Benjamin Hill

Brother Martin

MATH 412

6 February 2024

Reality Check 6: Tacoma Narrows Bridge

A mathematical model that attempts to capture the Tacoma Narrows Bridge incident was proposed by McKenna and Tuama. The goal is to explain how torsional oscillations can be magnified by forcing that is strictly vertical. Consider a roadway of width D hanging between two suspended cables, as in Figure 6.18(a).We will consider a two-dimensional slice of the bridge, ignoring the dimension of the bridge's length for this model, since we are only interested in the side-to-side motion. At rest, the roadway hangs at a certain equilibrium height due to gravity; let y denote the current distance the center of the roadway hangs below this equilibrium. Hooke's law postulates a linear response, meaning that the restoring force the cables apply will be proportional to the deviation. Let θ be the angle the roadway makes with the horizontal. There are two suspension cables, stretched $y - L\sin\theta$ and $y + L\sin\theta$ from equilibrium, respectively. Assume that a viscous damping term is given that is proportional to the velocity. Using Newton's law $F = ma$ and denoting Hooke's constant by K, the equations of motion for y and θ are as follows:

$$y'' = -dy' - \left[\frac{K}{m}(y - L\sin\theta) + \frac{K}{m}(y + L\sin\theta)\right]$$

$$\theta'' = -d\theta' + \frac{3\cos\theta}{L}\left[\frac{K}{m}(y - L\sin\theta) - \frac{K}{m}(y + L\sin\theta)\right]$$

However, Hooke's law is designed for springs, where the restoring force is more or less equal whether the springs are compressed or stretched. McKenna and Tuama hypothesize that cables pull back with more force when stretched than they push back when compressed. They replace the linear Hooke's law restoring force $f(y) = Ky$ with a nonlinear force $f(y) = \frac{K}{a}e^{-ay} - 1$. Both functions have the same slope K at $y = 0$; but for the nonlinear force, a positive y (stretched cable) causes a stronger restoring force than the corresponding negative y (slackened cable). Making this replacement in the preceding equations yields

$$y'' = -dy' - \frac{K}{m}\left[e^{a(y-L\sin\theta)} - 1 + e^{a(y+L\sin\theta)} - 1\right] + (0.2\,W\sin\omega t)$$

$$\theta'' = -d\theta' + \frac{3\cos\theta}{L}\,\frac{K}{ma}\left[e^{a(y-L\sin\theta)} - e^{a(y+L\sin\theta)}\right]$$

As the equations stand, the state $y = y' = \theta = \theta' = 0$ is an equilibrium. Now we turn on the wind, adding the forcing term $0.2\,W\sin\omega t$ (in parenthesis above) to the right-hand side of the y'' equation, where W is the wind speed in km/hr. This adds a strictly vertical oscillation to the bridge.

