Math 301 Proof Portfolio

Benjamin Hill - July 15, 2022

Introduction

The goal of the proof portfolio is to provide you a chance to reflect on the writing you have done throughout the semester, recognizing the growth you've made, as well as the breadth of topics/methods you've learned to work with.

Self Reflection

In the course of this semester, I learned my words have become very imprecise, and I made my vocabulary more dull to follow the crowd. However, now that I am nearing the end of this class, I am relearning to be conscientious of the words I use so that I can say exactly what I intend to say. It has also enabled me more fully to dissect complex ideas and understand them more deeply. Some people around me find that precision of language and dissecting of ideas annoying; but I feel those things are part of who I am. Therefore, what I have learned in this class will be applied in everything I do for the rest of my life, and knowing that fact makes me excited to learn more in the future about the methods of composing mathematical proofs.

Grade Proposal

A: I came to 80% of class periods and obtained a passing peer evaluation on both Week 7 and Week 13. I will have received a complete on at least 24 weekly write-ups. I have at least one weekly write-up that demonstrates mastery for the first four topics/methods and with at least 3 others (for a total of 10 topics/methods).

Topic/Method Mastery

1. proving two sets are equal
    - Problem 8 - First Proof That Two Sets Are Equal
    - Problem 9 - Second Proof That Two Sets Are Equal
    - Problem 22 - The Empty Set Is A Subset Of Every Set
    - Problem 30 - Associative Laws For Set Unions And Intersections
    - Problem 36 - Set Complement Rules 1 And 2
    - Problem 38 - Set Complements Rule 5
2. proving statements involving universal and existential quantifiers
    - Problem 25 - The Order Of Quantifiers Matters
3. proving that a sequence converges
    - Problem 49 - Showing A Sequence Converges
    - Problem 67 - Proving A Quotient Of Two Linear Sequences Converges
4. proving a statement is true by induction
    - Problem 33 - First Induction Problem
    - Problem 35 - Induction With The Sum Of The Squares Of The First N Natural Numbers
    - Problem 44 - Induction With Sum Of Odds
    - Problem 55 - Induction And An Inequality
5. proving something is an infimum or supremum
    - Problem 3 - Practice With Bounded Definitions
    - Problem 4 - Practice With Bounded Definitions 2
    - Problem 5 - Using The Completeness Axiom
6. proving something is a limit point of a set
    - Problem 6 - A Limit Point Of An Open Interval
    - Problem 7 - A Set With One Limit Point
    - Problem 24 - Limit Points Of Subsets Are Limit Points Of The Larger Set
7. proving something is a function, an injective function, and/or a surjective function.
    - Problem 42 - Which Relations Are Functions
    - Problem 43 - Practice With Injective And Surjective
8. proving a statement is true using an indirect proof (proof by contrapositive and/or contradiction)
    - Problem 22 - The Empty Set Is A Subset Of Every Set
    - Problem 52 - A Convergent Sequence Has a Unique Limit
    - Hill's Real Archimedian Expansion
9. proving facts about images and preimages of functions
    - Problem 58 - Image And Preimage Properties 1 And 2
10. proving properties about convergent sequences in general
    - Problem 52 - A Convergent Sequence Has a Unique Limit

All proofs

Solution

Between Any Two Real Numbers Is Another Real Number Benjamin

Contents [hide]

Problem 2: (Between Any Two Real Numbers Is Another Real Number) ↑

Suppose a and b are real numbers with $a < b$. Prove that there exists a real number c with $a < c < b$.

Solution ↑

Let a and b be any real numbers such that $a < b$. Since the two numbers are defined as not being equal, clearly the average of the two also cannot be equal to either, so let $\frac{a+b}{2} = c$, where c is some real number which is neither a nor b.

We will now show that $a < c < b$. Because b is greater than a, we know that $a + b > 2a$. Therefore, it is clear by rearranging that $\frac{a+b}{2} > a$. Likewise, we know that $a + b < 2b$, so it is also clear by rearranging that $\frac{a+b}{2} < b$. Thus, by substituting c for $\frac{a+b}{2}$, we see that $a < c < b$.

Therefore, for any numbers a and b that are not equal, there will always be another real number c between them.

Tags ↑

- Benjamin
- Complete
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Solution

Practice With Bounded Definitions Benjamin

Contents [hide]

Problem 3: (Practice With Bounded Definitions) ↑

Consider the set $S=[0,4)=\{x\in\mathbb{R}\mid 0\leq x \text{ and } x<4\}$.

1. Show that S is bounded below by giving a lower bound. Prove that the number you gave is a lower bound, and then state another lower bound different than the one you gave.
2. Of all possible lower bounds, which is the greatest lower bound. In other words, produce a lower bound m so that if m' is any lower bound, then we must have $m'\leq m$. Prove your answer.
3. Show that S is bounded above by giving an upper bound. Justify your answer.
4. Of all possible upper bounds, which is the least upper bound?

Solution ↑

Addressing part 1, given that $S=[0,4)$, by the definition given of bounds, because all elements of S are greater than or equal to 0, a lower bound for S is 0. Because no number between 0 and 4 is lower than -1, another lower bound of S is -1. For part 2, the smallest element of S is 0 by the definition of the interval, so no lower bound can be greater than 0 and remain a lower bound, because 0 would be lower than that bound and yet in S. This shows that any lower bound higher than 0 would defy the definition of a lower bound.

For part 3, because no number on the interval is greater than 4, an upper bound for S is 4. Therefore S must be bounded, and 4 is an upper bound. In part 4, we will show that the least upper bound of all possible upper bounds must be 4. Let m be an upper bound of S such that $m<4$. Because the interval S is defined to include all real numbers from 0 up to (but not including) 4, since m is less than 4 it is either in S or below all elements of S. Because of the proof shown in Problem 2, we know there is always some number $k\in\mathbb{R}$ such that $m<k<4$. Let $k\in\mathbb{R}$ such that $m<k<4$. In the case that $m\in S$, because all numbers between m and 4 are elements of S, we know also $k\in S$; and because there is an element of S greater than m, we know m cannot be an upper bound of S. In the case that m is below all elements of S, obviously there are elements of S greater than m, so it cannot be an upper bound of S. Thus m cannot be an upper bound because, as we have shown, no number less than 4 can be an upper bound. Since there cannot be any upper bound lower than 4, and since 4 is an upper bound, all upper bounds must be greater than or equal to 4, so 4 must be the supremum of S.

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Solution

Practice With Bounded Definitions 2 Benjamin

Contents [hide]

Problem 4: (Practice With Bounded Definitions 2) ↑

Let S be the set $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.

1. Give two different upper bounds for S. Then give the supremum of S.
2. Give two different lower bounds for S. Then give $\inf S$.

