a) $E(X)=\mu=(8)(15)=120$ weeks $\qquad \sigma=\sqrt{8(15)^2}=30\sqrt{2}\approx 42.4264$ weeks

$b)\qquad P(60\le X\le 120)=\int_{60}^{120}\frac{1}{15^8\Gamma(8)}x^7e^{-\frac{x}{15}}dx=\int_4^8\frac{1}{\Gamma(8)}y^7e^{-y}dy$

$$=-\left(y^7+7y^6+42y^5+210y^4+840y^3+2520y^2+5040y+5040\right)\frac{e^{-y}}{7!}\Big|_4^8$$

$$=\frac{261104e^{-4}-6805296e^{-8}}{7!}\approx 0.4959$$

$c)\qquad PX\ge 30)=\int_{30}^{\infty}\frac{1}{15^8\Gamma(8)}x^7e^{-\frac{x}{15}}dx=\int_2^{\infty}\frac{1}{7!}y^7e^{-y}dy$

$$=-\left(y^7+7y^6+42y^5+210y^4+840y^3+2520y^2+5040y+5040\right)\frac{e^{-y}}{7!}\Big|_2^{\infty}$$

$$=\frac{37200e^{-2}-0}{7!}\approx 0.9989$$

Question The time between failures of a laser machine is exponentially distributed with a mean of 25,000 hours. What is the expected time until the second failure? What is the probability that the time until the third failure exceeds 50,000 hours?

2.2.4 Beta Distribution

Beta Function

We define a beat function as $\mathrm{B}(\alpha,\beta)$ as $\mathrm{B}(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ with parameters $\alpha>0$ and $\beta>0$

Theorem:

i) $\mathrm{B}(\alpha,\beta)=\mathrm{B}(\beta,\alpha)$ ie by putting $y=1-x$ in the function

ii) $\mathrm{B}(\alpha,\beta)=\int_0^{\infty}\frac{u^{\alpha-1}}{(1+u)^{\beta-1}}du$ ie by putting $u=\frac{x}{1-x}$ or $x=\frac{u}{1+u}$ in the function

iii) $\mathrm{B}(\alpha,\beta)=2\int_0^{\pi/2}\cos^{2\alpha-1}t\sin^{2\beta-1}t\,dt$ ie by putting $x=\cos^2 t$ in the function

iv) $\mathrm{B}(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

v) $\Gamma(\tfrac{1}{2})=\sqrt{\pi}$ use $\mathrm{B}(\tfrac{1}{2},\tfrac{1}{2})=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}=\left[\Gamma(\tfrac{1}{2})\right]^2$ but

$\mathrm{B}(\tfrac{1}{2},\tfrac{1}{2})=2\int_0^{\pi/2}\cos^{2\times\frac{1}{2}-1}t\sin^{2\times\frac{1}{2}-1}t\,dt=2\int_0^{\pi/2}dt=\pi \qquad \Rightarrow \quad \Gamma(\tfrac{1}{2})=\sqrt{\pi}$

Note $\Gamma(\tfrac{1}{2})=\int_0^{\infty}x^{-1/2}e^{-x}dx=2\int_0^{\infty}e^{-u^2}du=\sqrt{\pi}$. Simply $\int_0^{\infty}e^{-x^2}dx=\tfrac{1}{2}\sqrt{\pi}$

Beta Distribution

Definition: A random variable X is said to have a standard beta distribution with parameters α and β if it's probability density function is given by

$$\text{f(x)} = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, 0 \le x \le 1 \text{ and f(x)} = 0 \textit{ elsewhere} \text{ we denote this as } X \sim Beta(\alpha,\beta)$$

Theorem: If X has a standard beta distribution with parameters α and β, then

$$\text{E(X)} = \mu = \frac{\alpha}{\alpha+\beta} \quad \text{and.} \quad \text{Var (X)} = \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

3. MOMENTS AND MOMENT-GENERATING FUNCTIONS

Definition: The k[th] **moment** of a r.v. X taken about zero, or about origin is defined to be $E[X^k]$ and denoted by μ_k' .

Definition: The k[th] **moment** of a r.v. X taken about its mean, or the k[th] **central moment** of X, is defined to be $E(X-\mu)^k$ and denoted by μ_k .

Definition: The **moment-generating function** (mgf), for a r.v. *X* denoted as $M_x(t)$ or simply $\text{M}(t)$ is given by $M(t) = E[e^{tx}]$

We say that an mgf for *X* exists if there is *b* > 0 such that $M(t) < \infty$ for $|t| \le b$.

Notice that $e^{tx} = 1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \frac{(tx)^4}{4!} + ...$ therefore

$$M(t) = E[e^{tx}] = \sum_{all\, x}\left(1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \frac{(tx)^4}{4!} + ...\right)P(X=x)$$

$$= \sum_{all\, x} P(X=x) + t\sum_{all\, x} xP(X=x) + \frac{t^2}{2!}\sum_{all\, x} x^2 P(X=x) + \frac{t^3}{3!}\sum_{all\, x} x^3 P(X=x) + ...$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + ... \text{ie a function of all moments about the origin}$$

Theorem: If $M(t)$ exists, then for any $k \in N$ $\left.\frac{d^k M(t)}{dt^k}\right|_{t-0} = \mu_k' = E(X^k)$

Proof

$$M(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + ...$$

$$M'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \frac{3t^2}{3!}\mu_3' + ... \quad \Rightarrow \quad M'(0) = \mu_1' = E(X)$$

$$M''(t) = \mu_2' + \frac{2t}{2!}\mu_3' + \frac{3t^2}{3!}\mu_4' + ... \quad \Rightarrow \quad M''(0) = \mu_2' = E(X^2)$$

Remark: The mgf of a particular distribution is unique and we can recognize the pdf if we are given the mgf.