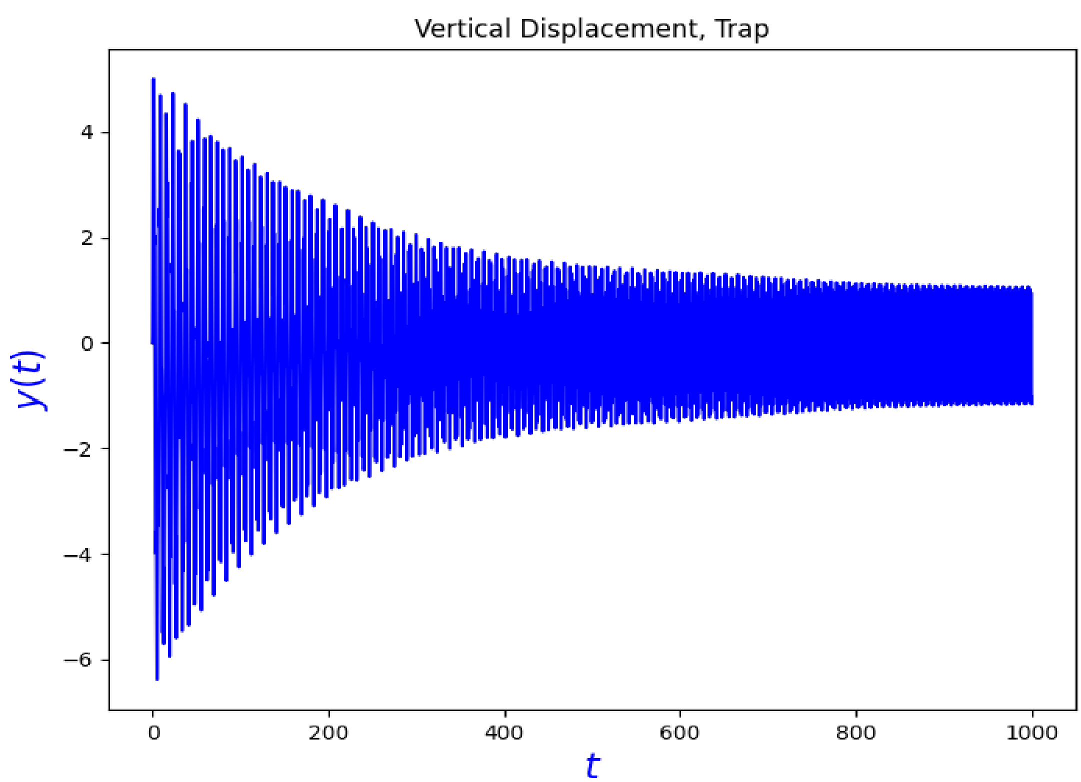
Useful estimates for the physical constants can be made. The mass of a one-foot length of roadway was about 2500 k g, and the spring constant K has been estimated at 1000 Newtons. The roadway was about 12 meters wide. For this simulation, the damping coefficient was set at $d = 0.01$, and the Hooke nonlinearity coefficient $a = 0.2$. Considering that an observer counted 38 vertical oscillations of the bridge in one minute shortly before the collapse, we set $\omega = 2\pi(38/60) = 3.97935$. These coefficients are only guesses, but they suffice to show