2.3 Trigonometry

In graphics, we use basic trigonometry in many contexts. Usually, it is nothing too fancy, and it often helps to remember the basic definitions.

2.3.1 Angles

Although we take angles somewhat for granted, we should return to their definition so we can extend the idea of the angle onto the sphere. An angle is formed between two half-lines (infinite rays stemming from an origin) or directions, and some convention must be used to decide between the two possibilities for the angle created between them as shown in Figure 2.6. An *angle* is defined by the length of the arc segment it cuts out on the unit circle. A common convention is that the smaller arc length is used, and the sign of the angle is determined by the order in which the two half-lines are specified. Using that convention, all angles are in the range $[-\pi, \pi]$.

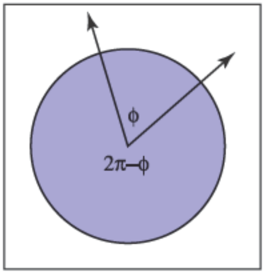

Figure 2.6. Two half-lines cut the unit circle into two arcs. The length of either arc is a valid angle "between" the two half-lines. Either we can use the convention that the smaller length is the angle, or that the two half-lines are specified in a certain order and the arc that determines angle ϕ is the one swept out counterclockwise from the first to the second half-line.

Each of these angles is *the length of the arc of the unit circle that is "cut" by the two directions*. Because the perimeter of the unit circle is 2π, the two possible angles sum to 2π. The unit of these arc lengths is *radians*. Another common unit is degrees, where the perimeter of the circle is 360°. Thus, an angle that is π radians is 180°, usually denoted 180°. The conversion between degrees and radians is

$$\text{Degrees} = \frac{180}{\pi}\text{ radians};$$

$$\text{Radians} = \frac{\pi}{180}\text{ degrees.}$$

2.3.2 Trigonometric Functions

Given a right triangle with sides of length a, o, and h, where h is the length of the longest side (which is always opposite the right angle), or *hypotenuse*, an important relation is described by the *Pythagorean theorem*:

$$a^2 + o^2 = h^2.$$

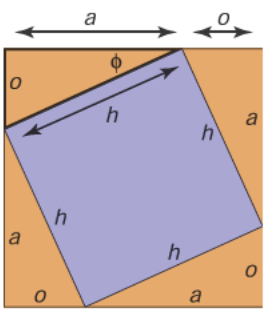

Figure 2.7. A geometric demonstration of the Pythagorean theorem.

You can see that this is true from Figure 2.7, where the big square has area $(a+o)^2$, the four triangles have the combined area $2ao$, and the center square has area h^2.

Because the triangles and inner square subdivide the larger square evenly, we have $2ao + h^2 = (a + o)^2$, which is easily manipulated to the form above.

We define *sine* and *cosine* of ϕ, as well as the other ratio-based trigonometric expressions:

$$\begin{aligned}\sin\phi &\equiv o/h;\\ \csc\phi &\equiv h/o;\\ \cos\phi &\equiv a/h;\\ \sec\phi &\equiv h/a;\\ \tan\phi &\equiv o/a;\\ \cot\phi &\equiv a/o.\end{aligned}$$

These definitions allow us to set up *polar coordinates*, where a point is coded as a distance from the origin and a signed angle relative to the positive x-axis (Figure 2.8). Note the convention that angles are in the range $\phi \in (-\pi, \pi]$, and that the positive angles are counterclockwise from the positive x-axis. This convention that counterclockwise maps to positive numbers is arbitrary, but it is used in many contexts in graphics so it is worth committing to memory.

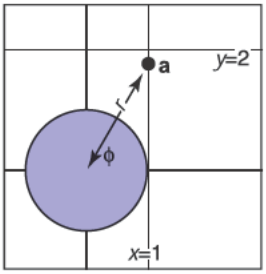

Figure 2.8. Polar coordinates for the point $(x_a, y_a) = (1, \sqrt{3})$ is $(r_a, \phi_a) = (2, \pi/3)$.

Trigonometric functions are periodic and can take any angle as an argument. For example, $\sin(A) = \sin(A + 2\pi)$. This means the functions are not invertible when considered with the domain $\mathbb{R}$. This problem is avoided by restricting the range of standard inverse functions, and this is done in a standard way in almost all modern math libraries (e.g., Plauger (1991)). The domains and ranges are

$$\begin{aligned}\text{asin} &: [-1, 1] \mapsto [-\pi/2, \pi/2];\\ \text{acos} &: [-1, 1] \mapsto [0, \pi];\\ \text{atan} &: \mathbb{R} \mapsto [-\pi/2, \pi/2];\\ \text{atan2} &: \mathbb{R}^2 \mapsto [-\pi, \pi].\end{aligned} \tag{2.2}$$

The last function, $\text{atan2}(s, c)$ is often very useful. It takes an s value proportional to $\sin A$ and a c value that scales $\cos A$ by the same factor and returns A. The factor is assumed to be positive. One way to think of this is that it returns the angle of a 2D Cartesian point (s, c) in polar coordinates (Figure 2.9).

