# NOTES ON MATH FOR DATA SCIENCE

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*The following collection was compiled while studying for the degree of masters of data science at City University of New York over the years 2018–present.*

## 1 ALGEBRA

### ARITHMETIC OPERATIONS

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{\frac{a}{b}}{c} = \frac{a}{bc}$$

$$\frac{a}{\frac{b}{c}} = \frac{ac}{b} \qquad \frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

$$\frac{a-b}{c-d} = \frac{b-a}{d-c} \qquad \frac{ab+ac}{a} = b + c,\ \forall a \neq 0$$

$$\frac{a+bc}{c} = \frac{a}{c} = \frac{b}{c} \qquad \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

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### EXPONENTS

$$x^a \cdot x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b}$$

$$(xy)^a = x^a y^a \qquad \frac{x}{y}^a = \frac{x^a}{y^a}$$

$$x^0 = 1^{\dagger} \qquad x^{-n} = \frac{1}{x}^{\dagger}$$

$$x^{\frac{1}{a}} = \sqrt[a]{x}^{\dagger} \qquad (xy)^a = x^a y^a$$

$$x^{-a} = \frac{1}{x^a} \qquad \left(\frac{x}{y}\right)^{-a} = \left(\frac{y}{x}\right)^a = \frac{y^a}{x^a}$$

$^{\dagger}\forall x \neq 0$

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### RADICALS

$$\sqrt[a]{x} = x^{\frac{1}{a}} \qquad \sqrt[a]{xy} = \sqrt[a]{x}\,\sqrt[a]{y}$$

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### LOGARITHMS

Defined as $y = \log_b x \equiv x = b^y$, with domain of $x > 0$

$$\log_b b = 1 \qquad \log_b 1 = 0$$

$$\log_b b^x = x \qquad b^{\log_b x} = x$$

$$\log_b (x^r) = r \log_b x$$

$$\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\log_b (xy) = \log_b x + \log_b y$$

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### FACTORING

$$x^2 - a^2 = (x+a)(x-a)$$

$$x^2 \pm 2ax + a^2 = (x \pm a)^2$$

$$x^2 + (a+b)x + ab = (x+a)(x+b)$$

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### FUNCTIONS

- A rule for a relationship between an input and an output quantity where each input *uniquely determines* an output value
- Must be of the form $y = f(x)$

| TYPE | FORM |
|---|---|
| Constant | $f(x) = c$ |
| Identity | $f(x) = x$ |
| Absolute | $f(x) = \lvert x \rvert$ |
| Quadratic | $f(x) = x^2$ |
| Cubic | $f(x) = x^3$ |
| Reciprocal | $f(x) = \frac{1}{x^2}$ |
| Square root | $f(x) = \sqrt{x}$ |

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### TRANSFORMATIONS OF FUNCTIONS

| Transformation | Form |
|---|---|
| Vertical shift | $f'(x) = f(x) + k$ |
| Horizontal shift | $f'(x) = f(x - k)$ |
| Horizontal reflection | $f'(x) = f(-x)$ |
| Vertical reflection | $f'(x) = -f(x)$ |
| Vertical stretch | $f'(x) = kf(x),\ k > 1$ |
| Vertical compression | $f'(x) = kf(x),\ k0 < k < 1$ |

### DISTANCE FORMULA

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

is the distance between points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$

### LINEAR FUNCTIONS

- Linear functions have a constant rate of change, $m$

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- Estimate horizontal line by solving $f(x) = 0$
- Find the point where two non-parallel lines meet: set $f(x) = g(x)$ and solve for $x$
- Estimate slope from two points by solving for $m$:

$$y_1 - y_2 = m(x_1 - x_2)$$

### QUADRATIC FUNCTIONS

| | |
|---|---|
| Standard form | $f(x) = ax^2 + bx + c$ |
| Vertex or transformation form | $f(x) = a(x - h)^2 + k$ |
| Quadratic eq. | $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |

- $b^2 - 4ac > 0 \Rightarrow$ Two real unequal solutions
- $b^2 - 4ac = 0 \Rightarrow$ Repeated real solution
- $b^2 - 4ac < 0 \Rightarrow$ Two complex solutions

### POLYNOMIAL FUNCTIONS

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^{n-k}$$

| | |
|---|---|
| $a_0$ | Constant term |
| $a_n$ | Polynomial coefficient |
| $a_n x^m$ | Term ($m$ is called *degree*) |

### RATIONAL FUNCTIONS

Can be written as a quotient of two polynomials $P(x)$ and $Q(x)$:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{\sum_{k=0}^{n} a_k x^{n-k}}{\sum_{k=0}^{n} b_k x^{n-k}}$$

- Horizontal Intercept is the inputs where the output is 0
- Vertical Intercept is where input is 0 (if defined)

### ASYMPTOTES

The "line" a function approaches but never touches

*vertical* A vertical line $x = a$ where the graph tends towards positive or negative infinity as the inputs approach $a$:

$$x \to a,\ f(x) \to \pm\infty$$

*horizontal* A horizontal line $y = b$ where the graph approaches the line as the input gets larger:

$$x \to \pm\infty,\ f(x) \to b$$

### ASYMPTOTES OF RATIONAL FUNCTIONS

*vertical* Where denominator = 0 but numerator $\neq 0$

*horizontal* Determined by respective degrees of numerator and denominator:

- Degree of denominator $>$ degree of numerator $\Rightarrow$ Horizontal asymptote at $y = 0$
- Degree of denominator $<$ degree of numerator $\Rightarrow$ No horizontal asymptote
- Degree of denominator = degree of numerator $\Rightarrow$ Horizontal asymptote is ratio of leading coefficients

