The Intersection of Statistics and Topology:

Confidence Sets

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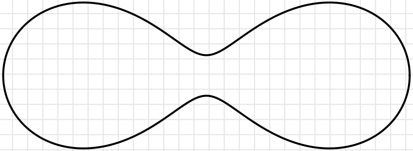
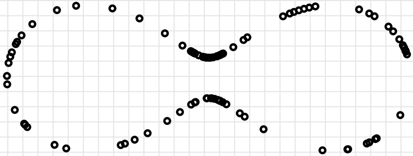
joint work with S. Balakrishnan, F. Chazal, F. Lecci,
A. Rinaldo, A. Singh, L. Wasserman

18 January 2014

How do we Interpret Data?

Data can be a fininte subset of $\mathbb{R}^D$.

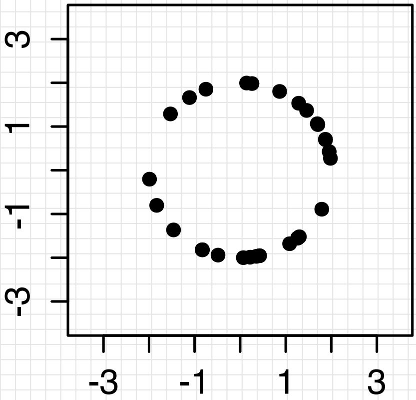




What is the homology / the structure of the underlying space?

How do we Interpret Data?

Induced Topological Space



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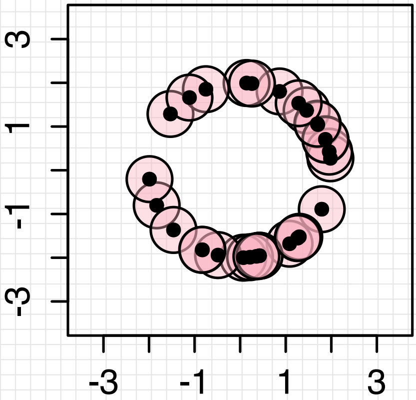

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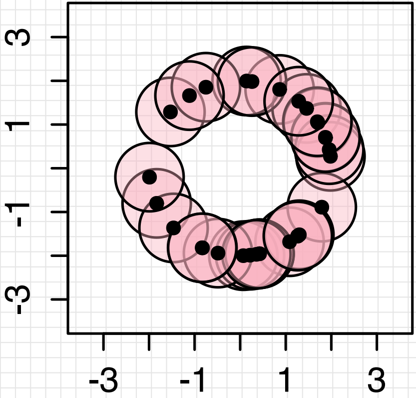

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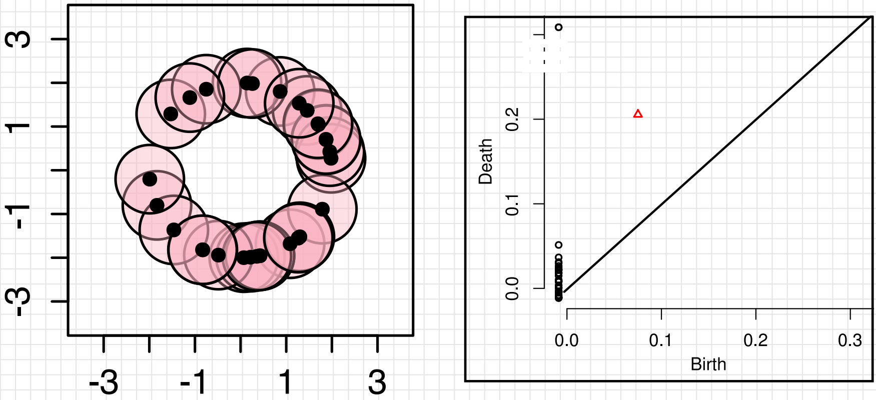

Objective

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Statistical Model

$\mathbb{M}$ is a manifold.
P is a probability distribution supported on $\mathbb{M}$.
Observe data $X_1, X_2, \ldots, X_n \sim P$.
Compute $\hat{\Theta}_n = \Theta(X_1, \ldots, X_n)$

