Section 13.4 Differentials

MATH211 Calculus III

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Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

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$$\begin{aligned}\Delta z &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\ &= (f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)) + (f(x_0, y_0 + \Delta y) - f(x_0, y_0))\end{aligned}$$

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$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)$$

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$$\begin{aligned}&f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)\\ &= \left(\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x}\right)\Delta x\end{aligned}$$

Linear Approximation

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$$\begin{aligned}&f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) \\ &= \left(\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x}\right)\Delta x \\ &= (f_x(x_0, y_0 + \Delta y) + \varepsilon_{11}(\Delta x, \Delta y))\,\Delta x\end{aligned}$$

Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\begin{aligned}\Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)\\ &= \left(f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)\right) + \left(f(x_0, y_0+\Delta y) - f(x_0, y_0)\right)\end{aligned}$$

$$\begin{aligned}&f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)\\ &= \left(\frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)}{\Delta x}\right)\Delta x\\ &= \left(f_x(x_0, y_0+\Delta y) + \varepsilon_{11}(\Delta x, \Delta y)\right)\Delta x\\ &= \Big[\overbrace{\Big(f_x(x_0, y_0) + \varepsilon_{12}(\Delta y)\Big)}^{f_x(x_0, y_0+\Delta y)} + \varepsilon_{11}(\Delta x, \Delta y)\Big]\Delta x\end{aligned}$$

where the underbrace labelled ε_1 groups $\varepsilon_{12}(\Delta y)) + \varepsilon_{11}(\Delta x, \Delta y)$.

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$$\begin{aligned}&f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) \\ &= \left(\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x}\right)\Delta x \\ &= (f_x(x_0, y_0 + \Delta y) + \varepsilon_{11}(\Delta x, \Delta y))\,\Delta x \\ &= \Big[\big(\overbrace{f_x(x_0, y_0) + \underbrace{\varepsilon_{12}(\Delta y)\big) + \varepsilon_{11}(\Delta x, \Delta y)}_{\varepsilon_1}}^{f_x(x_0, y_0 + \Delta y)}\Big]\Delta x \\ &= f_x(x_0, y_0)\Delta x + \varepsilon_1 \Delta x\end{aligned}$$

Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\begin{aligned}\Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)\\ &= \left(f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)\right) + \left(f(x_0, y_0+\Delta y) - f(x_0, y_0)\right)\end{aligned}$$

$$\begin{aligned}&f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)\\ &= \left(\frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y)}{\Delta x}\right)\Delta x\\ &= \left(f_x(x_0, y_0+\Delta y) + \varepsilon_{11}(\Delta x, \Delta y)\right)\Delta x\\ &= \Big[\overbrace{\big(f_x(x_0,y_0) + \varepsilon_{12}(\Delta y)\big)}^{f_x(x_0, y_0+\Delta y)} + \varepsilon_{11}(\Delta x,\Delta y)\Big]\Delta x \quad\text{with } \underbrace{\varepsilon_{12}(\Delta y) + \varepsilon_{11}(\Delta x,\Delta y)}_{\varepsilon_1}\\ &= f_x(x_0, y_0)\Delta x + \varepsilon_1 \Delta x\end{aligned}$$

$$\begin{aligned}f(x_0, y_0+\Delta y) - f(x_0, y_0) &= \left(\frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}\right)\Delta y\\ &= f_y(x_0, y_0)\Delta y + \varepsilon_2 \Delta y\end{aligned}$$

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Linear Approximation

Note that $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Consequently,

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ decays in a comparable order of $\|\Delta x\mathbf{i} + \Delta y\mathbf{j}\|^2$.