5 PROBABILITY

5.1 What is Probability?

Probability theory is the branch of mathematics that studies the possible outcomes of given events together with the outcomes' relative likelihoods and distributions. In common usage, the word “probability" is used to mean the chance that a particular event (or set of events) will occur expressed on a linear scale from 0 (impossibility) to 1 (certainty). Factually, It is the study of random or indeterministic experiments eg tossing a coin or rolling a die. If we roll a die, we are certain it will come down but we are uncertain which face will show up. Ie the face showing up is indeterministic. Probability is a way of summarizing the uncertainty of statements or events. It gives a numerical measure for the degree of certainty (or degree of uncertainty) of the occurrence of an event.

We often use P to represent a probability Eg P(rain) would be the probability that it rains. Inother cases Pr(.) is used instead of just P(.).

Definitions

- Experiment: A process by which an observation or measurement is obtained. Eg tossing a coin or rolling a die.
- Outcome: Possible result of a random experiment. Eg a 6 when a die is rolled once or a head when a coin is tossed.
- Sample space: Also called the probability space and it is a collection or set of all possible outcomes of a random experiment. Sample space is usually denoted by S or Ω or U
- Event: it’s a subset of the sample space. Events are usually denoted by upper case letters.

5.2 Approaches to Probability

There are three ways to define probability, namely classical, empirical and subjective probability.

5.2.1 Classical probability

Classical or theoretical probability is used when each outcome in a sample space is equally likely to occur. The underlying idea behind this view of probability is symmetry. Ie if the sample space contains n outcomes that are fairly likely then P(one outcome)=1/n.
The classical probability for an event A is given by

$$P(A)=\frac{\text{Number of outcomes in A}}{\text{Total number of outcomes in S}}=\frac{n(A)}{n(S)}$$

Eg Roll a die and observe that $P(A)=P(\text{rolling }3)=\frac{1}{6}$.

Example A fair die,is rolled once, write down the sample space S hence find the probability that the score showing up is ; a) a multiple of 3 b) a prime number.

Solution

$S=\{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ Multiples of 3 are 3 and 6 while prime numbers are 2, 3 and 5

Thus $P(\text{Multiple of }3)=\frac{2}{6}=\frac{1}{3}$ and $P(\text{prime number})=\frac{3}{6}=\frac{1}{2}$

5.2.2 Frequentist or Empirical probability

When the outcomes of an experiment are not equally likely, we can conduct experiments to give us some idea of how likely the different outcomes are. For example, suppose we are interested in measuring the probability of producing a defective item in a manufacturing process. The probability could be measured by monitoring the process over a reasonably long period of time and calculating the proportion of defective items.
In a nut shell Empirical (or frequrntist or statistical) probability is based on observed data.
The empirical probability of an event A is the relative frequency of event A, that is

$$P(A) = \frac{\text{Frequency of event A}}{\text{Total number of observations}}$$

Example 1 The following are the counts of fish of each type, that you have caught before.

Fish Types	Blue gill	Red gill	Crappy	Total
No of times caught	13	17	10	40

Estimate the probability that the next fish you catch will be a Blue gill.

$P(\text{Bluegill}) = \frac{13}{40} = 0.32$

Example 2A girl lists the number of male and female children her parent and her parent's brothers and sisters have. Her results were as tabulated below

	Males	Females
Her parents	2	5
Her mother's sisters	6	8
Her mother's brothers	4	8
Her father's sisters	5	8
Her father's brothers	7	7
Totals	24	36

a) Find the probability that, if the girl has children of her own, the 1st born will be a girl.
b) If the girl eventually has 10 children, how many are likely to be males?

Solution

a) Following the family pattern, $P(1^{st}\text{ born will be a girl}) = \frac{36}{60} = 0.6$
b) 40% of the children will be males. Thus 4 out of 10 children are likely to be males.

Remark: The empirical probability definition has a weakness that it depends on the results of a particular experiment. The next time this experiment is repeated, you are likely to get a somewhat different result. However, as an experiment is repeated many times, the empirical probability of an event, based on the combined results, approaches the theoretical probability of the event.