## EXPONENTIAL FUNCTIONS

Rate of change is as a percent, i.e., not an constant (absolute) rate. Takes the form:

$$f(x) = a(1+r)^x$$

or,

$$f(x) = ab^x, \; b = 1+r$$

Can always be rewritten in terms of logarithms:

$$b^a = c \equiv \log_b c = a$$

*continuous growth* Use $e = 2.718282\ldots$ for continuous growth, often natural phenomena

$$f(x) = ae^{rx}$$

where

- $a \equiv$ initial quantity
- $r \equiv$ continuous growth rate

*ALMOST ALWAYS USE CONTINUOUS e FORM!*

Solve by:

1. Isolate exponential expression, where possible
2. Take log on both sides
3. Use *exponent property* of logs to pull variables of out exponent
4. Use algebra to solve for variable

## GRAPHING EXPONENTIAL FUNCTIONS

- $a \equiv$ vertical intercept
  - $a > 0 \Rightarrow$ concave up
  - $a < 0 \Rightarrow$ concave down
- $b \equiv$ rate of growth
  - $b > 1 \Rightarrow$ growing
  - $0 < b < 1 \Rightarrow$ decaying
- Horizontal asymptote is where $y = 0$

## LOGARITHMIC FUNCTIONS

The inverse of exponential functions; use to solve exponential functions. Commonly used to express quantities that vary widely in size. Form:

$$\log_b x$$

which can be rewritten in terms of exponents:

$$b^a = c \equiv \log_b c = a$$

*inverse property of logs*

$$\begin{aligned} \log_b b^x &= x \\ b^{\log_b x} &= x \end{aligned}$$

*exponent property*

$$\log_b A^r = r\log_b A$$

# 2 PROBABILITY

### ASSUMPTIONS

$$0 \leqslant P(E) \leqslant 1$$
$$P(S) = 1$$
$$P(S) = \sum_{i=1}^{\infty} P(E_i)$$

Sample space S contains each event $E_i$, e.g., $E = \{$all outcomes in S starting with a 3$\}$

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### UNIONS AND INTERSECTIONS

$$E \cup F$$

is the union of the two sets E and F, i.e., the event where either E or F occurs. The intersection of two events, the outcomes in both E and F is

$$E \cap F.$$

| | |
|---|---|
| Commutative | $E \cup F = F \cup E$ |
| | $E \cap F = F \cap E$ |
| Associative | $(E \cup F) \cup G = E \cup (F \cup G)$ |
| | $(E \cap F) \cap G = E \cap (F \cap G)$ |
| Distributive | $(E \cup F) \cap G = (E \cap G) \cap (F \cap G)$ |
| | $(E \cap F) \cup G = (E \cup G)(F \cup G)$ |

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### INDEPENDENT EVENTS

Two events are independent if knowing the outcome of one provides no useful information about the outcome of the other.

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### MUTUALLY EXCLUSIVE EVENTS

$$P(A \cap B) = 0$$

are disjoint events, when A and B are mutually exclusive and there is no intersection—it is not possible for both to happen.

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### UNION AND ADDITION RULE

$$P(A \cup B) \equiv P(A \vee B) \equiv \{x : x \in A \vee x \in B\}$$

| | |
|---|---|
| Independent | $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ |
| Mutually exclusive | $P(A \cup B) = P(A) + P(B)$ |

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### INTERSECTION AND MULTIPLICATION RULES

$$P(A \cap B) \equiv P(A \wedge B) \equiv \{x : x \in A \wedge x \in B\}$$

| | |
|---|---|
| Independent | $P(A \cap B) = P(A) \cdot P(B)$ |
| Mutually exclusive | $P(A \cap B) = 0$ |
| Dependent | $P(A \cap B) = P(A) \cdot P(B\|A)$ |

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### COMPLEMENT AND SUBTRACTION RULE

$$\begin{aligned} P(A') \equiv P(A^c) \equiv P(\neg A) &\equiv \{x : x \notin A\} \equiv 1 - P(A) \\ P(A') &= 1 - P(A) \end{aligned}$$

Some implications:

$$P(A \cup A^c) = 1$$
$$P(A) = 1 - P(A^c)$$
$$P(A|B) = 1 - P(A^c|B)$$

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### CONDITIONAL PROBABILITY RULE

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

is the probability of the outcome of event A given condition B.

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### BAYES THEOREM

$$P(A|B) = \frac{P(A)\ P(B|A)}{P(A)\ P(B|A) + P(A)\ P(B|A)}$$

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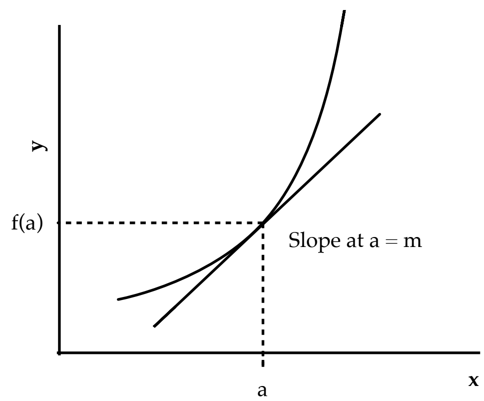
# 3 CALCULUS

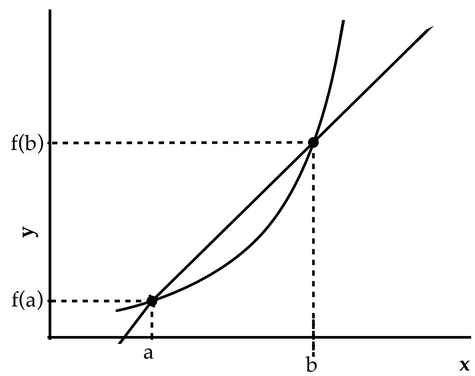
### TANGENT AND SECANT LINES

- Tangent line is the line on $f(x)$ at point $x = a$ that just touches the graph of the function

