# **Poking a Simplicial Complex**

### Benjamin Holmgren

- Montana State University, Bozeman, MT, USA
- benjamin.holmgren@student.montana.edu

#### Marco Huot

- Montana State University, Bozeman, MT, USA
- marco.huot@student.montana.edu

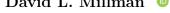
### Bradley McCov

- School of Computing, Montana State University, Bozeman, MT, USA
- bradley.mccoy@montana.edu

# Brittany Terese Fasy

- School of Computing and Dept. of Mathematical Sciences, Montana State University, Bozeman,
- MT, USA 13
- brittany.fasy@montana.edu

#### David L. Millman 🌘



- School of Computing, Montana State University, Bozeman, MT, USA
- david.millman@montana.edu

#### Abstract

Persistent homology has been used successfully to gain information about data. This success has 19 increased the demand for computing the homology of a simplicial complex. For large data sets, these computations are expensive. We present an educational video that illustrates how discrete Morse theory can be applied to simplify a simplicial complex without loosing any homological information.

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# Introduction

- The amount of high dimensional data being generated continues to increase rapidly. Persistent
- homology is useful in gaining incite into these data sets, as seen in [?,?,?,?] among others.
- In [?], a common algorithm for computing persistent homology is given with a runtime of
- $O(n^{\omega})$ , were  $\omega = \log_2(7)$  from matrix multiplication and n is the number of simplicies in the
- simplicial complex. Currently this is the best known bound. For large data sets this runtime
- is impractical. In [?], the authors show that discrete Morse theory can be used to reduce the
- size of the initial complex, while retaining all homological information. This preprocessing
- step leads to a faster algorithm for computing the homology of a simplicial complex that
- uses less space. 41
- In this video, we show how a simplicial complex can be simplified by a sequence of 42 homotopies called *elementary collapses*. These collapses are generated by a gradient vector

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field that is induced by a discrete Morse function. The definitions can be difficult to parse, but the geometry of the simplification is quite natural. Let  $\sigma$  be a simplex in a simplicial complex. Intuitively, elementary collapses eliminate  $\sigma$  by pairing  $\sigma$  with one of its faces or cofaces  $\tau$  and removing both from the complex. Our video illustrates this paring and elimination by showing a finger poking a simplicial complex. Our objective is to convey how discrete Morse theory can be used to simplify simplicial complexes without changing the homotopy type of the complex.

# 2 Background Definitions

In this section we provide definitions of the objects that appear in our video. In general our notation follows that of [?]. Let K be a simplicial complex. We denote a typical p-simplex by  $\sigma^p$  or  $\sigma$  if the dimension is clear.

The following definition is due to [?].

▶ **Definition 1.** Let K be a simplicial complex and suppose that there is a pair of simplices  $\{\sigma^{p-1}, \tau^p\}$  in K such that  $\sigma$  is a face of  $\tau$  and  $\sigma$  has no other cofaces. Then  $K - \{\sigma, \tau\}$  is a simplicial complex called an **elementary collapse** of K. The pair  $\{\sigma, \tau\}$  is called a **free** pair.

Moreover, K and  $K - \{\sigma, \tau\}$  have the same homotopy type. The concept that we hope to convey in our video is that an elementary collapse does not change the homotopy type of K and results in a simplified simplicial complex. But how do we know which simplices belong to a free pair? This is were discrete Morse theory is helpful.

▶ Definition 2. A function  $f: K \to \mathbb{R}$  is a discrete Morse function, if for every  $\sigma^p \in K$ , the following two conditions hold:

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66 1. |\{\tau^{(p+1)} > \sigma | f(\tau) \le f(\sigma)\}| \le 1,
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2.  $|\{\gamma^{(p-1)} < \sigma | f(\gamma) \ge f(\sigma)\}| \le 1$ .

A intuitive definition is given in [?], "the function generally increases as you increase the dimension of the simplices. But we allow at most one exception per simplex." Simplices with this exception deserve special attention.

▶ **Definition 3.** A simplex is **regular** if and only if either of the following hold

- 1. There exists  $\tau^{(p+1)} > \sigma$  with  $f(\tau) \leq f(\sigma)$
- 73 **2.** There exists  $\gamma^{(p-1)} < \sigma$  with  $f(\gamma) \ge f(\sigma)$ .

A simplex that is not regular is called **critical**. Conditions 1 and 2 in definition 2 cannot both be true. If  $\sigma \in K$  is regular then  $\sigma$  has a face  $\gamma$  with a greater function value or a coface  $\tau$  with a lesser function value but not both. We pair all regular simplices with the unique  $\gamma$  or  $\tau$  determined by the Morse function.

This leads to the definition an induced gradient vector field.

▶ Definition 4. Let f be a discrete Morse function on K. The induced gradient vector field  $V_f$  is

$$V_f := \{ (\sigma^p, \tau^{p+1}) : \sigma < \tau, f(\sigma) \ge f(\tau) \}.$$

if  $(\sigma, \tau) \in V_f, (\sigma, \tau)$  is called an **arrow** with **tail**  $\sigma$  and **head**  $\tau$ .

All arrows determine a free pair. Our video shows how we can collapse free pairs without changing the homotopy of K. To summerize, we begin with a simplicial complex K, then

assign real values to each simplex satisfying the definition of a morse function, f on K. Next, we pair all regular simplices in K with the simplex determined by the Morse function. This gives us a gradient vector field that determines free pairs. Finally, we collapse free pairs leaving us with the simplex consisting of critical simplices.

## 3 Video

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The video begins by defining a simplicial complex and giving an example that will be used throughout the video, K, which consists of a tetrahedron, two cycles, a triangle and two edges. We also give a non-example. We then attempt to give an intuitive feeling for simplicial homology as 'holes' of various dimensions. We explain that computing the Betti numbers involves considering all simplicies in K and that this is computationally expensive.

The next scene introduces discrete Morse functions. We illustrate the values of a Morse function on K. Then we depict how a discrete Morse function induces a gradient vector field on the simplicial complex.

Now the video shows a finger poking the simplicial complex on paired simplicies. The poked simplicies are removed and we see a simplified simplicial complex K' which is K with all free pairs collapsed. When the finger is done poking we are left with two connected triangles, which is the same homotopy type as our original simplicial complex K.