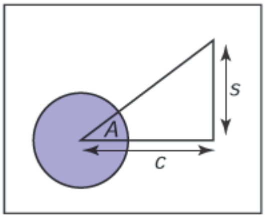

Figure 2.9. The function $\text{atan2}(s,c)$ returns the angle A and is often very useful in graphics.

2.3.3 Useful Identities

This section lists without derivation a variety of useful trigonometric identities.

Shifting identities:

$$\begin{aligned}\sin(-A) &= -\sin A\\ \cos(-A) &= \cos A\\ \tan(-A) &= -\tan A\\ \sin(\pi/2 - A) &= \cos A\\ \cos(\pi/2 - A) &= \sin A\\ \tan(\pi/2 - A) &= \cot A\end{aligned}$$

Pythagorean identities:

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A - \tan^2 A = 1$$
$$\csc^2 A - \cot^2 A = 1$$

Addition and subtraction identities:

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \sin B \cos A \\
\sin(A-B) &= \sin A \cos B - \sin B \cos A \\
\sin(2A) &= 2 \sin A \cos A \\
\cos(A+B) &= \cos A \cos B - \sin A \sin B \\
\cos(A-B) &= \cos A \cos B + \sin A \sin B \\
\cos(2A) &= \cos^2 A - \sin^2 A \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \\
\tan(2A) &= \frac{2 \tan A}{1 - \tan^2 A}
\end{aligned}$$

Half-angle identities:

$$\sin^2(A/2) = (1 - \cos A)/2$$
$$\cos^2(A/2) = (1 + \cos A)/2$$

Product identities:

$$\begin{aligned}
\sin A \sin B &= -(\cos(A+B) - \cos(A-B))/2 \\
\sin A \cos B &= (\sin(A+B) + \sin(A-B))/2 \\
\cos A \cos B &= (\cos(A+B) + \cos(A-B))/2
\end{aligned}$$

The following identities are for arbitrary triangles with side lengths a, b, and c, each with an angle opposite it given by A, B, C, respectively (Figure 2.10),

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{(Law of sines)}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \text{(Law of cosines)}$$

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} \qquad \text{(Law of tangents)}$$

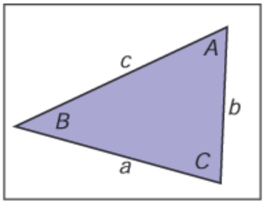

Figure 2.10. Geometry for triangle laws.

The area of a triangle can also be computed in terms of these side lengths:

$$\text{Triangle area} = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

2.3.4 Solid Angles and Spherical Trigonometry

Traditional trigonometry in this section deals with triangles on the plane. Triangles can be defined on non-planar surfaces as well, and one that arises in many fields, astronomy, for example, is triangles on the unit-radius sphere. These *spherical triangles* have sides that are segments of the *great circles* (unit-radius circles) on the sphere. The study of these triangles is a field called *spherical trigonometry* and is not used that commonly in graphics, but sometimes, it is critical when it does arise. We wont discuss the details of it here, but want the reader to be aware that area exists for when those problems do arise, and there are a lot of useful rules such as a spherical law of cosines and a spherical law of sines. For an example of the machinery of spherical trigonometry being used, see the paper on sampling triangle lights (which project to a spherical triangle) (Arvo, 1995b).

Of more central importance to computer graphics are *solid angles*. While angles allow us to quantify things like "what is the separation of those two poles in my visual field," solid angles let us quantify things like "how much of my visual field does that airplane cover." For traditional angles, we project the posts onto the unit circle and measure arc length between them on the unit circle. We work with angles often enough that many of us can forget this definition because it is all so intuitive to us now. Solid angles are just as simple, but they may seem more confusing because most of us learn about them as adults. For solid angles, we project the visible directions that "see" the airplane and project it onto the unit sphere and measure the area. This area is the solid angle in the same way the arc length is the angle. While angles are measured in radians and sum to 2π (the total length of a unit circle), solid angles are measured in *steradians* and sum to 4π (the total area of a unit sphere).