

Eigenvalues and Eigenvectors

MAT215 Intro to Linear Algebra

Instructor: Ben Huang

DUTCHESS

COMMUNITY COLLEGE

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

The simplest - scalar matrix:

$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

The simplest - scalar matrix:

$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

The second to the best - diagonal matrix:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

Prominent properties of diagonal matrices:

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

Prominent properties of diagonal matrices:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

Prominent properties of diagonal matrices:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

More generally,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

Consequently,

$$e^D = \sum_{k=0}^{\infty} \frac{1}{k!} D^k = \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix}$$

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

The almost as good - diagonalizable matrix:

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

The almost as good - diagonalizable matrix:

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

Properties:

$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

The almost as good - diagonalizable matrix:

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

Properties:

$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^5 & 0 \\ 0 & e^4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

A stronger version - orthogonally diagonalizable:

$$S = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

A stronger version - orthogonally diagonalizable:

$$S = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

Applications:

(a) Classify quadric curves and surfaces

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

A stronger version - orthogonally diagonalizable:

$$S = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

Applications:

- (a) Classify quadric curves and surfaces
- (b) Simplify the inertia tensor of a rigid body (classical mechanics)

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

A stronger version - orthogonally diagonalizable:

$$S = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

Applications:

- (a) Classify quadric curves and surfaces
- (b) Simplify the inertia tensor of a rigid body (classical mechanics)
- (c) Principal Component Analysis

The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

Advanced decompositions:

- Singular Value decomposition

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \end{bmatrix}^T$$

- Jordan canonical form

$$\begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 2 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 2 \\ 2 & 5 & 0 \end{bmatrix}^{-1}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Question: How to even start diagonalizing a matrix?

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Question: How to even start diagonalizing a matrix?

The Trick: Reverse Engineering!

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Question: How to even start diagonalizing a matrix?

The Trick: Reverse Engineering!

Suppose $A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$, where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [\mathbf{p}_1 \quad \mathbf{p}_2]$.

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Question: How to even start diagonalizing a matrix?

The Trick: Reverse Engineering!

Suppose $A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$, where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [\mathbf{p}_1 \quad \mathbf{p}_2]$.

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}\mathbf{p}_1$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Question: How to even start diagonalizing a matrix?

The Trick: Reverse Engineering!

Suppose $A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$, where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [\mathbf{p}_1 \quad \mathbf{p}_2]$.

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}\mathbf{p}_1$$

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Question: How to even start diagonalizing a matrix?

The Trick: Reverse Engineering!

Suppose $A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$, where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [\mathbf{p}_1 \quad \mathbf{p}_2]$.

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1} \mathbf{p}_1$$

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Question: How to even start diagonalizing a matrix?

The Trick: Reverse Engineering!

Suppose $A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$, where $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [\mathbf{p}_1 \quad \mathbf{p}_2]$.

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1} \mathbf{p}_1$$

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix}$$

$$A\mathbf{p}_1 = \begin{bmatrix} \lambda_1 p_{11} \\ \lambda_1 p_{21} \end{bmatrix} = \lambda_1 \mathbf{p}_1$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Definition Let A be a square matrix. λ is called an **eigenvalue** of A if there is a non-zero column vector \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v},$$

and \mathbf{v} is called an **eigenvector** of A associated with λ .

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Definition Let A be a square matrix. λ is called an **eigenvalue** of A if there is a non-zero column vector \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v},$$

and \mathbf{v} is called an **eigenvector** of A associated with λ .

Exercises on WeBWork

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

Definition Let A be a square matrix. λ is called an **eigenvalue** of A if there is a non-zero column vector \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v},$$

and \mathbf{v} is called an **eigenvector** of A associated with λ .

Exercises on WeBWork

A 3-D Example on GeoGebra

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v} \Leftrightarrow \lambda \mathbf{v} - A\mathbf{v} = \mathbf{0}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v} \Leftrightarrow \lambda \mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow (\lambda I - A)\mathbf{v} = \mathbf{0}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v} \Leftrightarrow \lambda \mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow \lambda I \mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow (\lambda I - A)\mathbf{v} = \mathbf{0}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v} \Leftrightarrow \lambda \mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow \lambda I \mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow (\lambda I - A)\mathbf{v} = \mathbf{0}$$

$$\mathbf{v} \neq \mathbf{0}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v} \Leftrightarrow \lambda \mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow \lambda I\mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow (\lambda I - A)\mathbf{v} = \mathbf{0}$$

$$\mathbf{v} \neq \mathbf{0} \Leftrightarrow \lambda I - A \text{ is nonsingular}$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v} \Leftrightarrow \lambda \mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow \lambda I\mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow (\lambda I - A)\mathbf{v} = \mathbf{0}$$

$$\mathbf{v} \neq \mathbf{0} \Leftrightarrow \lambda I - A \text{ is nonsingular} \Leftrightarrow \det(\lambda I - A) = 0.$$

Meet the Eigenvalues

B.H.

