Path-Connectivity of Fréchet Spaces of Graphs

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Introduction

The Problem

- Paths (curves) and graphs living in Euclidean ambient space

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- Define the Fréchet distance between elements in these spaces

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- Paths (curves) and graphs living in Euclidean ambient space
- Define the Fréchet distance between elements in these spaces
- What is the behavior of these topological spaces equipped with the Fréchet distance?
- Are these spaces path-connected?

Introduction

Six Different Spaces

- Paths and graphs continuously mapped to $\mathbb{R}^d$

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- Paths and graphs immersed in $\mathbb{R}^d$

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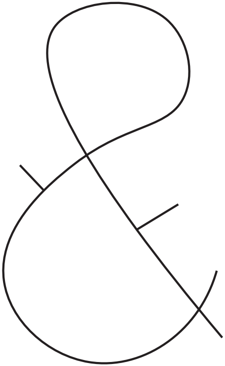
- Paths and graphs immersed in $\mathbb{R}^d$
- Paths and graphs embedded in $\mathbb{R}^d$

Spaces of Paths and Graphs

Continuous Mappings



- The set $\Pi_{\mathcal{C}}$ of all images of continuous maps $\alpha : [0,1] \to \mathbb{R}^d$
- Require that α is rectifiable

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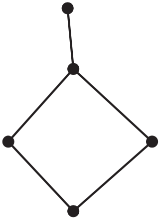

- Let G a finite multigraph
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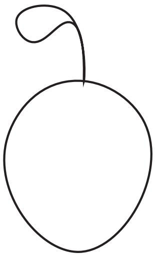

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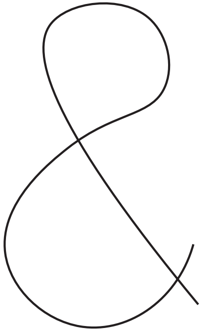
Immersions



- The set $\Pi_{\mathcal{I}}$ of all images of *immersions* $\alpha : [0, 1] \to \mathbb{R}^d$
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Spaces of Paths and Graphs

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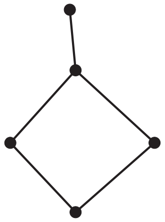

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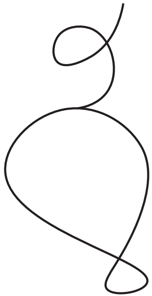

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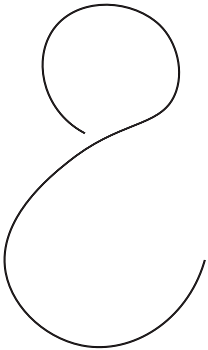

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Spaces of Paths and Graphs

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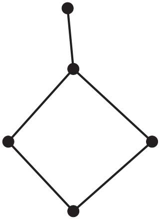

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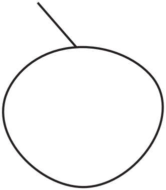

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The Fréchet Distance: Paths and Graphs

The Fréchet distance for Paths

Let $\alpha_0, \alpha_1 \in \Pi_{\mathcal{C}}$,

$$d_{FP}(\alpha_0, \alpha_1) := \min_{r\colon [0,1]\to[0,1]} \max_{t\in[0,1]} ||\alpha_0(t) - \alpha_1(r(t))||_2$$

Where r ranges over all reparametrizations of the unit interval.

The Fréchet Distance: Paths and Graphs

The Fréchet distance for Graphs

Let $(G, \phi), (G, \psi) \in \mathcal{G}_{\mathcal{C}}$.
For any homeomorphism $h : G \to G$ we use the induced L_∞ norm:

$$||\phi - \psi \circ h||_\infty = \max_{x \in G} |\phi(x) - \psi(h(x))|$$

to define the Fréchet distance:

$$d_{FG}\left((G, \phi), (G, \psi)\right) := \min_h ||\phi - \psi \circ h||_\infty$$

Path-Connectedness Property

The Problem

Are these spaces of continuous mappings, immersions, and embeddings path-connected under the Fréchet distance?

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That is, for a topological space X, can one always construct continuous $\Gamma : [0,1] \to X$ such that $\Gamma(0) = x_0, \Gamma(1) = x_1$ for any $x_0, x_1 \in X$?