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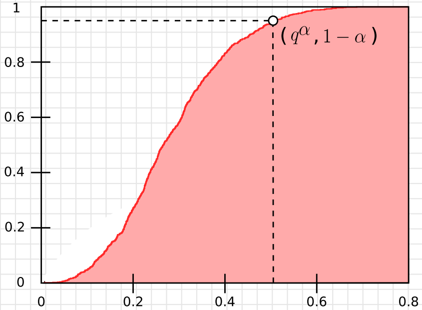
Computing a Confidence Interval

With Infinite Resources

Repeatedly sample *n* data points, obtaining:



Confidence Intervals

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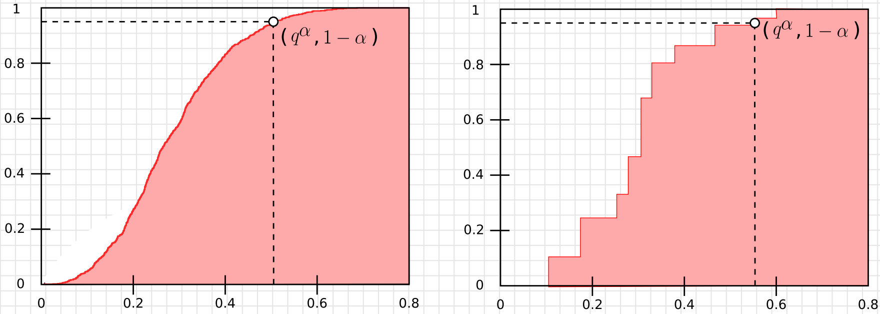

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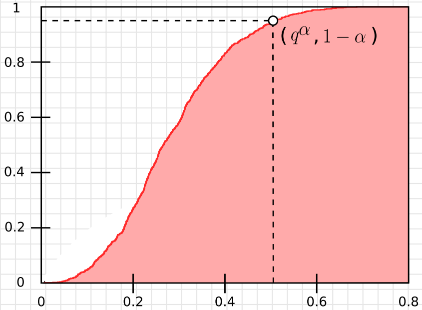

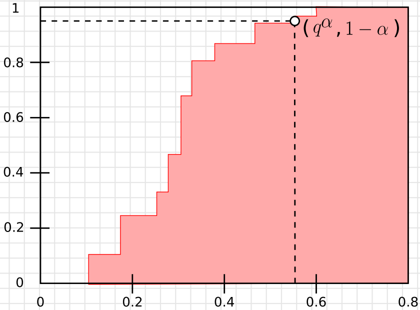

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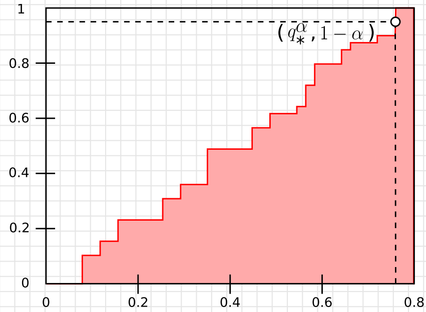

Bootstrapping Example

Estimating Densities

P has density p.

Smoothed Density: $p_h = p \star K_h$

KDE: $\hat{p}_h(x) = \frac{1}{n} \sum_1^n \frac{1}{h^D} K\left(\frac{||x - X_i||}{h}\right)$.

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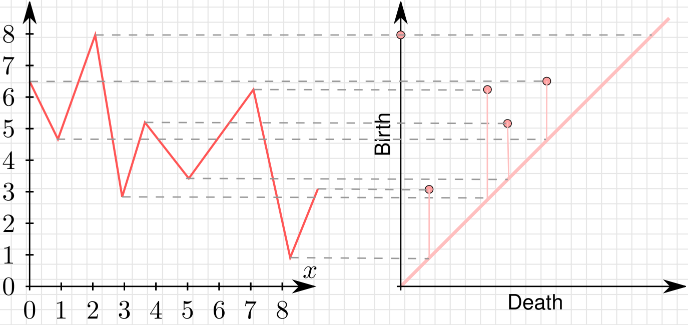
Bootstrap Theorem [FLRWBS]

$$\mathbb{P}(\sqrt{nh^D}||\hat{p}_h - p_h||_\infty > q_*^\alpha \mid X_1, \ldots, X_n) = \alpha + O\left(\sqrt{1/n}\right)$$

Persistent Homology

A Pairing of Critical Values.



