2.2 Solving Quadratic Equations

A *quadratic equation* has the form

$$Ax^2 + Bx + C = 0,$$

where x is a real unknown, and A, B, and C are known constants. If you think of a 2D xy plot with $y = Ax^2 + Bx + C$, the solution is just whatever x values are "zero crossings" in y. Because $y = Ax^2 + Bx + C$ is a parabola, there will be zero, one, or two real solutions depending on whether the parabola misses, grazes, or hits the x-axis (Figure 2.5).

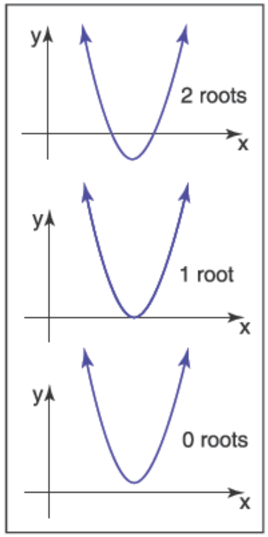

Figure 2.5. The geometric interpretation of the roots of a quadratic equation is the intersection points of a parabola with the x-axis.

To solve the quadratic equation analytically, we first divide by A:

$$x^2 + \frac{B}{A}x + \frac{C}{A} = 0.$$

Then, we "complete the square" to group terms:

$$\left(x + \frac{B}{2A}\right)^2 - \frac{B^2}{4A^2} + \frac{C}{A} = 0.$$

Moving the constant portion to the right-hand side and taking the square root give

$$x + \frac{B}{2A} = \pm\sqrt{\frac{B^2}{4A^2} - \frac{C}{A}}.$$

Subtracting $B/(2A)$ from both sides and grouping terms with the denominator $2A$ gives the familiar form:[1]

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{2.1}$$

Here, the "$\pm$" symbol means there are two solutions, one with a plus sign and one with a minus sign. Thus, 3 ± 1 equals "two or four." Note that the term that determines the number of real solutions is

$$D \equiv B^2 - 4AC,$$

which is called the *discriminant* of the quadratic equation. If $D > 0$, there are two real solutions (also called *roots*). If $D = 0$, there is one real solution (a "double" root). If $D < 0$, there are no real solutions.

For example, the roots of $2x^2 + 6x + 4 = 0$ are $x = -1$ and $x = -2$, and the equation $x^2 + x + 1$ has no real solutions. The discriminants of these equations are $D = 4$ and $D = -3$, respectively, so we expect the number of solutions given. In programs, it is usually a good idea to evaluate D first and return "no roots" without taking the square root if D is negative.

[1] A robust implementation will use the equivalent expression $2C/(-B \mp \sqrt{B^2 - 4AC})$ to compute one of the roots, depending on the sign of B (Exercise 7).