Hackenbush Unveiled: Exploring the Quirks of Combinatorial Game Theory

Step into the intriguing world of Hackenbush, a game with simple rules but unconventional numbers. In this presentation, we delve into combinatorial game theory through the lens of Hackenbush, uncovering the peculiar numbers that result from its gameplay. From surreal numbers to nimbers, infinitesimals to infinite numbers, we will explore how Hackenbush serves as a window into mathematical concepts in a novel and captivating manner. By highlighting its strangeness, this presentation aims to spark curiosity and appreciation for the wider field of combinatorial game theory, where simple games lead to unintuitive values and endless possibilities.

This graph [$\{0|1\}$] is equal to one half – it's not just represented by the number one half, but *equal* to it, and I'll prove it. I'm here to introduce you to a whole new world of math that I doubt many of you have seen, and it's a world of extremely strange numbers that defy our normal understanding of what a number is – and yet they're just as "valid" as the reals or the integers. I mean to show you today how playing games led to the discovery of more numbers than anybody had ever thought of before. The superb graphics I'll use are taken from a video that I'll tell you about at the end.

We will start by learning about a game called Hackenbush, in which this tree is a position. Hackenbush is a two-player game, and we'll call one player Left and the other Right for convenience. All bLue edges in the figure belong to Left while the Red ones belong to Right. Given a starting "tree" which is connected to the "ground", Left and Right [cycle through gameplay slides] take turns removing edges of their own color until one player has no valid moves, and any edges that are left disconnected from the ground fall and are also removed. The player who takes the last move wins. We will look at this game from Left's (or, Blue's) perspective, such that whenever Left wins it's a positive game and whenever Right wins it's a negative game. Therefore, Left's moves are considered positive (with its unit being called 1), and Right's are considered negative (with its units being called -1). Therefore, any positive integer n can be represented by n blue lines (either stacked up or adjacent), and any negative integer by n red lines.

Given these rules, one special case is the empty game. No matter who moves first, there is no legal move, so the second player wins by default. We will call this position zero. Any position in which specifically the second player to move wins is called a zero position. Here is another example of a zero position [$\{-1|1\}$], since no matter who moves first the second player simply mirrors their move and reduces it to an empty position, which means the first player to move always loses. Putting these two trees together is equivalent to adding their values, which is the Hackenbush version of calculating $1 + (-1) = 0$. In general, any game that is symmetric, with one tree having exactly the same shape as another but opposite colors, is a zero game. Thus we now have positive, negative, and zero positions.

I made a claim earlier that this tree is equal to one half. Now we're equipped to talk about why this is the case. Let's take this tree and duplicate it, and we'll see what we can discover about this new position. The only moves that Left has are to completely remove this tree or completely remove that tree, while Right can only take the top segment off of either tree. We can easily see here that no matter who moves first, Blue always wins, so we know this game is positive. Now that we know it's positive, we will try to balance it out to make a zero game, to find how large a positive number it is. We'll try adding a red segment, whose value is -1. Now, Blue still can only chop down a tree, but Red has a new option in their favor. We'll keep this new red line in reserve as a free move for Red. If Blue goes first, no matter which tree Blue cuts down, Red takes the top off the other; then Blue takes the trunk of that tree, and then Red takes their extra segment, winning the game. But then if Red moves first, they take off the top of one tree, which leaves Blue with a free move just like Red has; Blue chops down the other tree, Red takes their free move, Blue takes their free move, and Blue wins. Since the second player always wins, this position is zero – this means that the -1 perfectly balanced out two copies of the other tree, and therefore the value of this tree must be ½! Likewise, if we exchange the colors of the red and blue line segments, we get a position with the value -½, two of those trees being balanced to zero by one extra blue segment. It can be shown in a similar manner that this tree is equal to ¼, since two of them are balanced to zero by adding a tree with value -½. This same process easily extends to create all dyadic rationals. By adding and subtracting these fractions, we can create any real number.

Now we'll ease into some of the weirder stuff. Let's abstract away from the trees these numbers are derived from and look at the notation used to describe these positions. We will use the notation $\{L|R\}$, where L is Left's best move and R is Right's best move. For example, the standard empty zero position has no moves for either player, so we denote zero as $\{\ |\ \}$, where both L and R are empty. We can also write zero as $\{-1|1\}$, which is one of the positions we saw before, written in our new notation. This notation tells us that Blue's best move takes us to a position of value -1 while Red's best move leads to a 1 position. This means that neither player wants to move first, because the moment they do, the game will sway in the other's favor. For another example, here are the integers expressed in our new notation. Also, here's what the basic dyadic fractions look like.

Some of you might be wondering where other fractions, like thirds, fit into this system. Here is the infinite Hackenbush tree that represents 2/3, and you can prove it to yourself by putting three of these trees together and balancing them out with a -2 to get a zero game. Also, notice that the possible moves that Left can make create a sequence of dyadic fractions that approaches 2/3 from below; likewise, Right's possible moves create a sequence that approaches 2/3 from above. Here is a graphical representation of that fact. In this same way, any real number can be denoted with sequences of dyadic fractions that approach it from above and below, like this expression of $\tan\frac{\pi}{8}$.

Let's look at another infinite position. What's the value of this tree, made from an infinite number of blue segments? Since Blue can cut the tree off arbitrarily high, this tree can't have any integer as its value – it must be greater than all counting numbers. Since we've run out of names to give this number, let's call it ω. But notice, even this isn't the last number we can make with these trees. Making positions like these, we can indefinitely make other trees with infinite value which are still well-ordered, meaning these are infinities we can do arithmetic with! Likewise, we can also construct trees like this, whose values are strictly smaller than all dyadic fractions. The value of this tree we call ε, the unit infinitesimal. So, if you ever think that infinite numbers or infinitesimals don't exist, remember what I'm telling you now: here they are! We've created them! You might choose not to work with them, but they certainly exist. The collection of all values of blue-red Hackenbush games is called the surreal numbers; so you might be correct in saying infinitesimals aren't "real" because they're not in the reals, but they certainly are in the surreals.