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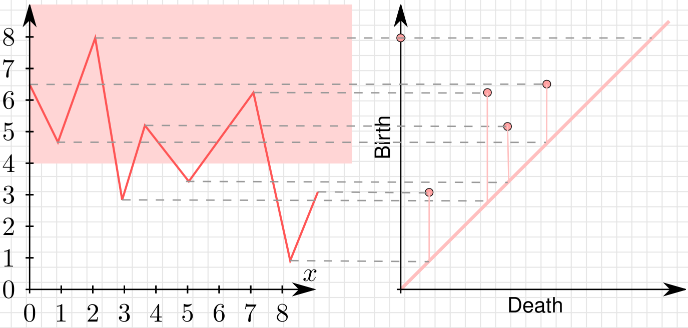

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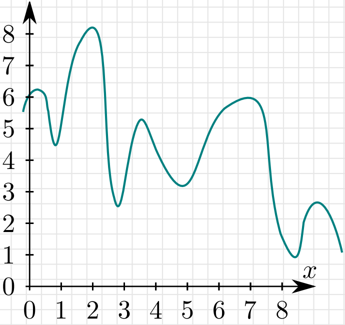

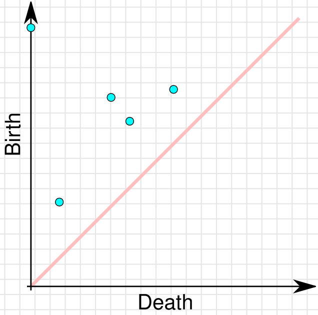

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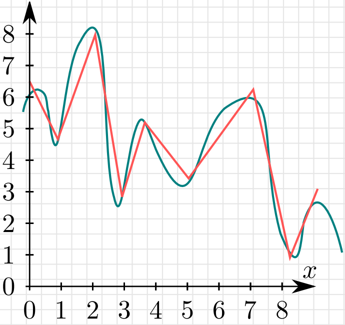

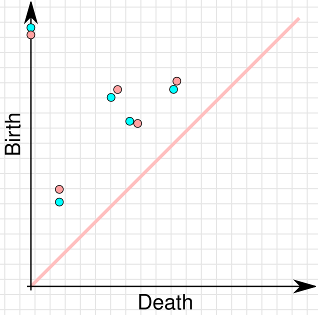

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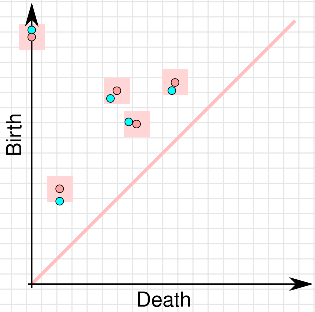
Bottleneck Distance

Given two persistence diagrams $\mathcal{P}$ and $\hat{\mathcal{P}}$, find the best *perfect matching* between the point sets.

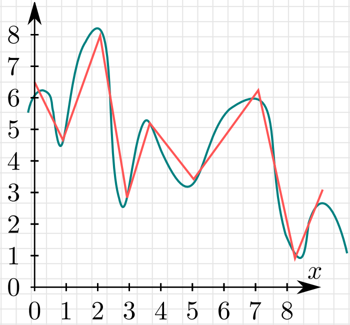
Minimize Cost

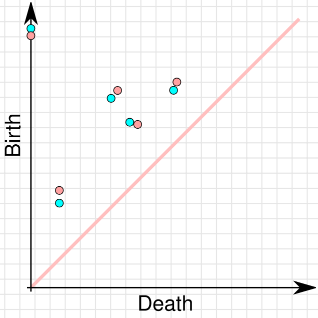
We wish to find

$$W_\infty = \min_M \{ \max_{(p,q)\in M} ||p - q||_\infty \}.$$



Stability of Matchings





Bottleneck Stability Theorem [CDGO]

$$||p - \hat{p}||_\infty \quad \geq \quad W_\infty(\mathcal{P}, \widehat{\mathcal{P}})$$