- The slope of a non-linear function at one point




- Secant line is the line between $A$ and $B$ on a curve




*Note*: As $A$ gets closer to $B$ the secant slope approaches the tangent line

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## AVERAGE RATE OF CHANGE

$$\mathrm{ARC} = f(a, a+\Delta) = \frac{\Delta y}{\Delta x} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

over interval $[a, a+\Delta x]$

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## DERIVATIVE

The instantaneous rate of change of a function $f(x)$ at point $x = a$

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

"the derivative of $y$ w.r.t. $x$" $\equiv \frac{dy}{dx} \equiv \lim_{\delta \to 0} \frac{\delta y}{\delta x}$

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## SIMPLE DERIVATIVES

| $y$ | $y'$ |
|---|---|
| $k$ | $0$ |
| $x$ | $1$ |
| $x^n$ | $nx^{n-1}$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| $\sqrt[n]{x}$ | $\frac{1}{n\sqrt[n]{x^{n-1}}}$ |
| $\frac{1}{x}$ | $\frac{-1}{x^2}$ |
| $e^x$ | $e^x$ |
| $a^x$ | $a^x \ln(a)$ |
| $x^x$ | $xx^{x-1} + x^x \ln(x)$ |
| $\ln(x)$ | $\frac{1}{x}$ |
| $\log_a(x)$ | $\frac{1}{x}\log_a(e)$ |

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## COMPOSITE DERIVATIVES

| $y$ | $y'$ |
|---|---|
| $u^n$ | $nu^{n-1}u'$ |
| $\sqrt{u}$ | $\frac{u'}{2\sqrt{u}}$ |
| $\sqrt[n]{u}$ | $\frac{u'}{n\sqrt[n]{u^{n-1}}}$ |
| $\frac{1}{u}$ | $\frac{-u'}{u^2}$ |
| $e^u$ | $e^u u'$ |
| $a^u$ | $a^u \ln(a) u'$ |
| $u^v$ | $vu^{v-1}u' + u^v \ln(u) v'$ |
| $\ln(u)$ | $\frac{u'}{u}$ |
| $\log_a(u)$ | $\frac{u'}{u} \log_a(e)$ |

## DERIVATIVE OPERATIONS

| | |
|---|---|
| Sum | $(f(x)+g(x))' = f'(x) + g'(x)$ |
| Difference | $(f-g)'(x) = f'(x) - g'(x)$ |
| Product | $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ |
| Quotient | $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$ |
| Chain rule | $(f(g))'(x) = f'(g(x))g'(x)$ |
| Inverse | $(f^{-1})'(x) = \frac{1}{f'(x)}$ |

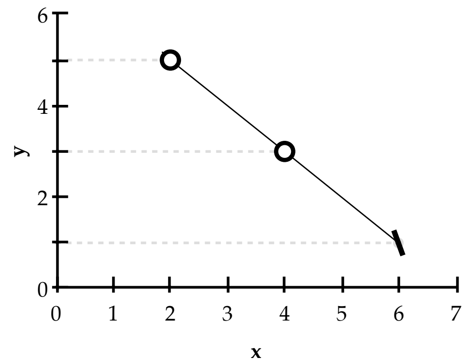
## LIMITS

$$\lim_{x \to c} f(x) = L$$

or, "the limit of $f(x)$, as $x$ approaches c, is L." $\lim_{x\to c} f(x)$ is a *single number* that describes the behavior of the function $f(x)$ *near* but not *at* the point $x = c$.

Introduced to make calculating rate of change at 0 feasible, by making the $\Delta$ so infinitesimal the difference is between it and 0 is negligible—"allows" division by 0

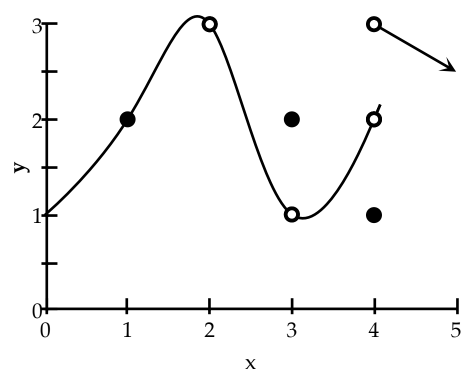
*Example*




where hollow points are undefined and solid points are defined

1. $\lim_{x\to 6} f(x) = 1$
2. $\lim_{x\to 4} f(x) = 3$
   - *Note*: even though $x = 4$ is undefined, we're only concerned with the area *around* 4, so we can still find the limit
3. $\lim_{x\to 2} f(x) = 5$

*Example: Determining Limits of Non-Linear Functions*




where hollow points are undefined and solid points are defined

1. $\lim_{x\to 1} f(x) = 2$—values where $x$ is close to but not equal to 1 are near 2
2. $\lim_{x\to 2} f(x) = 3$—even though $f(2)$ is undefined, only values *near* $f(2)$ are important
3. $\lim_{x\to 3} f(x) = 1$—even though $f(3)$ is actually 2
4. $\lim_{x\to 4} f(x)$ = *does not exist*: Can't determine a single number because $f(4)$ from the right is about 2, and from the left about 3

*Example: Determining Limits Using Algebra*

Factor equation to simplest form and plug in $c$ (assuming function at $c$ is defined):

$$\lim_{x\to 5} \frac{x^2-6x+8}{x-4} = \frac{25-30+8}{1} = 3$$

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### LIMITS OF BROKEN FUNCTIONS

Some functions are continuous but in an unusual way—they appear "broken" when graphed—and so there is a *left* and *right* limit

*left* "The limit coming from the left"; values of $f(x)$ as $f(x)$ nears $x$ and left of $c$, $x < c$

$$\lim_{x\to c^-} f(x) = L$$

*right* "This limit coming from the right"; values of $f(x)$ as $f(x)$ nears $x$ and right of $c$, $x > c$

$$\lim_{x\to c^+} f(x) = L$$

*Note*: If left and rights limits are not the same, limit *doesn't exist*.

