to gain more context. We compute

$$\{0|\mathbf{Off}\} + \{n|0\} = \left\{\{n|0\}, n + \{0|\mathbf{Off}\} \,\middle|\, \{0|\mathbf{Off}\}, \mathbf{Off} + \{n|0\}\right\}$$

$$= \left\{\{n|\mathbf{Off}\} \,\middle|\, \{\mathbf{Off}|\mathbf{Off}\}\right\}$$

$$= \left\{\{\ |\mathbf{Off}\} \,\middle|\, \{\mathbf{Off}|\ \}\right\}$$

If L moves first in this game, he moves to $\{\ |\mathbf{Off}\}$ and R moves to $\mathbf{Off}$, winning. If R moves first, she moves to $\{\mathbf{Off}|\ \}$, then L moves to $\mathbf{Off}$, and R wins. Therefore, this position has a negative value, and we will denote it in the other game as $\boldsymbol{negative}$. Thus we have the position

$$+_{\mathbf{On}} + -_n = \{\{\ |\boldsymbol{negative}\}|\{\boldsymbol{negative}|\ \}\}$$

If L moves first, he moves to $\{\ |\boldsymbol{negative}\}$ and R moves to $\boldsymbol{negative}$, winning. If R moves first, she moves to $\{\boldsymbol{negative}|\ \}$, L moves to $\boldsymbol{negative}$, and R wins. Because R wins regardless of who moves first, we know that $+_{\mathbf{On}} + -_n < 0$, and thus $+_{\mathbf{On}} < +_n$.

$+_n \| *$ for all $n > 0$

This implies that $+_n + * \parallel 0$. We compute

$$+_n + * = \{0|\{0|-n\}\} + \{0|0\}$$

$$= \{*, +_n | +_n, \{0|-n\} + \{0|0\}\}$$

$$= \left\{+_n \,\middle|\, \left\{*, \{0|-n\} \,\middle|\, -n + *, \{0|-n\}\right\}\right\}$$

$$= \left\{+_n \,\middle|\, \{\{0|-n\}|\ \}\right\}$$

If L moves first, he moves to $+_n$ and wins. If R moves first, she moves to $\{\{\ |-n\}|\ \}$, L moves to $\{\ |-n\}$, and R moves to $-n$ and wins. Because the first player to move wins, we know that $+_n + * \parallel 0$, and thus $+_n \| *$ for all $n > 0$.

$\mathbf{On} > x$ for any number $x \neq \mathbf{On}$

We will show that $\mathbf{On} - x$ is a win for L by analyzing cases.

- If $x \leq 0$, clearly $\mathbf{On} - x > 0$.
- If $x \parallel 0$, then $\mathbf{On} - x \parallel \mathbf{On} - 0 = \mathbf{On} > 0$.
- If $x = \mathbf{Off}$, then $\mathbf{On} - x = \mathbf{On} + \mathbf{On} = \mathbf{On} > 0$.
- If $x = \mathbf{On}$, then $\mathbf{On} - x = \mathbf{On} + \mathbf{Off} = \mathbf{dud}$.

Therefore, for all numbers $x \neq \mathbf{On}$, we have that $\mathbf{On} > x$.

Notes on other relationships

I attempted to determine whether $\mathbf{ace} < +_{\mathbf{On}}$ or $\mathbf{deuce} < \mathbf{ace}$, but I hit a roadblock in my analysis: I discovered that if L moves first in $\mathbf{ace} + -_{\mathbf{Off}}$ or $\mathbf{deuce} - \mathbf{ace}$, he wins, but if R moves first, the result is $\mathbf{dud}$. If I made no mistake, I think these numbers are so small that there is no longer any actual difference between their values. This gives credibility to what I've heard, that the number $+_{\mathbf{On}}$ is the smallest number in existence – perhaps it is impossible to prove that any given number is smaller than it, since they will end in a draw, and I know of no rules for relating values of draws. It is trivially easy to show that $+_{\mathbf{On}}$, $\mathbf{ace}$, and $\mathbf{deuce}$ are positive, but I know nothing about their relative sizes.