ranges of motion that tend to match photographic evidence of the bridge's final oscillations.

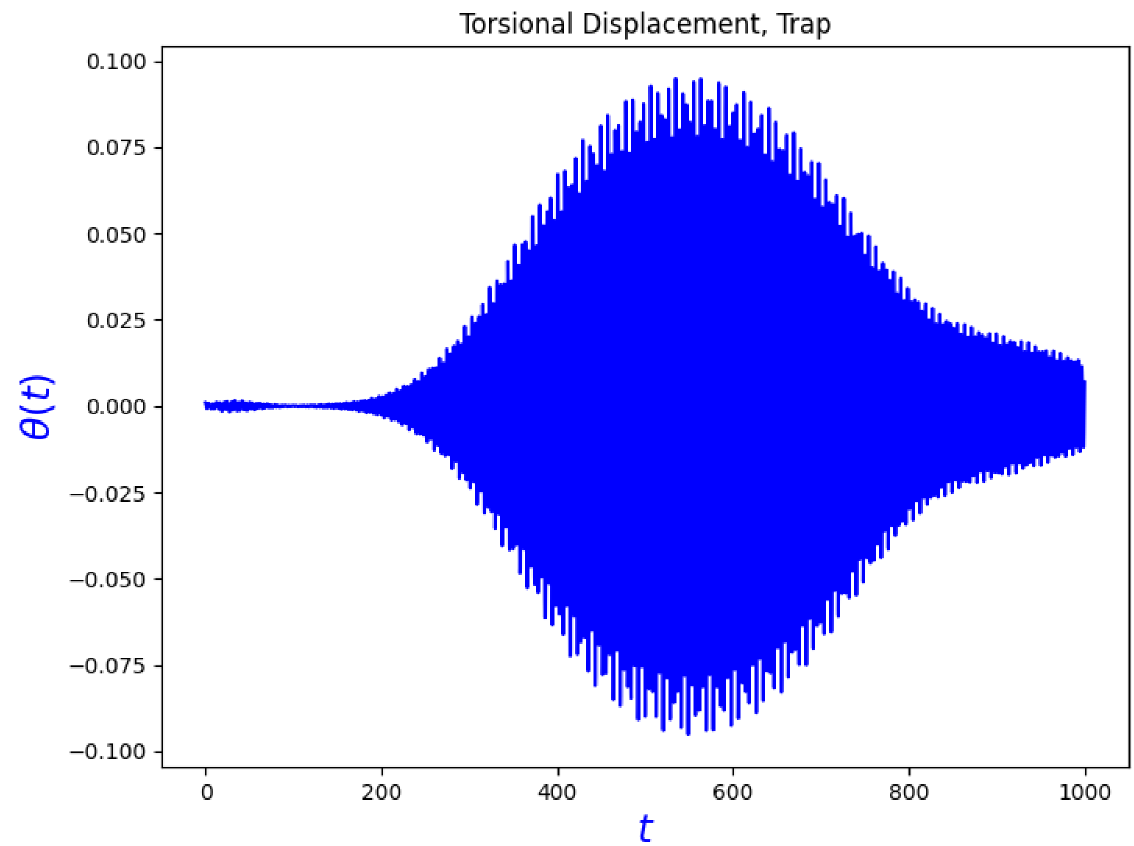
With the accompanying code, running the program with these parameter values allows us to see the phenomenon postulated earlier. If the angle θ of the roadway is set to any small nonzero value, vertical forcing causes θ to eventually grow to a macroscopic value, leading to significant torsion of the roadway. The interesting point is that there is no torsional forcing applied to the equation; the unstable "torsional mode'' is excited completely by vertical forcing.

