Refining Discrete Morse Functions

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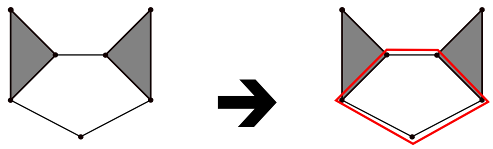
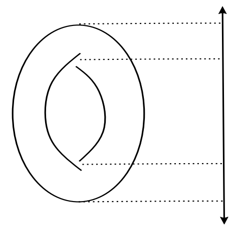
1 Introduction

Computational topology can broadly be described as the extension of techniques from the pure math field of topology to a discrete and algorithmic setting widely applicable to point cloud data. Computational topology often deals with data in the form of a simplicial complex, which may be arbitrarily high dimensional. One of the major goals of computational topology is to describe these simplicial complexes in terms of their connected components and their cycles in any dimension. In practice, these features frequently describe important attributes of data, with applications including mapping various networks, cancer screening, encryption, quantum physics, machine learning, and neuroscience.

I too am interested in this problem of capturing topological features in data, and am particularly captivated by a field known as discrete Morse theory. Morse theory arises out of classical topology as a set of methodologies to detect topological features by assigning a function to a topological space. Discrete Morse theory (DMT) attempts to construct similar functions, but in a computational setting. In other words, DMT seeks to generate topologically meaningful functions on a simplicial complex. Work has been done to algorithmically generate discrete Morse functions on a simplicial complex, but this specific area of research is relatively new and remarkably limited. I plan to expand upon the work that I, Brad McCoy, Dr. Brittany Fasy, and Dr. David Millman have conducted in the last year to improve upon existing algorithms which construct discrete Morse functions on a simplicial complex. In doing so, I aim to provide novel insights into how topological features may be detected in data. Consequently, I hope to move forward the viability of discrete Morse theory as an approach to the wide array of applications of computational topology mentioned above.



(a) Example simplicial complex with red cycle.



(b) An example smooth Morse function, whose critical points reveal topological features.

Figure 1: We want to detect topological features in a simplicial complex (such as the cycle in red on the bobcat shaped complex), using functions similar to classical Morse theory height functions.

2 Background

Milnor's classical Morse theory provides tools for investigating the topology of smooth manifolds [13]. In [8], Forman showed that many of the tools for continuous functions can be applied in the discrete setting. Inferences about the topology of a simplicial complex can be made from the number of critical cells in a Morse function on the complex. Given a Morse function one can interpret the function in many ways. Switching interpretations is often revealing. In previous work, we've found it particularly useful to think of a discrete Morse function in three different ways. Algebraically, a Morse function is a function from the faces of a complex to the real numbers, subject to certain inequalities. Topologically, a Morse function is a pairing of the faces such that the removal of any pair does not change the topology of the complex. Combinatorially, a Morse function is an acyclic matching in the Hasse diagram of the complex, where unmatched faces correspond to critical cells.

In [7], we provided a new algorithm we call ExtractRightChild to compute a rudimentary discrete Morse function on a simplicial complex which improved the time complexity of pre-existing algorithms. To do so, we primarily focused on generating discrete Morse functions in a combinatorial setting, via matchings in the Hasse diagram. We were pleased with our result, which builds on algorithms proposed in [11], but our algorithm can be refined. In particular, ExtractRightChild may output a large number of critical cells, which means that there may exist possible Morse functions which are more strongly indicative of the topology of a complex. This is the main area of work for the upcoming year- to propose efficient new algorithms which further refine the output of ExtractRightChild. Along with this, a major area of research is also to bound the number of critical cells produced by ExtractRightChild. In each of these research objectives, we aim to propose increasingly reliable indicators of the topology of a simplicial complex, which may pose

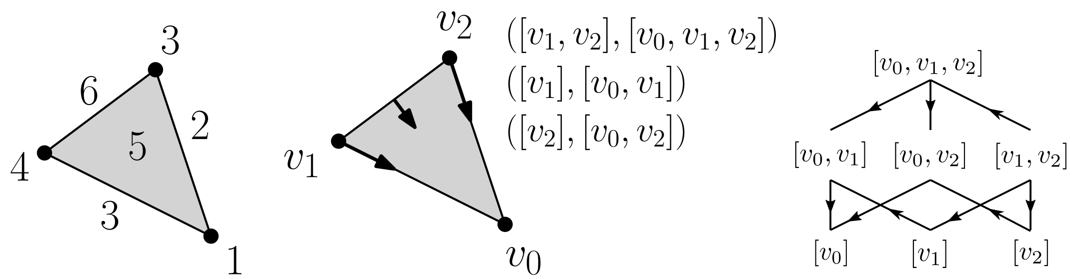
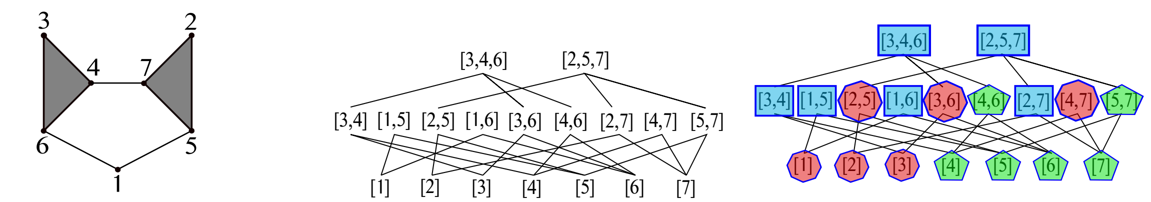


Figure 2: The three mathematically equivalent "flavors" of discrete Morse functions we're concerned with. From left to right, these are algebraic, topological, and combinatorial.