5.2.3 Subjective Probability:

Subjective probabilities result from intuition, educated guesses, and estimates. For example: given a patient's health and extent of injuries a doctor may feel that the patient has a 90% chance of a full recovery. Subjectivity means two people can assign different probabilities to the same event.

Regardless of the way probabilities are defined, they always follow the same laws, which we will explore in the following Section.

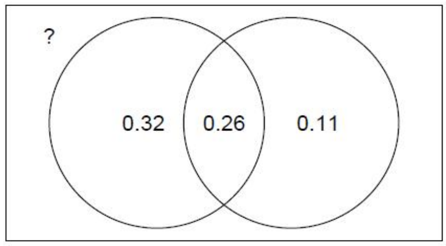
Exercise

1) What is the probability of getting a total of 7 or 11, when two dice are rolled?
2) Two cards are drawn from a pack, without replacement. What is the probability that both are greater than 2 and less than 8?
3) A permutation of the word "white" is chosen at random. Find the probability that it begins with a vowel. Also find the probability that it ends with a consonant.
4) Find the probability that a leap year will have 53 Sundays.
5) Two tetrahedral (4-sided) symmetrical dice are rolled, one after the other. Find the probability that; a) both dice will land on the same number. b) each die will land on a number less than 3 c) the two numbers will differ by at most 1.
   Will the answers change if we rolled the dice simultaneously?

Ways to represent probabilities:

1) *Venn diagram*; We may write the probabilities inside the elementary pieces within a Venn diagram. For example, $P(AB') = 0.32$ and

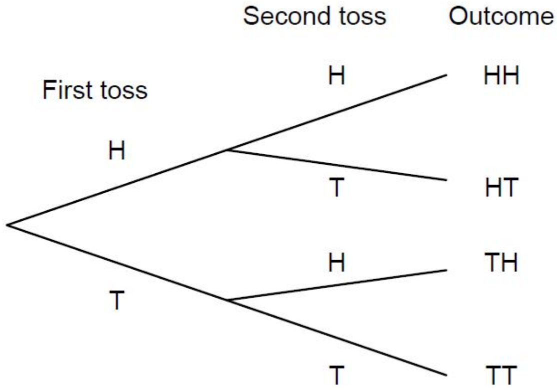
$P(A) = P(AB) + P(AB') = 0.58$ [why?] The relative sizes of the pieces do not have to match the numbers.



2) *Two-way table;* this is a popular way to represent statistical data. The cells of the table correspond to the intersections of row and column events. Note that the contents of the table add up across rows and columns of the table. The bottom-right corner of the table contains P(S) = 1

	B	B'	
A	0.26	0.32	0.58
A'	0.11	?	0.42
	0.37	0.63	1

3) *Tree diagram*; Tree diagrams or probability trees are simper clear ways of representing probabilistic information. A tree diagram may be used to show the sequence of choices that lead to the complete description of outcomes. For example, when tossing two coins, we may represent this as follows
A tree diagram is also often useful for representing conditional probabilities



5.3 Review of set notation

Unions of Events: The event $A \cup B$ read as 'A union B' consists of the outcomes that are contained within at least one of the events A and B. The probability of this event $P(A \cup B)$; is the probability that events A and/or B occurs.

Intersection of events: The event $A \cap B$ (or simply AB) read as 'A intersection B' consists of outcomes that are contained within both events A and B. The probability of this event is the probability that both events A and B occur [but not necessarily at the same time]. Here after we will abbreviate intersection as AB.

Complement: The complement of event A, (denoted A'), is the set of all outcomes in a sample that are not included in the event A.

Set notation

Suppose a set S consists of points labelled 1, 2, 3 and 4. We denote this by $S = \{1, 2, 3, 4\}$. . If $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, then A and B are subsets of S, denoted by $A \subset S$ and $B \subset S$ (B is contained in S). We denote the fact that 2 is anelement of A by $2 \in A$.

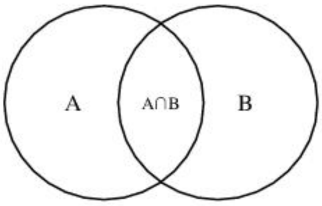
The union of A and B, $A \cup B = \{1, 2, 3, 4\}$. If $C = \{4\}$, then $A \cup C = \{1, 2, 4\}$. The intersection $A \cap B = AB = \{2\}$: The complement ofA,is $A' = \{3, 4\}$.

