Continuous Joint Distributions

A two-dimensional r.v (X,Y) is said to be continuous if each of its components X and Y is a continuous r.v. If both X and Y have continuous density functions then (X,Y) is said to have a bivariate density function.

- Let us assume that the continuous random vector (X,Y) takes values in a region R in $\mathbb{R}^2$ such that

$$P\{(X,Y)\in R\} = 1$$

- Suppose that this unit prob is continuously distributed over the region R. The aim is to obtain an expression for $P\{(X,Y)\in A\}$ where $A \subset R$ defined by $A = \{(x,y) : a < x < b,\ c < y < d\}$

where a, b, c, d are real numbers.

- Therefore

$$P\{(X,Y)\in A\} = P\,(a<X<b,\ c<Y<d)$$

$$= \int_A f(x,y)\,dx\,dy$$

A 2-dimensional or bivariate density function $f(x,y)$ for the random vector (X,Y) is a non-negative real-valued function defined on $\mathbb{R}^2$ such that

(i) $P\{(X,Y)\in\mathbb{R}^2\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

(ii) $P\{a<X\le b,\ c<Y\le d\} = P\{a\le X<b,\ c\le Y<d\}$

$$= P\{a\le X\le b,\ c\le Y\le d\}$$

$$= P\{a<X<b,\ c<Y<d\}$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Do this first

(iii) For $R\in\mathbb{R}^2$, $P\{(X,Y)\in R\} = \iint_R f(x,y)\,dx\,dy$

The function $f(x,y)$ defined above is the joint probability density function

Example 1

Consider a bivariate function defined by

$\frac{1}{8}$

$$f(x,y) = \begin{cases} K(6-x-y), & 0<x<2,\ 2<y<4 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine the value of constant K so that $f(x,y)$ is a joint p.d.f of X and Y

(b) Hence evaluate (i) $P(X<1, Y<3)$
(ii) $P(X+Y<3)$

Solution

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

Do this first!

$$\int_{2}^{4}\int_{0}^{2} K(6-x-y)\,dx\,dy = 1$$

$$K\int_{2}^{4}\left[6x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2} dy = 1$$

$$K\int_{2}^{4}\left[(6\times 2) - \frac{2^2}{2} - (2\times y)\right] dy = 1$$

$$K\int_{2}^{4}(12-2-2y)\,dy = 1$$

$$K\int_{2}^{4}(10-2y)\,dy = 1$$

$$K\left[10y - \frac{2y^2}{2}\right]_{y=2}^{y=4} = 1$$

$$k\left[40-16-(20-4)\right]=1$$

$$k[8]=1$$

$$k=1/8$$

$$\therefore\ f(x,y)=\begin{cases}1/8\,(6-x-y), & 0<x<2,\ 2<y<4\\ 0, & \text{elsewhere}\end{cases}$$