(a) Bobcat simplicial complex with pre-existing data on the vertices

(b) Equivalent Hasse diagram

(c) Sample output of ExtractRightChild

Figure 3: An example run of ExtractRightChild, where output matchings are shown by the heads of arrows in blue, the tails of arrows in green, and critical cells in red. This is a case where we have more critical cells than are exactly indicative of the topology. (One critical vertex and one critical edge would be the optimal output for this example.)

wider implications throughout the field of computational topology in the future.

Discrete Morse theory has been particularly effective when combined with persistent homology to analyze data, as is done in [11, 1, 2, 6, 4, 3, 5]. This is also an active avenue of research, to tie discrete Morse theory to exciting new applications in persistent homology. When dealing with data, we so far worked with the additional constraint that vertices have function values assigned. For complexes without any preassigned function values, Joswig and Pfetsch showed that finding a Morse function with a minimum number of critical cells is NP-Hard [10]. Algorithms that find Morse functions with relatively few critical cells have been explored in [12, 14, 9]. This presents yet another area to pursue in the upcoming year, which is to construct Morse functions on data with more sparsely pre-assigned function values, like for instance on higher dimensional simplices. Furthermore, all of our recent results have been purely theoretical, and their implementation in a programming setting would undoubtedly be of use in the future and is another focus of the project.

3 Methods

In order to successfully build upon the previous work I've conducted with other members of the compTAG group at MSU, I plan to continue working as collaboratively with my groupmates as possible, despite being in the midst of a pandemic. Over the past few months, work on these types of problems has been accomplished using collaborative whiteboard tools in a remote setting. Our group seems to have settled on using Miro, an online whiteboard environment, which has helped us to regain the levels of productivity we've had in person in previous years. More specifically, we will continue weekly hour long group meetings where I will meet on Miro with Brad McCoy (a PhD student), Dr. Brittany Fasy, and Dr. David Millman (my research advisors). In these meetings, we start by sharing recent developments and proofs we've written in the last week, then we often work together to iron out our most recent results and to brainstorm next steps, and then establish our goals for the upcoming week. These weekly meetings have been quite effective thus far, and I expect us to continue efficiently producing new results in this adjusted online format. Then throughout the week, if any one of us is working on a particular proof and would like to collaborate, we use Slack for group messaging, and often will get back into Miro to work out proofs together. Furthermore, our entire compTAG group participates in weekly seminars, which discuss group developments and new areas of research. These meetings, which are anticipated to happen via Zoom for the academic year, are also an effective venue to learn and share results.

As for the specific outline of the project, the focus generally will be to create an algorithm which refines the output of ExtractRightChild, attempt to bound its output, implement our algorithms, and then formally write up our results for a conference in the spring (we will most likely try to be published again in the Canadian Conference on Computational Geometry). Along with this, throughout the process, we will be open minded towards new, exciting applications of discrete Morse theory, particularly in the realm of persistent homology. A slightly more rigorous schedule is as follows:

1. Create a refining algorithm for extractrightchild with time complexity improvements on prior algorithms: 10 weeks
2. Formally prove performance of new algorithm in relation to pre-existing algorithms: 4 weeks
3. Formally bound the number of critical cells produced by our new algoritm: 8 weeks
4. Implement our algorithm: 2 weeks

5. Formally write up and submit our results to a computational geometry conference: 4 weeks

Due to the highly flexible nature of the research I'm conducting, this schedule is certainly subject to change. However, based on prior results of a similar nature that my collaborators and I have generated, all of these tasks should be within reason to be accomplished roughly in the above timeframe. Along with this, due to the uncertainty of the world that we live in, research of this nature which is independent of physical experimentation or on-campus facilities seems to be a relatively reliable academic pursuit.

4 Collaboration with Faculty Sponsor

As mentioned briefly above, the methodologies associated with this kind of theoretical research lend themselves well to collaboration with my research advisors. We will continue our weekly meetings for the duration of the year, and I will also be an active member of our group's weekly seminars. This project ties in closely with the work of both of my advisors, as discrete Morse theory is an increasingly influential field in computational topology. Both of my mentors do a great deal of research in persistent homology, where these results will be directly applicable. I have no doubt that with the guidance of my mentors this will be another excellent and productive year of research. They have already equipped me well in the past not only with the tools to effectively produce theoretical results, but also to have successful experiences in the context of gaining publications and participating in conferences in the field. I am incredibly excited to see what we are able to accomplish in another year of collaboration.

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