Example 1

The mgf of a r.v Y is given by $M(t) = \frac{1}{6}e^t + \frac{1}{3}e^{2t} + \frac{1}{2}e^{3t}$ Find the mean and variance of Y

Solution

$$E(Y) = M'(0) = \left(\tfrac{1}{6}e^t + \tfrac{2}{3}e^{2t} + \tfrac{3}{2}e^{3t}\right)\Big|_{t-0} = \tfrac{1}{6} + \tfrac{2}{3} + \tfrac{3}{2} = \tfrac{7}{3}$$

$E(Y^2) = M''(0) = \left(\frac{1}{6}e^t + \frac{4}{3}e^{2t} + \frac{9}{2}e^{3t}\right)\Big|_{t-0} = \frac{1}{6} + \frac{4}{3} + \frac{9}{2} = 6$

$Var(Y) = E(Y^2) - \mu^2 = 6 - \left(\frac{7}{3}\right)^2 = \frac{5}{9}$

Example 2 Find the mgf of a r.v $X \sim \text{Po}(\lambda)$

Solution

$$M(t) = E\left[e^{tx}\right] = \sum_{x=0}^{\infty}\left[e^{tx}\right]\frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\left(\lambda e^t\right)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda\left(e^t-1\right)}$$

Now $E(X) = M'(0) = \lambda e^t e^{\lambda\left(e^t-1\right)}\Big|_{t-0} = \lambda$

Example 3 Find the mgf of a r.v X whose pmf is given by $f(x) = \begin{cases} \frac{1}{6}\left(\frac{5}{6}\right)^x & \text{for } x = 0,1,2,3..... \\ 0 & elsewhere \end{cases}$

hence obtain the mean and variance of X

Solution

$$.\,M(t) = E\left[e^{tx}\right] = \sum_{x=0}^{\infty} e^{tx}\,\frac{1}{6}\left(\frac{5}{6}\right)^x = \sum_{x=0}^{\infty}\frac{1}{6}\left(\frac{5}{6}e^t\right)^x = \frac{\frac{1}{6}}{1-\frac{5}{6}e^t} = \frac{1}{6-5e^t} = \left(6-5e^t\right)^{-1}$$

$M'(t) = 5e^t\left(6-5e^t\right)^{-2}$ and $M''(t) = 5e^t\left(6-5e^t\right)^{-2} + 50e^{2t}\left(6-5e^t\right)^{-3}$

$\Rightarrow E(X) = 5e^t\left(6-5e^t\right)^{-2}\Big|_{t-0} = 5$ and $E(Y^2) = \left[5e^t\left(6-5e^t\right)^{-2} + 50e^{2t}\left(6-5e^t\right)^{-3}\right]_{t=0} = 55$

$Var(X) = E(X^2) - \mu^2 = 55 - 5^2 = 30$

Exercise

1) The mgf of a r.v Y is given by; a) $M(t) = e^{2t^2+3t}$ b) $M(t) = \exp\left\{\frac{1}{2}\sigma^2 t^2 + t\mu\right\}$ Find the mean and variance of Y
2) A r.v X has a gamma distribution with parameters α and β, Find the mgf of X hence obtain the mean and variance of X

3.1 The Mgf of a Sum of Independent Random Variables

The mgf of the sum of n independent random variable is the product of their individual mgf's
The mean (variance) of the sum of n independent random variable is the sum of their individual means (variances).
The mgf of a r.v X about $X = a$ is given by $M_{x,a}(t) = E\left[e^{t(x-a)}\right] = e^{-at}E\left[e^{tx}\right] = e^{-at}M_x(t)$

4 NORMAL DISTRIBUTION

4.1 Introduction

The normal, or Gaussian, distribution is one of the most important distributions in probability theory. It is widely used in statistical inference. One reason for this is that sums of random variables often approximately follow a normal distribution.