b (i) $P(X<1, Y<3)$

$$'' = \int_0^1\int_2^3 1/8\,(6-x-y)\,dy\,dx$$

$$'' = 1/8\int_0^1\left[6y-xy-\frac{y^2}{2}\right]_{y=2}^{y=3}dx$$

$$'' = 1/8\int_0^1\left(18-3x-\frac{9}{2}-12+2x+2\right)dx$$

$$'' = 1/8\int_0^1\left(\frac{7}{2}-x\right)dx$$

$$'' = \frac{1}{8}\left[\frac{7}{2}x-\frac{x^2}{2}\right]_{x=0}^{x=1}$$

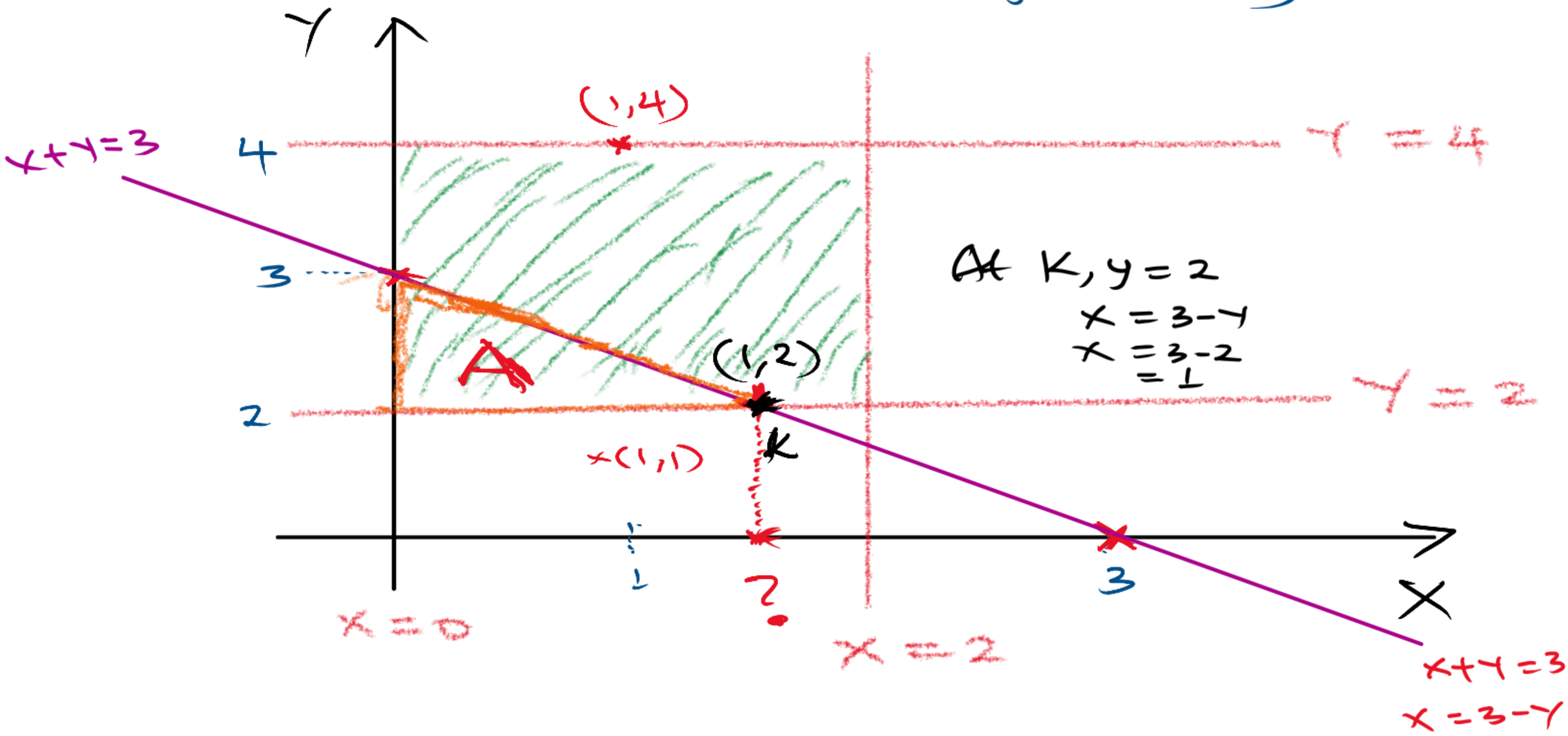
$$'' = \frac{1}{8}\left[\frac{7}{2}-1/2\right]$$

$$'' = 3/8$$

b (ii) $P(X+Y<3)$

$$f(x,y)=\begin{cases}\frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4\\ 0, & \text{otherwise}\end{cases}$$

Represent the limits graphically



$X+Y=3$

x	y
0	3
3	0

$(0,3),\ (3,0)$

$$P(X+Y<3)=\int_2^3\int_0^{3-y}\frac{1}{8}(6-x-y)\,dx\,dy \quad \text{--- } (*)$$

Alternatively,

$x+y=3$

$y=3-x$

$$P(X+Y<3)=\int_0^1\int_2^{3-x}\frac{1}{8}(6-x-y)\,dy\,dx \quad \text{--- } (**)$$

Prove that $(*)$ and $(**)$ give the same result.

Exercise 1

Exercise 2

Consider a joint pdf of X and Y given by

$$f(x,y) = \begin{cases} c\,(x^2+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) Value of c

(ii) $P(X < 1/2, Y > 1/2)$

(iii) $P(1/4 < X < 3/4)$

(iv) $P(Y < 1/2)$

(v) $P(Y > 1/2)$