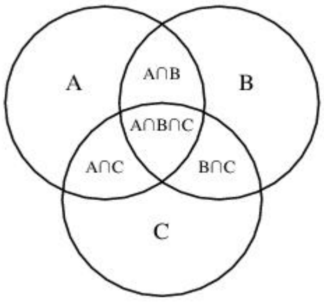
Distributive laws; $A \cap (B \cup C) = AB \cup AC$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan's Law; $(A \cup B)' = A'B'$ and $(AB)' = A' \cup B'$

Venn diagram

Venn diagram is a diagram that shows all possible logical relations between a finite collection of sets.The sets A and B are represented as circles; operations between them (intersections, unions and complements) can also be represented as parts of thediagram.





Exercise

1) Use the Venn diagrams to illustrate Distributive laws and De Morgan's law.
2) Simplify the following (Draw the Venn diagrams to visualize)
   a) $(A')'$ b) $(AB)' \cup A$ c) $AB \cup AB'$ d) $(A \cup B\cup) \cap B$
3) Represent by set notation and exhibit on a Venn diagram the following events
   a) both A and B occur
   b) exactly one of A, B occurs
   c) A and B but not C occur s
   d) at least one of A, B, C occurs
   e) at most one of A, B, C occurs
4) The sample space consists of eight capital letters (outcomes), A, B, C ,...,H. Let V be the event that the letter represents a vowel, and L be the event that theletter is made of straight lines. Describe the outcomes that comprise; a) VL b) $V \cup L'$ c) $V'L'$
5) Out of all items sent for refurbishing, 40% had mechanical defects, 50% hadelectrical defects, and 25% had both.Denoting A = fan item has a mechanical defect andB = fan item has an electrical defect, fill the probabilities into the Venndiagram and determine the quantities listed below. a) $P(A)$ b) $P(AB)$ c) $P(A'B)$ d) $P(A'B')$ e) $P(A \cup B)$ f) $P(A' \cup B')$ g) $P[(A \cup B)']$
6) A sample of mutual funds was classified according to whether a fund was up or down last year (A and A') and whether it was investing in internationalstocks (B and B'). The probabilities of these events and their intersectionsare represented in the two-way table below. Fill in all the question marks hence find the probability of $A \cup B$

	B	B'	
A	0.33	?	?
A'	?	?	0.52
	0.64	?	1

5.4Rules of Probability

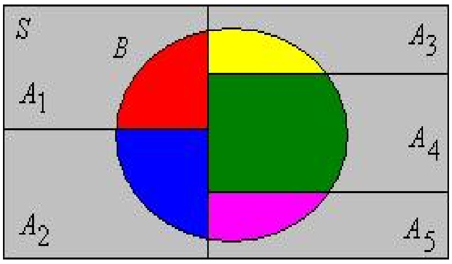
1) For any event Ain a sample space S, $0 \leq P(A) \leq 1$. ConsequentlyIf A is empty (has no elements), then $P(A) = P(\phi) = 0$ and if A =S then $P(A) = P(S) = 1$
2) If A is an event in the sample space S, then A'(read as 'A complement') is an event in S but outside A. $P(A) + P(A') = 1 \Rightarrow P(A') = 1 - P(A)$
3) If the sample space S contains n disjoint events $E_1, E_2, \ldots, E_n$, then

$$P(E_1) + P(E_2) +, \ldots P(E_n) = \sum_{i=1}^{n} P(E_i) = 1$$

4) Let A and B be two events such that $A \subseteq B$, then $P(A) \leq P(B)$
5) For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(AB)$ where $P(AB) = P(A \cap B)$. Extension of this rule leads to the **Inclusion–Exclusion Principle.** This principle is a way to extend the general addition rule to 3 or more events. Here we will limit it to 3 events.
   $P(A \cup B \cup C) = P(A) + P(B) - P(AB) - P(AC) - P(BC) + P(ABC)$

6) **Law of Partitions:** The law of partitions is a way to calculate the probability of an event. Let $A_1, A_2, \ldots, A_k$ form a partition of the sample space S.then, for any events B

$$P(B) = P(A_1B) + P(A_2B) + ... + P(A_kB) = \sum_{i=1}^{k}$$



Example 1 The Probability that John passes a Maths exam is 4/5 and that he passes a Chemistry exam is 5/6. If the probability that he passes both exams is 3/4, find the probability that he will pass at least one exam.