Putting It All Together

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Confidence Sets for Persistence Diagrams

$$\mathbb{P}(W_\infty(\mathcal{P}, \widehat{\mathcal{P}}) \leq \frac{q_*^\alpha}{\sqrt{nh^D}}) \quad \geq \quad 1 - \alpha - O\left(\sqrt{1/n}\right)$$

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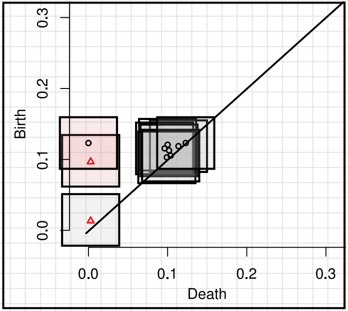
Asymptotic Confidence Sets for Persistence Diagrams

$$\lim_{n\to\infty} \mathbb{P}(W_\infty(\mathcal{P}, \widehat{\mathcal{P}}) \leq \frac{q_*^\alpha}{\sqrt{nh^D}}) \quad \geq \quad 1 - \alpha$$

Visualizing Confidence Intervals



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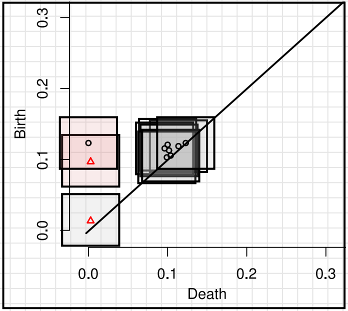
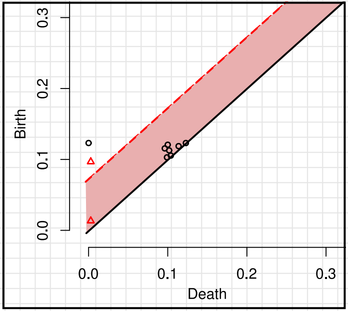
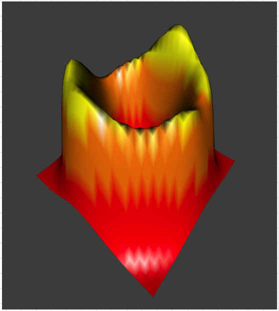




Density Function Examples

Uniform Distribution on Unit Circle



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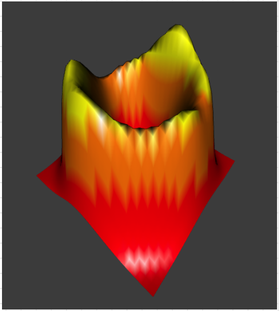

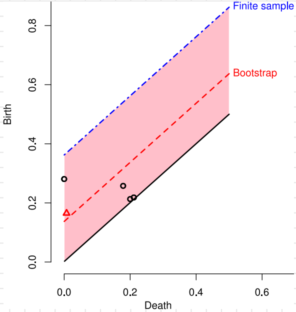

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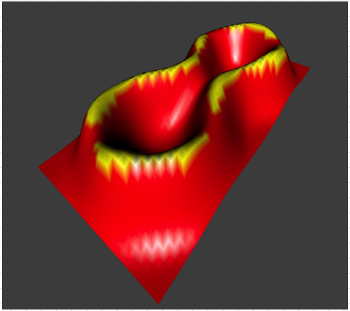

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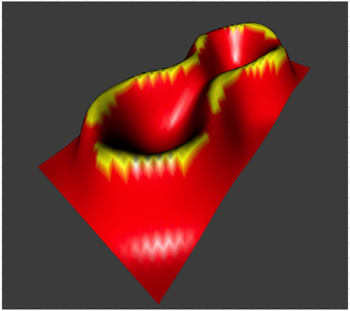