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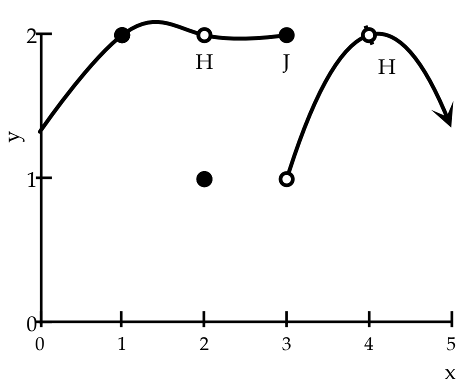
### CONTINUITY

A function $f$ is continuous at $x = a$ iff $\lim_{x\to a} f(x) = f(a)$, i.e., if the limit of $x$ at $a$ is equals $f(a)$, i.e., no breaks or jumps

*Example*




where H indicates a hole—where the graph is defined but could be made continuous by changing the point—and J a jump—where the left and right limits are not the same.

- Continuous at 1 since $\lim_{x\to 1} f(x) = f(1) = 2$
- Not continuous at 2, 3, or 4
  - $\lim_{x\to 2} f(x) = 2 \neq f(2) = 1$
  - $\lim_{x\to 3} f(x) = \text{doesn't exist} \neq f(3) = 2$
  - $\lim_{x\to 4} f(x) = 2 \neq f(4) = \text{undefined}$

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### CALCULATING DERIVATIVES

*Example: Using Formal Definition*

Find derivative of $f(x) = 2x^2 - 16x + 35$.

1. Assemble using the formal definition

$$\frac{\left[2(x-h)^2 - 16(x+h) + 35\right] - \left[2x^2 - 16x - 35\right]}{h}$$

2. Factor—*cannot plug $h = 0$ because no division by zero!*

$$= \frac{2x^2 + 4xh + 2h^2 - 16x - 16h + 35 - 2x^2 + 16x - 35}{h}$$

$$= \lim_{h\to 0} \frac{4xh + 2h^2 - 16h}{h}$$

3. Factor out $h$ in numerator to cancel $h$ in denominator

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{h(4x+2h-16)}{h} \\ &= \lim_{h\to 0} 4x + 2h - 16 \\ &= 4x - 16 \end{aligned}$$

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### IMPLICIT DIFFERENTIATION

The process to find $y' = f'(x)$ when $f(x)$ is difficult or impossible to use with explicit differentiation, by assuming $y$ is a function of $x$:

*Example*

Implicitly differentiate $x^2 + y^2 = 25$

1. Differentiate each side, treating $y$ as a function

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}25$$

$$\Rightarrow \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + 2yy' = 0$$

2. Algebraically solve for $y'$

$$\begin{aligned} 2yy' &= -2x \\ \Rightarrow \quad y' &= \frac{-2x}{2y} \\ \Rightarrow \quad y' &= -\frac{x}{y} \end{aligned}$$

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### DEFINITE INTEGRAL

The definite integral of a positive function $f(x)$ over an interval $[a, b]$ is the area between $f$, the $x$-axis, $x = a$, and $x = b$.

$$\int_a^b f(x)\,dx$$

where $a$ and $b$ are the "limits of integration" and $f(x)$ is the integrand

$$\begin{aligned} \int_a^a f(x)\,dx &= 0 \\ \int_a^b f(x)\,dx &= -\int_b^a f(x)\,dx \\ \int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \end{aligned}$$

where $a < c < b$

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### INTEGRATION OPERATIONS

| | |
|---|---|
| Sum | $\int u + v\,dx = \int u\,dx + \int v\,dx$ |
| Difference | $\int u - v\,dx = \int u\,dx - \int v\,dx$ |
| Product | $\int af(x)\,dx = a \int f(x)\,dx$ |
| Parts | $\int u\,dv = uv - \int v\,du$ |
| Substitution | $\int f(u)u'\,dx = \int f(u)\,du$ |

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### GROWTH, CONCAVITY, AND EXTREMA

*growth*

- $\forall x \in I\ f'(x) \geqslant 0 \Rightarrow f$ is increasing in I.
- $\forall x \in I\ f'(x) \leqslant 0 \Rightarrow f$ is decreasing in I.

*concavity*

- $\forall x \in I\ f''(x) \geqslant 0 \Rightarrow f$ is concave up in I.
- $\forall x \in I\ f''(x) \leqslant 0 \Rightarrow f$ is concave down in I.

*extrema* If $f'(a) = 0$ (critical point)

- $f''(a) < 0 \Rightarrow f$ has a local maximum at $x = a$.

Figure 1: Concave up (left) and concave down (right)

- $f''(a) > 0 \Rightarrow f$ has a local minimum at $x = a$.

