On Computing Discrete Morse Functions Using Vertex Data

Brittany Terese Fasy
School of Computing & Department of Mathematical Sciences, Montana State University, United States.

Benjamin Holmgren
Montana State University & Montana Math Man

Bradley McCoy
School of Computing, Montana State University

David L. Millman
Blocky, USA.

Binhai Zhu
School of Computing & Department of Mathematical Sciences, Montana State University, United States.

Abstract

Discrete Morse theory provides a way of studying simplicial complexes akin to studying flows over smooth surfaces. Discrete Morse functions assign a real value to each simplex, and then pair simplices based on homology-preserving gradients. The unpaired "critical" cells either represent an essential homology class of the underlying topological space, or are a vestige of the function itself. We consider a variant of the optimization problem: MinMM, which is to find a function over a given simplicial complex K that minimizes the number of critical simplices. We study this problem through the lens of King et al. (2005), which extends a Morse matching to K given an injective function $f_0 : K_0 \to \mathbb{R}$ on the vertices of K. We call this variant ExtMM, which is to find a Morse function consistent to a given f_0 minimizing the number of critical cells. Though MinMM is NP-hard and W[P]-Hard to approximate, it is unclear if vertex data could benefit the computation of Morse functions. We give a linear time algorithm solving ExtMM for the restricted case where the input is a triangulation of a two-manifold. In general, we show that the same NP-hardness and W[P]-hardness results for MinMM also apply to ExtMM in dimensions greater than or equal 2, even under reasonable "niceness" assumptions of a given f_0. Despite these hardness results, in higher dimensions we provide a linear time gradient descent heuristic for MinMM that improves upon randomized methods to compute Morse functions, making clear theoretical improvements for Costa-Farber complexes. We illustrate these improvements on the Lutz triangulation library.

2012 ACM Subject Classification Theory of computation → Computational geometry; Mathematics of computing → Geometric topology

Keywords and phrases discrete Morse theory, persistence

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Acknowledgements .

1 Introduction

In classical Morse theory, continuous functions are assigned to smooth manifolds in order to study their topology [31]. For example, the Betti numbers of a manifold can be computed by examining critical points of the continuous functions. In [20], Forman defines analogous tools in the discrete setting; leading to the field of *discrete Morse theory*. Discrete Morse theory has been fruitful when paired with persistent homology [2, 4, 9, 10, 13, 14, 19, 28], where it is often used to reduce the size and complexity of data in a topologically faithful manner. In this work, we study discrete Morse functions on simplicial complexes. In particular, we consider discrete Morse functions from three perspectives: the algebraic, the combinatorial, and the topological. Algebraically, a Morse function is a function from the cells of a simplicial complex to $\mathbb{R}$, subject to specific inequalities as cells increase in dimension. Combinatorially, a discrete Morse function is a constrained matching in the Hasse diagram of the complex, where unmatched cells are called critical. Topologically, a Morse function takes the form of a gradient vector field on a simplicial complex. These gradient vector fields are composed of matchings between cells such that the collapse of any given matching does not alter the homotopy type of the complex.

The number of critical i-cells is an upper bound to the rank of the ith homology group (i.e., the ith Betti number). As a consequence, there is great interest in algorithms that minimize the number of critical simplices in a Morse function on a complex [7, 16, 22, 23, 26, 30]. Doing so gives not only the Betti numbers, but also the exact instructions of how to collapse paired simplices until only critical simplices remain. The problem of minimizing the number of critical simplices is known in the literature as "Minimum Morse Matching", or MinMM. Joswig and Pfetsch established that MinMM is NP-hard [26]. Moreover, recent work has demonstrated the inapproximability of generating discrete Morse functions on complexes that are not subcomplexes of two manifolds. For K of dimension greater than two with n simplices, MinMM is NP-Hard to approximate within a factor of $O(n^{1-\varepsilon})$, and if K has dimension two, MinMM is hard to approximate within a factor of $2^{\log^{(1-\varepsilon)} n}$ for any $\varepsilon > 0$ [5, 6].

Because of the hardness of MinMM, King et al. propose the problem of constructing a discrete Morse function given an injective function $f_0 : K_0 \to \mathbb{R}$ on the vertices of K in [28]. In doing so, [28] introduces the question: "Does the use of vertex data simplify the computation of discrete Morse functions?" Unfortunately, in this work we show that the answer to the above question is in general, no. We introduce formally the natural extension problem to MinMM with vertex data, and give a $\Theta(n)$ algorithm for 2-manifolds based on algorithms to compute homology generators on surfaces [1, 8, 11, 12, 17, 18, 35]. We show that the extension problem is NP-hard in higher dimensions.