Hierarchy

Eigenvalues

Diagonalization

How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v} \Leftrightarrow \lambda \mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow \lambda I\mathbf{v} - A\mathbf{v} = \mathbf{0} \Leftrightarrow (\lambda I - A)\mathbf{v} = \mathbf{0}$$

$$\mathbf{v} \neq \mathbf{0} \Leftrightarrow \lambda I - A \text{ is nonsingular} \Leftrightarrow \det(\lambda I - A) = 0.$$

Definition The polynomial $p(\lambda) = \det(\lambda I - A)$ is called the **characteristic polynomial** of A . Note that the roots of $p(\lambda)$ are precisely the eigenvalues of A .

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

Example.

Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. Diagonalize A .

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

Example.

Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. Diagonalize A .

Solution:

Step 1: Find the eigenvalues of A .

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) =$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

Example.

Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. Diagonalize A .

Solution:

Step 1: Find the eigenvalues of A .

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) = \lambda^2 - 7\lambda = 0;$$

$$\lambda_1 = 0, \lambda_2 = 7.$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

Example.

Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. Diagonalize A .

Solution:

Step 1: Find the eigenvalues of A .

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) = \lambda^2 - 7\lambda = 0;$$

$$\lambda_1 = 0, \lambda_2 = 7.$$

Step 2: Find a basis for each eigenspace.

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

Example.

Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. Diagonalize A .

Solution:

Step 1: Find the eigenvalues of A .

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) = \lambda^2 - 7\lambda = 0;$$

$$\lambda_1 = 0, \lambda_2 = 7.$$

Step 2: Find a basis for each eigenspace.

$$\lambda_1 = 0:$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

Example.

Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. Diagonalize A .

Solution:

Step 1: Find the eigenvalues of A .

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) = \lambda^2 - 7\lambda = 0;$$

$$\lambda_1 = 0, \lambda_2 = 7.$$

Step 2: Find a basis for each eigenspace.

$$\lambda_1 = 0:$$

$$(0I - A)\mathbf{v} = \mathbf{0}$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

Example.

Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. Diagonalize A .

Solution:

Step 1: Find the eigenvalues of A .

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) = \lambda^2 - 7\lambda = 0;$$

$$\lambda_1 = 0, \lambda_2 = 7.$$

Step 2: Find a basis for each eigenspace.

$$\lambda_1 = 0:$$

$$(0I - A)\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

Example.

Let $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$. Diagonalize A .

Solution:

Step 1: Find the eigenvalues of A .

$$\det(\lambda I - A) = \det \left(\begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) = \lambda^2 - 7\lambda = 0;$$

$$\lambda_1 = 0, \lambda_2 = 7.$$

Step 2: Find a basis for each eigenspace.

$$\lambda_1 = 0:$$

$$(0I - A)\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \text{ thus } \mathcal{B}_1 = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}.$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

$$\lambda_2 = 7:$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

$$\lambda_2 = 7:$$

$$(7I - A)\mathbf{v} = \mathbf{0}$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

$$\lambda_2 = 7:$$

$$(7I - A)\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

$$\lambda_2 = 7:$$

$$(7I - A)\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \text{ thus } \mathcal{B}_2 = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}.$$

Diagonalize a Matrix

B.H.

Hierarchy

Eigenvalues

Diagonalization

$$\lambda_2 = 7:$$

$$(7I - A)\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \text{ thus } \mathcal{B}_2 = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}.$$

Step 4: Since the number of basis vectors is the same as the dimension of the ambient space (\mathbb{R}^2), A is diagonalizable, and

$$D = P^{-1}AP,$$

$$\text{where } D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}, P = \begin{bmatrix} \frac{1}{2} & -3 \\ 1 & 1 \end{bmatrix}.$$