Consider the nth order linear homogeneous equation with constant coefficients:

$$L(y) = y^{(n)} + \sum_{i=0}^{n-1} a_i y^{(i)} = 0.$$

Suppose ϕ is a solution to this equation. Define

$$\|\phi(x)\| = \left(\sum_{i=0}^{n-1} \left(\phi^{(i)}(x) \right)^2 \right)^{1/2}.$$

We begin with finding an estimate of $\|\phi(x)\|$ in some explicit terms. Let $u(x) = \|\phi(x)\|^2 = \sum_{i=0}^{n-1} \left(\phi^{(i)}(x) \right)^2$, then

$$u'(x) =$$

$$|u'(x)| \leq$$

Since ϕ is assumed to be a solution of $L(y) = 0$, we have

$$\phi^{(n)}(x) = \sum_{i=0}^{n-1} -a_i \phi^{(i)}(x)$$

$$|\phi^{(n)}(x)| \leq \sum_{i=0}^{n-1} |a_i||\phi^{(i)}(x)|$$

Consequently, substituting in $u'(x)$, we have

We now apply the elementary inequality $2|b||c| \leq |b|^2 + |c|^2$ to obtain

Now, denote $k = 1 + \sum_{i=0}^{n-1} |a_i|$, then

$$|u'| \leq 2ku,$$

or equivalently,

$$-2ku \leq u' \leq 2ku. \tag{1}$$

Problems.

1. Based on inequality (1), prove that

$$\|\phi(x_0)\|e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\|e^{k|x-x_0|}. \tag{2}$$

2. Let $\{c_i\}_{i=0}^{n-1}$ be any n constants, and let x_0 be any real number. We want to prove that there exists at most one solution ϕ of $L(y) = 0$ satisfying $\{y^{(i)}(x_0) = c_i\}_{i=0}^{n-1}$. We break the proof into two steps:

   (a) Suppose both ϕ and ψ were two solutions of $L(y) = 0$ satisfying $\{y^{(i)}(x_0) = c_i\}_{i=0}^{n-1}$. Prove that $\chi = \phi - \psi$ satisfies $L(y) = 0$ and $\{\chi^{(i)}(x_0) = 0\}_{i=0}^{n-1}$.

   (b) Use inequality (2) to prove that $\|\chi\| = 0$, which implies $\chi(x) = 0$ for all x and hence $\phi = \psi$.