The spaces $(\Pi_{\mathcal{C}}, d_{FP})$ and $(\mathcal{G}_{\mathcal{C}}(G), d_{FG})$ are path-connected

Proof Sketch

Let $\alpha_0, \alpha_1 \in \Pi_{\mathcal{C}}$. Construct a path from α_0 to α_1 by interpolating along the pointwise matchings defining $d_{FP}(\alpha_0, \alpha_1)$.

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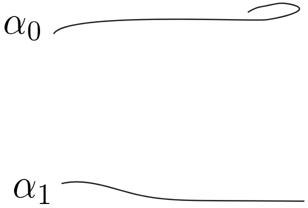

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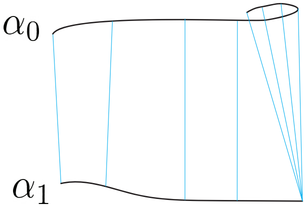

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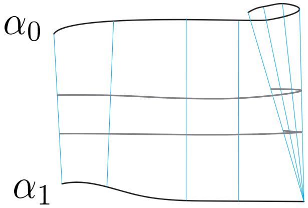

The spaces $(\Pi_{\mathcal{I}}, d_{FP})$ and $(\mathcal{G}_{\mathcal{I}}(G), d_{FG})$ are path-connected

Proof Sketch

Let $\alpha_0, \alpha_1 \in \Pi_{\mathcal{I}}$. Construct a path from α_0 to α_1 by interpolating as before, but carefully avoid degeneracies with a self-crossing maneuver.

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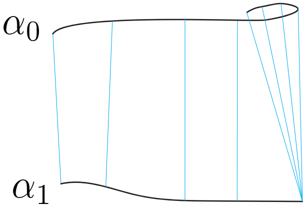

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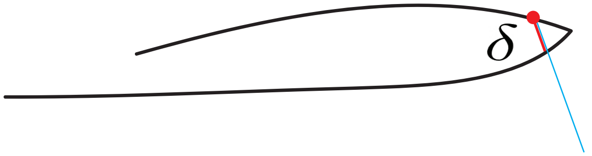
Proof Sketch

Let $\alpha_0, \alpha_1 \in \Pi_{\mathcal{I}}$. Construct a path from α_0 to α_1 by interpolating as before, but carefully avoid degeneracies with a self-crossing maneuver.

conduct self-crossings when sufficiently close at a crossing point



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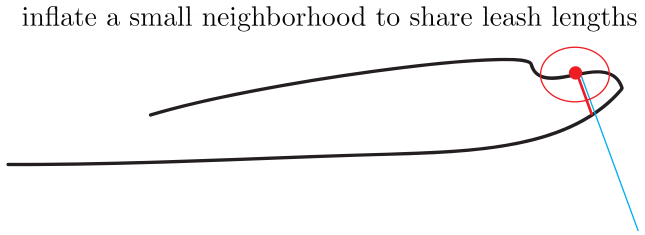

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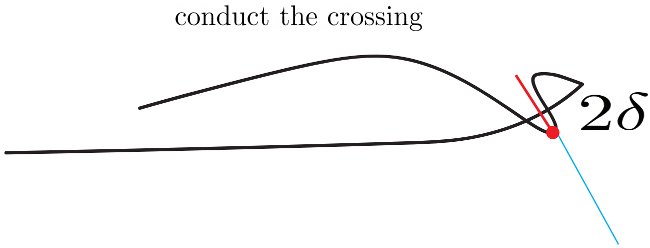
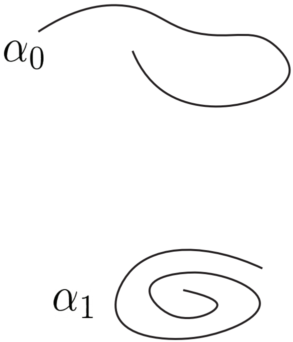


The space $(\Pi_{\mathcal{E}}, d_{FP})$ is path-connected

Proof Sketch

Let $\alpha_0, \alpha_1 \in \Pi_{\mathcal{E}}$. There exists a canonical path from α_0 to α_1:



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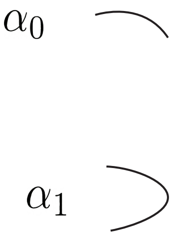
Let $\alpha_0, \alpha_1 \in \Pi_{\mathcal{E}}$. There exists a canonical path from α_0 to α_1:

Condense each path toward its center.



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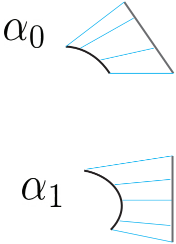
Proof Sketch

Let $\alpha_0, \alpha_1 \in \Pi_{\mathcal{E}}$. There exists a canonical path from α_0 to α_1:

Due to rectifiability, we are guaranteed that each path will become 'straight enough' for direct interpolation to a straight segment.