Solution

Let M be the event thet John passes Math exam, and C be the event thet John passes Chemistryexam.

$$P(\text{John passes at least one exam}) = P(M \cup C) = P(M) + P(C) - P(MC) = \tfrac{4}{5} + \tfrac{5}{6} - \tfrac{3}{4} = \tfrac{53}{60}$$

Example 2A fair 6 sided die is rolled twice and the sum of the scores showing up noted. Define A, B and C to be the event that the sum of the scores is greater than 7, a multiple of 3 and a prime number respectively. Show that $P(A \cup B) = P(A) + P(B) - P(AB)$ and also find $P(A \cup C)$, $P(BC)$ and $P(BC')$

Solution

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(A) = \tfrac{15}{36} = \tfrac{5}{12} = P(C)$ and $P(B) = \tfrac{12}{36} = \tfrac{1}{3}$

$A \cap B$ means the set of all multiples of 3 that are greater than 7. Clearly $P(A \cap B) = \tfrac{5}{36}$

$A \cup B$ means the set of all values that are multiples of 3 and/or greater than 7. Clearly $P(A \cup B) = \tfrac{22}{36} = \tfrac{11}{18} = P(A) + P(B) - P(AB)$

$A \cap C$ means the set of all multiples of 3 that are prime number. Clearly $P(A \cap C) = \tfrac{2}{36} = \tfrac{1}{18}$

$$P(A \cup C) = P(A) + P(C) - P(AC) = \tfrac{5}{12} + \tfrac{5}{12} - \tfrac{1}{18} = \tfrac{7}{9}$$

$B \cap C$ means the set of all greater than 7 that are prime numbers. Clearly

$P(BC) = \tfrac{2}{36} = \tfrac{1}{18}$ $B \cap C'$ means the set of all greater than 7 that are not prime numbers. Clearly $P(BC') = \tfrac{11}{36}$

Exercise

1) Which of the following is a probability function defined on $S = \{E_1, E_2, E_3\}$
   a) $P(E_1) = \tfrac{1}{4}, P(E_2) = \tfrac{1}{3}$ and $P(E_3) = \tfrac{1}{2}$ b) $P(E_1) = \tfrac{1}{3}, P(E_2) = \tfrac{1}{6}$ and $P(E_3) = \tfrac{1}{2}$
   c) $P(E_1) = \tfrac{2}{3}, P(E_2) = -\tfrac{1}{3}$ and $P(E_3) = \tfrac{2}{3}$ d) $P(E_1) = 0, P(E_2) = \tfrac{1}{3}$ and $P(E_3) = \tfrac{2}{3}$
2) As a foreign language, 40% of the students took Spanish and 30% took French, while 60% took at least one of these languages. What percent of students took both Spanish and French?
3) In a class of 100 students, 30 are in mathematics. Moreover, of the 40 females in the class, 10 are in Mathematics. If a student is selected at random from the class, what is the probability that the student will be a male or be in mathematics?

4) The probability that a car stopped at a road brook will have faulty breaks is 0.23, the probability that it will have badly worn out tyres is 0.24 and the probability that it will have faulty brakes and/or badly worn out tyres is 0.38. Find the probability that a car which has just been stopped will have both faulty brakes and badly worn out tyres.
5) Given two events A and B in the sane sample space such that $P(A) = 0.59$, $P(B) = 0.3$ and $P(AB) = 0.21$. Find; a) $P(A \cup B)$ b) $P(A' B)$ c) $P(AB')$ d) $P(A \cup B')$
6) Let A and B be two events in the sane sample space such that $P(A \cup B) = \frac{3}{4}$, $P(B') = \frac{2}{3}$ and $P(AB) = \frac{1}{4}$. Find $P(B)$, $P(A)$ and $P(AB')$
7) Suppose that $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$ Find; a) $P(A \cup B)$ b) $P(A' B')$ c) $P[A' \cap (A \cup B)]$ d) $P[A \cup (A' B)]$
8) A die is loaded such that even numbers are twice as likely as odd numbers. Find the probability that for a single toss of this die the spot showing up is greater than 3
9) A point is selected at random inside an equilateral triangle of sides 3 units. Find the probability that its distance to any corner is greater than 1 unit.

