Section 11.4 Cross Product

MATH211 Calculus III

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MESSIAH UNIVERSITY

DEPARTMENT OF
COMPUTING, MATHEMATICS
AND PHYSICS

Knowledge Checks

Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. What is the cross product $\mathbf{u} \times \mathbf{v}$?

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What is the special relation between the directions of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u}$ or $\mathbf{v}$?

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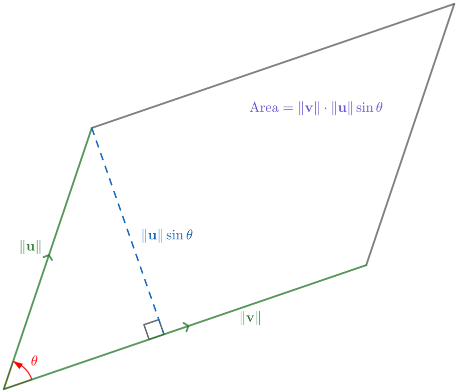
What is the special relation between the directions of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u}$ or $\mathbf{v}$?
$\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Knowledge Checks

How is $\|\mathbf{u} \times \mathbf{v}\|$ related to $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and θ?



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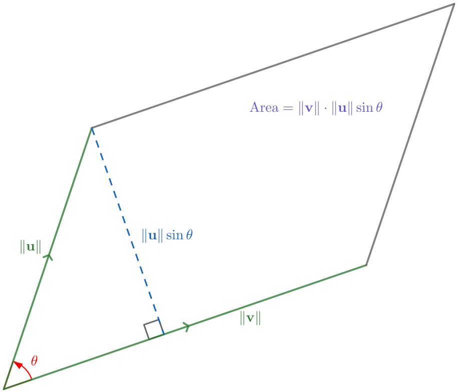
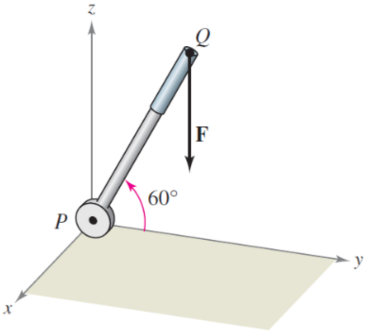
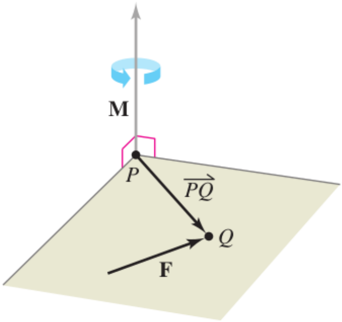


$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\| \sin\theta =$ the area of the parallelogram.

Knowledge Checks

What is the torque?





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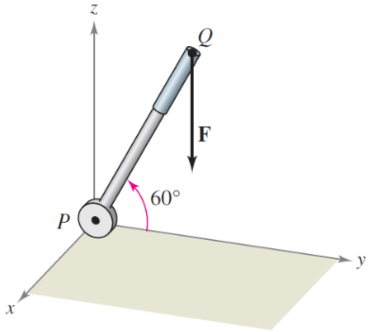

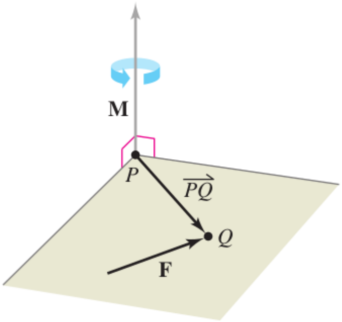

$$\mathbf{M} \text{ (or } \tau) = \overrightarrow{PQ} \times \mathbf{F}.$$

Knowledge Checks

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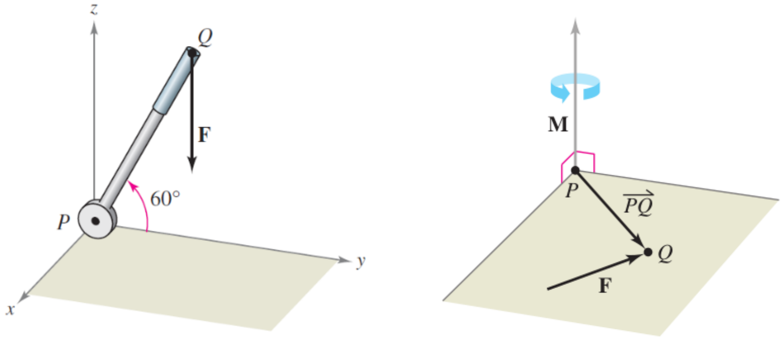
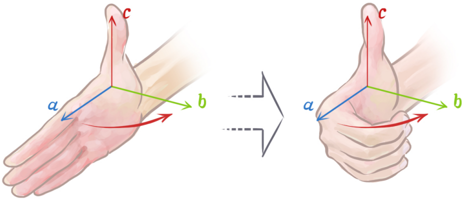
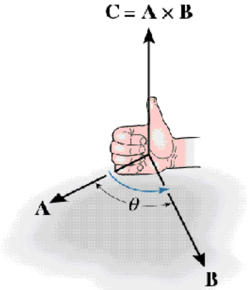


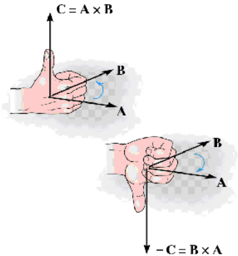
$$\mathbf{M} \text{ (or } \tau\text{)} = \overrightarrow{PQ} \times \mathbf{F}.$$

Remark: The torque is a vector, NOT a number.

The Right Hand Rule







The Cross Product of the Standard Unit Vectors

Exercise. According to the right hand rule and the magnitude formula, find

- $\mathbf{i} \times \mathbf{j}$.
- $\mathbf{j} \times \mathbf{i}$.
- $\mathbf{j} \times \mathbf{k}$.
- $\mathbf{k} \times \mathbf{j}$.
- $\mathbf{k} \times \mathbf{i}$.
- $\mathbf{i} \times \mathbf{k}$.

The General Formula

Suppose $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. If distributivity is to be respected, we must have the following.

The General Formula

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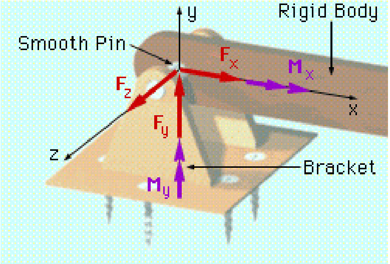
$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= \quad u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\
&\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\
&\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \\
&= \quad (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}.
\end{aligned}
$$

Pin Support



Suppose a force $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is acting on the lever at $(1, 1, 1)$.

- Find the torque of $\mathbf{F}$ about the origin.
- If the lever is stuck, find the force $\langle F_x, F_y, F_z \rangle$ at the pin support.
- If the lever can rotate freely about pin, find the couple moment $\langle M_x, M_y, M_z \rangle$ at the pin support.