Section 11.4 Cross Product

MATH211 Calculus III

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MESSIAH UNIVERSITY

DEPARTMENT OF
COMPUTING, MATHEMATICS
AND PHYSICS

Knowledge Checks

Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. What is the cross product $\mathbf{u} \times \mathbf{v}$?

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$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

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What is the special relation between the directions of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u}$ or $\mathbf{v}$?

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Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. What is the cross product $\mathbf{u} \times \mathbf{v}$?

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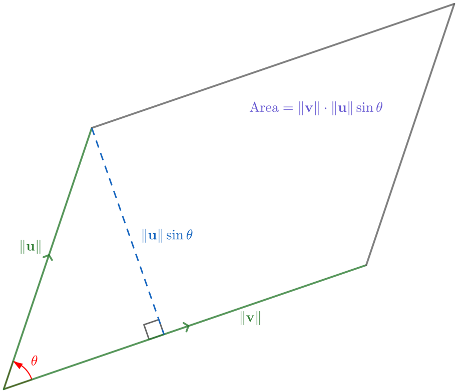
What is the special relation between the directions of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u}$ or $\mathbf{v}$?

$\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Knowledge Checks

How is $\|\mathbf{u} \times \mathbf{v}\|$ related to $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and θ?



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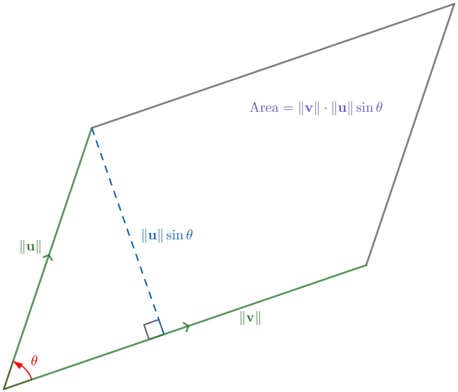
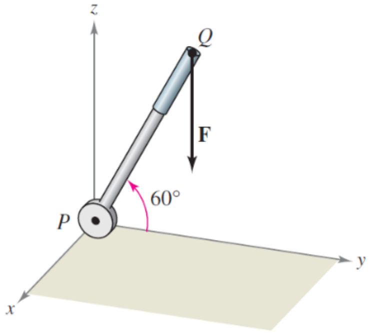
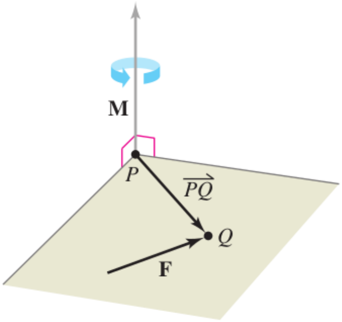


$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \; \|\mathbf{v}\| \sin\theta = \text{the area of the parallelogram.}$$

Knowledge Checks

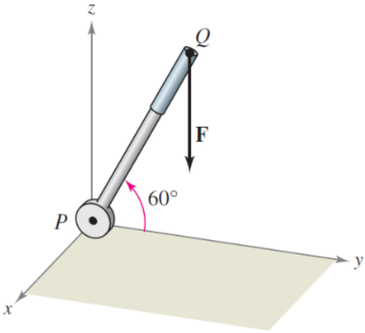
What is the torque?

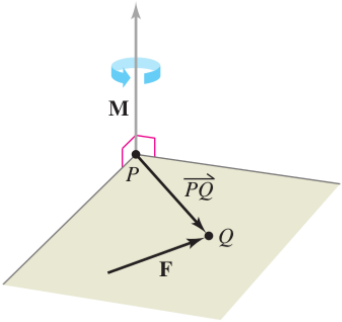




Knowledge Checks

What is the torque?





$$\mathbf{M} \text{ (or } \tau) = \overrightarrow{PQ} \times \mathbf{F}.$$

Knowledge Checks

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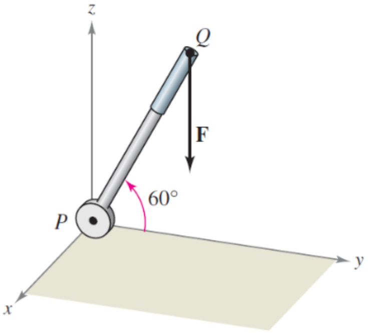

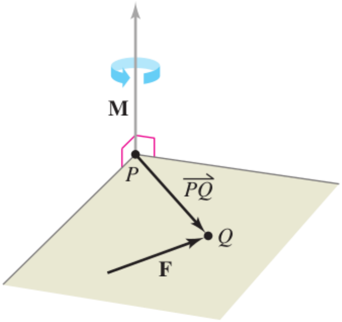

$$\mathbf{M} \text{ (or } \tau) = \overrightarrow{PQ} \times \mathbf{F}.$$

Remark: The torque is a vector, NOT a number.

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Knowledge Check

B.H.

Determinants:

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} =$$

Knowledge Check

Determinants:

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1.$$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} =$$

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$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$=$$

Knowledge Check

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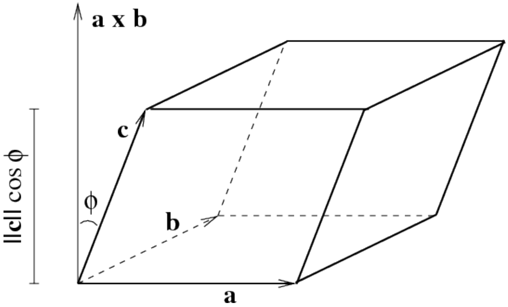
$$\begin{aligned} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= u_1(v_2 w_3 - v_3 w_2) - u_2(v_1 w_3 - v_3 w_1) \\ &\quad + u_3(v_1 w_2 - v_2 w_1). \end{aligned}$$

Knowledge Checks

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space.