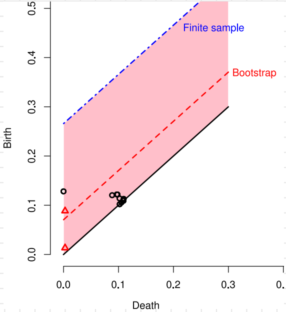

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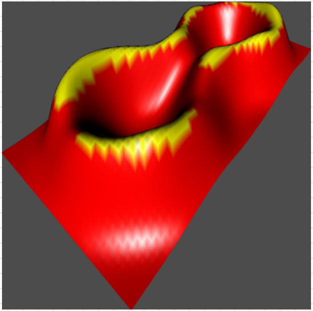
Cassini Curve with Outliers



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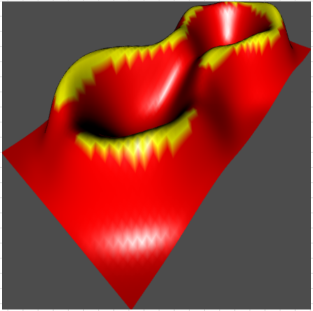

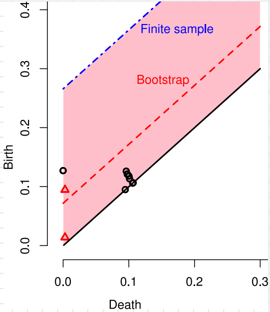

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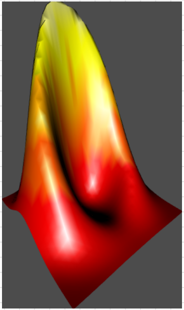

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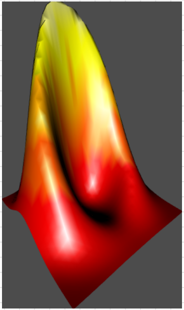
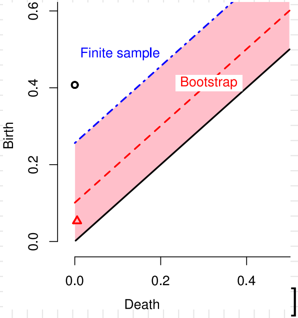
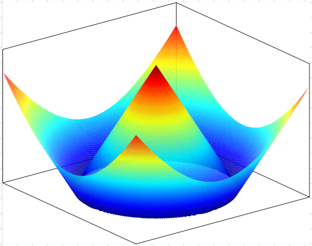




Distance to a Subset

$d_{\mathbb{M}}(a) = \inf_{x \in \mathbb{M}} ||x - a||$
$\mathcal{P}_1 = \mathrm{Dgm}_p^-(d_{\mathbb{X}})$



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P has continuous density p.
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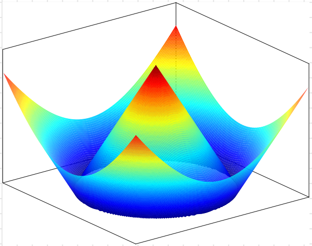


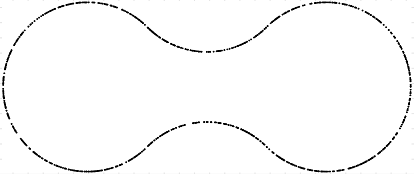
Subsampling

Confidence Interval from Subsampling [**F**LRWBS]

Assume that $p(x)$ is bounded away from zero.
Then, almost surely, for all large n,

$$\mathbb{P}\left(W_\infty(\mathcal{P}_1, \widehat{\mathcal{P}}_1) > c_n\right) \leq \alpha + \frac{2^d}{n \log n} + O\left(\sqrt{\frac{b_n \log n}{n}}\right)$$

Varying α



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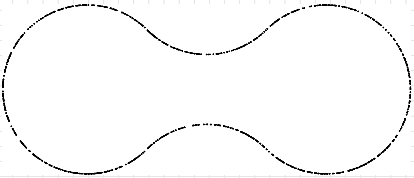
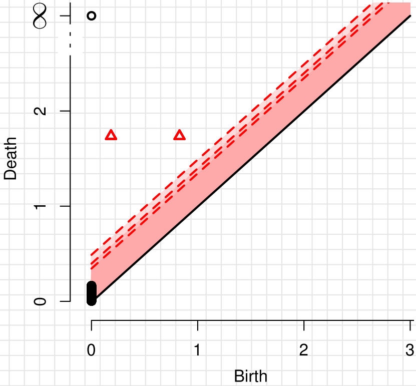




$$\alpha = 0.001, 0.05, 0.25$$

Two More Methods

$\mathcal{S}_n = \mathcal{S}_{1,n} \bigsqcup \mathcal{S}_{2,n}$.