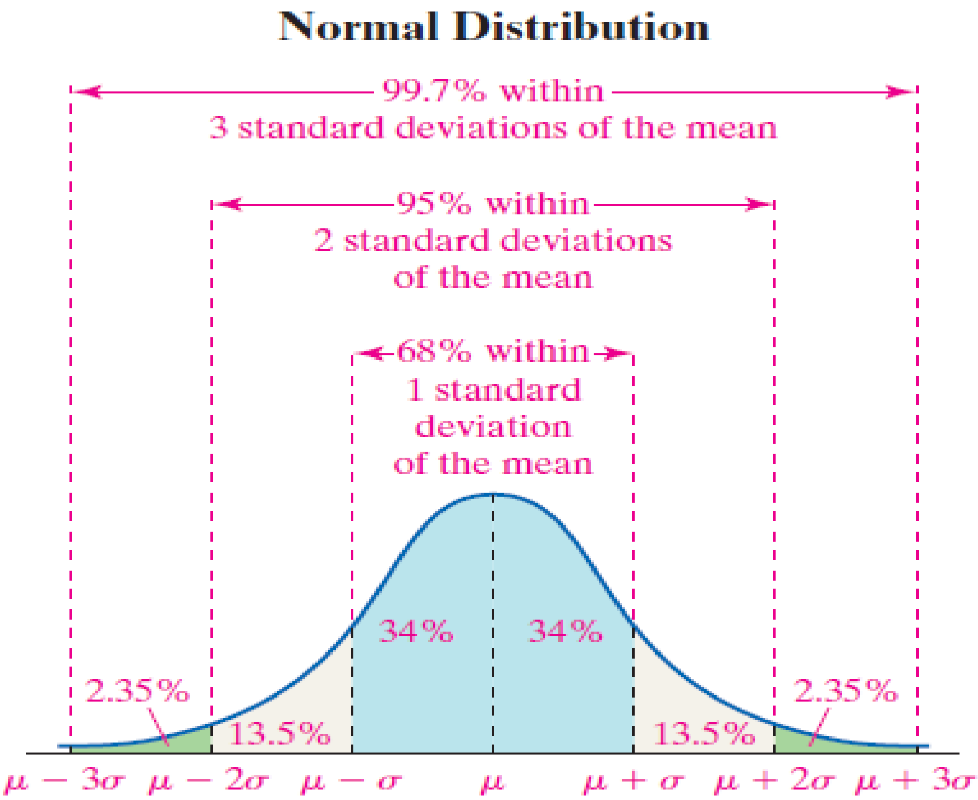
Definition A r.v X has a normal distribution with parameters μ and σ^2, abbreviated $X \sim N\left(\mu, \sigma^2\right)$ if it has probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} \text{ for } -\infty < x < \infty \text{ and } \sigma > 0$$

Where μ is the mean and σ is the standard deviation.

4.1.1 Properties of normal distribution

1) The normal distribution curve is bell-shaped and symmetric, about the mean
2) The curve is asymptotic to the horizontal axis at the extremes.
3) The highest point on the normal curve is at the mean, which is also the median and mode.
4) The mean can be any numerical value: negative, zero, or positive
5) The standard deviation determines the width of the curve: larger values result in wider, flatter curves
6) Probabilities for the normal random variable are given by areas under the curve. The total area under the curve is 1 (0.5 to the left of the mean and 0.5 to the right).
7) It has inflection points at $\mu-\sigma$ and $\mu+\sigma$.
8) Empirical Rule:
   a) 68.26% of values of a normal random variable are within ± 1 standard deviation of its mean. ie $P(\mu-\sigma \leq X \leq \mu+\sigma)=0.6826$
   b) 95.44% of values of a normal random variable are within ± 2 standard deviation of its mean. ie $P(\mu-2\sigma \leq X \leq \mu+2\sigma)=0.9544$
   c) 99.72% of values of a normal random variable are within ± 3 standard deviation of its mean. ie $P(\mu-3\sigma \leq X \leq \mu+3\sigma)=0.9972$



4.2 Standard Normal Probability Distribution

A random variable having a normal distribution with a mean of 0 and a variance of 1 is said to have a **standard normal** probability distribution

Definition The random variable Z is said to have the standard normal distribution if $Z \sim N(0,1)$. Therefore, the density of Z, which is usually denoted $\phi(z)$ is given by;

$$\phi(z)=\frac{1}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}z^2\right\} \text{ for } -\infty<z<\infty$$

The cumulative distribution function of a standard normal random variable is denoted $\Phi(z)$, and is given by

$$\Phi(z) = \int_{-\infty}^{z} \phi(\mathrm{t})dt$$

4.2.1 Computing Normal Probabilities

It is very important to understand how the standardized normal distribution works, so we will spend some time here going over it. There is no simple analytic expression for $\Phi(z)$ in terms of elementary functions. but the values of $\Phi(z)$ has been exhaustively tabulated. This greatly simplifies the task of computing normal probabilities..

Table 1 below reports the cumulative normal probabilities for normally distributed variables in standardized form (i.e. Z-scores). That is, this table reports $\mathrm{P(Z \le z)} = \Phi(z)$). For a given value of Z, the table reports what proportion of the distribution lies below that value. For example, $\mathrm{P(Z \le 0)} = \Phi(0) = 0.5$; half the area of the standardized normal curve lies to the left of $Z = 0$.

Theorem: It may be useful to keep in mind that

i) $\mathrm{P(Z > z)} = 1 - \Phi(z)$ complementary law

ii) $\mathrm{P(Z \le -z)} = \mathrm{P(Z \ge z)} = 1 - \Phi(z)$ ie due to symmetry

$\Rightarrow \quad \Phi(z) + \Phi(-z) = 1$ Since $\mathrm{P(Z \le z)} + \mathrm{P(Z \ge z)} = 1$

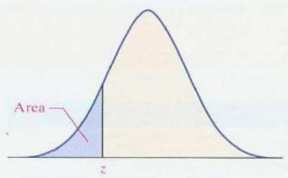
iii) $\mathrm{P(a \le} z \mathrm{\le b)} = \Phi(b) - \Phi(a)$

iv) $\mathrm{P(\text{-}a \le} z \mathrm{\le a)} = 2\Phi(a) - 1$ since

$\mathrm{P(\text{-}a \le} z \mathrm{\le a)} = \Phi(a) - \Phi(-a) = \Phi(a) - [1 - \Phi(a)] = 2\,\Phi(a) - 1$

v) If we now make $\Phi(a)$ the subject, then $\Phi(a) = \frac{1}{2}[1 + \mathrm{P(\text{-}a \le} z \mathrm{\le a)}]$

Table 1



The table gives the area under the standard normal curve to the left of a specified Z-score as shown in the figure.

