Section 13.4 Differentials

MATH211 Calculus III

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MESSIAH UNIVERSITY

DEPARTMENT OF COMPUTING, MATHEMATICS AND PHYSICS

Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$

$ = (f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)) + (f(x_0, y_0 + \Delta y) - f(x_0, y_0))$

Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$
$$= (f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)) + (f(x_0, y_0 + \Delta y) - f(x_0, y_0))$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)$$

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$$= (f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)) + (f(x_0, y_0 + \Delta y) - f(x_0, y_0))$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)$$
$$= \left(\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} \right) \Delta x$$

Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$
$$= (f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)) + (f(x_0, y_0 + \Delta y) - f(x_0, y_0))$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)$$
$$= \left(\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} \right) \Delta x$$
$$= (f_x(x_0, y_0 + \Delta y) + \varepsilon_{11}(\Delta x, \Delta y)) \Delta x$$

Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$
$$= (f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)) + (f(x_0, y_0 + \Delta y) - f(x_0, y_0))$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)$$
$$= \left(\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} \right) \Delta x$$
$$= (f_x(x_0, y_0 + \Delta y) + \varepsilon_{11}(\Delta x, \Delta y)) \, \Delta x$$
$$= \Big[\big(\overbrace{f_x(x_0, y_0) + \underbrace{\varepsilon_{12}(\Delta y)}}^{f_x(x_0, y_0 + \Delta y)} \big) + \varepsilon_{11}(\Delta x, \Delta y) \Big] \Delta x$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\varepsilon_1}$$

Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$
$$= (f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)) + (f(x_0, y_0 + \Delta y) - f(x_0, y_0))$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)$$
$$= \left(\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} \right) \Delta x$$
$$= (f_x(x_0, y_0 + \Delta y) + \varepsilon_{11}(\Delta x, \Delta y)) \Delta x$$
$$= \Big[\big(\overbrace{f_x(x_0, y_0) + \underbrace{\varepsilon_{12}(\Delta y)}}^{f_x(x_0, y_0 + \Delta y)} \big) + \varepsilon_{11}(\Delta x, \Delta y) \Big] \Delta x$$
$$\underbrace{\phantom{f_x(x_0, y_0) + \varepsilon_{12}(\Delta y)}}_{\varepsilon_1}$$
$$= f_x(x_0, y_0) \Delta x + \varepsilon_1 \Delta x$$

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Linear Approximation

Suppose f is a function of (x, y) and both f_x and f_y are continuous.

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$
$$= (f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)) + (f(x_0, y_0 + \Delta y) - f(x_0, y_0))$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)$$
$$= \left(\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)}{\Delta x} \right) \Delta x$$
$$= (f_x(x_0, y_0 + \Delta y) + \varepsilon_{11}(\Delta x, \Delta y)) \Delta x$$
$$= \Big[\big(\overbrace{f_x(x_0, y_0) + \underbrace{\varepsilon_{12}(\Delta y)}}^{f_x(x_0, y_0 + \Delta y)} \big) + \varepsilon_{11}(\Delta x, \Delta y) \Big] \Delta x$$
$$\underbrace{\phantom{f_x(x_0, y_0) + \varepsilon_{12}(\Delta y)}}_{\varepsilon_1}$$
$$= f_x(x_0, y_0) \Delta x + \varepsilon_1 \Delta x$$

$$f(x_0, y_0 + \Delta y) - f(x_0, y_0) = \left(\frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \right) \Delta y$$
$$= f_y(x_0, y_0) \Delta y + \varepsilon_2 \Delta y$$

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Note that $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Consequently,

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ decays in a comparable order of $\|\Delta x \mathbf{i} + \Delta y \mathbf{j}\|^2$.