Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix}.$$

Find the matrix A of the orthogonal projection onto W.

Solution: ① Apply the Gram-Schmidt Process to find an orthogonal basis.

(i) $\vec{w_1} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

(ii) $\vec{w_2} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} - \text{proj}_{w_1} \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -2 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix}$

② Find the images of the orthogonal projection of the standard basis of $\mathbb{R}^4$.

$P(e_1) = \frac{-1}{4} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{20} \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.2 \\ -0.1 \\ -0.4 \end{bmatrix}$

$P(e_2) = \frac{-1}{4} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \frac{-1}{20} \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.3 \\ -0.4 \\ -0.1 \end{bmatrix}$

$P(e_3) = \frac{1}{4} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \frac{3}{20} \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -0.1 \\ -0.4 \\ 0.7 \\ -0.2 \end{bmatrix}$

$P(e_4) = \frac{1}{4} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} + \frac{-3}{20} \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -0.4 \\ -0.1 \\ -0.2 \\ 0.7 \end{bmatrix}$

③ Form the standard matrix

$$A = \begin{bmatrix} P(e_1) & P(e_2) & P(e_3) & P(e_4) \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & -0.1 & -0.4 \\ 0.2 & 0.3 & -0.4 & -0.1 \\ -0.1 & -0.4 & 0.7 & -0.2 \\ -0.4 & -0.1 & 0.2 & 0.7 \end{bmatrix}$$

Given $\vec{v} = \begin{bmatrix} -7 \\ -10 \\ 7 \\ 2 \end{bmatrix}$, find the closest point to $\vec{v}$ in the subspace W spanned by $\begin{bmatrix} 6 \\ -4 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 1 \\ 3 \\ -22 \end{bmatrix}$.

Solution: According to the property, it's just the orthogonal projection of $\begin{bmatrix} -7 \\ -10 \\ 7 \\ 2 \end{bmatrix}$ onto W.

Since $\begin{bmatrix} 6 \\ -4 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \\ 3 \\ -22 \end{bmatrix} = -24 - 4 + 6 + 22 = 0$, they form an orthogonal basis, hence

$$\text{Proj}_W \begin{bmatrix} -7 \\ -10 \\ 7 \\ 2 \end{bmatrix} = \frac{10}{57} \begin{bmatrix} 6 \\ -4 \\ 2 \\ -1 \end{bmatrix} + \frac{-5}{510} \begin{bmatrix} -4 \\ 1 \\ 3 \\ -22 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1058}{969} \\ -\frac{1379}{1938} \\ \frac{623}{1938} \\ \frac{13}{323} \end{bmatrix}$$