TABLE II

Standard Normal Distribution

Z-values.

Numbers in the body of the table represent area under the standard normal curve.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.000
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.000
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.000
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.000
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.001
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.001
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.001
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.002
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.003
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.004

Example 1

Given $\mathrm{Z \sim N(0,1)}$, find;

a) $\mathrm{P(Z \le z)}$ if $z = 1.65$, -1.65, 1.0, -1.0

b) $\mathrm{P(Z > z)}$ for $z = 1.02$, -1.65

c) $\mathrm{P(0.365 \le} z \le 1.75)$

d) $\mathrm{P(\text{-}0.696 \le} z \le 1.865)$

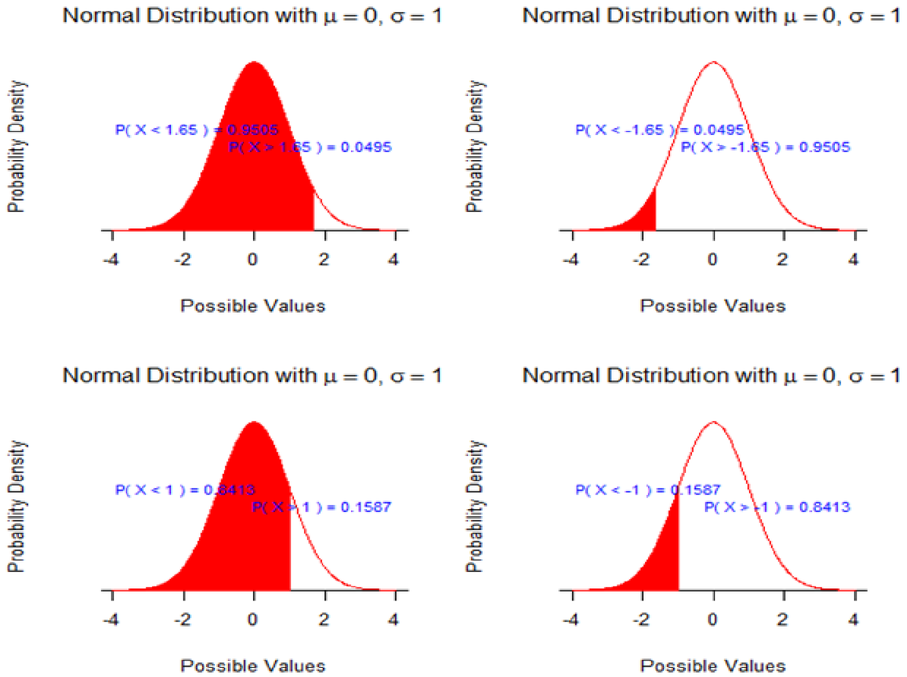
e) $\mathrm{P(\text{-}2.345 \le} z \le -1.65)$

f) $\mathrm{P(}|z| \le 1.43)$

Solution

a) Look up and report the value for $\Phi(z)$ from the standard normal probabilities table

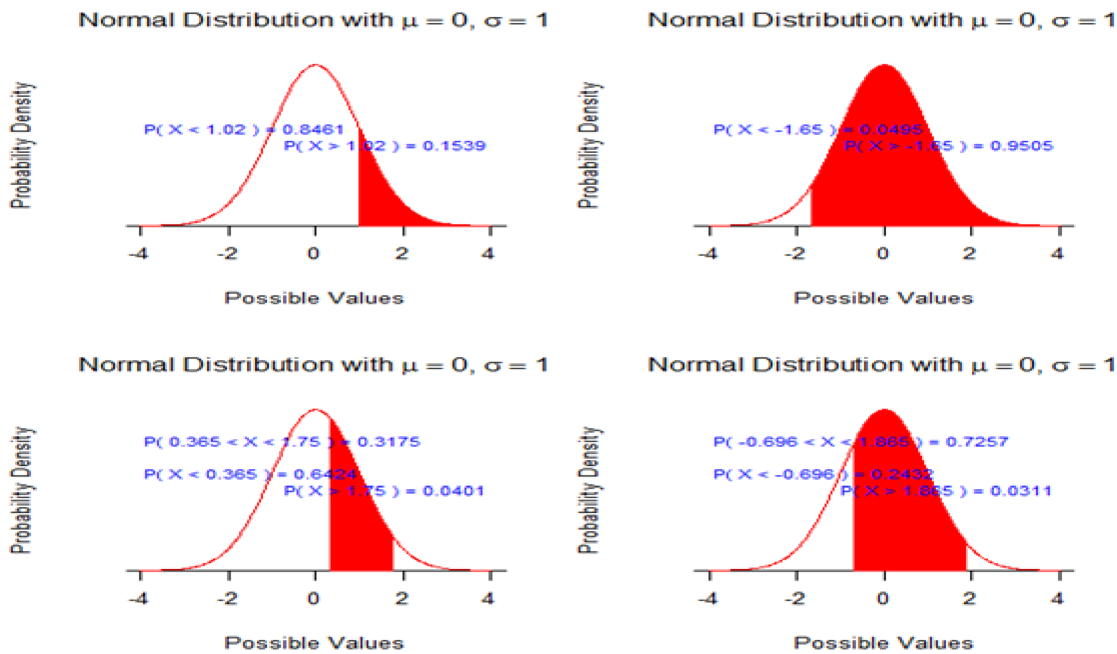
$\mathrm{P(Z \le 1.65)} = \Phi(1.65) = 0.9505 \quad \Phi(-1.65) = 0.0495 \quad \Phi(1.0) = 0.8413 \quad \Phi(-1.0) = 0.1587$