Definition: (Odds of an event) It's the ratio of the probability of an eventoccurring to that of the event not happening. If A is an event then the odds of A is given by $\frac{P(A)}{P(A')} = \frac{P(A)}{1-P(A)}$

ExampleFind $P(A)$ and $P(A')$ if the odds of event A is $\frac{5}{4}$

Solution

$\frac{P(A)}{1-P(A)} = \frac{5}{4} \Rightarrow 4(1 - P(A)) = 5P(A) \Rightarrow 4 = 9P(A) \Rightarrow P(A) = \frac{4}{9}$ and $P(A') = \frac{5}{9}$

Question: Find $P(E)$ and $P(E')$ if the odds of event E is (i) $\frac{3}{4}$ (ii) $\frac{a}{b}$

5.5 Relationship between Events

i) Mutually exclusive events: Two events A and B are said to be mutually exclusive if they cannot occur simultaneously. That is if the occurrence of A totally excludes the occurrence of B. Effectively events A and B are said to be mutually exclusive if they disjoint. ie $A \cap B = \phi \Rightarrow P(AB) = 0$
ii) Exhaustive events: Disjoint events whose union equals the sample space.
iii) Independent events: Two events A and B are said to be independent if the occurrence of A does not affect the occurrence of B. If events A and B are independent then $P(AB) = P(A) \times P(B)$

Remark: Three events A,B and C are said to be jointly independent if and only if

- $P(AB) = P(A) \times P(B)$, $P(AC) = P(A) \times P(C)$ and $P(BC) = P(B) \times P(C)$ (iethey are pairwise independent) and
- if $P(ABC) = P(A) \times P(B) \times P(C)$

Note it does not necessarily mean that ifevents A,B and C are pairwise independent then they are jointly independent

Example 1 Roll a fair die twice and define A to be the event that the sum of the scores showing up is greater than 7, B be the event that the sum of the scores showing up is a multiple of 3 and C be the event that the sum of the scores showing up is a prime number. Which of the events A,B and C are independent? Are the 3 events jointly independent?

Solution

From the above example, $P(A) = P(C) = \frac{5}{12}$, $P(B) = \frac{1}{3}$, $P(AB) = \frac{5}{36}$ and $P(AC) = P(BC) = \frac{1}{18}$

Since $P(AB) \neq 0$ events A and B are not mutually exclusive. Similarly events A & C and B and C are not mutually exclusive

$P(A) \times P(B) = \frac{5}{12} \times \frac{1}{3} = \frac{5}{36} = P(AB) \Rightarrow$ A and B are independent events.

$P(A) \times P(C) = \frac{5}{12} \times \frac{5}{12} = \frac{25}{144} \neq P(AC) \Rightarrow$ A and C are dependent events.

$P(B) \times P(C) = \frac{1}{3} \times \frac{5}{12} = \frac{5}{36} \neq P(BC) \Rightarrow$ B and C are dependent events.

The 3 events are not jointly independent since pairwise independence is not satisfied.

Example 2 Three different machines in a factory have the following probabilities of breaking down during a shift.

Machine	A	B	C
probability	$\frac{4}{15}$	$\frac{3}{10}$	$\frac{2}{11}$

Find the probability that in a particular shift,;
a) All the machines will break down
b) None of the machines will break down.