Despite this and the number of other hardness results in discrete Morse theory, there is a rich literature of simple heuristics that achieve nearly optimal discrete Morse functions. We introduce a new refinement step for randomized methods to compute discrete Morse functions. We give an randomized gradient-descent algorithm to approximate MinMM, which outperforms previous randomized methods both theoretically and experimentally.

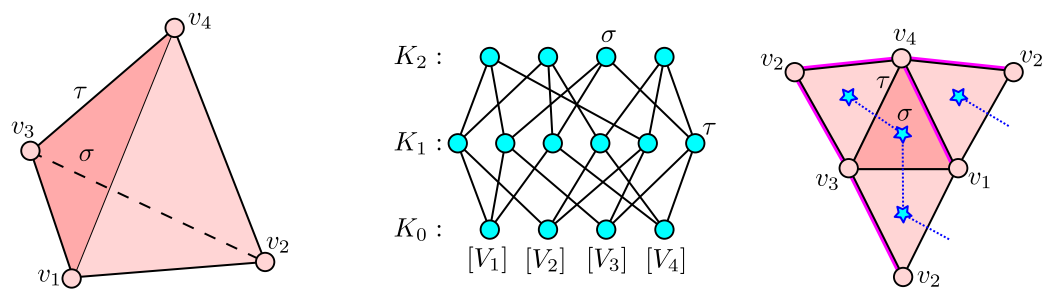
2 Preliminaries

In this section, we provide the definitions and notation used throughout the paper. We provide essential definitions here and additional definitions from computational topology in Section **??**, but for a general survey of discrete Morse theory, see [29, 33].

2.1 Simplicial Complexes and Two-Manifolds

Let K be an abstract simplicial complex with n simplices. For a simplex $\sigma \in K$, we denote the dimension of σ as $\dim(\sigma)$ and we define $d = \dim(K)$ to be the maximum dimension of any simplex in K (in which case, we call K a d-simplex). Thinking of a simplex as a set of vertices, we write $v \in \sigma$ if v is a vertex of σ. The *star* of v in K, denoted $\mathrm{star}_K(v)$, is the set of all simplices of K containing v. The *closed star* of v in K, denoted $\overline{\text{star}}_K(v)$, is the closure of $\text{star}_K(v)$. We denote the set of i-simplices of K as K_i and note that (K_0, K_1) is a graph whose vertices are the zero-simplices of K and whose edges are the one-simplices of K. We call this graph the *one-skeleton* of K. As a shorthand, we adopt the notation of Forman and write $\sigma^{(i)} \in K_i$ to mean an i-simplex. Often, it is useful to discuss *adjacent* vertices on the one-skeleton of K. For a vertex $v \in K_0$, the set of vertices adjacent to v (sharing an edge with v in the one-skeleton of K), is denoted $N(v)$. We write $\tau^{(i)} \prec \sigma^{(j)}$ for $i < j$ to say that τ is a *face* of σ, meaning that the vertices of τ are a subset of those of σ: $\tau^{(i)} = \{v_1, v_2, ..., v_i\} \subset \sigma^{(j)} = \{v_1, v_2, ..., v_i, ..., v_j\}$.

We often study a simplicial complex K combinatorially through its graph representation or *Hasse diagram*, which can be helpful algoritmically. The Hasse diagram $\mathcal{H}$ for K is a graph whose vertices correspond to the simplices of K, two simplices τ and σ are connected by an edge if $\tau^{(n-1)}$ is a codimension-one face of σ^n (that is, $\tau \prec \sigma$ and the two differ by a single dimension). See Figure 1 for an example of a simplicial complex that is a triangulated sphere, and its corresponding Hasse diagram.



(a) A Two-Complex K **(b)** The Hasse diagram **(c)** Tree-cotree decomposition

Figure 1 A simplicial complex K, along with its Hasse diagram, and its tree-cotree partition. In (a), K is a triangulation of the sphere using four 2-faces, six edges, and four vertices. Each k-simplex is a vertex in the Hasse diagram for K, shown in (b). We arrange the vertices in rows corresponding to their dimensions; with two-simplices in the top row, one-simplices in the middle row, and zero-simplices in the bottom row. Since $\tau^{(1)} \prec \sigma^{(2)}$ are codimension-one, an edge exists between $\tau^{(1)}$ and $\sigma^{(2)}$. In the tree-cotree partition in (c), the tree T is highlighted in pink, the cotree R has blue stars as vertices and dashed lines as edges, and the tree-cotree partition is $(T, R, \emptyset)$.