Remember, any time you make a claim, you must prove that your claim is correct.

Solution ↑

For part 1, both 2 and 3 are upper bounds of S because all elements of S are lower than them. We will now show that the supremum of S is 1. Let n be any natural number. Because S is made with only natural numbers in the denominator, we know that for any choice of n, the fraction $\frac{1}{n}$ will never be larger than 1 because 1 is the smallest natural number, and with larger choices of n the fraction $\frac{1}{n}$ is always smaller than 1. Therefore, no element of S is larger than 1, so 1 is an upper bound. Now, since 1 is the greatest element of S and is also an upper bound, we know that no upper bound can be less than 1, so 1 must be the supremum of S.

For part 2, both -1 and -2 are lower bounds of S because, since the natural numbers are never negative, all elements of S are higher than them. We will now show that the infimum of S is 0. Because no natural number is negative, we know no element of S can be negative, so 0 must be a lower bound of S. Now let $k > 0$. Since k is strictly greater than 0, we will choose some number $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < k$. Since $\frac{1}{n} \in S$, we know k cannot be a lower bound of S, so 0 must be the infimum of S.

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Solution

Using The Completeness Axiom Benjamin

Contents [hide]

Problem 5: (Using The Completeness Axiom) ↑

Consider the solutions to the inequality $x^2 < 2$. We can write this using set builder notation as $S = \{x \mid x^2 < 2\}$.

1. Is S bounded above? Prove your claim.
2. What does the completeness axiom say about this set?
3. Give $\sup S$, or explain why there is no supremum of S.

Solution ↑

Let $S = \{x \mid x^2 < 2\}$. We will show that S is bounded above. Let $x \in S$. Because $x \in S$, we know that $x < \sqrt{2}$. Since $x < \sqrt{2}$ is true for all $x \in S$, we know that $\sqrt{2}$ is an upper bound of S. By the Completeness Axiom, this shows that the set S has a supremum.

Now we will show that $\sqrt{2}$ must be the supremum of S. Suppose that $\sqrt{2} - t$ is an upper bound of S with $t > 0$. By the Archimedian Property, we know that there exists some $n \in \mathbb{N}$ such that $\sqrt{2} - \frac{1}{n} > \sqrt{2} - t$. Since $0 < \sqrt{2} - \frac{1}{n} < \sqrt{2}$, we know that $\left(\sqrt{2} - \frac{1}{n}\right)^2 < 2$, and therefore $\sqrt{2} - \frac{1}{n} \in S$. Because $\sqrt{2} - \frac{1}{n} \in S$, we know $\sqrt{2} - t$ cannot be an upper bound for S. Thus we see that $\sqrt{2}$ is the supremum of S.

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Solution

A Limit Point Of An Open Interval Benjamin

Problem 6: (A Limit Point Of An Open Interval) ↑

Contents [hide]

Let $S = (0, 1)$ which is the open interval from 0 to 1 that does not include the end points. Prove that $p = 1$ is a limit point of S. Then state another limit point of S.

Solution ↑

Let $I = (a, b)$ such that $1 \in I$, with $a, b \in \mathbb{R}$. Suppose $a \geq 0$ and let $x = \frac{a+1}{2}$. Therefore we know that $a < x < 1 < b$, which means $x \in S$. Because $1 < b$, we know $a < x < b$, which means $x \in I$. Thus 1 is a limit point of S. Now suppose $a < 0$ and let $x = \frac{1}{2}$. Because $0 < \frac{1}{2} < 1$, we know $x \in S$. Since $a < 0$ and $1 < b$, we know $a < x < b$, which means $x \in I$. Thus 1 is a limit point of S.

Now we will show that 0 is also a limit point of S. Let $I = (a, b)$ such that $0 \in I$, with $a, b \in \mathbb{R}$. Suppose $b \leq 1$ and let $x = \frac{0+b}{2}$. Therefore we know that $0 < x < b < 1$, which means $x \in S$. Because $a < 0$, we know $a < x < b$, which means $x \in I$. Thus 0 is a limit point of S. Now suppose $b > 1$ and let $x = \frac{1}{2}$. Because $0 < \frac{1}{2} < 1$, we know $x \in S$. Since $a < 0$ and $1 < b$, we know $a < x < b$, which means $x \in I$. Thus 0 is a limit point of S.

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Solution

A Set With One Limit Point Benjamin

Contents [hide]

Problem 7: (A Set With One Limit Point) ↑

Let $S = \left\{ \dfrac{1}{n} \mid n \in \mathbb{N} \right\}$, the collection of fractions of the form $\frac{1}{n}$ where n is a natural number. Prove that $p = 0$ is a limit point of S.

Solution ↑

Let $I = (a, b)$ such that $0 \in I$. Using the Archimedian Property, pick an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < b$. Because $a < 0$, we know $a < \frac{1}{n} < b$, which means $\frac{1}{n} \in I$. We also know $\frac{1}{n} \in S$, so we conclude that 0 must be a limit point of S.

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Solution

First Proof That Two Sets Are Equal Benjamin

Contents [hide]

Problem 8: (First Proof That Two Sets Are Equal) ↑

Let $I = (5, 9]$. Consider the sets
$S = \{x \mid x \text{ is an upper bound of } I\}$ and $A = \{x \mid x \geq 9\}$. Prove that $S = A$.

- Start by proving that $S \subseteq A$. So show that every element of S is an element of A.
- Then prove that $A \subseteq S$.

Solution ↑

First we will show that $S \subseteq A$. Let $s \in S$. We know that s is an upper bound of I, by definition of S. By Isaac's Bounded Theorem, it is clear that 9 is the supremum of I, so all upper bounds of I are greater than or equal to 9. Because s is an upper bound of I, we know $s \geq 9$, so we see $s \in A$.

Now we will show that $A \subseteq S$. Let $a \in A$. We know that $a \geq 9$, by definition of A. Because 9 is the supremum of I, we know by the Reeve Bounding Corollary that all numbers greater than 9 are also upper bounds. Since a cannot be less than 9, we know that a must be an upper bound of I, and is therefore in S. Because all elements of both sets are contained in each other, we say that the sets S and A are equal.

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Solution

Second Proof That Two Sets Are Equal Benjamin

Contents [hide]

Problem 9: (Second Proof That Two Sets Are Equal) ↑

Let $I = (5, 9]$. Consider the sets $T = \{x \mid x \text{ is a lower bound of } I\}$ and $B = \{x \mid x \leq 5\}$. Prove that $T = B$.

Solution ↑

First we will show that $T \subseteq B$. Let $t \in T$. We know that t is a lower bound of I, by definition of T. By Isaac's Bounded Theorem, it is clear that 5 is the infimum of I, so all lower bounds of I are less than or equal to 5. Because t is a lower bound of I, we know $t \leq 5$, so we see $t \in B$.