b) $P(Z > z) = \Phi(-z)$ Thus $P(Z > 1.02) = \Phi(-1.02) = 0.1515 \quad P(Z > -1.65) = \Phi(1.65) = 0.9505$

c) $P(0.365 \le z \le 1.75) = \Phi(1.75) - \Phi(0.365) = 0.9599 - 0.6350 = 0.3249$

d) $P(-0.696 \le z \le 1.865) = \Phi(1.865) - \Phi(-0.696) = 0.9689 - 0.2432 = 0.3249 = 0.7257$



e) $P(-2.345 \le z \le -1.65) = \Phi(-1.65) - \Phi(-2.345) = 0.0505 - 0.0095 = 0.0410$

f) $P(|z| \le 1.43) = P(-1.43 \le z \le 1.43) = 2\Phi(1.43) - 1 = 2(0.9236) - 1 = 0.8472$

Example 2 If $Z \sim N(0,1)$, find the value of t for which;

a) $P(Z \le t) == 0.6026, \; 0.9750, \; 0.3446$

b) $P(Z > t) == 0.4026, \; 0.7265, \; 0.5446$

c) $P(-0.28 \le z \le t) = 0.2665$

d) $P(-t \le z \le t) = 0.9972, \; 0.9505, \; 0.9750$

Solution

Here we find the probability value in Table I, and report the corresponding value for Z.

a) $\Phi(t) = 0.6026 \Rightarrow t = 0.26 \quad \Phi(t) = 0.950 \Rightarrow t = 1.96 \quad \Phi(t) = 0.3446 \Rightarrow t = -0.40$

b) $P(Z > t) = 0.4026 \Rightarrow \Phi(t) = 0.5974 \Rightarrow t = 0.25$

$P(Z > t) = 0.7265 \Rightarrow \Phi(t) = 0.2735 \Rightarrow t = -0.60$

$P(Z > t) = 0.5446 \Rightarrow \Phi(t) = 0.4554 \Rightarrow t = -0.11$

c) $P(-0.28 \le z \le t) = \Phi(t) - \Phi(-0.28) = 0.2665 \Rightarrow \Phi(t) = 0.3897 + 0.2665 \Rightarrow t = 0.40$

d) $P(-t \le z \le t) = 2\,\Phi(t) - 1 = 0.9972 \Rightarrow \Phi(t) = 0.9986 \Rightarrow t = 2.99$

$P(-t \le z \le t) = 2\,\Phi(t) - 1 = 0.9505 \Rightarrow \Phi(t) = 0.9753 \Rightarrow t = 1.96$

$P(-t \le z \le t) = 2\,\Phi(t) - 1 = 0.9750 \Rightarrow \Phi(t) = 0.9875 \Rightarrow t = 2.24$

Exercise

1...Given $Z \sim N(0,1)$, find;

a) $P(Z \le z)$ if $z = 1.95, -1.89, 1.074, -1.53$

b) $P(Z > z)$ for $z = 1.72, -1.15$

c) $P(0 \le z \le 1.05)$

d) $P(-1.396 \le z \le 1.125)$

e) $P(-1.96 \le z \le -1.65)$

f) $P(|z| \le 2.33)$

2...If $Z \sim N(0,1)$, find the value of z for which;

a) $P(Z \le a) = = 0.973, \ 0.6693, \ 0.4634$

b) $P(Z > a) == 0.3719, 0.9545, 0.7546$

c) $P(-1.21 \le z \le t) = 0.6965$

d) $P(|z| \le t) = 0.9544, 0.9905, 0.3750$

4.3 The General Normal Density

Consider $Z \sim N(0,1)$ and let $X = \mu + \sigma Z$ for $\sigma > 0$. Then $X \sim N(\mu, \sigma^2)$ But we know that

$f(x) = \frac{1}{\sigma}\phi\left(\frac{X-\mu}{\sigma}\right)$ from which the claim follows. Conversely, if $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. It is also easily shown that the cumulative distribution function satisfies

$$F(x) = \Phi\left(\frac{X-\mu}{\sigma}\right)$$

and so the cumulative probabilities for any normal random variable can be calculated using the tables for the standard normal distribution..