If K is a two-manifold, the dual graph G of K is a graph whose vertices represent the two-simplices of K_2, and edges represent two two-simplices that share a common codimension-one face (i.e., two triangles that share an edge in K). Given a spanning tree T of the one-skeleton of K, the *restricted dual graph D of K with respect to T* is the dual graph of G obtained by removing all edges whose duals are edges in T. Let R be a spanning tree of D, and let $X = D \setminus R$; then we obtain a *tree-cotree partition* of K, (T, R, X). The sets of edges T, R and X partition K_1; see [17,18,35]. Tree-cotree partitions have been extended to a number of related algorithmic results [1,8,11,12] and efficient data structures [17].

2.2 Discrete Morse Theory

We consider three views of discrete Morse functions; switching among them reveals useful properties.

▶ **Definition 2.1** ((Algebraic) Discrete Morse Function). *A function $f : K \to \mathbb{R}$ is a* discrete Morse function *if, for every $\sigma \in K$:*

1. $|\{\alpha^{(i-1)} \prec \sigma^{(i)} | f(\alpha) \geq f(\sigma)\}| \leq 1$
2. $|\{\sigma^{(i)} \prec \beta^{(i+1)} | f(\beta) \leq f(\sigma)\}| \leq 1$

A standard intuitive definition is given in [33, p. 49], "the function generally increases as you increase the dimension of the simplices. But we allow at most one exception per simplex." Let $\tau^{(i-1)}$ and $\sigma^{(i)}$ be one such pair that realizes an exception. Then, we call $(\tau^{(i-1)}, \sigma^{(i)})$ a *matched pair*, with $\sigma^{(i)}$ the *head* and $\tau^{(i-1)}$ the *tail*. In [21], Forman proves that each simplex in K can be a member of at most one matched pair. If $\sigma \in K$ is a member of a matched pair, we call σ *regular*. A simplex $\sigma \in K$ is called *critical* if it is not regular. See Figure 2(b) for an example.

The matchings that result from Definition 2.1 lead naturally to the *topological* definition of a discrete Morse function. In a topological Morse function, we simply record the matching information in a discrete gradient vector field and let go of function values. Visually, matchings are drawn as an arrow from the lower-dimensional simplex in a matching to the higher-dimensional one. See Figure 2(c). We write M as the set of all regular cells and C as the set of all critical simplices. As a shorthand, we write M^T as the set of all tails in M, and M^H as the set of heads in M. Since each simplex is in at most one matched pair, the sets M^T, M^H, and C partition the simplices of K. The pair (M, C) is called the *gradient vector field (GVF)* on K induced by f. Collapsing simplices along the gradient preserves the homology of K. This topological definition is equivalent to Definition 2.1 in the following sense:

▶ **Lemma 2.2** (Topological Morse Functions). *Let f and g be two algebraic discrete Morse functions, denoting the GVF induced by f as (M_f, C_f) and the GVF induced by g as (M_g, C_g). If f and g induce the same partial order on the simplices of K, then $(M_f, C_f) = (M_g, C_g)$.*

Proof. In Definition 2.1, the set of tuples M is defined precisely by the order of simplices induced by f. As a consequence, $M_f = M_g$. Since $M_f = M_g$ and (M, C) partitions K, we also have $C_f = C_g$. ◀

A GVF on K also gives a combinatorial representation by decorating the Hasse diagram $\mathcal{H}$ of K with directions. An edge between $\tau^{(i-1)} \prec \sigma^{(i)}$ in $\mathcal{H}$ is directed up (from the vertex representing $\tau^{(i-1)}$ to the one representing $\sigma^{(i)}$) if $(\tau^{(i-1)}, \tau^{(i)}) \in M$. Otherwise, edges in $\mathcal{H}$ are directed down. Then, by construction, for $i > 0$, $\sigma^{(i)} \in K_i$ is critical if and only if every edge in $\mathcal{H}$ between $\sigma^{(i)}$ and an $i-1$ cell directs down in dimension, and there are no edges directed up in dimension from σ to some $i+1$ cell. If $\sigma^{(0)} \in K_0$ is a vertex, only the second property is needed to have $\sigma^{(0)} \in C$. If $\sigma^{(d)} \in K_d$ is of dimension $d = \dim(K)$, only the first property is needed to have $\sigma^{(d)} \in C$. Each criteria is to say, $\sigma \in C$ iff σ is unmatched. See Figure 2(d) for an example, and Appendix **??** for further elaboration on discrete Morse functions in the combinatorial setting.