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### INCREASING AND DECREASING FUNCTIONS

- $f(x)$ is *increasing* iff $\forall\ x_1, x_2$ in interval I is such that $x_1 < x_2$ and $f(x_1) < f(x_2)$
- $f(x)$ is *decreasing* iff $\forall\ x_1, x_2$ in interval I is such that $x_1 > x_2$ and $f(x_1) > f(x_2)$
- Determine all intervals where $f(x)$ is in/decreasing:
  1. Find all critical points (via first deriative)
  2. For each critical point, select a number $a$ in that range and see if $f'(a)$ is positive or negative

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### INFLECTION POINTS

- Where the second derivative changes signs
- The point(s) on a graph where the concavity of a function changes from up to down
- functions can be increasing (positive derivative) or decreasing (negative derivative) regardless of concavity,

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### MAXIMA AND MINIMA

A *critical point* is a point where either $f'(a) = 0$ or $f'(a)$ is undefined, and is a *candidate* of being a local or global extreme

*Local*

*maximum* at $a$ if $f(a) \geqslant f(x)\ \forall x$ near $a$

*minimum* at $a$ if $f(a) \leqslant f(x)\ \forall x$ near $a$

*extreme* at $a$ if $f(a)$ is a local maximum or minimum

*Global*

*maximum* at $a$ if $f(a) \geqslant f(x)\ \forall x$ in domain of $f$

*minimum* at $a$ if $f(a) \leqslant f(x)\ \forall x$ in domain of $f$

*extreme* at $a$ if $f(a)$ is a global maximum or minimum

*Example*

Find the critical point of $f(x) = x^3 - 6x^2 + 9x + 2$

1. Find $f'(x)$

$$\begin{aligned} f'(x) = 3x^2 - 12x + 9 \\ &= 3(x^2 - 4x + 3) \\ &= 3(x-1)(x-3) \end{aligned}$$

2. Find where $f'(x) = 0$, which is 1 and 3
3. Put $x = 1$ and $x = 3$ into $f(x)$ to find the critical points

$$\begin{aligned} (1, f(1)) &= (1, 6) \\ (3, f(2)) &= (3, 2) \end{aligned}$$

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### ANTIDERIVATIVES

- *An* antiderivative of a function $f(x)$ is any function $F(x)$ such that $F'(x) = f(x)$
- *The* antiderivative is an entire family of functions, written $F(x) + c$
- Also known as the *indefinite integral* (with no limit markers):

$$\int f(x)\ dx$$

*Example*

*An* antiderivative of $\int 2x\ dx$ is $x^2 - 5.2$; *the* antiderative is $x^2 + C$

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### DEFINITE V. INDEFINITE INTEGRALS

Indefinite integrals do not have limits to integration where definite integrals do

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### INTEGRATION BY SUBSTITUTION

A method to algebraically manipulate an integrand so it is amenable to antiderivative rules; especially useful when there is a product in the integral.
Substitute $u$ for $g(x)$ where necessary, making $\frac{du}{dx} = g'(x)$, so $du = g'(x)\ dx$. Since

$$\frac{du}{dx} = g'(x) \equiv du = g'(x)\ dx$$

we can substitute so that

$$\int f'(g(x))g'(x)\ dx \equiv \int f'(u)\ du$$

Now integrate $f'(u)\ du$. (Note that $g(x) \equiv u$ and $g'(x)\ dx \equiv du$.)

1. Set one part of the integrand to $u$, one "level" into the integral
2. Compute $du = \frac{du}{dx}\ dx$ (the derivative of $u$)
3. Convert $x$'s to $u$'s in original integral, even including in $dx$
4. Integrate new $u$ integral
5. Substitute $u$'s back to $x$'s in integral

*Example*

Integrate $\int (x+1)^3\ dx$

1. Rearrange so that $u = x + 1$ and $du = 1\ dx$

$$= \int (x+1)^3 \cdot 1\ dx$$

2. Substitute in $u$ and $du$

$$\int u^3\ du$$

3. Integrate

$$= \frac{u^4}{4} + C$$

4. Add $u$ back in

$$= \frac{(x+1)^4}{4} + C$$

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### INTEGRATION BY PARTS

Integrate a complex function by rewriting it as a product of two simpler functions $u$ and $du$, using two possible forms:

$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ \int_a^b u\,dv &= uv|_a^b - \int_a^b v\,du\end{aligned}$$

### 3.1 *Example: First Form*

Integrate $\int xe^x\ dx$:

1. Break into two parts: $u = x$ and $dv = e^x\ dx$
2. Calculate the derivative of $u$, $du$, and $v$, the integral of $dv$

$$\begin{aligned}du &= \left(\frac{d}{dx}x\right)\ dx = 1\ dx \\ v &= \int dv = \int e^x\ dx = e^x\end{aligned}$$

3. Using the first formula, noting the prior forms from 1 and 2

$$\begin{aligned}\int u\ dv &= \int xe^x\ dx \\ &= xe^x - \int e^x\ dv \\ &= xe^x - e^x + C\end{aligned}$$

### 3.2 *Example: Second Form*

Integrate $\int_1^4 6x^2 \ln x\ dx$:

1. Break into two parts: $u = \ln x$ and $dv = 6x^2$
2. Calculate the derivative of $u$, $du$, and the integral of $dv$, $v$:

$$\begin{aligned}du &= \frac{d}{dx}\ln x = \frac{1}{x}\ dx \\ v &= \int 6x^2\ dx = 6\int x^2\ dx = 6 \cdot \frac{x^3}{3} = 2x^3\end{aligned}$$

3. Use the second formula

$$\begin{aligned}\int_1^4 6x^2 \ln x\ dx &= 2x^3 \ln x|_1^4 - \int_1^4 2x^3 \frac{1}{x}\ dx \\ &= 2x^3 \ln x|_1^4 - 3x^2|_1^4\end{aligned}$$

4. Find the integral the usual way:

$$\begin{aligned}&\left[(2 \cdot 4^3 \ln(4)) - (2 \cdot 1^3 \ln(1))\right] - \left[(3 \cdot 4^2) - (3 \cdot 1^2)\right] \\ &= 128 \cdot \ln(4) - 45 \\ &\approx 132.446\end{aligned}$$

---

ANTIDERIVATIVE RULES

| | |
|---|---|
| $\int a\,dx$ | $ax + C$ |
| $\int x^n\,dx$ | $\frac{u'}{2\sqrt{u}}$ |
| $\int e^x\,dx$ | $e^x + C$ |
| $\int a^x\,dx$ | $\frac{a^x}{\ln(a)} + C$ |
| $\int \frac{1}{x}\,dx$ | $\ln\lvert x\rvert + C$ |

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# 4 Statistics

### Expected value

$$E(X) = \mu = \sum_{i=1}^{k} x_i \; P(X = x_i)$$

for a discrete random variable with $k$ possible values.

