

# Eigenvalues and Eigenvectors

MAT215 Intro to Linear Algebra

Instructor: Ben Huang

**DUTCHESS**  

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**COMMUNITY COLLEGE**

# The Hierarchy of Matrices

B.H.

Hierarchy

Eigenvalues

Diagonalization

The simplest - scalar matrix:

$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

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The simplest - scalar matrix:

$$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

The second to the best - diagonal matrix:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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Prominent properties of diagonal matrices:

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Prominent properties of diagonal matrices:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

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Prominent properties of diagonal matrices:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

More generally,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

Consequently,

$$e^D = \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix}$$

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The almost as good - diagonalizable matrix:

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

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Properties:

$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$



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$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^5 & 0 \\ 0 & e^4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

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A stronger version - orthogonally diagonalizable:

$$S = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

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Applications:

(a) Classify quadric curves and surfaces

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Applications:

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- (b) Simplify the inertia tensor of a rigid body (classical mechanics)

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Applications:

- (a) Classify quadric curves and surfaces
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- (c) Principal Component Analysis

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Advanced decompositions:

- Singular Value decomposition

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \end{bmatrix}^T$$

- Jordan canonical form

$$\begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 2 \\ 2 & 5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 2 \\ 2 & 5 & 0 \end{bmatrix}^{-1}$$

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**Question:** How to even start diagonalizing a matrix?

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**The Trick:** Reverse Engineering!

Suppose  $A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$ , where  $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = [\mathbf{p}_1 \quad \mathbf{p}_2]$ .

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$$A\mathbf{p}_1 = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}\mathbf{p}_1$$

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**Definition** Let  $A$  be a square matrix.  $\lambda$  is called an **eigenvalue** of  $A$  if there is a non-zero column vector  $\mathbf{v}$  such that

$$A\mathbf{v} = \lambda\mathbf{v},$$

and  $\mathbf{v}$  is called an **eigenvector** of  $A$  associated with  $\lambda$ .

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Exercises on WeBWork

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Exercises on WeBWork

A 3-D Example on GeoGebra



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$$\lambda \mathbf{v} = A\mathbf{v}$$

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How to find the eigenvalues?

$$\lambda \mathbf{v} = A\mathbf{v} \Leftrightarrow \lambda \mathbf{v} - A\mathbf{v} = \mathbf{0}$$

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$$\mathbf{v} \neq \mathbf{0}$$

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$$\mathbf{v} \neq \mathbf{0} \Leftrightarrow \lambda I - A \text{ is nonsingular} \Leftrightarrow \det(\lambda I - A) = 0.$$

**Definition** The polynomial  $p(\lambda) = \det(\lambda I - A)$  is called the **characteristic polynomial** of  $A$ . Note that the roots of  $p(\lambda)$  are precisely the eigenvalues of  $A$ .

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**Example.**

Let  $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$ . Diagonalize  $A$ .

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**Solution:**

Step 1: Find the eigenvalues of  $A$ .

$$\det(\lambda I - A) = \det \left( \begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) =$$

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$$\det(\lambda I - A) = \det \left( \begin{bmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{bmatrix} \right) = \lambda^2 - 7\lambda = 0;$$

$$\lambda_1 = 0, \lambda_2 = 7.$$

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$$\begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



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$$\lambda_1 = 0, \lambda_2 = 7.$$

Step 2: Find a basis for each eigenspace.

$$\lambda_1 = 0:$$

$$(0I - A)\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \text{ thus } \mathcal{B}_1 = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}.$$

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$$\lambda_2 = 7:$$

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$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \text{ thus } \mathcal{B}_2 = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}.$$

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Step 4: Since the number of basis vectors is the same as the dimension of the ambient space ( $\mathbb{R}^2$ ),  $A$  is diagonalizable, and

$$D = P^{-1}AP,$$

$$\text{where } D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}, P = \begin{bmatrix} \frac{1}{2} & -3 \\ 1 & 1 \end{bmatrix}.$$