Note: the windspeed observed on the day of the bridge collapse was about 64 km/hr (see https://en.wikipedia.org/wiki/Tacoma_Narrows_Bridge_(1940)).

Part 1

We will explore this situation, using the trapezoid method to approximate the oscillations with wind speed $W = 80\ km/hr$ and initial conditions $y = y' = \theta' = 0, \theta = 0.001$. The following graphs depict the vertical displacement and torsion of the bridge over time.

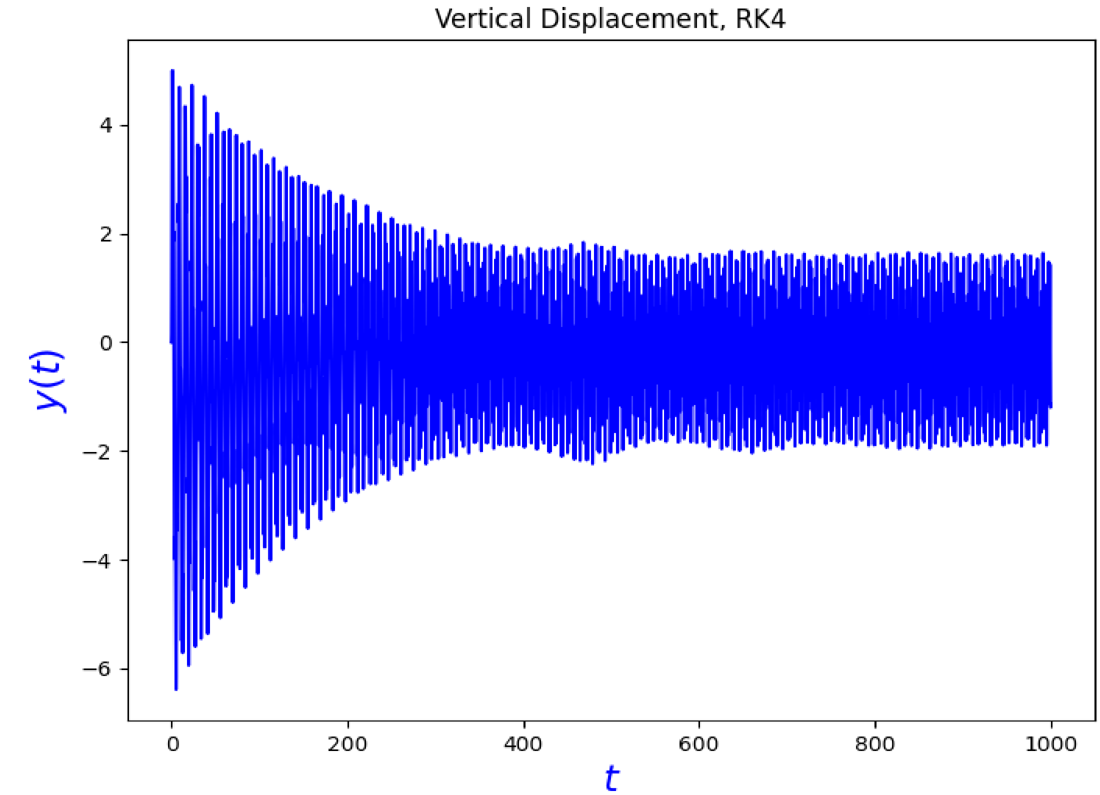


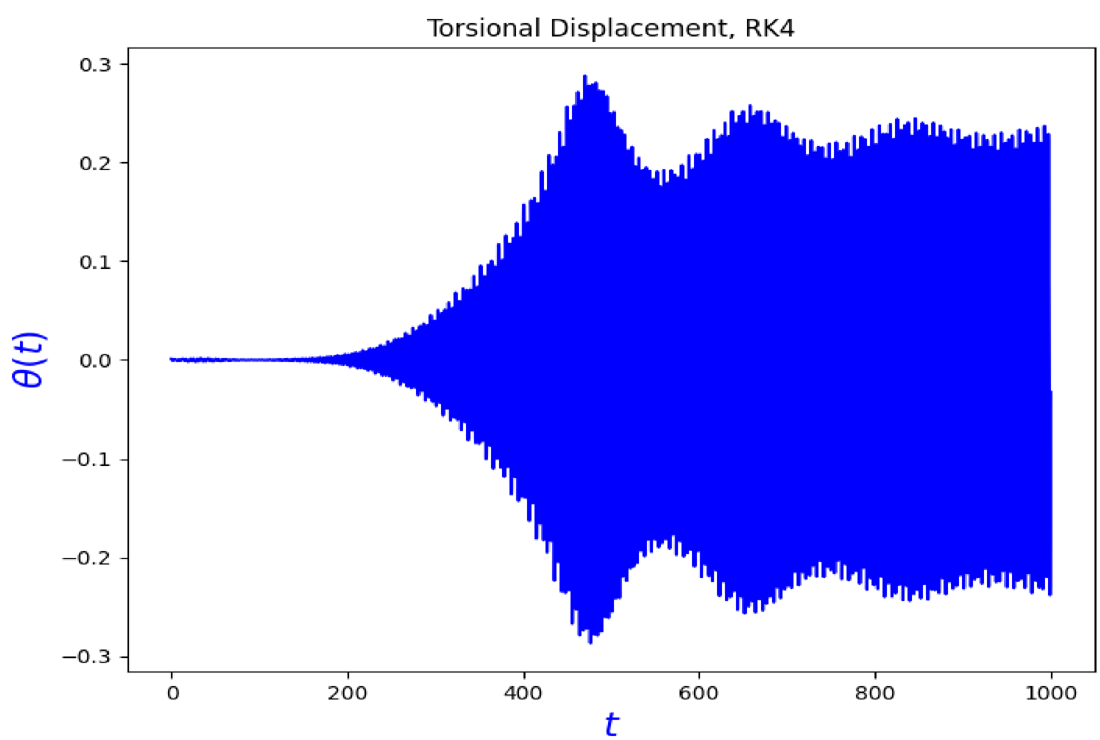


The bridge is said to be stable in the torsional dimension if small disturbances in θ die out; unstable if they grow far beyond original size. For this wind speed, we can see that the initial disturbance in the angle grows enormously, so the bridge is unstable under these conditions.

Part 2

We will now improve the accuracy of our approximation by replacing the trapezoid method with the fourth-order Runge-Kutta method. This yields the following graphs of the displacements.


Vertical Displacement, RK4
4
2
0
−2
−4
−6
$y(t)$
0
200
400
600
800
1000
t


Torsional Displacement, RK4
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
$\theta(t)$
0
200
400
600
800
1000
t

From the torsion graph we can see even more clearly than before that this bridge is unstable in these conditions, since the oscillations do not decrease significantly with time, once they start.

