Extending Discrete Morse Theory to Neuron Reconstruction

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Background

In all, our project can be divided into a series of tasks, as follows:

1. To begin, we need to actually analyze and fully understand the TMD algorithm as proposed by Hess et al. in terms of inputs, outputs, and time/space complexity. Specifications of the inputs, outputs and the time complexity of the TMD algorithm are included in Hess' work, but there is no rigorous analysis of the space complexity of the algorithm. We've provided a rigorous statement of all of these items below.

2. We also need to show that we can generate a discrete Morse function which can do an equivalent task to the filtration used in the TMD algorithm. Functionally, we need to show correctness if a Morse function were used in place of the filtration. To do this, we need to modify the definitions inherent in a discrete Morse function to fit the critical cells on a neuron related to branching.

3. We need to show that using Morse theory yields some advantages. Namely, we need to show that using Morse theory retains the original time complexity of the TMD algorithm, but using Morse theory yields savings with regards to space complexity.

4. We also hope to show additional features that using a discrete Morse function allows. Specifically, we want to show that using Morse theory allows for the pinpointing of specific regions within a neuron, which may relate to neuron functionality.

Progress

Thus far, we've successfully accomplished tasks 1-3. Our progress for each is loosely as follows:

1. The algorithm's input, output, and runtime are clearly expressed on page 7 of Hess et al. For an analysis of the runtime, we claim that the space complexity of the algorithm is linear with respect to the number of nodes on the neuron. We prove this by considering the number of nodes in an active branch of the neuron. At worst, this is equivalent to the total number of nodes. As a lower bound, there could possibly be only one node in an active branch. But, no matter what, there must be a maximal active branch with a considerable fraction of the nodes. Meaning that the number of active nodes has a linear relationship to the total number of nodes, and the space complexity of the TMD algorithm is order of n.

2. The filtration specified in the TMD algorithm essentially uses a filtration to find the persistence of branches throughout the neuron. To generate an equivalent Morse function, we first

need to define underlying vertex data present in the image of a neuron. Letting N denote the set of nodes in the neuron image, we define an underlying function $f : N \to \mathbb{R}$ where f defines the path-based distance from a source node to any other given node throughout the neuron. Then, we're able to assign a modified a discrete Morse function to the complex. The function is modified in that critical cells are defined by any vertex which is larger than degree 2. (Because the complex is guaranteed to be an acyclic graph, we know that only critical vertices are possible). Then we gain matchings from a Morse function revealing of the flow into any critical cell in the complex. In doing so, we gain branchwise distance in the number of collapses needed to reach critical cells, and in doing so gain a metric of persistence in each individual branch. Thus, we gain all of the features given by the filtration in the TMD algorithm in using a modified Morse function.

3. From the proposed setup, we gain that we can generate a discrete Morse function in linear time with the underlying vertex data. Notably, this discrete Morse function is guaranteed to be "high performing" in that the critical cells are guaranteed to be directly reflective of the areas where the neuron branches, since we guarantee that the underlying vertex data used by the algorithm proposed by Holmgren et al. is increasing along the path from the source to any given node, and that the function f is injective. And, since the underlying vertex data is also guaranteed to be sorted radially along the neuron, we are able to improve the functionality of Holmgren et al. Indeed, the runtime in Holmgren et al. is guaranteed to remain linear with respect to the simplices of the neuron (i.e. roughly double the number of vertices for a graph), so the TMD algorithm has equivalent runtime. However, the algorithm by Holmgren et al. when adapted to a lexicographically sorted Hasse diagram of the underlying neuron is able to operate with constant space complexity. Improving the space complexity of the TMD algorithm.

Remaining Work

As for the remainder of the project, we still need to work on specifying how to collapse simplices in an assigned Morse function to highlight structural differences within different kinds of neurons. We have a rough idea of how to go about this, but need to elaborate fully on the theory behind this process. And, we need to more formally write up our developments thus far, especially our framing of the problem and our provided improvements with regards to its space complexity.

Works cited

A Topological Representation of Branching Neuronal Morphologies, Hess et al.

If You Must Choose Among Your Children, Pick the Right One, Fasy, Holmgren et al.