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### General variance formula

$$Var(X) = \sigma^2 = \sum_{j=1}^{k} (x_j - \mu)^2 \; P(X = x_j),$$

or, the sum of the squared deviations $(x_j - \mu)^2$ weighted by the corresponding probabilities $P(X = x_1), \ldots, P(X = x_k)$.

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### General standard deviation

$$\sigma = \sqrt{\sigma^2} = \sqrt{Var(X)}$$

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### Linear combinations of variables

$$Z = aX + bY$$

is a linear combination of the independent, random variables $X$ and $Y$ (often $a$ and $b$ are $1$ or $-1$).

$$\begin{aligned} E(Z) &= a \times E(X) + b \times E(Y) \\ Var(Z) &= a^2 \times Var(X) + b^2 \times Var(Y) \end{aligned}$$

---

### Probability density function (PDF)

$$P(a \leqslant X \leqslant b) = \int_a^b f(x) \; dx$$

is a PDF of $X$, for any two numbers $a$ and $b$ where $a \leqslant b$. I.e., the probability that $X$ takes on a value in the interval $[a, b]$ is the area above this interval and below the graph of the density curve.

- $P(X = c) = 0$ for any constant (bins are infinitesimally small)
- $\sum P(x_i) = 1$

---

### Normal distribution v. standard normal

There is any entire family of distributions that can be called normal, but the prototypical distribution with mean of 0 and standard deviation of 1 is called the standard normal. Formally defined by its PDF as:

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

*Properties*

1. Symmetric around mean
2. Mean = mode = median
3. Denser at center than in tails

Consequently,

- 68 percent of distribution is within one standard deviation of the mean
- 95 percent of distribution is within approximately two standard deviations of the mean

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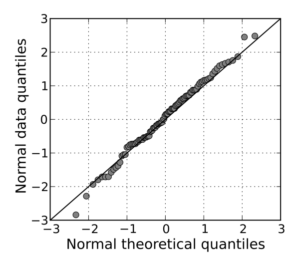
### Evaluating normality




- A normal probability plot using quartiles can be used to evaluate how closely a given distribution adheres to normality, where the straight line is a perfect normal curve
- As N increases, the deviation from normality will decrease

---

### Z scores

$$Z = \frac{x - \mu}{\sigma}$$

converts any value from a normal distribution to its corresponding value on the standard normal distribution

- Describes the number of standard deviations a point is from the mean $\mu$
- Z scores to the left of $\mu$ are negative, and positive to the right of $\mu$

---

### Z SCORES: PROBABILITIES ON NORMAL DISTRIBUTION

*Ex. What is the probability* $X > A$, *given* $X \sim N(\mu = 1500, \sigma = 300)$*?*

$$Z = \frac{x - \mu}{\sigma} = \frac{1630 - 1500}{300} = 0.43$$

This is 0.6664 on Z table, so 66.64 percent of X is to the left of A so:

$$1 - 0.6664 = 0.3336$$

The probability $X > A$ is 33.36 percent.

*Ex. Given* $A = 1400$ *and* $X \sim N(\mu = 1500, \sigma = 300)$, *what is the percentile corresponding to A?*

$$Z = \frac{x - \mu}{\sigma} = \frac{1400 - 1500}{300} = -0.33$$

The corresponding value on the Z table is 0.3707, so A is the 37th percentile.

*Ex. Given* $p = .40$ *and* $X \sim N(\mu = 70, \sigma = 3.3)$, *what is the value corresponding to percentile* $p$*?*

Lookup $p$ on Z table, getting a $Z = -0.25$. Work backwards:

$$-0.25 = Z = \frac{x - \mu}{\sigma} = \frac{x - 70}{3.3}$$

and solve for $x = 69.18$.

*Ex. What is the probability* $X$ *is between* $A$ *and* $B$, *given* $X \sim N(\mu, \sigma)$*?*

Using Z-scores method, find the area to the left of A and to the right of B, then $A - B = 1 -$ area left of $A -$ are to right of B.

---

### BERNOULLI DISTRIBUTION

$$P(X = x) = \begin{cases} p \text{ for } x = 1 \\ 1 - p \text{ for } x = 0 \end{cases}$$

describes the distribution of individual trials with two possible outcomes, success or failure, described by proportion of successes $0 \leqslant p \leqslant 1$:

$$\begin{aligned} \hat{p} &= \frac{|\,\text{successes}\,|}{|\,\text{failures}\,|} \\ \mu &= p \\ \sigma^2 &= p(1 - p) \end{aligned}$$

- The probability of success after $n$ trials is $(1 - p)^{n-1} \times p$

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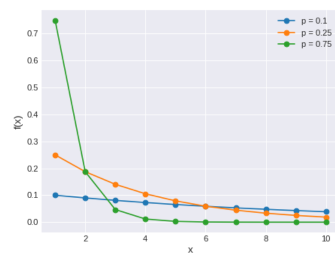
### BERNOULLI: GEOMETRIC DISTRIBUTION



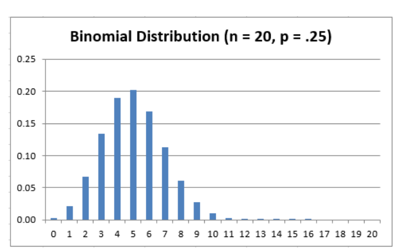

describes the wait time until a success for *independent* Bernoulli random variables; or, the probability of observing the k-th success by the n-th trial

$$\begin{aligned} \mu &= \frac{1}{p} \\ \sigma^2 &= \frac{1 - p}{p^2} \end{aligned}$$

- Higher $p$ means fewer trials until success
- Can never be approximated by a normal distribution

---

### BINOMIAL DISTRIBUTION



describes the probability of having exactly $k$ successes in $n$ independent Bernoulli trials (with probability of success $p$):

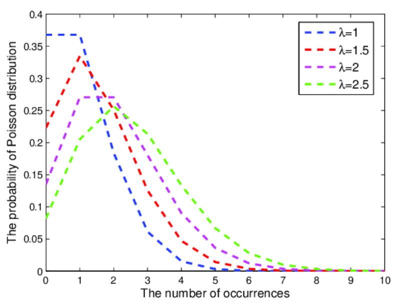
$$\begin{aligned} P(x = k \mid n, \mu, \sigma) &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \end{aligned}$$

Parameters, can be used to approximate to normal when $n$ is sufficiently large and $np$ and $n(1-p)$ are both greater than or equal to 10:

$$\begin{aligned} \mu &= np \\ \sigma^2 &= np(1-p) \end{aligned}$$

---

### POISSON DISTRIBUTION




$$P(X = x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Describes the number of events in a larger population over a unit of time with rate $\lambda$:

$$\begin{aligned} \mu &= \lambda \\ \sigma^2 &= \lambda \end{aligned}$$

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### INFERENTIAL STATISTICS

The body of thought governing the inferences of populations from samples, and how these sample statistics can vary.

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### STANDARD ERROR

$$SE = \frac{\sigma}{\sqrt{n}}$$

is the standard deviation of distributions of sample statistics, when population $\sigma$ is known. If it is unknown, and if $n > 30$, substitute sample standard deviation $s$

- SE decreases as $n$ increases
- SE decreases as $\sigma$ (or $s$) decreases

---

### CONFIDENCE INTERVALS

$$\bar{x} \pm z \times SE$$

- $\bar{x}$ is the sample statistic, such as sample mean
- $z \times SE$ is the *margin of error*
- $z$ is the desired confidence level, e.g., $z = 1.96$ for a 95 percent confidence interval

*Interpretation.* "We are Z percent confident the true population *statistic* is between A and B"; or, "Z percent of samples will have a *sample statistic* between A and B."

---

### CENTRAL LIMIT THEOREM

Given a population with a finite mean $\mu$ and a finite non-zero variance $\sigma^2$, the sampling distribution of the mean approaches a normal distribution with a mean of $\mu$ and a variance of $\frac{\sigma^2}{N}$, as N, the sample size, increases—regardless of the shape of the parent population.

---

### HYPOTHESIS TESTING

| One-Sided | Two-Sided |
|---|---|
| $H_0 : x = A$ | $H_0 : x = A$ |
| $H_A : x > / < A$ | $H_A : x \neq A$ |

The process of comparing two point estimates, to determine if any difference between them is "real" or the result of natural variance in samples.

- *Type I* errors, or false positives, occur when $H_0$ is true, but rejected
- *Type II* errors, or false negatives, occur when $H_A$ is true, and $H_0$ is not rejected

*Quantifying Risk*

- The risk of Type I errors is quantified by $\alpha$, i.e., the probability the point estimate is more than $z^*$ standard deviations away from the true population parameter
- The p-value is the probability of observing data at least as favorable to the alternative hypothesis, i.e., as "extreme," as the present data set, if $H_0$ is actually true
- If the p-value is less than the chosen $\alpha$, data is sufficient to reject $H_0$

---

## P-VALUE CALCULATIONS

*One-Sided*

*Two-Sided*

---

## SAMPLE PROPORTIONS

Population parameter $\pi$ is sampled:

$$\mu = \hat{p} = \frac{\sum_n^{i=1} x_i}{n}$$
$$SE_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$$

where $0 \leqslant \hat{p} \leqslant 1$ and $x_i = \{0, 1\}$

---

## SAMPLE PROPORTIONS: CONFIDENCE INTERVALS

1. Assess normality:
   - At least 10 observations for each $\{0, 1\}$
   - Sample is less than 10 percent of population and observations are independent
2. Calculate standard error $SE_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$
3. Determine $z^*$, e.g., 1.96
4. Put together point estimate and margin of error:

$$\hat{p} \pm z^* \times SE_{\hat{p}}$$

---

## SAMPLE PROPORTIONS: HYPOTHESIS TESTS

**FIX**

$$H_0 : \hat{p} = 0.5$$
$$H_A : \hat{p} > / \neq 0.5$$

1. Evaluate normality
2. Compute $SE_{\hat{p}}$ *using null hypothesis*:

   $$SE_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}},$$

   often $\sqrt{\frac{0.5(1-0.5)}{n}}$
3. Calculate Z-score using hypotheses:

   $$\frac{\hat{p} - \hat{p}_0}{SE_{\hat{p}}}$$

4. Convert Z to p-value and decide whether to reject the null or fail to

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## SAMPLE PROPORTIONS: SAMPLE SIZE

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## DIFFERENCE OF PROPORTIONS

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## DIFFERENCE OF PROPORTIONS: CONFIDENCE INTERVALS

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## DIFFERENCE OF PROPORTIONS: HYPOTHESIS TESTS

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## DIFFERENCE OF PROPORTIONS: POOLED PROPORTION

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## $\chi^2$ GOODNESS OF FIT

$$\chi^2 = \sum_{k=1}^{N} \frac{(\text{observed}_k - \text{expected}_k)^2}{\text{expected}_k}$$