Definition A variable X is said to be standardized if it has been adjusted (or transformed) such that its mean equals 0 and its standard deviation equals 1. Standardization can be accomplished using the formula for a z-score: $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$. The z-score represents the number of standard deviations that a data value is away fromthe m ean.

Let $X \sim N(\mu, \sigma^2)$ then $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$ where $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

Example 1 A r.v $X \sim N(50, 25)$ compute $P(45 \le z \le 60)$

Solution

$\mu = 50$ and $\sigma = 5 \Rightarrow Z = \frac{x-50}{5} \sim N(0,1)$

$P(45 \le X \le 60) = P\left(\frac{45-50}{5} \le Z \le \frac{60-50}{5}\right) = \Phi(2) - \Phi(-1) = 0.9772 - 0.1587 = 0.8185$

Example 2 Suppose $X \sim \mathrm{N}(30,16)$. Find; a) $P(X<40)$ b) $P(X>21)$ c) $P(30<X<35)$

Solution

$X \sim \mathrm{N}(30,16) \Rightarrow Z=\frac{X-30}{4} \sim \mathrm{N}(0,1)$

a) $\mathrm{P}(X \leq 40)=\mathrm{P}\left(Z \leq \frac{40-30}{4}\right)=\Phi(2.5)=0.9938$

b) $P(X>21)=\mathrm{P}\left(Z>\frac{21-30}{4}\right)=\mathrm{P}(Z>-2.25)=\mathrm{P}(Z \leq 2.25)=\Phi(2.25)=0.9878$

c) $P(30<X<35)=P\left(\frac{30-30}{4} \leq Z \leq \frac{35-30}{4}\right)=P(0<Z<1.25)=0.8944-0.5=0.3944$

Example 3

The top 5% of applicants (as measured by GRE scores) will receive scholarships. If $\mathrm{GRE} \sim \mathrm{N}(500,100^2)$, how high does your GRE score have to be to qualify for a scholarship?

Solution

Let $\mathrm{X}=\mathrm{GRE}$. We want to find x such that $\mathrm{P}(\mathrm{X} \geq \mathrm{x})=0.05$ this is too hard to solve as it stands - so instead, compute $Z=\frac{X-500}{100} \sim \mathrm{N}(0,1)$ and find z for the problem,

$$\mathrm{P}(\mathrm{Z} \geq z)=1-\Phi(z)=0.05 \quad \Rightarrow \quad \Phi(z)=0.95 \quad \Rightarrow \quad z=1.645$$

To find the equivalent x, compute $\mathrm{X}=\mu+\sigma \mathrm{Z} \Rightarrow x=500+100(1.645)=66.5$

Thus, your GRE score needs to be 665 or higher to qualify for a scholarship.

Example 4

Family income is believed to be normally distributed with a mean of $25000 and a standard deviation on $10000. If the poverty level is $10,000, what percentage of the population lives in poverty? A new tax law is expected to benefit “middle income” families, those with incomes between $20,000 and $30,000. What percentage of the population will benefit from the law?

Solution

Let X = Family income. We want to find P(X ≤ $10,000)., so

$X \sim \mathrm{N}(25000,10000^2) \Rightarrow Z=\frac{X-25000}{10000} \sim \mathrm{N}(0,1)$

$\mathrm{P}(X \leq 10{,}000)=\mathrm{P}(Z \leq -1.5)=\Phi(-1.5)=0.0668$.

Hence, a slightly below 7% of the population lives in poverty.

$\mathrm{P}(20{,}000 \leq X \leq 30{,}000)=\mathrm{P}(-0.5 \leq Z \leq 0.5)=2\Phi(0.5)-1=2 \times 0.6915-1-0.383$

Thus, about 38% of the taxpayers will benefit from the new law.

Exercise

1) Suppose $X \sim \mathrm{N}(130,25)$. Find; a) $P(X<140)$ b) $P(X>120)$ c) $P(130<X<135)$
2) The random variable X is normally distributed with mean 500 and standard deviation 100. Find; (i) $P(X<400)$, (ii) $P(X>620)$ (iii) the 90th percentile (iv) the lower and upper quartiles. Use graphs with labels to illustrate your answers.
3) A radar unit is used to measure speeds of cars on a motorway. The speeds are normally distributed with a mean of 90 km/hr and a standard deviation of 10 km/hr. What is the probability that a car picked at random is travelling at more than 100 km/hr?
4) For a certain type of computers, the length of time bewteen charges of the battery is normally distributed with a mean of 50 hours and a standard deviation of 15 hours. John owns one of these computers and wants to know the probability that the length of time will be between 50 and 70 hours
5) Entry to a certain University is determined by a national test. The scores on this test are normally distributed with a mean of 500 and a standard deviation of 100. Tom wants to be admitted to this university and he knows that he must score better than at least 70% of

the students who took the test. Tom takes the test and scores 585. Will he be admitted to this university?