Solution

Since the events of breaking down of machines are independent, probability that all the machines will break down is given by $P(ABC) = \frac{4}{15} \times \frac{3}{10} \times \frac{2}{11} = \frac{4}{275}$

The probability that none of the machines will break down is $P(ABC) = \frac{11}{15} \times \frac{7}{10} \times \frac{9}{11} = \frac{21}{50}$

Exercises

1) In a game of archery the probability that A hits the target is $\frac{1}{3}$ and the probability that B hits the target is $\frac{2}{5}$. What is the probability that the target will be hit?
2) Toss a fair coin 3 times and let A be the event that two or more heads appears, B be the event that all outcomes are the same and C be the event that at most two tails appears. Which of the events A,B and C are independent? Are the 3 events jointly independent?
3) A fair coin and a fair die are rolled together once. Let A be the event that a head and an even number appears, B be the event that a prime number appears and C be the event that a tail and an odd number appears.
   a) Express explicitly the event that i) A and B occurs ii) Only B occurs iii) B and C occur
   b) Which of the events A, B and C are independent and which ones are mutually exclusive?
4) The allocation of Mary's portfolio consists of 25% in bonds and 75% in stocks. Suppose in a one-year period, the probabilities for Mary to make profits in bonds and stocks are 0.9 and 0.4 respectively.
   a) Find the probability that Mary's portfolio turns out to be profitable in one year.
   b) Given that Mary has a loss in her portfolio in one year, find the relative proportion of the loss in stocks.
5) A die is loaded so that the probability of a face showing up is proportional to the face number. Write down the probability of each sample point. If A is the event that an even number appears, B is the event that a prime number appears and C is the event that an odd number appears.
   a) Find the probability that: i) A and/or B occurs ii) A but not B occurs iii) B and C occurs d) A and/or C occurs
   b) Which of the events A, B and C are independent and which ones are mutually exclusive?

Theorem 1: If events A and B are independent, then A and B' are also independent
Proof

Decomposing A into two disjointevents AB and AB'. We can write
$P(A) = P(AB) + P(AB') \Rightarrow P(AB') = P(A) - P(AB) = P(A) - P(A) \times P(B)$ since events A and B are independent. Thus $P(AB') = P(A)[1 - P(B)] = P(A) \times P(B') \Rightarrow$ A and B' are independent

Theorem 2: If events A and B are independent, then A' and B' are also independent
Proof
Decomposing B' into two disjoint events AB' and $A'B'$. We can write
$P(B') = P(AB') + P(AB') \Rightarrow P(A'B') = P(B') - P(AB') = P(B') - P(A) \times P(B')$ since events A and B' are independent. (From theorem 1 above) Thus
$P(A'B') = [1 - P(A)]P(B') = P(A') \times P(B') \Rightarrow$ A' and B' are also independent

5.6Counting Rules useful in Probability

In some experiments it is helpful to list the elements of the sample space systematically by means of a tree diagram,. In many cases, we shall be able to solve a probability problem by counting the number of points in the sample space without actually listing each element.
Theorem (Multiplication principle)
If one operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 n_2$ ways.
Eg How large is the sample space when a pair of dice is thrown?
Solution; The first die can be thrown in $n_1 = 6$ ways and the second in $n_2 = 6$ Ways. Therefore, the pair of dice can land in $n_1 n_2 = 36$ possibleways.
The above theorem can naturally be extended to more than two operations: ifwe have $n_1, n_2, \ldots, n_k$ consequent choices, then the total number of ways is $n_1 \times n_2 \times \ldots \times n_k$

Permutations
Permutations refer to an arrangement of objects when the order matters (for example, letters in a word).The number of permutations of n distinct objects taken r at a time is ${}_nP_r = \frac{n!}{(n-r)!}$

ExampleFrom among ten employees, three are to be selected to travel to three out-of-town plants A, B, and C, one to each plant. Since the plants are located in different cities, the order in which the employees are assigned to the plants is an important consideration. In how many ways can the assignments be made?
Solution;
Because order is important, the number of possible distinct assignments is ${}_{10}P_3 = 720$
In other words, there are ten choices for plant A, but then only nine for plant B, and eight for plant C. This gives a total of 10(9)(8) ways of assigning employees to the plants.