Theorem (Concentration of Measure)

There exists $\hat{t}_{cm} = \hat{t}_{cm}(\alpha, d, n, \mathcal{S}_{1,n})$ *such that*

$$\mathbb{P}\left(W_\infty(\mathcal{P}_1, \widehat{\mathcal{P}}_1) > \hat{t}_{cm}\right) \leq \alpha + O\left(\left(\frac{\log n}{n}\right)^{1/d+2}\right).$$

Theorem (Method of Shells)

There exists $\hat{t}_s = \hat{t}_s(\alpha, d, n, K, \mathcal{S}_{1,n})$ *such that*

$$\mathbb{P}\left(W_\infty(\mathcal{P}_1, \widehat{\mathcal{P}}_1) > \hat{t}_s\right) \leq \alpha + O\left(\left(\frac{\log n}{n}\right)^{1/d+2}\right).$$

These Methods are Different

Concentration of Measure

$\hat{t}_{cm}$ is found by solving the following for t:

$$\frac{2^{d+1}}{t^d \hat{\rho}_{1,n}} \exp\left(-\frac{nt^d \hat{\rho}_{1,n}}{2}\right) = \alpha.$$

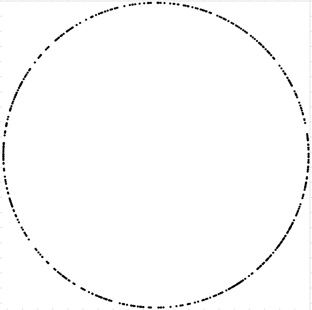
Shells

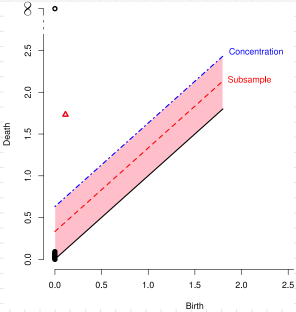
$\hat{t}_s$ is found by solving the following for t:

$$\frac{2^{d+1}}{t^d} \int_{\hat{\rho}_n}^{\infty} \frac{\hat{g}(v)}{v} \exp\left(-\frac{nvt^d \hat{\rho}_{1,n}}{2}\right) dv = \alpha.$$