6) A large group of students took a test in Physics and the final grades have a mean of 70 and a standard deviation of 10. If we can approximate the distribution of these grades by a normal distribution, what percent of the student; (a) scored higher than 80? (b) should pass the test (grades≥60)? (c) should fail the test (grades<60)?
7) A machine produces bolts which are N(4 0.09) where measurements are in cm. Bolts are measured accurately and any bolt smaller than 3.5 cm or larger than 4.4 cm is rejected. Out of 500 bolts how many would be accepted? Ans 430
8) Suppose IQ ~ N (100, 22.5).a woman wants to form an Egghead society which only admits people with the top 1% IQ score. What should she have to set the cut-off in the test to allow this to happen? Ans 134.9
9) A manufacturer does not know the mean and standard deviation of ball bearing he is producing. However a sieving system rejects all the bearings larger than 2.4 cm and those under 1.8 cm in diameter. Out of 1,000 ball bearings, 8% are rejected as too small and 5.5% as too big. What is the mean and standard deviation of the ball bearings produced? Ans mean=2.08 sigma=0.2

4.4 Normal Approximation to the Binomial Distribution.

4.4.1 Introduction

Suppose a fair coin is tossed 10 times, wharf is the probability of observing: a) exactly 4 heads b) at most 4 heads?

Solution

Let X be the r.v the number of heads observed then $X \sim \text{Bin}(10, 0.5)$

$\Rightarrow \; P(X = x) = {}_{10}C_x \times (0.5)^x (0.5)^{10-x} = {}_{10}C_x \times \left(\tfrac{1}{2}\right)^{10}$ for x = 0,1,2,….,10

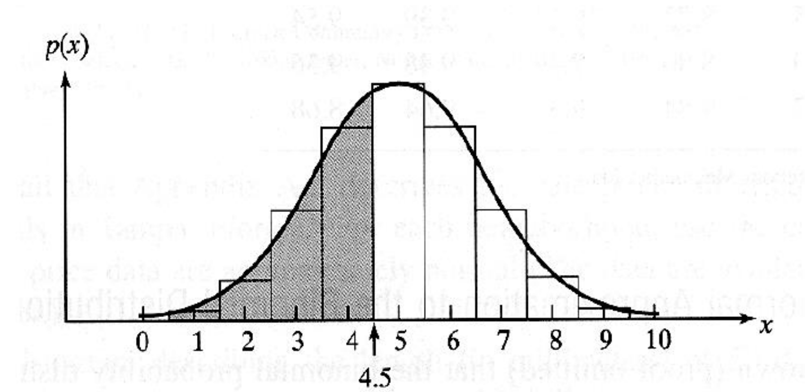
a) $\Rightarrow \; P(X = 4) = {}_{10}C_4 \times \left(\tfrac{1}{2}\right)^{10} = \dfrac{105}{512} \approx 0.2051$

b) $P(X \le 4) = \left[{}_{10}C_0 + {}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + {}_{10}C_4\right](0.5)^{10} = \dfrac{193}{512} \approx 0.3770$

4.4.2 Normal approximation:

Many interesting problems can be addressed via the binomial distribution. However, for large n, it is sometimes difficult to directly compute probabilities for a binomial (*n, p*) random variable, X. **Eg**: Compute $P(X \le 12)$ for 25 tosses of a fair coin. Direct calculations can get cumbersome very quickly. . Fortunately, as n becomes large, the binomial distribution becomes more and more symmetric, and begins to converge to a normal distribution. That is, for a large enough n, a binomial variable X is approximately ~ N(*np, npq*). Hence, the normal distribution can be used to approximate the binomial distribution.

To get a feel for why this might work, let us draw the probability histogram for 10 tosses of a fair coin. The histogram looks bell-shaped, as long as the number of trials is not too small



In general, the distribution of a binomial random variable may be accurately approximated by that of a normal random variable, as long as $np \ge 5$ and $nq \ge 5$, and assuming that a .continuity correction is made to account for the fact that we are using a continuous distribution (the normal) to approximate a discrete one (the binomial).
In approximating the distribution of a binomial random variable X, we will use the normal distribution with mean $\mu = np$ and variance $\sigma^2 = npq$ where $q = 1 - p$. Why are these reasonable choices of μ, σ^2?

4.4.3 Continuity Correction

In the binomial, $P(X \le a) + P(X \ge a+1) = 1$ whenever a is an integer. But if we sum the area under the normal curve corresponding to $P(X \le a) + P(X \ge a+1)$, this area does not sum to 1 because the area from a to (a + 1) is missing.

The usual way to solve this problem is to associate 1/2 of the interval from a to a + 1 with each adjacent integer. The continuous approximation to the probability $P(X \le a)$ would thus be $P(X \le a + \frac{1}{2})$, while the continuous approximation to $P(X \ge a+1)$ would be $P(X \ge a + \frac{1}{2})$. This adjustment is called a continuity correction. More specifically,

$$\underbrace{P(X \le x) = P(X < x+1)}_{\text{Binomialdistribution}} \to \underbrace{P(X \le x + 0.5)}_{\text{Normal approximation}} = \underbrace{P\left(z \le \tfrac{x+0.5-np}{\sqrt{npq}}\right)}_{\text{Standardd Normal approx}}$$

$$\underbrace{P(X \ge x) = P(X > x-1)}_{\text{Binomialdistribution}} \to \underbrace{P(X \ge x - 0.5)}_{\text{Normal approximation}} = \underbrace{P\left(z \le \tfrac{x-0.5-np}{\sqrt{npq}}\right)}_{\text{Standardd Normal approx}}$$

