Eigenvalues and Eigenvectors

MAT215 Intro to Linear Algebra

Instructor: Ben Huang

DUTCHESS COMMUNITY COLLEGE

The Hierarchy of Matrices

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The second to the best - diagonal matrix:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

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More generally,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{k} = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

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Consequently,

$$e^D = \sum_{k=0}^{\infty} \frac{1}{k!} D^k = \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix}$$

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The almost as good - diagonalizable matrix:

$$A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

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A stronger version - orthogonally diagonalizable:

$$S = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

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Applications:

(a) Classify quadric curves and surfaces

(b) Simplify the inertia tensor of a rigid body (classical mechanics)

(c) Principal Component Analysis

The Hierarchy of Matrices

Advanced decompositions:

- Singular Value decomposition

$$\begin{bmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{18} & 2/3\\ 1/\sqrt{2} & -1/\sqrt{18} & -2/3\\ 0 & 4/\sqrt{18} & -1/3\end{bmatrix}^T$$

- Jordan canonical form

$$\begin{bmatrix}-2 & 2 & 1\\ -7 & 4 & 2\\ 5 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 1 & 1\\ -1 & -1 & 2\\ 2 & 5 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 1\\ -1 & -1 & 2\\ 2 & 5 & 0\end{bmatrix}^{-1}$$

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Definition Let A be a square matrix. λ is called an **eigenvalue** of A if there is a non-zero column vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v},$$

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A 3-D Example on GeoGebra

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Definition The polynomial $p(\lambda) = \det(\lambda I - A)$ is called the **characteristic polynomial** of A. Note that the roots of $p(\lambda)$ are precisely the eigenvalues of A.

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \text{ thus } \mathscr{B}_1 = \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\}.$$

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \text{ thus } \mathscr{B}_2 = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}.$$

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Step 4: Since the number of basis vectors is the same as the dimension of the ambient space (R^2), A is diagonalizable, and

$$D = P^{-1}AP,$$

where $D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{2} & -3 \\ 1 & 1 \end{bmatrix}$.