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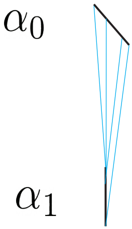
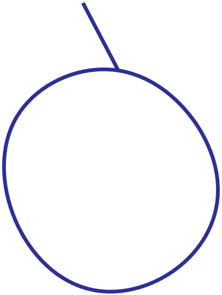
After attaining straight segments, interpolation is sufficient.



The space $(\mathcal{G}_{\mathcal{E}}(G), d_{FG})$ is not path-connected in low dimensions

Counterexample in dimension 2

Suppose G is a graph comprising a cycle and a dangling edge. Then the space $(\mathcal{G}_{\mathcal{E}}(G), d_{FG})$ is not path-connected:



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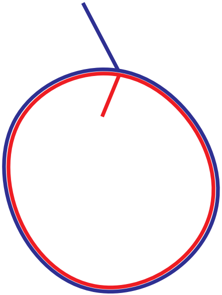

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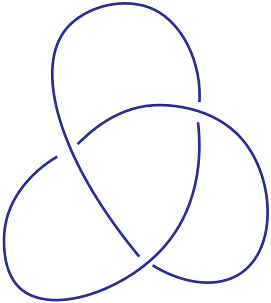

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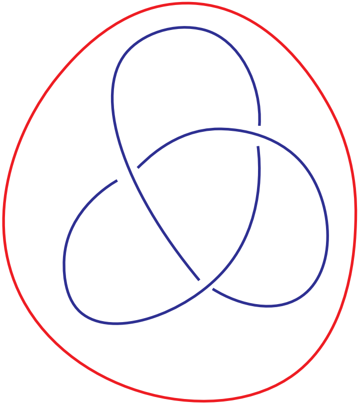

The space $(\mathcal{G}_{\mathcal{E}}(G), d_{FG})$ is path-connected in high dimensions

Path-connectedness property in dimension ≥ 4

- Every tame knot in $\mathbb{R}^d$ for $d \geq 4$ has a sequence of Reidemeister moves unravelling to the unknot.

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- Due to rectifiability, any $\phi \in \mathcal{G}_{\mathcal{E}}(G)$ is made up of tame knots.

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Path-connectedness property in dimension ≥ 4

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- Due to rectifiability, any $\phi \in \mathcal{G}_{\mathcal{E}}(G)$ is made up of tame knots.
- Hence, interpolate along leashes, and at a self-crossing event conduct the corresponding Reidemeister move.

Path-Connectedness of Balls

Definition

Is it always possible to create a path in these spaces without increasing the initial distance?

Path-Connectedness of Balls

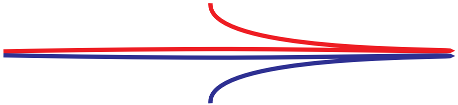
Continuous Mappings and Immersions

The paths constructed for $\Pi_{\mathcal{C}}, \mathcal{G}_{\mathcal{C}}, \Pi_{\mathcal{I}}$, and $\mathcal{G}_{\mathcal{I}}$ never strictly increase the Fréchet distance.

Path-Connectedness of Balls

Embeddings

This is not the case for $\Pi_{\mathcal{E}}$ restricted to dimension 2:



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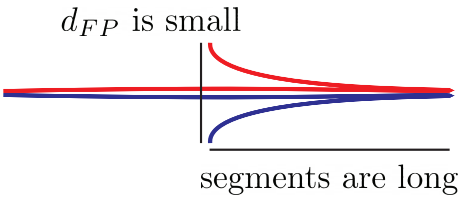

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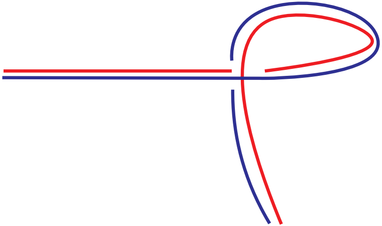

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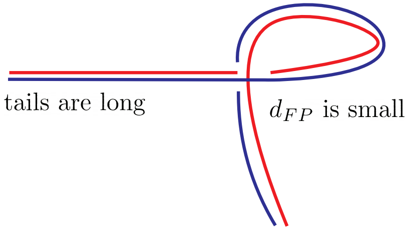

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Thank You

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