$$\underbrace{P(a \le X \le b) = P(a-1 < X < b+1)}_{\text{Binomialdistribution}} \to = \underbrace{P(a - 0.5 \le X \le b + 0.5)}_{\text{Normal approximation}} = P\left(\tfrac{a-0.5-np}{\sqrt{npq}} \le z \le \tfrac{b+0.5-np}{\sqrt{npq}}\right)$$

$$\underbrace{P(X = x) = P(x-1 < X < x+1)}_{\text{Binomialdistribution}} \to \underbrace{P(x - 0.5 \le X \le x + 0.5)}_{\text{Normal approximation}} = \underbrace{P\left(\tfrac{x-0.5-np}{\sqrt{npq}} \le z \le \tfrac{x+0.5-np}{\sqrt{npq}}\right)}_{\text{Standardd Normal approx}}$$

NB: For the binomial distribution, the values to the right of each = sign are primarily included for illustrative purposes. The equalities which hold in the binomial distribution do not hold in the normal distribution, because there is a gap between consecutive values of a. The normal approximation deals with this by “splitting” the difference.
For example, in the binomial, $P(X \le 6) = P(X < 7)$ since 6 is the next possible value of X that is less than 7. In the normal, we approximate this by finding $P(X \le 6.5)$. And, in the binomial, $P(X \ge 6) = P(X > 5)$, because 6 is the next value of X that is greater than 5. In the normal, we approximate this by finding $P(X \ge 5.5)$

Returning to the case of coin tossing Suppose we wish to find $P(X \le 4)$, the probability that the binomial r.v is less than or equal to 4. In the diagram above, the bars represent the binomial distribution with n = 10, p = 0.5. The superimposed curve is a normal density f(x). The mean of the normal is $\mu = np = 5$ and the standard deviation is $\sigma = \sqrt{10(0.5)(0.5)} \approx 1.58$ Using the normal approximation, we need to calculate the probability that our normal r.v is less than or equals to 4.5. ie

$\underbrace{P(X \le 4)}_{Binomial} = \underbrace{P(X \le 4.5)}_{Normal} = \underbrace{P\left(Z \le \frac{4.5-5}{1.58}\right)}_{std\ normal} = \Phi(-0.3162) = 0.3759$ which is very close to the actual answer of 0.377

Example 1 Suppose 50% of the population approves of the job the governor is doing, and that 20 individuals are drawn at random from the population. Solve the following, using the normal approximation to the binomial. What is the probability that;

a) exactly 7 people will support the governor?
b) at least 7 people will support the governor?
c) more than 11 people will support the governor?
d) 11 or fewer will support the governor?

Solution

Note that $n = 20,\ p = 0.5 \Rightarrow \mu = np = 10$ and $\sigma = \sqrt{npq} = \sqrt{5}$ Since $np \ge 5$ and $nq \ge 5$, it is probably safe to assume that $X \sim N(10, 5)$

a) $\underbrace{P(X = 7)}_{Binomial} = \underbrace{P(6.5 \le X \le 7.5)}_{Normal} = \underbrace{P\left(\frac{6.5-10}{\sqrt{5}} \le Z \le \frac{7.5-10}{\sqrt{5}}\right)}_{std\ normal} = \underbrace{P(-1.565 \le Z \le -1.118)}_{std\ normal}$

$= \Phi(-1.118) - \Phi(-1.565) = 0.1318 - 0.0588 = 0.0730$

b) $\underbrace{P(X \le 7)}_{Binomial} = \underbrace{P(X \le 7.5)}_{Normal} = \underbrace{P(Z \le -1.118)}_{std\ normal} = 0.1318$

c) $\underbrace{P(X > 11)}_{Binomial} = \underbrace{P(X \ge 11.5)}_{Normal} = \underbrace{P(Z \ge 0.6708)}_{std\ normal} = 1 - \Phi(0.6708) = 1 - 0.7488 = 0.2512$

d) $\underbrace{P(X \le 11)}_{Binomial} = \underbrace{P(X \le 11.5)}_{Normal} = \underbrace{P(Z \le 0.6708)}_{std\ normal} = \Phi(0.6708) = 0.7488$

Example 2

In each of 25 races, the Democrats have a 60% chance of winning. What are the odds that the Democrats will win 19 or more races? Use the normal approximation to the binomial

Solution

Note that $n = 25,\ p = 0.6 \Rightarrow \mu = np = 15$ and $\sigma = \sqrt{npq} = \sqrt{6}$ Since $np > 5$ and $nq > 5$, it is probably safe to assume that $X \sim N(15, 6)$. .

Using the normal approximation to the binomial,

$\underbrace{P(X \ge 19)}_{Binomial} = \underbrace{P(X \ge 18.5)}_{Normal} = \underbrace{P(Z \ge 1.4289)}_{std\ normal} = 1 - \Phi(1.4289) = 1 - 0.9235 = 0.0765$

Hence, Democrats have a little less than an 8% chance of winning 19 or more races.

Example 3 Tomorrow morning Iberia flight to Madrid can seat 370 passengers. From past experience, Iberia knows that the probability is 0.90 that a given ticket-holder will show up for the flight. They have sold 400 tickets, deliberately overbooking the flight. How confident can Iberia be that no passenger will need to be .bumped (denied boarding)?

Solution: