Orthogonal Projection

Linear Algebra

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Theorem

Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ *be an orthogonal basis of the subspace* W*. The orthogonal projection of* $\mathbf{v}$ *onto* W *is*

$$\sum_{i=1}^{m} \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i$$

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Proof: Let $\mathbf{w} = \sum_{i=1}^{m} w_i \mathbf{u}_i$ be an arbitrary vector in W. We want to show $\left(\mathbf{v} - \sum_{i=1}^{m} \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i \right) \cdot \mathbf{w} = 0$.

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$$\begin{aligned}
\left(\mathbf{v} - \sum_{i=1}^{m} \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i \right) \cdot \mathbf{w} &= \left(\mathbf{v} - \sum_{i=1}^{m} \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i \right) \cdot \sum_{i=1}^{m} w_i \mathbf{u}_i \\
&= \sum_{i=1}^{m} w_i \mathbf{v} \cdot \mathbf{u}_i - \sum_{i=1}^{m} w_i \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i \cdot \mathbf{u}_i
\end{aligned}$$

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$$\begin{aligned} \left(\mathbf{v} - \sum_{i=1}^{m} \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i \right) \cdot \mathbf{w} &= \left(\mathbf{v} - \sum_{i=1}^{m} \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i \right) \cdot \sum_{i=1}^{m} w_i \mathbf{u}_i \\ &= \sum_{i=1}^{m} w_i \mathbf{v} \cdot \mathbf{u}_i - \sum_{i=1}^{m} w_i \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\cancel{\mathbf{u}_i \cdot \mathbf{u}_i}} \right) \cancel{\mathbf{u}_i \cdot \mathbf{u}_i} \end{aligned}$$

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$$\begin{aligned}
\left(\mathbf{v} - \sum_{i=1}^{m} \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i \right) \cdot \mathbf{w} &= \left(\mathbf{v} - \sum_{i=1}^{m} \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \right) \mathbf{u}_i \right) \cdot \sum_{i=1}^{m} w_i \mathbf{u}_i \\
&= \sum_{i=1}^{m} w_i \mathbf{v} \cdot \mathbf{u}_i - \sum_{i=1}^{m} w_i \left(\frac{\mathbf{v} \cdot \mathbf{u}_i}{\cancel{\mathbf{u}_i \cdot \mathbf{u}_i}} \right) \cancel{\mathbf{u}_i \cdot \mathbf{u}_i} \\
&= 0.
\end{aligned}$$

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Proof: Let $\mathbf{u} = \mathbf{v} - \operatorname{proj}_W \mathbf{v}$, and $\mathbf{w} \in W$. Since $\mathbf{v} - \mathbf{w} = \mathbf{u} + (\operatorname{proj}_W \mathbf{v} - \mathbf{w})$,

$$\begin{aligned}
\|\mathbf{v} - \mathbf{w}\|^2 &= (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \\
&= (\mathbf{u} + (\operatorname{proj}_W \mathbf{v} - \mathbf{w})) \cdot (\mathbf{u} + (\operatorname{proj}_W \mathbf{v} - \mathbf{w})) \\
&= \mathbf{u} \cdot \mathbf{u} + (\operatorname{proj}_W \mathbf{v} - \mathbf{w}) \cdot (\operatorname{proj}_W \mathbf{v} - \mathbf{w}) \\
&\geq \|u\|^2.
\end{aligned}$$