Part 3

The system is torsionally stable for $W = 50\ km/hr$. We will find the magnification factor for very small initial angles, $\theta(0) = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}$, which means we will find the ratio of the maximum angle $\theta(t), 0 \le t < \infty$, to $\theta(0)$. Computing it thus, we find that the magnification factors are 19.1849, 19.0567, 19.0554, and 19.0554, respectively, which means that the magnification factor is remarkably consistent in relative size for small initial angles.

Part 4

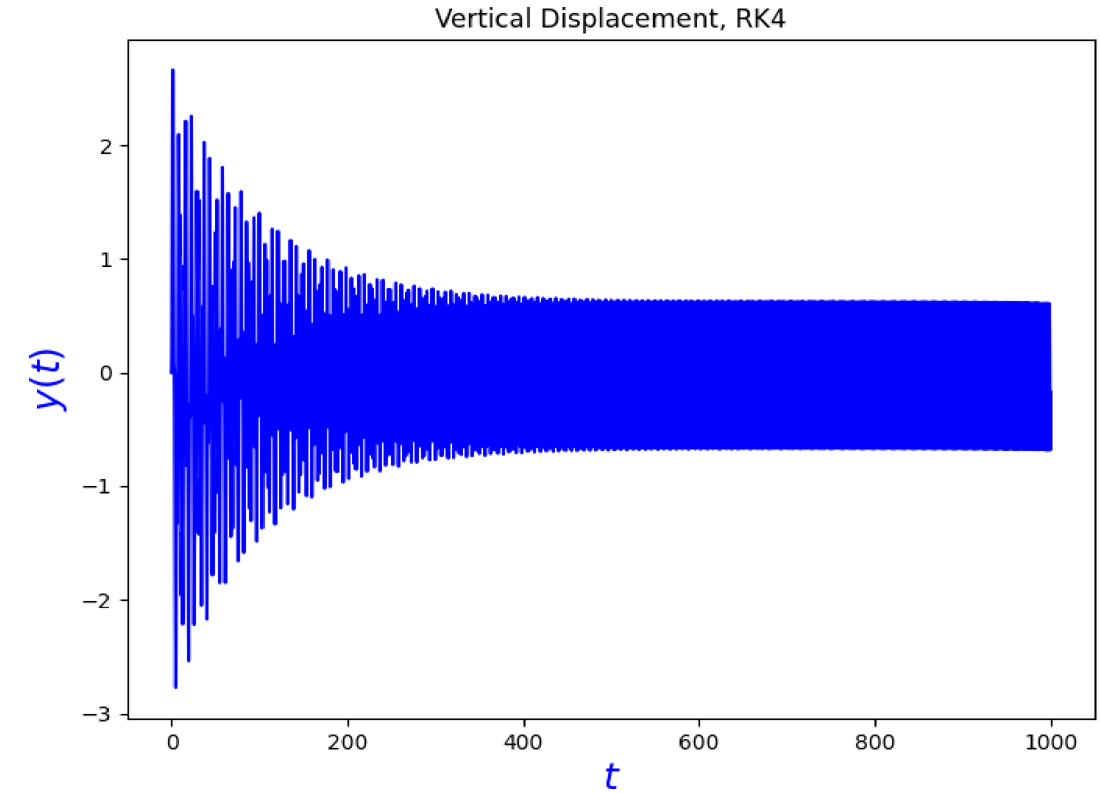
Now we will find the minimum wind speed W for which the small disturbance of $\theta(0) = 10^{-3}$ has a magnification factor of 100 or more. In this case, the method used was simple trial and error, computing the magnification factor for various choices of W until the correct one appeared. This led to the conclusion that the minimum wind speed that has a magnification factor of 100 or more is approximately 58.993 km/h. For the same set of initial angles as in Part 3, the respective magnification factors are 100.0301, 75.2685, 75.1504, and 75.1492, which is decently consistent but not as consistent as it was with the wind speed of 50 km/h.