Now we will show that $B \subseteq T$. Let $b \in B$. We know that $b \leq 5$, by definition of B. Because 5 is the infimum of I, we know by the Reeve Bounding Corollary that all numbers less than 5 are also lower bounds. Since b cannot be greater than 5, we know that b must be a lower bound of I, and is therefore in T. Because all elements of both sets are contained in each other, we say that the sets T and B are equal.

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Solution

Which Dominoes Remain Standing Benjamin

Contents [hide]

Problem 10: (Which Dominoes Remain Standing) ↑

Suppose that Jon has set up an infinite number of dominoes, with the dominoes numbered $1, 2, 3, \ldots$. The dominoes are set up so that if the kth domino falls, then the $(k+1)$st domino will also fall. So if the 7th domino falls, then the 8th must fall as well. Jon knocks down the first domino, which starts causing other dominos to fall. Which dominos fall? Which dominoes remain standing? Make sure you prove your result. The well ordering principle will come in handy.

Suggestion: Use set builder notation to help you, so let $F = \{n \in \mathbb{N} \mid \text{domino } n \text{ fell}\}$ and $S = \{n \in \mathbb{N} \mid \text{domino } n \text{ remains standing}\}$. Then make some claims and prove they are correct.

Solution ↑

Let $S = \{n \in \mathbb{N} \mid \text{domino } n \text{ remains standing}\}$. Suppose by way of contradiction that some domino did not fall. This means $S \neq \emptyset$. By the well ordering principle, let k be the least element of S. In other words, domino k is the first domino that is left standing. It is given that the first domino falls, so $k \neq 1$. Hence we know there is some domino $(k-1)$ which did not fall over, which means $(k-1) \notin S$. Since $(k-1) \notin S$, we know domino $(k-1)$ fell, which must mean that domino k also fell, which is a contradiction. Thus we see that for all $n \in \mathbb{N}$, domino n must have fallen, and therefore S cannot contain any elements.

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Solution

De Morgans Laws With Truth Tables Benjamin

Contents [hide]

Problem 11: (De Morgan's Laws With Truth Tables) ↑

Let P and Q be statements or open sentences. Start by completing the truth table below to give the truth values for $P \vee Q$, $\sim(P \vee Q)$, $(\sim P) \vee (\sim Q)$, and $(\sim P) \wedge (\sim Q)$.

P	Q	$P \vee Q$	$\sim(P \vee Q)$	$\sim P$	$\sim Q$	$(\sim P) \vee (\sim Q)$	$(\sim P) \wedge (\sim Q)$
T	T						
T	F						
F	T						
F	F						

1. Use your truth table to prove that $\sim(P \vee Q)$ and $(\sim P) \wedge (\sim Q)$ are logically equivalent.
2. Construct a similar truth table to prove that $\sim(P \wedge Q)$ and $(\sim P) \vee (\sim Q)$ are logically equivalent .

In other words, when you have finished this problem you will have shown that
the negation of a disjunction is the conjunction of the negations, and
the negation of a conjunction is the disjunction of the negations.

Solution ↑

P	Q	$P \vee Q$	$\sim(P \vee Q)$	$\sim P$	$\sim Q$	$(\sim P) \vee (\sim Q)$	$(\sim P) \wedge (\sim Q)$
T	T	T	F	F	F	F	F
T	F	T	F	F	T	T	F
F	T	T	F	T	F	T	F
F	F	F	T	T	T	T	T

The first and second columns establish permutations of the truth values of P and Q. The third column shows $P \vee Q$, which (as shown in the chart) is always true unless both P and Q are false. The fourth column is the negation of the third, so its truth values will always be opposite those of the third column. The fifth and sixth columns are the negations of P and Q, respectively, and are thus opposite the respective values in the first and second columns. The values in the seventh column are always true unless both $\sim P$ and $\sim Q$ are false. The values in the eighth column, on the other hand, will always be false unless both $\sim P$ and $\sim Q$ are true.

Therefore, referencing the chart, we see that $\sim(P \vee Q)$ in the fourth column is logically equivalent to $(\sim P) \wedge (\sim Q)$ in the eighth column, by definition of logical equivalence: they both have the same truth values for all possible values of their component statements.

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$(\sim P) \vee (\sim Q)$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

In this chart, the fifth column is taken directly from the seventh column of the previous chart. The values in the third column will only be true if both P and Q are true - which means the values in the fourth column are only false when both P and Q are true, because it is the negation of the third column.

Again, referencing this table, we see that $\sim(P \wedge Q)$ and $(\sim P) \vee (\sim Q)$ are logically equivalent because all their truth values are the same for all respective values of their component statements.

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Solution

The Empty Set Is A Subset Of Every Set Benjamin

Contents [hide]

Problem 22: (The Empty Set Is A Subset Of Every Set) ↑

Prove or disprove. An empty set is a subset of every set. In other words, "If S is a set, then we have $\emptyset \subseteq S$."

- Then use this fact to prove that the empty set is unique. (Suppose that $\emptyset_1$ and $\emptyset_2$ are both empty sets, and then prove that $\emptyset_1 = \emptyset_2$.)

Solution ↑

Let S be a set. We will show that $\emptyset \subseteq S$. Suppose by way of contradiction that $\emptyset \nsubseteq S$, which means there is some element $x \in \emptyset$ that is not contained in S. Since $\emptyset$ contains no elements, this is impossible and the statement $\emptyset \nsubseteq S$ is false. Therefore, $\emptyset \subseteq S$.

Now we will show that the empty set is unique — in other words, given two empty sets $\emptyset_1$ and $\emptyset_2$, we will show $\emptyset_1 = \emptyset_2$. Let $\emptyset_1$ and $\emptyset_2$ be empty sets. To show they are equal to each other, we must show that they are subsets of each other. As shown above, $\emptyset \subseteq S$ for any set S, and since $\emptyset_1$ and $\emptyset_2$ are both empty sets, it follows that $\emptyset_1 \subseteq \emptyset_2$ and $\emptyset_2 \subseteq \emptyset_1$. Thus we know that $\emptyset_1 = \emptyset_2$.

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Solution

Limit Points Of Subsets Are Limit Points Of The Larger Set Benjamin

Contents [hide]

Problem 24: (Limit Points Of Subsets Are Limit Points Of The Larger Set) ↑

Suppose that A and B are subsets of the real numbers. Prove that if $A \subseteq B$ and p is a limit point of A, then p is a limit point of B.