Now let's change the game a bit. We will add green edges, which either player can cut on their turn. With that change, what is the value of this position? It's very much like zero because the result of the game depends on who goes first: if Left goes first, Left wins; and if Right goes first, Right wins. Therefore, we know that this position is neither positive nor negative, because neither player is guaranteed a win in general. But it's also distinctly not zero, because zero is defined as a position where the second player wins, but here the first player wins. Such a position is common in games like this, so its value has been given the name $*$ ("star"), which we denote with an asterisk. Therefore, with the inclusion of these green edges, not only are we able to construct all the surreal numbers, we can also construct this new number $*$ that is greater than all negatives and less than all positives, as though it inhabits the same place as zero; and yet it's specifically not equal to zero! This number $*$ is "confused with" zero, which simply refers to the fact that the first player wins as opposed to the second player. Likewise, we have other trees like these that are all confused with zero and with each other, but strictly not equal to zero nor to each other. These star numbers are called "nimbers", after a game called Nim. Also, for any given number x, we have that $x + * \parallel x$, or is "x-ish". Yes, that's a technical term. Also, x plus one given nimber and x plus a different nimber are confused with each other and with x.

What a mess! Why did we make up numbers that are so unintuitive? Now is a good time to explain that we are not finding which numbers describe which game positions, we're doing the opposite, defining numbers as being positions in a game, not as quantities. The number $*$ does not represent any real-world quantity. However, as a result of this new definition, we've opened up a whole new world of numbers that expands enormously beyond the reals, creating whole new unbounded collections of infinite numbers, infinitesimals, and confused numbers. I think it's enlightening to further challenge our preconceptions about numbers, so let's look at more examples from this weird new world.

Consider this position. This game's value turns out to be lower than every dyadic fraction, and it's even smaller than all ε numbers, the infinitesimals. This number is so incredibly small that it's actually confused with $*$, even though it's not confused with zero! We've run out of names again, so we'll call this new number $\uparrow$ ("up"), denoted with an up-arrow. Its negative, naturally, is called $\downarrow$ ("down"), and $\uparrow + \downarrow = 0$.

Now consider this position, a flower with infinitely many red petals. Notice that Blue can win instantly by cutting the flower down, but Red can never make any progress since no matter which branch they cut it makes no difference. This is an even stranger position than any we've seen before, since the only option for Red is to go to a position that equals the initial position. Therefore, its value is circularly defined, which is usually appalling to mathematicians. But let's push past that objection anyway and see the strangeness that unfolds. This number is greater than all multiples of $\uparrow$, but it's also smaller than all powers of ε. Again, we lack names for these numbers, so we'll call this one "**over**", since it's over up. Naturally, its negative is called "**under**".

Now consider this position, an infinite set of independent blue line segments. Similar to the previous number we looked at, Blue's only option is to move to a position that is identical to the original position. Therefore this number is also circularly defined, like the previous one was. But is this number equal to ω, which we saw earlier? No, because if we cut off the tree form of ω (or any other omega number) the result is a position that has strictly less than the original value. In this game, the number of lines never decreases, so this position is a truer infinity, one that behaves much like our standard concept of infinity, since adding or subtracting any finite tree makes no difference to its value. Also, adding its negative makes an indeterminate form. This position is given the name "**on**", and its negative is called "**off**". Also, the position $\mathbf{on} + \mathbf{off}$ is given the name "**dud**", which serves as an acronym for “deathless universal draw”, since no matter how long this game is played no one will ever win or lose. This number is also defined circularly.

Now you get a speed-round of weirdness, with some of these numbers not even having a clear graphical representation. We have **over** and **under** as we saw before, and also **upon** and **downon**, **hot**, **ono** and **oof**, **hi** and **lo**, **tiny** and **miny**, 0&**off**, 1&**over**, **ace** and **deuce** and **trey**, **joker**, “club”, “diamond”, “2 of hearts”, and “3 of spades”. Clearly, this is only the beginning of a vast, messy universe of new numbers – and isn’t it fascinating? This analysis reveals so many new numbers that we have to resort to names like “**oof**” and “**ono**” – which is exactly how some of you are feeling listening to all this, no doubt.

This study of the relationships between positions and their values is called combinatorial game theory. The game of Hackenbush, which we’ve been using here, is only a single combinatorial game, and there are infinitely many others that can be defined and analyzed as we’ve done here. If you have further questions for me, please come and ask! I’d love to expound more on these weird and fascinating numbers, and there’s so much more to learn in this wide new field. Credit goes to Owen Maitzen for the visuals – his video on Hackenbush is superb, and I recommend you watch it if any of this has piqued your interest. With that I close, and thank you for your time.

Details I’m not covering here:

- Surreal “days of creation” (“birthdays”)
- L and R can be simplified to include only the best move each for Left and Right, but here I’m treating L and R like they naturally contain only the best move.
- Numbers are added together “modulo isomorphism”, which is why summation here acts like summation of reals.
- Nimbers are their own negatives.
- Nimber addition follows XOR binary addition rules.
- Details on tinies and minies.
- Winning Ways for your Mathematical Plays, by Conway, Berlekamp, and Guy, which is considered the “Bible” of combinatorial game theory, and it’s written in a remarkably whimsical and accessible way.