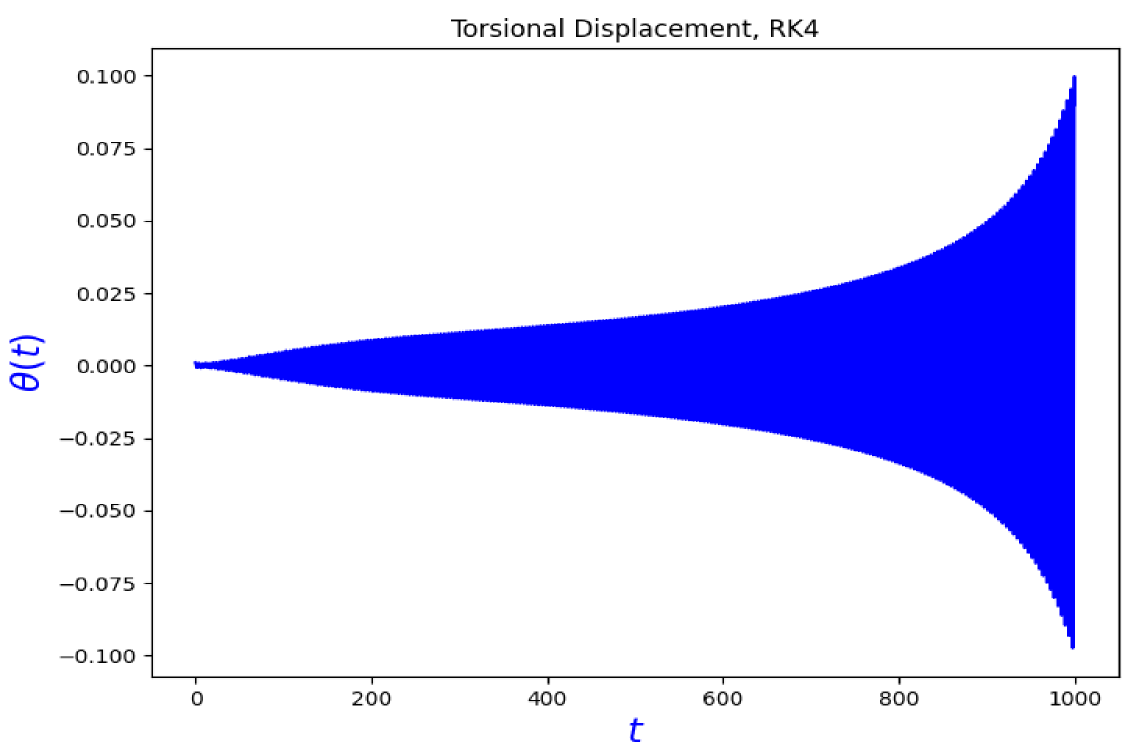
Part 5

Now, we will design a method for computing the minimum wind speed in Step 4, to within $0.5 \times 10^{-3}\ km/hr$. This can be done easily using a form of the Bisection Method for root-finding. The method is as follows: given a starting interval of wind speeds to test, take the midpoint of the interval and compute its magnification factor. If the factor is greater than 100, redefine the upper bound of the interval to be at that midpoint; otherwise, redefine the lower bound to be at that midpoint. The resulting interval become the starting point for the next iteration, and this is done until the width of the interval is less than 10^{-3}. The result is the minimum wind speed for which the magnification factor is 100 or more, as desired. This method in the code yields a wind speed of 58.9923 km/h, which agrees quite closely with our brute force attempt in Part 4.

Part 6

Now we will explore the effect of increasing the damping coefficient. Double the current value of d (to $d = 0.02$) and change ω to 3 to adjust for the new d. We compute the new critical W to be 26.5605, which is less than half the wind speed from Part 5. This is strange, because intuition would lead us to suppose that a greater damping force would make the bridge more stable, but it seems it now grows unstable at an even lower speed. The following graphs depict the oscillations.


Vertical Displacement, RK4
y(t)
t
2
1
0
−1
−2
−3
0
200
400
600
800
1000


Torsional Displacement, RK4
θ(t)
t
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
0
200
400
600
800
1000

As a result of this finding, I suppose that to make the bridge less susceptible to torsion, the damping force would have to be lessened, not increased. Perhaps that means making the bridge more flexible to absorb more of the flexing energy, rather than simply locking it down harder. That's unintuitive, but our analysis here seems to support that conclusion.