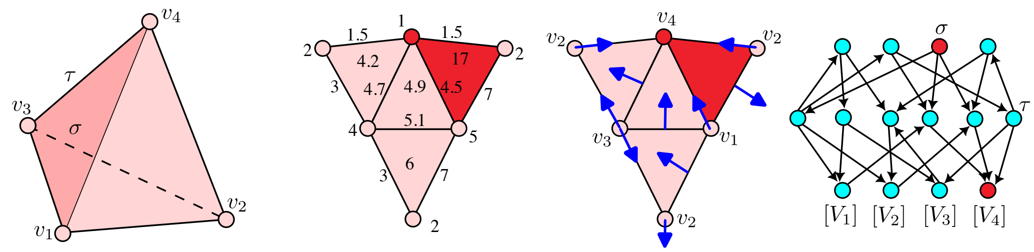
2.3 Computational Problems in Discrete Morse Theory

A fundamental problem in discrete Morse theory is how to compute a discrete Morse function that minimizes the number of critical cells. That is,

▶ **Problem 2.3** (Minimum Morse Matching, MinMM). *Given a simplicial complex K, assign a gradient vector field (M, C) to K that minimizes the number of critical simplices, $|C|$.*

MinMM is well known to be NP-Hard, due Joswig and Pfetsch [26].

▶ Remark 2.4 (MinMM Hardness of Approximation [5]). Along with being NP-Hard, Bauer et al. show that MinMM is W[P]-hard to approximate if $d > 2$, with respect to solution size.



(a) A two-complex. **(b)** Algebraic. **(c)** Topological. **(d)** Combinatorial.

Figure 2 The so-called algebraic, topological, and combinatorial interpretations of a Morse function respectively on K as a triangulated sphere. Critical simplices are red.

Because of the hardness of MinMM, King et al. propose methods in [28] to compute a GVF using real-valued data on the vertices of K. Doing so leads to an interesting variant of MinMM, and motivates the question of whether or not vertex data is indeed helpful in computing discrete Morse functions. The paradigm of [28] is the focus of this paper, where we use an injective function $f_0 : K_0 \to \mathbb{R}$ on the vertices of K to compute optimal or near-optimal Morse matchings. We propose the natural extension problem to MinMM, given a function on the vertices:

▶ **Problem 2.5** (Extended Minimum Morse Matching, ExtMM). *Given a simplicial complex K and an injective real-valued function $f_0 : K_0 \to \mathbb{R}$ on the vertices of K, assign a discrete Morse function f to K such that $f|_{K_0} = f_0$, minimizing the number of critical simplices while retaining this property.*

Note that there is technically less freedom in ExtMM than in MinMM; the criterion that $f|_{K_0} = f_0$ along with the Morse inequalities forces matchings in K_1, which in turn forces matchings in higher dimensions. Intuitively, one might infer that this implies that ExtMM is in fact easier than MinMM. This paper investigates how much easier computing a Morse function is if we have help from data on the vertices.

In both [19, 28], a discrete Morse function is obtained by extending f_0 into higher dimensions; for details see Appendix **??**. In [28], this extended Morse function is then refined toward a solution to MinMM. In [19], the insight is made that f_0 induces a lexicographical ordering on the higher dimensional simplices of K, which can be computed in $\Theta(dn)$. This paper similarly exploits the lexicographical ordering of simplices from a given f_0. In doing so, we relate the results of King et al. to existing randomized methods introduced by Kahle in [27] using the so-called *apparent pairs gradient* from random assignments of lexicographical order on K. The apparent pairs gradient has been successful in persistent homology applications [3, 25], and more recently has even been studied through the lens of probability theory on random clique complexes [34], and on Costa-Farber complexes [6].

We now define lexicographical order among simplices in the usual sense, where we write $lex(\sigma) > lex(\sigma')$ if σ is lexicographically larger than σ':

▶ **Definition 2.6** (Lexicographical Order). *Let $i \in \{0, 1, ..., d\}$, and let $\sigma, \sigma' \in K_i$. Using f_0, define the sets $U = \{f_0(u_1), f_0(u_2), ..., f_0(u_i)\}$ and $V = \{f_0(v_1), f_0(v_2), ..., f_0(v_i)\}$ for every $u_k \in \sigma, v_k \in \sigma'$ with $k \in \{1, 2, ..., i\}$. Assuming that U and V are sorted from largest to smallest, we define the* lexicographical order *of σ, σ' by $lex(\sigma) > lex(\sigma')$ if $u_j > v_j$ at an index $1 \le j \le i$, and the values of U and V are equivalent at all prior indices.*

Using the lexicographical order, we define the *lexicographical matching* $\mathcal{M}$:

(a) An example input to MinMM, with f_0. **(b)** ExtractRightChild (K,f_0) output.

