Poking a Simplicial Complex

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Abstract

Persistent homology has been used successfully to gain information about data. This success has increased the demand for computing the homology of a simplicial complex. For large data sets, these computations are expensive. We present an educational video that illustrates how discrete Morse theory can be applied to simplify a simplicial complex without loosing any homological information.

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1 Introduction

The amount of high dimensional data being generated continues to increase rapidly. Persistent homology is useful in gaining incite into these data sets, as seen in [?, ?, ?, ?] among others. In [?], a common algorithm for computing persistent homology is given with a runtime of $O(n^{\omega})$, were $\omega = \log_2(7)$ from matrix multiplication and n is the number of simplicies in the simplicial complex. Currently this is the best known bound. For large data sets this runtime is impractical. In [?], the authors show that discrete Morse theory can be used to reduce the size of the initial complex, while retaining all homological information. This preprocessing step leads to a faster algorithm for computing the homology of a simplicial complex that uses less space.

In this video, we show how a simplicial complex can be simplified by a sequence of homotopies called *elementary collapses*. These collapses are generated by a gradient vector field that is induced by a discrete Morse function. The definitions can be difficult to parse, but the geometry of the simplification is quite natural. Let σ be a simplex in a simplicial complex. Intuitively, elementary collapses eliminate σ by pairing σ with one of its faces or cofaces τ and removing both from the complex. Our video illustrates this paring and elimination by showing a finger poking a simplicial complex. Our objective is to convey how discrete Morse theory can be used to simplify simplicial complexes without changing the homotopy type of the complex.

2 Background Definitions

In this section we provide definitions of the objects that appear in our video. In general our notation follows that of [?]. Let K be a simplicial complex. We denote a typical $p-$simplex by σ^p or σ if the dimension is clear.

The following definition is due to [?].

▶ **Definition 1.** *Let K be a simplicial complex and suppose that there is a pair of simplices $\{\sigma^{p-1}, \tau^p\}$ in K such that σ is a face of τ and σ has no other cofaces. Then $K - \{\sigma, \tau\}$ is a simplicial complex called an **elementary collapse** of K. The pair $\{\sigma, \tau\}$ is called a **free pair**.*

Moreover, K and $K - \{\sigma, \tau\}$ have the same homotopy type. The concept that we hope to convey in our video is that an elementary collapse does not change the homotopy type of K and results in a simplified simplicial complex . But how do we know which simplices belong to a free pair? This is were discrete Morse theory is helpful.

▶ **Definition 2.** *A function $f : K \to \mathbb{R}$ is a **discrete Morse function**, if for every $\sigma^p \in K$, the following two conditions hold:*

1. $|\{\tau^{(p+1)} > \sigma | f(\tau) \leq f(\sigma)\}| \leq 1$,
2. $|\{\gamma^{(p-1)} < \sigma | f(\gamma) \geq f(\sigma)\}| \leq 1$.

A intuitive definition is given in [?], “the function generally increases as you increase the dimension of the simplices. But we allow at most one exception per simplex." Simplices with this exception deserve special attention.

▶ **Definition 3.** *A simplex is **regular** if and only if either of the following hold*

1. *There exists $\tau^{(p+1)} > \sigma$ with $f(\tau) \leq f(\sigma)$*
2. *There exists $\gamma^{(p-1)} < \sigma$ with $f(\gamma) \geq f(\sigma)$.*

A simplex that is not regular is called **critical.** Conditions 1 and 2 in definition 2 cannot both be true. If $\sigma \in K$ is regular then σ has a face γ with a greater function value or a coface τ with a lesser function value but not both. We pair all regular simplices with the unique γ or τ determined by the Morse function.

This leads to the definition an induced gradient vector field.

▶ **Definition 4.** *Let f be a discrete Morse function on K. The **induced gradient vector field** V_f is*

$$V_f := \{(\sigma^p, \tau^{p+1}) : \sigma < \tau, f(\sigma) \geq f(\tau)\}.$$

*if $(\sigma, \tau) \in V_f, (\sigma, \tau)$ is called an **arrow** with **tail** σ and **head** τ.*

All arrows determine a free pair. Our video shows how we can collapse free pairs without changing the homotopy of K. To summerize, we begin with a simplicial complex K, then assign real values to each simplex satisfying the definition of a morse function, f on K. Next, we pair all regular simplices in K with the simplex determined by the Morse function. This gives us a gradient vector field that determines free pairs. Finally, we collapse free pairs leaving us with the simplex consisting of critical simplices.

3 Video

The video begins by defining a simplicial complex and giving an example that will be used throughout the video, K, which consists of a tetrahedron, two cycles, a triangle and two edges. We also give a non-example. We then attempt to give an intuitive feeling for simplicial homology as 'holes' of various dimensions. We explain that computing the Betti numbers involves considering all simplicies in K and that this is computationally expensive.

The next scene introduces discrete Morse functions. We illustrate the values of a Morse function on K. Then we depict how a discrete Morse function induces a gradient vector field on the simplicial complex.

Now the video shows a finger poking the simplicial complex on paired simplicies. The poked simplicies are removed and we see a simplified simplicial complex K' which is K with all free pairs collapsed. When the finger is done poking we are left with two connected triangles, which is the same homotopy type as our original simplicial complex K.