- $k$ mutually exclusive classes
- $n$ observations of $x_i$
- one parameter, degrees of freedom $df$
- follows the chi-square distribution if null hypothesis is true

Summarizes how strongly observed count data deviates from the expected, or null, counts—larger values of $\chi^2$ indicate stronger deviation

*Does a statistical model fit this sample?*

1. Develop hypotheses:
   a) $H_0$: Sample follows distribution D
   b) $H_A$: Sample does not follow distribution D
2. Check assumptions
   a) Each expected count must be at least 5
   b) Can use binning to get around this
3. Establish expected counts (expected proportion of total count in each bin):

$$E_k = \mathtt{expected}_k \times n$$

4. Compute $\chi^2$ statistic
5. Validate assumptions hold to apply $\chi^2$ to $\chi^2$ distribution
6. Using $k-1$ degrees of freedom, use $\chi^2$ table to compute a p-value
7. Decide to reject or fail to reject $H_0$

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## $\chi^2$: p-VALUE

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## TWO-WAY TABLES: INDEPENDENCE

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## SAMPLE STATISTICS: MEAN AND VARIANCE

- $\mu_M = \mu$ is the mean of the sampling distribution of means
- $\sigma^2_M = \frac{\sigma^2}{N}$ is the variance of the sampling distribution of the mean
- $\sigma_M = \frac{\sigma}{\sqrt{N}}$ is the standard error of the sampling distribution of the mean

As N increases, variance of sample mean decreases

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## SAMPLE STATISTICS: DIFFERENCE IN MEAN

Two samples from a population the size $n_1$ and $n_2$, calculate the means $M_1$ and $M_2$, and the difference is $M_1 - M_2$

$$\begin{aligned}
\mu_{M_1-M_2} &= M_1 - M_2 \\
\sigma^2_{M_1-M_2} &= \sigma^2_{M_1} + \sigma^2_{M_2} \\
\sigma_{M_1-M_2} &= \sqrt{\frac{\sigma^2_1}{n_1} + \frac{\sigma^2_2}{n_2}}
\end{aligned}$$

When variance and sample size are the same, standard error becomes:

$$\sigma_{M_1-M_2} = \sqrt{\frac{\sigma^2_1}{n_1} + \frac{\sigma^2_2}{n_2}i} = \sqrt{\frac{\sigma^2}{n} + \frac{\sigma^2}{n}} = \sqrt{\frac{2\sigma^2}{n}}$$

If $n_1 \neq n_2$ then variance becomes:

$$\sigma^2_{M_1-M2} = \frac{\sigma^2_1}{n_1} + \frac{\sigma^2_2}{n_2}$$

*What is the probability that the mean of sample 1 will exceed that of sample 2 by* N *or more?*

1. Find mean: $\mu_{M_1-M_2} = M_1 - M_2$
2. Find standard error: $\sigma_{M_1-M_2}$
3. Find area underneath distribution of sample 1 to the right of the mean of sample 2 plus N

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## SAMPLE STATISTICS: r AND $\rho$

- Not normally distributed—right-skewed—because correlation cannot exceed 1
- As $\rho$ increases, the more right-skewed the distribution

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## SAMPLE STATISTICS: PROPORTION $\pi$

Sampling proportion is closely related to the binomial distribution—the total number of successes—where p is the distribution of the mean number of successes

$$\begin{aligned}
\mu_p &= \pi \\
\sigma_p &= \frac{\sqrt{N\pi(1-\pi)}}{N} = \sqrt{\frac{\pi(1-\pi)}{N}}
\end{aligned}$$

*Find probability* p *is greater than* A

Given N and population proportion $\pi$:

1. Find mean of $p = \pi$
2. Calculate standard error as above
3. Conduct as normal distribution given N is sufficiently large and $\pi$ is not too close to 0 or 1

### ESTIMATION

The process of estimating population parameters from sample statistics. Usually results in a point estimate as well as interval estimates called confidence intervals.

### DEGREES OF FREEDOM

## 5 REGRESSION: OLS

### FORM

$$y = \beta_0 + \beta_1 x$$

describes the true, unobserved model, while

$$\hat{y} = b_0 + b_1 x$$

describes the estimated model. The estimate $\hat{y}$ describes the average value around which subjects where $x = x_i$ will cluster.

### RESIDUALS

$$e_i = y_i - \hat{y}_i$$

is the residual of the $i$th observations $(x_i, y_i)$, the differences between the observed response $(y_i)$ and the prediction $\hat{y}$

### CORRELATION

$$r = \frac{1}{n-1} \sum_{i=1}^{n} \frac{x_i - \bar{x}}{s_x} \frac{y_i - \bar{y}}{s_y}$$

describes the strength of the linear relationship between two variables $x$ and $y$, where $0 \leqslant r \leqslant 1$, and $s$ is sample standard deviation

### LEAST SQUARES CRITERION

$$\arg\min \sum_{i=1}^{n} e_i^2 \equiv e_1^2 + e_2^2 + \ldots + e_n^2$$

describes the best fitting line, the *least squares line*, i.e., minimizes the sum of squared residuals. To calculate:

$$b_1 = \frac{s_y}{s_x} r$$

$$y - \hat{y} = b_1 (x - \hat{x})$$

where $x_0 = \bar{x}$ and $y_0 = \bar{y}$

*Assumptions*

1. *Linearity*. Data must show a linear trend
2. *Near normal residuals*

3. *Constant variability*. The variability of points around the least-squares line must be constant, e.g., the scale of $e$ cannot increase as $x$ increases producing a fanning pattern

4. *Independent observations*.