Figure 3 A simplicial complex K with an accompanying injective function $f_0 : K_0 \to \mathbb{R}$ on the vertices where ExtractRightChild (K,f_0) does not minimize $|C|$.

$$\mathcal{M} := \{(\tau,\sigma) \mid lex(\tau) = \max_{\alpha^{(i-1)} \prec \sigma^{(i)}} lex(\alpha^{(i-1)}), lex(\sigma) = \min_{\sigma^{(i)} | \tau \prec \sigma^{(i)}} lex(\sigma^{(i)})\}$$

We remark that due to the injectivity of f_0, $lex(\sigma) = lex(\sigma')$ if and only if $\sigma = \sigma'$. This paper builds on the techniques of Algorithm **??**, ExtractRightChild, from [19], which uses lexicographical orderings induced by f_0 on a given simplicial complex K to compute discrete Morse functions in $\Theta(dn)$ time where $d = dim(K)$ and n is the number of simplices of K, and links lexicographical matchings to the original formulation in King et al.

▶ **Observation 2.7.** *Notice that a lexicographical matching is often not a solution to MinMM. For a simple example, see Figure 3. In fact, it is possible to even have ExtractRightChild (K,f_0) producing $O(n)$ critical cells, as occurs in Figure* **??**.

3 An Algorithm Solving ExtMM for Two-Manifolds

In this section, we give a $\Theta(n)$-time algorithm solving ExtMM for two-manifolds with n simplices.

Our algorithm relies on invariants of spanning trees and their cotrees on triangulations, which are defined in Section 2.1. For an example, see Figure 1c. .

We now prove the optimality of Algorithm 1, demonstrating that the algorithm solves ExtMM for two-manifolds.

▶ **Lemma 3.1** (Algorithm 1 Minimizes $|C_1|$). *Algorithm 1 minimizes the number of critical edges in its output GVF while maintaining that $f|_{K_0} = f_0$.*

Proof. Let (T,R,X) be a tree-cotree decomposition produced by the algorithm, then $|C_1| = |X|$. Lemma 2 in [17], shows that the loops $\{(T,e)|e \in X\}$ are the fundamental cycles of the surface. By Theorem 2A.1 in [24], $H_1(K,\mathbb{Z}_2)$ is the abelianization of the fundamental group and $\beta_1 = |X|$. Thereafter, the only edges added to C_1 are contained in T. Since T is a minimum spanning tree, this minimizes the edges in T that inconsistent with the gradient induced by f_0 and therefore are assigned critical. ◀

Similar methods also compute the generators of the fundamental group in linear time [12,35].

▶ **Lemma 3.2** (MorseDual Recovers $|C_2| = \beta_2$). *MorseDual minimizes the number of critical faces in its output GVF.*

Proof. In Line 29, a single face is marked as critical and $|C_2| = 1$. By Poincaré duality (Corollary 65.5 of [32]), $H_2(K, \mathbb{Z}_2) \cong \mathbb{Z}_2$, regardless of orientability, and $\beta_2 = 1$. ◀

▶ **Theorem 3.3** (Algorithm 1 is $O(n \log n)$.)**.** *Algorithm 1 runs in $O(n \log n)$ time using $O(n)$ space*

Proof. Finding a global minimum r and passing every edge outward from r in Line **??** is linear in the number of simplices, n, just by iterating over every vertex to find the minumum, and then doing a BFS on the 1-skeleton of K. Computing a minimum spanning tree in Line **??** is $O(|K_1| * \log(|K_0|))$ running Prim's algorithm with a binary heap on the 1-skeleton of K. Otherwise, the algorithm follows the same procedure layed out in [**?**], which is $O(n)$. Hence, the most expensive step in Algorithm 1 is the requirement of a minimum spanning tree, and Algorithm 1 is $O(n \log n)$. ◀

Interestingly, for surfaces ExtMM actually appears more difficult than MinMM, due to required consistency of f_0 to $f|_{K_0}$ and the need for a minimum spanning tree. This is somewhat counterintuitive, since we are given a rough structure to follow from f_0, but insodoing we lose the freedom to choose pairings among faces and edges arbitrarily which is a necessary criterion for optimal tree-cotree algorithms on general surfaces. .