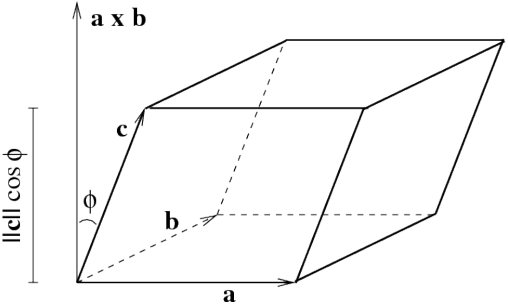
$$\mathbf{u} \times \mathbf{v} := \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$
$$= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.$$

Triple scalar product



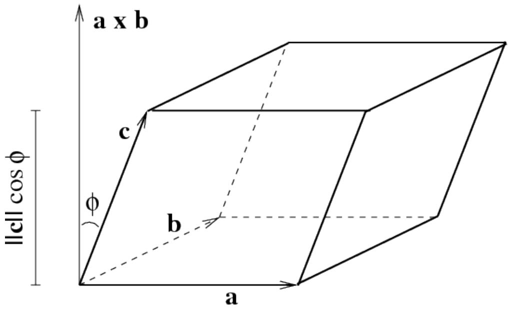
Volume(parallelepiped)

Triple scalar product



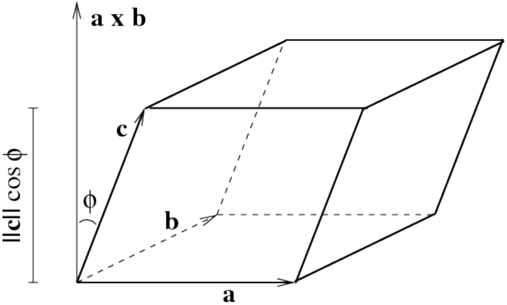
$$\text{Volume(parallelepiped)} = |\|\mathbf{a} \times \mathbf{b}\| \|\mathbf{c}\| \cos\theta|$$

Triple scalar product



$$\text{Volume(parallelepiped)} = |\|\mathbf{a} \times \mathbf{b}\|\|\mathbf{c}\| \cos\theta| \; = |\mathbf{c} \cdot \|\mathbf{a} \times \mathbf{b}\||$$

Triple scalar product

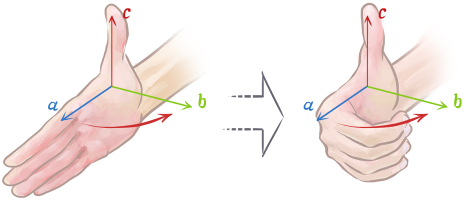


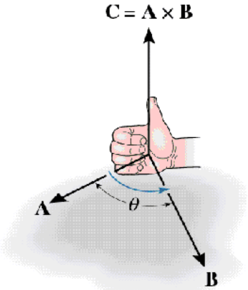
$$\text{Volume(parallelepiped)} = |\|\mathbf{a}\times\mathbf{b}\|\|\mathbf{c}\|\cos\theta| \ = |\mathbf{c}\cdot\|\mathbf{a}\times\mathbf{b}\||$$

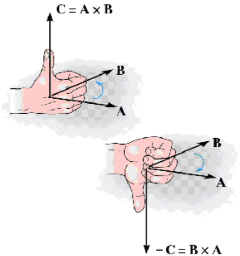
$$= \ |c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)|$$

$$= \text{absolute value}\left(\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\right)$$

The Right Hand Rule







The Cross Product of the Standard Unit Vectors

Exercise. According to the right hand rule and the magnitude formula, find

- $\mathbf{i} \times \mathbf{j}$.
- $\mathbf{j} \times \mathbf{i}$.
- $\mathbf{j} \times \mathbf{k}$.
- $\mathbf{k} \times \mathbf{j}$.
- $\mathbf{k} \times \mathbf{i}$.
- $\mathbf{i} \times \mathbf{k}$.

The General Formula

Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. If distributivity is to be respected, we must have the following.

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Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. If distributivity is to be respected, we must have the following.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= \end{aligned}$$

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Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. If distributivity is to be respected, we must have the following.

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\ &= \quad u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k}\end{aligned}$$

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Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. If distributivity is to be respected, we must have the following.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= \quad u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\
&\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k}
\end{aligned}$$

The General Formula

Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. If distributivity is to be respected, we must have the following.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= \quad u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\
&\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\
&\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k}
\end{aligned}$$

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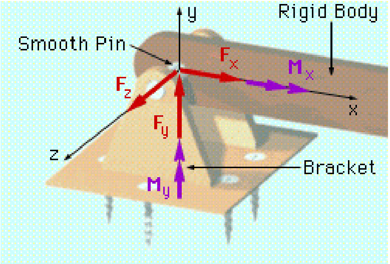
Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. If distributivity is to be respected, we must have the following.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= \quad u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\
&\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\
&\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \\
&= \quad (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.
\end{aligned}$$

Pin Support



Suppose a force $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is acting on the lever at $(1, 1, 1)$.

- Find the torque of $\mathbf{F}$ about the origin.
- If the lever is stuck, find the force $\langle F_x, F_y, F_z \rangle$ at the pin support.
- If the lever can rotate freely about pin, find the couple moment $\langle M_x, M_y, M_z \rangle$ at the pin support.