Solution ↑

Let A and B be sets of real numbers with $A \subseteq B$, with p being a limit point of A. Let I be an open interval such that $p \in I$. Because p is a limit point of A, we know there is some $x \in A$ such that $x \in I$ with $x \neq p$. Let $x \in A$ such that $x \in I$ with $x \neq p$. Because $x \in A$ we also know $x \in B$. Thus we know that p must also be a limit point of B.

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Solution

The Order Of Quantifiers Matters Benjamin

Contents [hide]

Problem 25: (The Order Of Quantifiers Matters) ↑

Translate each of the following into an English sentence. Then determine the truth value of each statement.

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y+1>x$.
2. $\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}$, we have $y+1>x$.

Solution ↑

For the first statement, we can rewrite it in English as, "For all $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that $y+1>x$." By Hill's Archimedian Expansion, the reals are not bounded above, which means for any choice of $x \in \mathbb{R}$, there is always another real number greater than x, which can be expressed as $y+1$. Therefore, this statement is true.

For the second statement, we can rewrite it in English as, "There exists a $y \in \mathbb{R}$ such that, for all $x \in \mathbb{R}$, we have $y+1>x$." This is false by Hill's Archimedian Expansion, because the reals are unbounded so there is no number $y+1 \in \mathbb{R}$ that is greater than all other real numbers.

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Solution

Associative Laws For Set Unions And Intersections Benjamin

Problem 30: (Associative Laws For Set Unions And Intersections) ↑

Contents [hide]

Let A, B, and C be sets.

1. Prove that $A \cup (B \cup C) = (A \cup B) \cup C$.
2. Prove that $A \cap (B \cap C) = (A \cap B) \cap C$.

Solution ↑

We will show that $A \cup (B \cup C) = (A \cup B) \cup C$. To show they are equal, we must show that each set is a subset of the other. First let $x \in A \cup (B \cup C)$. This means that either $x \in A$, or $x \in B$ or $x \in C$. Therefore, it is also clear that $x \in A$ or $x \in B$, or $x \in C$, which can be written symbolically as $x \in (A \cup B) \cup C$. Therefore, we know that $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. A similar argument shows that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$, so we have shown $A \cup (B \cup C) = (A \cup B) \cup C$.

Now we will show that $A \cap (B \cap C) = (A \cap B) \cap C$. First let $x \in A \cap (B \cap C)$. This means that $x \in A$, and $x \in B$ and $x \in C$. Therefore, it is also clear that $x \in A$ and $x \in B$, and $x \in C$, which can be written symbolically as $x \in (A \cap B) \cap C$. Therefore, we know that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$. A similar argument shows that $(A \cap B) \cap C \subseteq A \cap (B \cap C)$, so we have shown $A \cap (B \cap C) = (A \cap B) \cap C$.

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Solution

Induction With The Sum Of The Squares Of The First N Natural Numbers Benjamin

Contents [hide]

Problem 35: (Induction With The Sum Of The Squares Of The First n Natural Numbers) ↑

Use induction to prove that $1^2+2^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$ for every $n\in\mathbb{N}$.

Solution ↑

For every $n\in\mathbb{N}$, let S_n be the statement $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$. We will show by induction that S_n is true for all $n\in\mathbb{N}$.

First we will show that S_1 is true. The statement S_1 is $1^2=\frac{1(1+1)(2(1)+1)}{6}$, and manipulating the right-hand side of the equation we see that

$$\begin{aligned}\frac{1(1+1)(2(1)+1)}{6}&=\frac{1(2)(3)}{6}\\&=\frac{6}{6}\\&=1\\&=1^2.\end{aligned}$$

Thus we see that S_1 is true. Now we assume that S_k is true for some $k\in\mathbb{N}$. We will now show this implies that S_{k+1} is also true, which means we must prove that

$$1^2+2^2+\cdots+k^2+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

Because of our assumption that $1^2+2^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$, we know we must prove that

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.$$

Manipulating the left-hand side of the equation, we see that

$$\begin{aligned}
\frac{k(k+1)(2k+1)}{6} + (k+1)^2 &= \frac{2k^3+3k^2+k}{6} + k^2+2k+1 && \text{(foil)}\\
&= \frac{2k^3+3k^2+k}{6} + \frac{6k^2+12k+6}{6} && \text{(find common denominator)}\\
&= \frac{2k^3+9k^2+13k+6}{6} && \text{(combine fractions)}\\
&= \frac{(k^2+3k+2)(2k+3)}{6} && \text{(factor)}\\
&= \frac{(k+1)(k+2)(2k+3)}{6} && \text{(factor)}\\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} && \text{(expand).}
\end{aligned}$$

Thus we see that if S_k is true, clearly S_{k+1} must also be true. By the principle of mathematical induction we conclude that S_n is true for all $n \in \mathbb{N}$.

Tags ↑

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Solution

Set Complements Rules 1 And 2 Benjamin

Contents [hide]

Problem 36: (Set Complements Rules 1 And 2) ↑

Prove rules 1 and 2 of Theorem (Rules For Set Complements).

Solution ↑

We will show that $A \setminus A = \emptyset$. Despite the fact that $A \setminus A$ is the set of elements of A that are not in A, we know there is no $x \in A$ that is not in A. Since there are no elements in $A \setminus A$, we know it is an empty set, and the empty set is unique, so $A \setminus A = \emptyset$.

We will now show that $A \setminus \emptyset = A$. First let $x \in A \setminus \emptyset$. By definition of set complements, we know that $x \in A$ and also $x \notin \emptyset$. Since $x \in A$, we know that $A \setminus \emptyset \subseteq A$. Now let $x \in A$. Since $A \setminus \emptyset = \{a \mid a \in A \text{ and } a \notin \emptyset\}$, to show that $x \in A \setminus \emptyset$ we must show that $x \in A$ and also $x \notin \emptyset$. We already defined $x \in A$, so that is true. Now, since there are no elements in the empty set, there does not exist any number y such that $y \in \emptyset$, so we know $x \notin \emptyset$. This shows that $A \subseteq A \setminus \emptyset$. Thus we see that $A \setminus \emptyset = A$.

Tags ↑

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Solution

Set Complements Rule 5 Benjamin

Problem 38: (Set Complements Rule 5) ↑

Prove rule 5 of Theorem (Rules For Set Complements).

Contents [hide]

Solution ↑

First, let $x \in A \setminus (B \setminus C)$, it must be that $x \in A$ and also $x \notin B \setminus C$. For some number $y \in B \setminus C$, it must be that $y \in B$ and also $y \notin C$. This means that since $x \notin B \setminus C$, it must be that $x \notin B$ or $x \in C$. Therefore, since $x \in A \setminus (B \setminus C)$ we know that $x \in A$ and also $x \notin B$, or that $x \in A$ and $x \in C$. This can be rewritten as $x \in (A \setminus B) \cup (A \cap C)$, which shows that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$.