4 NP-Hardness of ExtMM in High Dimensions

In this section, we show that ExtMM is NP-Hard in two dimensions or higher (if K is allowed to be a non-manifold). To do so, we make a modification to the conventional reduction showing the NP-Completeness of MinMM given by Joswig and Pfetsch [26]. Recall that the strategy used in [26] is to reduce an instance of the *collapsibility problem*, introduced by Eğecioğlu and Gonzalez [15], to an instance of MinMM. Here, we proceed analogously, reducing an arbitrary instance of the collapsibility problem to an instance of ExtMM. That problem's hardness (phrased through the lens of discrete Morse theory) is as follows:

▶ **Theorem 4.1.** *Given a connected pure 2-dimensional simplicial complex K, which is embeddable in $\mathbb{R}^3$, and a nonnegative integer k, it is $\mathcal{NP}$-complete in the strong sense to decide whether there exists a Morse matching with at most k critical 2-faces.*

Let $\Gamma(K) = (K_0, K_1 \setminus \{\sigma^{(1)} \in M^T\})$ denote the 1-skeleton of a simplicial complex K subtracting every edge paired with a 2-face in a Morse matching M. Recall the hardness result given in [26] hinges on the following lemma:

▶ **Lemma 4.2.** *The graph $\Gamma(K)$ is connected.*

Using Lemma 4.2, we can then assign in polynomial time $f_0 : K_0 \to \mathbb{R}$ by picking an arbitrary vertex $v \in K_0$, and doing a breadth-first search outward from v, enumerating vertices in the order they are touched. Then, due to the Morse inequalities, setting $f_0 = f|_{K_0}$ forces that matchings in K_1 must follow the induced gradient from f_0. Since every vertex has a path to v decreasing in f_0, we have a Morse matching on $\Gamma(K)$ which is optimal (with only a single critical vertex). Note, every Morse matching has at least one critical vertex, and an optimal matching has critical vertices in one-to-one correspondence with connected components. As in [26], we have computed an optimal matching on $\Gamma(K)$, so the number of critical edges $|C_1|$ can only decrease, while the number of critical i-faces for $i \geq 2$ is unchanged. Hence, we have an analogous corollary to Corollary 4.3 in [26]:

▶ **Corollary 4.3.** *Let M be a Morse matching on K. Then we can compute a Morse matching M' in polynomial time which has exactly one critical vertex and the same number of critical faces of dimension 2 or higher as M, such that $|C'| \leq |C|$ and $f|_{K_0} = f_0$.*

Finally, we obtain the following analogous result to [26]:

▶ **Theorem 4.4.** *Given a simplicial complex K and a nonnegative integer c, it is strongly $\mathcal{NP}$-complete to decide whether there exists a solution to* ExtMM *with at most c critical faces, even if K is connected, pure, 2-dimensional, and can be embedded in $\mathbb{R}^3$.*

Proof. We proceed as in [26]. Let (K, k) be an input to the collapsibility problem. From Theorem 4.4 of [26], there exists a Morse matching with exactly $|C_2| = k$ iff there exists a Morse matching with at most $g(k) := 2(k+1) - \chi(K)$ critical cells altogether, where χ is the Euler characteristic. Due to Corollary 4.3, this Morse matching is also consistent with f_0 (i.e. $f|_{K_0} = f_0$), and is therefore a solution to ExtMM. Now, $g(k)$ is computable in polynomial time, so from an instance of the collapsibility problem we have a polynomial time reduction to an instance of ExtMM, and from a solution to ExtMM we indeed have a solution to the collapsibility problem. Finally, Theorem 4.1 yields that ExtMM is $\mathcal{NP}$-complete. ◀

Not only are the hardness results in [26] applicable to ExtMM, but so are the stronger parameterized inapproximability results given in [6]. There, we can take the modified Dunce cap used as a gadget to reduce Circuit-SAT to MinMM, and in linear time generate an $f_0 : K_0 \to \mathbb{R}$ such that the induced gradient is the same. See . Otherwise using the same gadget, the reduction is identical.

5 Discussion

Given a simplicial complex K, this paper studies the problem of MinMM, which is to find a discrete Morse function on K that minimizes the number of critical simplices. The problem is approached through the lens of King et al., which additionally requires that an injective function $f_0 : K_0 \to \mathbb{R}$ is given on the vertices of K. We give a linear time algorithm solving MinMM for two-manifolds, which is the first improvement since 2005 on the methods of King et al. in [28]. In doing so, we demonstrate that the framework introduced in King et al., which computes a GVF using a given injective $f_0 : K_0 \to \mathbb{R}$ does not aid in efficiently computing a discrete Morse function in the case for two-manifolds. It is difficult to imagine the existence of a faster algorithm, as one would expect that every simplex would need to be visited at least once to construct a gradient vector field.

This paper also examines MinMM in higher dimensions when given an injective f_0 on the vertices of K. Using simple heuristics exploiting lexicographical orderings resulting from f_0, we provide an approximation of MinMM that is within a small additive factor by assigning a randomized f_0 to the vertices of K on a substantial class of complexes. In particular, these are complexes with few regions of torsion, where there *exists* an injective $f_0^* : K_0 \to \mathbb{R}$ such that ExtractRightChild (K, f_0^*) is a solution to MinMM. We additionally introduce a Morse-theoretic gradient descent heuristic to manipulate a given f_0 that approaches f_0^*. Our gradient descent substantially limits the expected number of critical simplices that result from ExtractRightChild (K, f_0). Despite the inapproximability of MinMM, we provide a brief experiment demonstrating the remarkably strong performance of ExtractRightChild when combined with MorseGradientDescent in practice. This leads us to a far-reaching randomized algorithm approximating MinMM within a constant additive factor on a realistic class of complexes.

Extensions to this work abound, and include the integration of randomized Morse theoretic heuristics in persistent homology applications. Our hope is that these results will have a sizable impact on the viability of computational techniques in Morse theory, and by extension, that computational topology broadly will become more powerful as a consequence.

References

1. Pankaj K. Agarwal, Herbert Edelsbrunner, John Harer, and Yusu Wang. Extreme elevation on a 2-manifold. In *Proceedings of the Twentieth Annual Symposium on Computational Geometry (SoCG)*, pages 357–365, 2004.
2. Ulrich Bauer. *Persistence in Discrete Morse Theory*. PhD thesis, Niedersächsische Staats-und Universitätsbibliothek Göttingen, 2011.
3. Ulrich Bauer. Ripser: efficient computation of vietoris–rips persistence barcodes. *Journal of Applied and Computational Topology*, 5(3):391–423, 2021.
4. Ulrich Bauer, Carsten Lange, and Max Wardetzky. Optimal topological simplification of discrete functions on surfaces. *Discrete and Computational Geometry*, 47(2):347–377, 2012.
5. Ulrich Bauer and Abhishek Rathod. Hardness of approximation for morse matching. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2663–2674, 2019.
6. Ulrich Bauer and Abhishek Rathod. Parameterized inapproximability of morse matching. arXiv:2109.04529, 2021.
7. Bruno Benedetti and Frank H. Lutz. Random Discrete Morse Theory and a New Library of Triangulations. *Experimental Mathematics*, 23(1):66–94, 2014.
8. Kree Cole-McLaughlin, Herbert Edelsbrunner, John Harer, Vijay Natarajan, and Valerio Pascucci. Loops in Reeb graphs of 2-manifolds. In *Proceedings of the Nineteenth Annual Symposium on Computational Geometry (SoCG)*, pages 344–350, 2003.
9. Lidija Čomić and Leila De Floriani. Dimension-independent simplification and refinement of Morse complexes. *Graphical Models*, 73(5):261–285, 2011.
10. Tamal Dey, Jiayuan Wang, and Yusu Wang. Graph reconstruction by discrete Morse theory. In *Proceedings of the Thirty-Fourth Annual ACM-SIAM Symposium on Computational Geometry (SoCG)*, pages 31:1–31–13, 2018.
11. Tamal K. Dey. A new technique to compute polygonal schema for 2-manifolds with application to null-homotopy detection. In *Proceedings of the Tenth Annual Symposium on Computational Geometry*, (SoCG), page 277–284, New York, NY, USA, 1994. Association for Computing Machinery.
12. Tamal K. Dey, Fengtao Fan, and Yusu Wang. An efficient computation of handle and tunnel loops via Reeb graphs. *ACM Transactions on Graphics (TOG)*, 32(4):1–10, 2013.
13. Herbert Edelsbrunner, John Harer, and Afra Zomorodian. Hierarchical Morse-Smale complexes for piecewise linear 2-manifolds. *Discrete and Computational Geometry*, 30(1):87–107, 2003.
14. Herbert Edelsbrunner, David Letscher, and Afra Zomorodian. Topological persistence and simplification. *Discrete and Computational Geometry*, 28:511–533, 2002.
15. Ömer Eğecioğlu and Teofilo F. Gonzalez. A computationally intractable problem on simplicial complexes. *Computational Geometry*, 6(2):85–98, 1996. URL: `https://www.sciencedirect.com/science/article/pii/0925772195000151`, `doi:https://doi.org/10.1016/0925-7721(95)00015-1`.
16. Alexander Engström. Discrete Morse Functions from Fourier Transforms. *Experimental Mathematics*, 18(1):45–54, January 2009.
17. David Eppstein. Dynamic generators of topologically embedded graphs. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 599–608, 2003.
18. Jeff Erickson. One-dimensional computational topology lecture notes. Lecture 10: Tree-CoTree Decompositions, 2020.

19 Brittany Fasy, Benjamin Holmgren, Bradley McCoy, and David Millman. If you must choose among your children, pick the right one. In *Canadian Conference on Computational Geometry (CCCG 2020)*, August 2020.

20 Robin Forman. Discrete Morse theory for cell complexes. *Advances in Mathematics*, 134:90–145, 1998.

21 Robin Forman. A user's guide to discrete Morse theory. *Séminaire Lotharingien de Combinatoire*, 42:Art. B48c, 35pp, 2002.

22 Attila Gyulassy, Peer-Timo Bremer, Bernd Hamann, and Valerio Pascucci. A practical approach to morse-smale complex computation: Scalability and generality. *IEEE Transactions on Visualization and Computer Graphics*, 14(6):1619–1626, 2008.

23 Shaun Harker, Konstantin Mischaikow, Marian Mrozek, Vidit Nanda, Hubert Wagner, Mateusz Juda, and Paweł Dłotko. The efficiency of a homology algorithm based on discrete morse theory and coreductions. *Proceedings of the Third International Workshop on Computational Topology in Image Context*, pages 41–47, 2010.

24 Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

25 Gregory Henselman and Robert Ghrist. Matroid filtrations and computational persistent homology. *arXiv: Algebraic Topology*, 2016. URL: `https://api.semanticscholar.org/CorpusID:16354165`.

26 Michael Joswig and Marc Pfetsch. Computing optimal Morse matchings. *SIAM Journal on Discrete Mathematics (SIDMA)*, 20(1):11–25, 2006.

27 Matthew Kahle. Random geometric complexes. *Discrete & Computational Geometry*, 45:553–573, 2011.

28 Henry King, Kevin Knudson, and Nevža Mramor. Generating discrete Morse functions from point data. *Experimental Mathematics*, 14:435–444, 2005. MR2193806.

29 Kevin Knudson. *Morse Theory: Smooth and Discrete*. World Scientific Publishing Company, 2015.

30 Thomas Lewiner, Hélio Lopes, and Geovan Tavares. Toward optimality in discrete Morse theory. *Experimental Mathematics*, 12:271–285, 2003.

31 John Milnor. *Morse Theory*. Princeton University Press, Princeton, New Jersey, 1963.

32 James Munkres. *Elements of algebraic topology*. CRC Press, Boca Raton London New York, 1993.

33 Nicholas Scoville. *Discrete Morse Theory*. American Mathematical Society, Providence, Rhode Island, 2019.

34 Tadas Temčinas, Vidit Nanda, and Gesine Reinert. Multivariate central limit theorems for random clique complexes. arXiv:2112.08922v3, 2022.

35 Gert Vegter and Chee K. Yap. Computational complexity of combinatorial surfaces. In *Proceedings of the Sixth Annual Symposium on Computational Geometry*, pages 102–111, 1990.

Algorithm 1

Input: K, a triangulation of a two-manifold
Output: a GVF over K minimizing C over all GVFs over K

Find the root $r \in K_0$ s.t. $f_0(r) = \min_{v \in K_0} f_0(v)$
Pass every edge outward from r, assigning edge weight 0 if f_0 increases and 1 otherwise.
Compute a minimum spanning tree $T = (K_0, E_T)$ of K using directed edge weights.
Compute the restricted dual graph $G^* = (K_2^*, E^* := K_1^* \setminus E_T)$
$C \leftarrow \emptyset$ ▷ critical cells
$M \leftarrow \emptyset$ ▷ matched cells
For each cell in K, add an attribute 'marked' and set it to False
Let T' denote the sub-tree of T that is unmarked (stored implicitly)
while $\exists$ unmarked leaf node v in T' **do** ▷ match cells of T
    Let e be the edge that connects v to the rest of T'.
    Mark e and v
    Add (v, e) to M
end while
$v \leftarrow$ unmarked vertex of K_0
Add v to C.
Let G' denote the sub-graph of G^* whose vertices/edges are unmarked in K.
while $\exists$ unmarked cells of K **do**
    while $\exists$ unmarked degree-one vertex v^* in G' **do**
        Let e^* be the edge that connects v^* to the rest of G'.
        Let (e, f) be the dual to (e^*, v^*)
        Mark e and f
        Add (e, f) to M
    end while
    if $\exists$ unmarked edge $e^* \in E^*$ **then** ▷ e^* must be in a cycle
        Mark e^*
        Add e^* to C
    end if
    if A single unmarked $f \in F$ remains **then**
        Add f to C
    end if
end while