Distance Function Examples

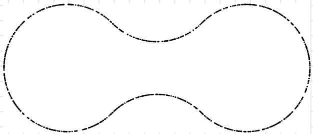
Uniform Distribution on Unit Circle

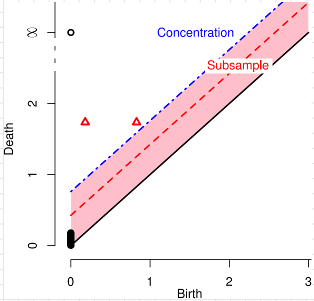




Distance Function Examples

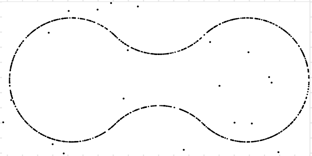
Uniform Distribution on Cassini Curve

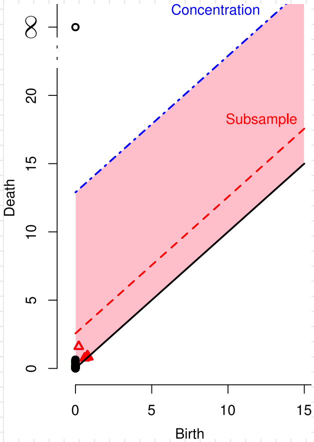




Distance Function Examples

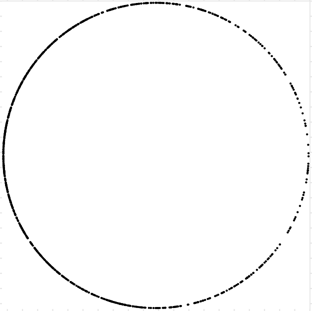
Cassini Curve with Outliers

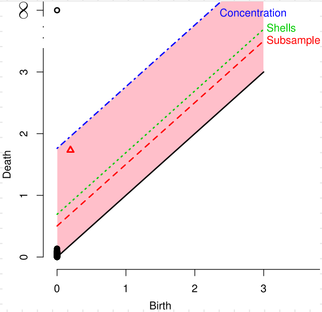




Distance Function Examples

Normal Distribution on Unit Circle





Summary

Recalling the Problem

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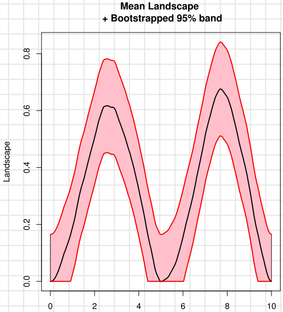
- **Find c_n such that $\mathbb{P}\left(W_\infty(\mathcal{P}, \widehat{\mathcal{P}_n}) > c_n\right) \leq \alpha$.**
- The pair $\widehat{\mathcal{P}_n}$ and $[0, c_n]$ define a confidence set for $\mathcal{P}$.

Ongoing Research

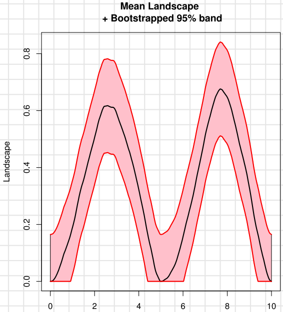
Ongoing Research

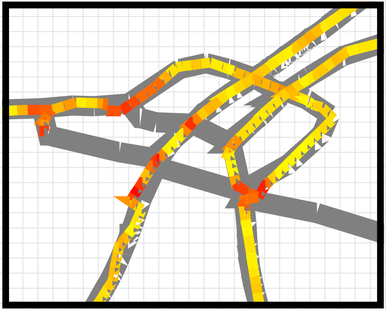


Functional Analysis

Confidence Bands for Landscapes
joint w/ F. Chazal, F. Lecci,
A. Rinaldo, L. Wasserman

Ongoing Research





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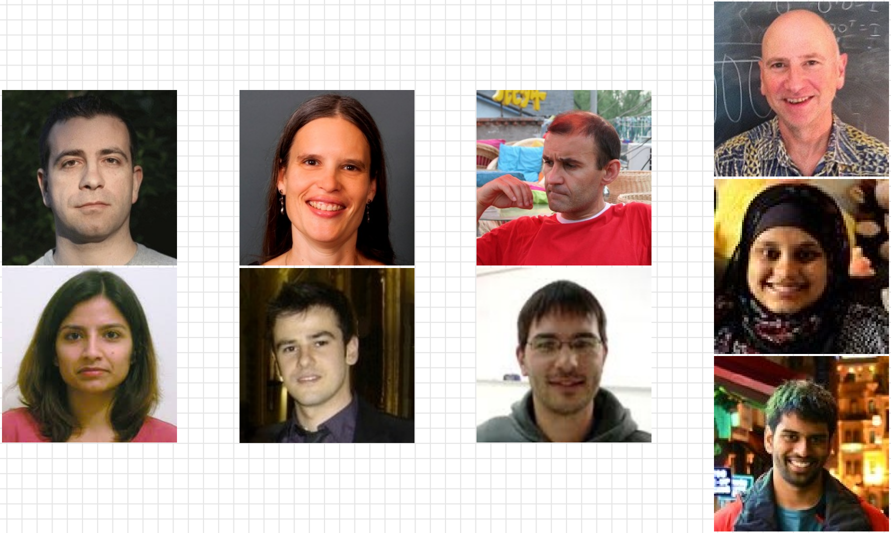
Really Great Upcoming Talk

Carola Wenk
Map Construction & Comparison
3:30 Here!

Collaborator Collage



Thank you!

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References

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