Now, let $x \in (A \setminus B) \cup (A \cap C)$. We know $x \in A$ and also $x \notin B$, or both $x \in A$ and $x \in C$. Note that $x \in A$ in both cases. Since the statement "$x \in B$ and also $x \notin C$" is clearly the negation of "$x \notin B$ or $x \in C$", we know $x \notin B \setminus C$. Because $x \in A$ and also $x \notin B \setminus C$, this can be rewritten as $x \in A \setminus (B \setminus C)$. Therefore we see that $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$, and because both sets are subsets of the other we know $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

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Solution

Which Relations Are Functions Benjamin

Contents [hide]

Problem 42: (Which Relations Are Functions) ↑

In each item below, you are given a relation R between a set A and a set B. Prove that the relation is a function from A into B or give a counter example to show that the relation is not a function from A into B.

1. Let R be the relation between $\mathbb{N}$ and $\mathbb{Z}$ given by $(n,m) \in \mathrm{R}$ if and only if $n^2 = m$.
2. Let R be the relation between $\mathbb{N}$ and $\mathbb{Z}$ given by $(n,m) \in \mathrm{R}$ if and only if $n = m^2$.
3. Let R be the relation between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ given by $((x,y),z) \in \mathrm{R}$ if and only if $x + y = z$.
4. Let R be the relation between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ given by $((x,y),z) \in \mathrm{R}$ if and only if $x/y = z$.

Solution ↑

Let R be the relation between $\mathbb{N}$ and $\mathbb{Z}$ given by $(n,m) \in \mathrm{R}$ if and only if $n^2 = m$. We will show that R is a function. We will show that for all $n \in \mathbb{N}$ there exists a unique $m \in \mathbb{Z}$ such that $(n,m) \in \mathrm{R}$. Let $n \in \mathbb{N}$ and let $m_1 = n^2$. Because $n \in \mathbb{N}$, clearly $m_1 \in \mathbb{Z}$, so we know $(n, m_1) \in \mathrm{R}$. Now let $m_2 \in \mathbb{Z}$ with $(n, m_2) \in \mathrm{R}$. Because $(n, m_2) \in \mathrm{R}$, we know $m_2 = n^2$. Thus we see that $m_1 = n^2 = m_2$, so R must be a function.

Now let R be the relation between $\mathbb{N}$ and $\mathbb{Z}$ given by $(n,m) \in \mathrm{R}$ if and only if $n = m^2$. We will show that R is not a function, which means we will show that there exists some $n \in \mathbb{N}$ such that there does not exist a unique $m \in \mathbb{Z}$. Let $n = 4$ and $m_1 = -2$ and $m_2 = 2$. We see that $n = 4 = (-2)^2 = 2^2$, and therefore because $-2 \neq 2$ (in other words, $m_1 \neq m_2$), we see that R cannot be a function.

Now let R be the relation between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ given by $((x,y),z) \in \mathrm{R}$ if and only if $x + y = z$. We will show that R is a function. Let $x, y \in \mathbb{R}$ and $z_1 = x + y$. Clearly $x + y = z_1$, so we know $z_1 \in \mathbb{R}$. Since clearly $(x,y) \in \mathbb{R} \times \mathbb{R}$ and $z_1 \in \mathbb{R}$, we know that $((x,y), z_1) \in \mathrm{R}$. Now let $z_2 \in \mathbb{R}$ such that $((x,y), z_2) \in \mathrm{R}$. Since $((x,y), z_2) \in \mathrm{R}$, we know $x + y = z_2$, which means that $z_1 = x + y = z_2$. Therefore R must be a function.

Now let R be the relation between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ given by $((x,y),z) \in \mathrm{R}$ if and only if $x/y = z$. We will show that R is not a function, which means we will show that there exists some $(x,y) \in \mathbb{R} \times \mathbb{R}$ such that there does not exist a unique $z \in \mathbb{R}$. Let $x = 1$ and $y = 0$. The expression $1/0$ is undefined, which proves there does not exist a unique $z \in \mathbb{R}$ such that $((1,0),z) \in \mathrm{R}$. Therefore, R cannot be a function.

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Solution

Practice With Injective And Surjective Benjamin

Problem 43: (Practice With Injective And Surjective) ↑

Contents [hide]

For each function below, state the domain and codomain, determine if the function is injective, and then determine if the function is surjective.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.
2. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^2$.
3. Let $f : \mathbb{R} \to [0, \infty)$ be defined by $f(x) = x^2$.
4. Let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2$.

As always, remember to justify each claim you make.

Solution ↑

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. The domain and codomain of f are both $\mathbb{R}$. We will show that f is not injective, which means we will show there exist $a, b \in \mathbb{R}$ such that $f(a) = f(b)$ and $a \neq b$. Let $a = 2$ and $b = -2$. We see that $f(a) = 2^2 = 4$ and $f(b) = (-2)^2 = 4$, but $a \neq b$. Therefore f cannot be injective.

We will now show that f is not surjective, which means we will show there exists some $y \in \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have that $y \neq f(x)$. Let $y = -1$ and $x \in \mathbb{R}$. Since x^2 cannot be negative, we know that $f(x) = x^2 \neq -1$, and therefore $y \neq f(x)$. Thus we see that f is not surjective.

Now let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^2$. The domain of f is $[0, \infty)$ and the codomain is $\mathbb{R}$. We will show that f is injective, which means we will show that for every $a, b \in [0, \infty)$ we have that $f(a) = f(b)$ implies $a = b$. Let $a, b \in [0, \infty)$. Suppose that $f(a) = f(b)$, or in other words $a^2 = b^2$. Since $a^2 = b^2$, by taking the square root of both sides we know that $a = \pm b$. Since both a and b are defined as being positive, we drop the negative and find that $a = b$. Thus, if $f(a) = f(b)$, then $a = b$. Therefore f is injective.

We will now show that f is not surjective, which means we will show there exists some $y \in \mathbb{R}$ such that for every $x \in [0, \infty)$ we have that $y \neq f(x)$. Let $y = -1$ and $x \in [0, \infty)$. Since x^2 cannot be negative, we know that $f(x) = x^2 \neq -1$, and therefore $y \neq f(x)$. Thus we see that f is not surjective.

Now let $f : \mathbb{R} \to [0, \infty)$ be defined by $f(x) = x^2$. The domain of f is $\mathbb{R}$ and the codomain is $[0, \infty)$. We will show that f is not injective, which means we will show there exist $a, b \in \mathbb{R}$ such that $f(a) = f(b)$ and $a \neq b$. Let $a = 2$ and $b = -2$. We see that $f(a) = 2^2 = 4$ and $f(b) = (-2)^2 = 4$, but $a \neq b$. Therefore f cannot be injective.

We will now show that f is surjective, which means we will show that for every $y \in [0, \infty)$ there exists an $x \in \mathbb{R}$ such that $y = f(x)$. Let $y \in [0, \infty)$ and $x = \sqrt{y}$. Note that since y is positive, we know $\sqrt{y} \in \mathbb{R}$ and thus $x \in \mathbb{R}$. Since $f(x) = x^2$ and $x = \sqrt{y}$, we see that $f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = y$, so f must be surjective.

Now let $f : [0, \infty) \to [0, \infty)$ be defined by $f(x) = x^2$. The domain and codomain of f are both $[0, \infty)$. We will show that f is injective, which means we will show that for every $a, b \in [0, \infty)$ we have that $f(a) = f(b)$ implies $a = b$. Let $a, b \in [0, \infty)$. Suppose that $f(a) = f(b)$, or in other words $a^2 = b^2$. Since $a^2 = b^2$, by taking the square root of both sides we know that $a = \pm b$. Since both a and b are defined as being positive, we drop the negative and find that $a = b$. Thus, if $f(a) = f(b)$, then $a = b$. Therefore f is injective.

We will now show that f is surjective, which means we will show that for every $y \in [0, \infty)$ there exists an $x \in [0, \infty)$ such that $y = f(x)$. Let $y \in [0, \infty)$ and $x = \sqrt{y}$. Note that since y is positive, we know $\sqrt{y} \in [0, \infty)$ and thus $x \in [0, \infty)$. Since $f(x) = x^2$ and $x = \sqrt{y}$, we see that $f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = y$, so f must be surjective.

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Solution

Induction With Sum Of Odds Benjamin

Contents [hide]

Problem 44: (Induction With Sum Of Odds) ↑

Notice that

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \\ 1+3+5+7+9 &= 25. \end{aligned}$$

Make a conjecture about the sum of odd numbers by giving a general formula for what you see above. Then prove that your formula is valid by using mathematical induction.

Solution ↑

Considering the set of equations given, we conjecture that a general formula that describes the relationship is given by the statement $1+3+5+\cdots+(2n-1)=n^2$ with $n \in \mathbb{N}$. We will prove by induction that this statement is true for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let S_n be the statement $1+3+5+\cdots+(2n-1)=n^2$. Clearly S_1 is true because $2(1)-1=1$, as shown in the given equations. Now we will assume that S_k is true for some $k \in \mathbb{N}$, which means $1+3+5+\cdots+(2k-1)=k^2$. We will now show that this implies S_{k+1} is also true, which means we must prove that

$$1+3+5+\cdots+(2k-1)+(2(k+1)-1)=(k+1)^2.$$

Manipulating the left-hand side of the equation, we see that

$$\begin{aligned} 1+3+5+\cdots+(2k-1)+(2(k+1)-1) &= k^2+(2(k+1)-1) && \text{(replace with assumption that } S_k \text{ is true)} \\ &= k^2+2k+1 && \text{(distribute)} \\ &= (k+1)(k+1) && \text{(factor)} \\ &= (k+1)^2. \end{aligned}$$

Thus we see that if S_k is true, clearly S_{k+1} must also be true. By the principle of mathematical induction we conclude that S_n is true for all $n \in \mathbb{N}$.

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Solution

Showing A Sequence Converges Benjamin

Contents [hide]

Problem 49: (Showing A Sequence Converges) ↑

Consider the sequence $(a_n) = \left(\frac{n+1}{n}\right)$. Prove that (a_n) converges to $L = 1$.

Solution ↑

Let $(a_n) = \left(\frac{n+1}{n}\right)$. We will show that $(a_n) \to L$ with $L = 1$. This means we must prove that for every $\varepsilon > 0$, there exists a real number M such that for every $n \in \mathbb{N}$, we have that $n > M$ implies $|a_n - 1| < \varepsilon$. Let $\varepsilon > 0$ and $M = \frac{1}{\varepsilon}$ and $n \in \mathbb{N}$. Suppose $n > M$. Considering the expression $|a_n - 1|$, we compute

$$\begin{aligned}
|a_n - 1| &= \left|\left(\frac{n+1}{n}\right) - 1\right| && \text{(replace } a_n \text{ with the sequence)} \\
&= \left|\frac{n+1}{n} - \frac{n}{n}\right| && \text{(find common denominator)} \\
&= \left|\frac{n+1-n}{n}\right| && \text{(simplify)} \\
&= \frac{1}{n} && \text{(simplify; } n \text{ is positive)} \\
&< \frac{1}{M} && (n > M) \\
&= \varepsilon && (M = \frac{1}{\varepsilon}).
\end{aligned}$$

Thus we see that $n > M$ implies $|a_n - 1| < \varepsilon$, which completes the proof that the sequence (a_n) converges to $L = 1$.

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Solution

A Convergent Sequence Has A Unique Limit Benjamin

Problem 52: (A Convergent Sequence Has A Unique Limit) ↑

Let (a_n) be a convergent sequence of real numbers. Prove that (a_n) converges to a unique real number.

Contents [hide]

Note: The general way to prove something is unique is to suppose that there are two of those things, and then prove they must be equal. We need to prove that if (a_n) converges to both A and B, then we must have $A = B$. The contrapositive of this statement may be easier to work with, or possibly a proof by contradiction instead.

Solution ↑

Let (a_n) be a convergent sequence. We will show that the limit of convergence of (a_n) is unique. Suppose that L_1 and L_2 are limits of convergence of (a_n). Suppose by way of contradiction that $L_1 \neq L_2$. Without loss of generality, suppose that $L_1 > L_2$. As a reminder, because $(a_n) \to L_1$, we know that for every $\varepsilon > 0$, there exists a real number M_1 such that for every $n \in \mathbb{N}$, we have that $n > M_1$ implies $|a_n - L_1| < \varepsilon$. Let $\varepsilon = \frac{L_1 - L_2}{2}$, and note that $\varepsilon > 0$. Because $\varepsilon > 0$ and $(a_n) \to L_1$, we now pick M_1 such that for every $n \in \mathbb{N}$, we have that $n > M_1$ implies $|a_n - L_1| < \varepsilon$. In a similar fashion, pick M_2 such that for every $n \in \mathbb{N}$, we have that $n > M_2$ implies $|a_n - L_2| < \varepsilon$.

Let $n \in \mathbb{N}$ such that $n > M_1$ and $n > M_2$; this means that $|a_n - L_1| < \varepsilon$ and $|a_n - L_2| < \varepsilon$. Since $|a_n - L_1| < \varepsilon$, we know that $a_n - L_1 < \varepsilon$ and therefore $a_n < L_1 + \varepsilon$; we also know that $-(a_n - L_1) < \varepsilon$, which means $a_n - L_1 > -\varepsilon$ and therefore $a_n > L_1 - \varepsilon$. Thus we see that $L_1 - \varepsilon < a_n < L_1 + \varepsilon$, which means that a_n is in the epsilon neighborhood of L_1, which can be written as $N_\varepsilon(L_1) = (L_1 - \varepsilon, L_1 + \varepsilon)$. Using similar logic, we know a_n is likewise in the neighborhood of L_2, written $N_\varepsilon(L_2) = (L_2 - \varepsilon, L_2 + \varepsilon)$. Because $\varepsilon = \frac{L_1 - L_2}{2}$, we compute

$$\begin{aligned}
L_1 - \varepsilon &= L_1 - \frac{L_1 - L_2}{2} \\
&= L_1 - \frac{L_1}{2} + \frac{L_2}{2} \\
&= \frac{L_1}{2} + \frac{L_2}{2} \\
&= L_2 + \frac{L_1}{2} - \frac{L_2}{2} \\
&= L_2 + \frac{L_1 - L_2}{2} \\
&= L_2 + \varepsilon.
\end{aligned}$$

This shows that $a_n < L_2 + \varepsilon = L_1 - \varepsilon < a_n$, which is a contradiction, which must mean that $L_1 = L_2$. Therefore, the limit of convergence of the sequence (a_n) is unique.

Tags ↑

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Solution

Induction And An Inequality Benjamin

Problem 55: (Induction And $2^n \geq n^2$) ↑

Use induction to prove that for every $n \in \mathbb{N}$ except for $n = 3$, we have $2^n \geq n^2$.

Contents [hide]

Solution ↑

For all $n \in \mathbb{N}$, let S_n be the statement $2^n \geq n^2$. We will show that S_n is true for all $n \in \mathbb{N} \setminus \{3\}$.

First we will show that S_3 is false. Clearly $2^3 \geq 3^2$ is false because $8 \not\geq 9$. Now we will show that S_1 and S_2 and S_4 are true. Consider S_1. We know that $2^1 \geq 1^2$ is true because $2 \geq 1$. Now consider S_2. We know that $2^2 \geq 2^2$ is true because $4 \geq 4$. Now consider S_4. We know that $2^4 \geq 4^2$ is true because $16 \geq 16$.

Now we will show that S_n is true for $n > 3$. We will assume the statement S_k is true, for some $k \in \mathbb{N}$ and $k \geq 4$, which means $2^k \geq k^2$. Now we must prove that S_{k+1} is true, which means we must prove $2^{k+1} \geq (k+1)^2$. Manipulating the left-hand side of the inequality, we find that

$$\begin{aligned}
2^{k+1} &= 2(2^k) && \text{(exponent rules)} \\
&\geq 2(k^2) && \text{(replace with assumption that } 2^k \geq k^2\text{)} \\
&= k^2 + k^2 && \\
&= k^2 + k(k) && \\
&\geq k^2 + 4k && (k \geq 4) \\
&= k^2 + (2k + 2k) && \\
&> k^2 + (2k + 1) && (k > \frac{1}{2}) \\
&= (k+1)^2 && \text{(factor).}
\end{aligned}$$

Therefore, clearly $2^{k+1} \geq (k+1)^2$. Thus, by the principle of mathematical induction, we see that S_n is true for all $n \in \mathbb{N} \setminus \{3\}$.

Tags ↑

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Solution

Image And Preimage Properties 1 And 2 Benjamin

Contents [hide]

Problem 58: (Image And Preimage Properties 1 And 2) ↑

Prove properties 1 and 2 for images and preimages. So prove for a function $f : X \to Y$ both of the following.

1. If $A \subseteq X$, then $A \subseteq f^{-1}(f(A))$.
2. If $B \subseteq Y$, then $f(f^{-1}(B)) \subseteq B$.

Then give an example of a function $f : X \to Y$ and subsets $A \subseteq X$ and $B \subseteq Y$ where $A \neq f^{-1}(f(A))$ and $B \neq f(f^{-1}(B))$.

Definition (Image $f(A)$ and Preimage $f^{-1}(B)$) ↑

Consider the function $f : X \to Y$. Let A be a subset of the domain X and let B be a subset of the codomain Y.

- The image of A under f is the subset of Y defined by

$$\begin{aligned} f(A) &= \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \\ &= \{f(a) \mid a \in A\}. \end{aligned}$$

  This means that $y \in f(A)$ if and only if $y = f(a)$ for some $a \in A$.
- The preimage (or inverse image) of B under f is the subset of X defined by

$$\begin{aligned} f^{-1}(B) &= \{x \in X \mid f(x) = b \text{ for some } b \in B\} \\ &= \{x \in X \mid f(x) \in B\}. \end{aligned}$$

  This means that $x \in f^{-1}(B)$ if and only if $f(x) = b$ for some $b \in B$ if and only if $f(x) \in B$. Note that when the set B contains a single element, then we write $f^{-1}(b)$ rather than $f^{-1}(\{b\})$.

Solution ↑

Let f be a function such that $f : X \to Y$. First we will show that if $A \subseteq X$, then $A \subseteq f^{-1}(f(A))$. Suppose $A \subseteq X$ and let $a \in A$. Because $A \subseteq X$, we know $a \in X$. By definition of image, we know $f(A) = \{f(a) \mid a \in A\}$, so clearly $f(a) \in f(A)$. By the definition of preimage, we know that if $f(x) \in f(A)$, then $x \in f^{-1}(f(A))$. Because we have shown $f(a) \in f(A)$, we know that $a \in f^{-1}(f(A))$. This concludes our proof that $A \subseteq X$ implies $A \subseteq f^{-1}(f(A))$.

Now we will show that if $B \subseteq Y$, then $f(f^{-1}(B)) \subseteq B$. Suppose $B \subseteq Y$ and let $y \in f(f^{-1}(B))$. Because $y \in f(f^{-1}(B))$, we know $y = f(x)$ for some $x \in f^{-1}(B)$, by definition of image of $f^{-1}(B)$. Pick such an x. Because $x \in f^{-1}(B)$, we know $f(x) = b$, for some $b \in B$. Pick such a b. Since $y = f(x) = b$, we know $y = b \in B$. Thus we see that $f(f^{-1}(B)) \subseteq B$.

Now we will examine an example of a function $f : X \to Y$ and subsets $A \subseteq X$ and $B \subseteq Y$ where $A \neq f^{-1}(f(A))$ and $B \neq f(f^{-1}(B))$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos(x)$, and let $A = \{0\}$ and $B = \{2\}$. Clearly $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Because A and B each contain only one element, let $a = 0$ and $b = 2$, with $a \in A$ and $b \in B$. We know that $f(a) = f(0) = \cos(0) = 1$. Therefore, we also see that $f^{-1}(f(a)) = f^{-1}(1) = \{x \mid \cos(x) = 1\} = \{0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots\}$; clearly the set $\{0, \pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots\}$ is not equal to $\{0\}$, so we know $A \neq f^{-1}(f(A))$. Similarly, we know that $f^{-1}(b) = f^{-1}(2) = \{x \mid \cos(x) = 2\}$. Because $\cos(x) \neq 2$ for all $x \in \mathbb{R}$, the set of outputs is the empty set. Thus we have that $f(f^{-1}(b)) = f^{-1}(\emptyset) = \emptyset$. Since $B \neq \emptyset$, we know $B \neq f(f^{-1}(B))$.

Tags ↑

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Solution

Proving A Quotient Of Two Linear Sequences Converges Benjamin

Problem 67: (Proving A Quotient Of Two Linear Sequences Converges) ↑

Contents [hide]

Prove that $\left(\frac{2n+1}{3n+4}\right)$ converges to $\frac{2}{3}$.

Hint: Given $\varepsilon > 0$, solve the equality $|a_M - L| = \varepsilon$ for M, which should help you find a value M you can choose to satisfy the definition of converges. So start by solving $\left|\frac{2M+1}{3M+4} - \frac{2}{3}\right| = \varepsilon$ for M.

Solution ↑

Let (a_n) be the sequence defined as $(a_n) = \left(\frac{2n+1}{3n+4}\right)$. We will show that $(a_n) \to \frac{2}{3}$. Let $\varepsilon > 0$ and $M = \frac{1}{\varepsilon}$. Let $n \in \mathbb{N}$ and suppose $n > M$. We compute

$$\begin{aligned}
\left|a_n - \frac{2}{3}\right| &= \left|\frac{2n+1}{3n+4} - \frac{2}{3}\right| \\
&= \left|\frac{3(2n+1)}{3(3n+4)} - \frac{2(3n+4)}{3(3n+4)}\right| && \text{(find common denominator)} \\
&= \left|\frac{-5}{9n+12}\right| \\
&= \frac{5}{9n+12} \\
&< \frac{5}{9n} \\
&< \frac{5}{5n} \\
&= \frac{1}{n} \\
&< \frac{1}{M} && (n > M) \\
&= \varepsilon.
\end{aligned}$$

Because $\left|a_n - \frac{2}{3}\right| < \varepsilon$, we conclude that $(a_n) \to \frac{2}{3}$.

Tags ↑

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Solution

Hills Real Archimedian Expansion Benjamin

Theorem (Hills Real Archimedian Expansion) ↑

For each set $S \in \{\mathbb{R}, \mathbb{Z}, \mathbb{Q}\}$, the set S is not bounded above nor below. The set $\mathbb{N}$ is also not bounded above.

Contents [hide]

Solution ↑

We will prove that the set of natural numbers $\mathbb{N}$ is not bounded above. Suppose by way of contradiction that the set of natural numbers is bounded above. Let m be the supremum of $\mathbb{N}$. Because m is the supremum of $\mathbb{N}$, we know that $m-1$ is not an upper bound for $\mathbb{N}$. This means we can pick a natural number n with $m-1<n$. Hence we know $m<n+1$. However, notice that $n+1 \in \mathbb{N}$, which means we have a natural number larger than m, which is impossible based on the definition of m. This means that the set of natural numbers is not bounded above.

The natural numbers are bounded below because 1 is the smallest element of $\mathbb{N}$.

Because the set of natural numbers is not bounded above, for every $y \in \mathbb{R}$ we know y is not an upper bound for $\mathbb{N}$. This means for every real number y, there exists $n \in \mathbb{N}$ such that $n>y$.

We will prove that the set of real numbers $\mathbb{R}$ is not bounded above nor below. Suppose by way of contradiction that the set of real numbers is bounded. Let m be the supremum of $\mathbb{R}$. Because m is the supremum of $\mathbb{R}$, we know that $m-1$ is not an upper bound for $\mathbb{R}$. This means we can pick a real number n with $m-1<n$. Hence we know $m<n+1$. However, notice that $n+1 \in \mathbb{R}$, which means we have a real number larger than m, which is impossible based on the definition of m. This means that the set of real numbers is not bounded above.

Now let m be the infimum of $\mathbb{R}$. Because m is the infimum of $\mathbb{R}$, we know that $m+1$ is not a lower bound for $\mathbb{R}$. This means we can pick a real number n with $m+1>n$. Hence we know $m>n-1$. However, notice that $n-1 \in \mathbb{R}$, which means we have a real number smaller than m, which is impossible based on the definition of m. This means that the set of real numbers is not bounded below.

Because the set of real numbers is not bounded above nor below, for every $y \in \mathbb{R}$ we know y is not an upper bound nor a lower bound for $\mathbb{R}$. This means for every real number y, there exist m and n in $\mathbb{R}$ such that $m<y<n$.

We will prove that the set of integers $\mathbb{Z}$ is not bounded above nor below. Suppose by way of contradiction that the set of integers is bounded. Let m be the supremum of $\mathbb{Z}$. Because m is the supremum of $\mathbb{Z}$, we know that $m-1$ is not an upper bound for $\mathbb{Z}$. This means we can pick a real number n with $m-1<n$. Hence we know $m<n+1$. However, notice that $n+1 \in \mathbb{Z}$, which means we have an integer larger than m, which is impossible based on the definition of m. This means that the set of integers is not bounded above.

Now let m be the infimum of $\mathbb{Z}$. Because m is the infimum of $\mathbb{Z}$, we know that $m+1$ is not a lower bound for $\mathbb{Z}$. This means we can pick an integer n with $m+1>n$. Hence we know $m>n-1$. However, notice that $n-1 \in \mathbb{Z}$, which means we have an integer smaller than m, which is impossible based on the definition of m. This means that the set of integers is not bounded below.

Because the set of integers is not bounded above nor below, for every $y \in \mathbb{Z}$ we know y is not an upper bound nor a lower bound for $\mathbb{Z}$. This means for every real number y, there exist m and n in $\mathbb{R}$ such that $m < y < n$.

We will prove that the set of rational numbers $\mathbb{Q}$ is not bounded above nor below. Because all rational numbers can be written in the form $\frac{p}{q}$, with p and q in $\mathbb{Z}$ and $q \neq 0$, and since $\mathbb{Z}$ is not bounded above nor below, we see that $\mathbb{Q}$ also cannot be bounded above nor below.

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