

Section 13.5 Chain Rules for Functions of Several Variables

MATH211 Calculus III

Instructor: Ben Huang



DEPARTMENT OF
COMPUTING, MATHEMATICS
AND PHYSICS

Chain Rules

Section 13.5

B.H.

Let $w = f(x, y)$, where f is a differentiable function of x and y ,
 $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of
 t . Note that w is a composite function of t .

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$$\frac{dw}{dt} = \lim_{\Delta t \rightarrow 0} \frac{w(t + \Delta t) - w(t)}{\Delta t}$$

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$$\begin{aligned}\frac{dw}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{w(t + \Delta t) - w(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t))}{\Delta t}\end{aligned}$$

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 &= \lim_{\Delta t \rightarrow 0} \frac{f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t))}{\Delta t}
 \end{aligned}$$

Since f is differentiable, from Section 13.4,

$$\begin{aligned}
 &f(x(t + \Delta t), y(t + \Delta t)) - f(x(t), y(t)) \\
 &= \frac{\partial w}{\partial x}(x, y)\Delta x + \frac{\partial w}{\partial y}(x, y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y,
 \end{aligned}$$

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where $\Delta x = x(t + \Delta t) - x(t)$, $\Delta y = y(t + \Delta t) - y(t)$, and $\varepsilon_1, \varepsilon_2 \xrightarrow{(\Delta x, \Delta y) \rightarrow (0, 0)} 0$.

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Consequently,

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y} \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} \\ &\quad + \lim_{\Delta t \rightarrow 0} \varepsilon_1 \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \rightarrow 0} \varepsilon_2 \lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t}\end{aligned}$$

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