Section 13.5 Chain Rules for Functions of Several Variables

MATH211 Calculus III

Instructor: Ben Huang

MESSIAH UNIVERSITY

DEPARTMENT OF
COMPUTING, MATHEMATICS
AND PHYSICS

Chain Rules

Let $w = f(x, y)$, where f is a differentiable function of x and y, $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Note that w is a composite function of t.

Chain Rules

Let $w = f(x, y)$, where f is a differentiable function of x and y, $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Note that w is a composite function of t.

Chain Rules

Let $w = f(x, y)$, where f is a differentiable function of x and y, $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Note that w is a composite function of t.

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{w(t + \Delta t) - w(t)}{\Delta t}$$

Chain Rules

Let $w = f(x, y)$, where f is a differentiable function of x and y, $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Note that w is a composite function of t.

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{w(t + \Delta t) - w(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t))}{\Delta t}
\end{aligned}$$

Chain Rules

Let $w = f(x, y)$, where f is a differentiable function of x and y, $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Note that w is a composite function of t.

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{w(t+\Delta t) - w(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}\end{aligned}$$

Since f is differentiable, from Section 13.4,

$$\begin{aligned}& f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t)) \\ =& \frac{\partial w}{\partial x}(x, y)\Delta x + \frac{\partial w}{\partial y}(x, y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,\end{aligned}$$

Chain Rules

Let $w = f(x, y)$, where f is a differentiable function of x and y, $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t. Note that w is a composite function of t.

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \lim_{\Delta t\to 0}\frac{w(t+\Delta t)-w(t)}{\Delta t}\\ &= \lim_{\Delta t\to 0}\frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}\end{aligned}$$

Since f is differentiable, from Section 13.4,

$$\begin{aligned}&f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))\\ =&\frac{\partial w}{\partial x}(x,y)\Delta x + \frac{\partial w}{\partial y}(x,y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y,\end{aligned}$$

where $\Delta x = x(t+\Delta t) - x(t)$, $\Delta y = y(t+\Delta t) - y(t)$, and $\varepsilon_1, \varepsilon_2 \xrightarrow{(\Delta x,\Delta y)\to(0,0)} 0$.

Chain Rules

Consequently,

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{\partial w}{\partial x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} \\ &\quad + \lim_{\Delta t\to 0}\varepsilon_1 \lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \lim_{\Delta t\to 0}\varepsilon_2 \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t}\end{aligned}$$

Chain Rules

Consequently,

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}t} &= \frac{\partial w}{\partial x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} \\
&\quad + \lim_{\Delta t\to 0}\varepsilon_1 \lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \lim_{\Delta t\to 0}\varepsilon_2 \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} \\
&= \frac{\partial w}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + 0\cdot\frac{\mathrm{d}x}{\mathrm{d}t} + 0\cdot\frac{\mathrm{d}y}{\mathrm{d}t}
\end{aligned}$$

Chain Rules

Consequently,

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \\
&\quad + \lim_{\Delta t \to 0} \varepsilon_1 \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \varepsilon_2 \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \\
&= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\
&= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\end{aligned}$$

MESSIAH UNIVERSITY
DEPARTMENT OF COMPUTING, MATHEMATICS AND PHYSICS

Chain Rules

Consequently,

$$
\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} \\
&\quad + \lim_{\Delta t\to 0}\varepsilon_1 \lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t} + \lim_{\Delta t\to 0}\varepsilon_2 \lim_{\Delta t\to 0}\frac{\Delta y}{\Delta t} \\
&= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + 0\cdot\frac{dx}{dt} + 0\cdot\frac{dy}{dt} \\
&= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}
\end{aligned}
$$