Section 13.4 Differentials

MATH211 Calculus III

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Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\begin{aligned}\Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)\\ &= (f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)) + (f(x_0, y_0+\Delta y) - f(x_0, y_0))\end{aligned}$$

$$\begin{aligned}&f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)\\ &= \left(\frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)}{\Delta x}\right)\Delta x\\ &= (f_x(x_0, y_0+\Delta y) + \varepsilon_{11}(\Delta x, \Delta y))\,\Delta x\\ &= \left[\overbrace{(f_x(x_0, y_0) + \varepsilon_{12}(\Delta y))}^{f_x(x_0,y_0+\Delta y)} + \varepsilon_{11}(\Delta x, \Delta y)\right]\Delta x\\ &= f_x(x_0, y_0)\Delta x + \overbrace{\varepsilon_1}^{\varepsilon_{12}+\varepsilon_{11}}\,\Delta x\end{aligned}$$

$$\begin{aligned}f(x_0, y_0+\Delta y) - f(x_0, y_0) &= \left(\frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}\right)\Delta y\\ &= f_y(x_0, y_0)\Delta y + \varepsilon_2\Delta y\end{aligned}$$

Linear Approximation

Note that $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Consequently,

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ decays in a comparable order of $\|\Delta x \mathbf{i} + \Delta y \mathbf{j}\|^2$.