Combinations
The term combination refers to the arrangement of objects when order does not matter. For example, choosing 4 books to buy at the store in any order will leave you with the same set of books. The number of distinct subsets or combinations of size r that can be selected from n distinct objects, (r _ n), is given by ${}_nC_r = \frac{n!}{r!(n-r)!}$

Example 1 In the previous example, suppose that three employees are to be selectedfrom among the ten available to go to the same plant. In how many wayscan this selection be made?
Solution

Here, order is not important; we want to know how many subsetsof size r = 3 can be selected from n = 10 people. The result is ${}_{10}C_3 = 120$

Example 2 In a poker consisting of 5 cards, find the probability of holding 2 acesand 3 jacks.
Solution
The number of ways of being dealt 2 aces from 4 is ${}_4C_2 = 6$ andthe number of ways of being dealt 3 jacks from 4 is ${}_4C_3 = 4$

The total number of 5-card poker hands, all of which are equally likely is ${}_{52}C_5 = 2{,}598{,}960$
Hence, the probability of getting 2 aces and 3 jacks in a 5-card poker hand is

$$P(C) == \frac{6 \times 4}{2,\ 598,\ 960} \approx 0.0000092344 \quad 6$$

Example3A university warehouse has received a shipment of 25 printers, of which 10 are laser printers and 15 are inkjet models. If 6 of these 25 are selected at random to be checked by a particular technician, what is the probability that; a) exactly 3 of these selected are laser printers? b) at least 3 inkjet printers?
Solution
First choose 3 of the 15 inkjet and then 3 of the 10 laser printers.
There are ${}_{15}C_3$ and ${}_{10}C_3$ ways to do it, and therefore

$$P(\text{exactly 3 of the 6}) = \frac{{}_{15}C_3 \times {}_{10}C_3}{{}_{25}C_6} = 0.3083$$

$$P(\text{at least 3}) = \frac{{}_{15}C_3 \times {}_{10}C_3}{{}_{25}C_6} + \frac{{}_{15}C_4 \times {}_{10}C_2}{{}_{25}C_6} + \frac{{}_{15}C_5 \times {}_{10}C_1}{{}_{25}C_6} + \frac{{}_{15}C_6 \times {}_{10}C_0}{{}_{25}C_6} = 0.8530$$

Exercises
1) An incoming lot of silicon wafers is to be inspected for defectives by an engineer in a microchip manufacturing plant. Suppose that, in a tray containing 20 wafers, 4 are defective. Two wafers are to be selected randomly for inspection. Find the probability that neither is defective.
2) A person draws 5 cards from a shuffled pack of cards. Find the probability that the person has at least; a) 3 aces. b) 4 cards of the same suit.
3) A California licence plate consists of a sequence of seven symbols: number, letter, letter, letter, number, number, number, where a letter is any one of 26 letters and a number is one among *0,1,…9*. Assume that all licence plates are equally likely. What is the probability that; a) all symbols are different? b) all symbols are different and the first number is the largest among the numbers?
4) A bag contains 80 balls numbered 1.... 80. Before the game starts, you choose 10 different numbers from amongst 1.... 80 and write them on a piece of paper. Then 20 balls are selected (without replacement) out of the bag at random. What is the probability that;
   a) all your numbers are selected?
   b) none of your numbers is selected?
   c) exactly 4 of your numbers are selected?
5) A full deck of 52 cards contains 13 hearts. Pick 8 cards from the deck at random (a) without replacement and (b) with replacement. In each case compute the probability that you get no hearts.
6) Three people enter the elevator on the basement level. The building has 7 floors. Find the probability that all three get off at different floors.

7) In a group of 7 people, each person shakes hands with every other person. How many handshakes did occur?
8) A marketing director considers that there's “overwhelming agreement" in a 5-member focus group when either 4 or 5 people like or dislike the product. If, in fact, the product's popularity is 50% (so that all outcomes are equallylikely), what is the probability that the focus group will be in “overwhelmingagreement" about it? Is the marketing director making a judgement error in declaring such agreement “overwhelming"?
9) A die is tossed 5 times. Find the probability that we will have 4 of a kind.
10) A tennis tournament has $2n$ participants, n Swedes and n Norwegians. First, n people are chosen at random from the $2n$ (with no regard to nationality) and then paired randomly with the other n people. Each pair proceeds to play one match. An outcome is a *set* of n (ordered) pairs, giving the winner and the loser in each of the n matches. (a) Determine the number of outcomes. (b) What do you need to assume to conclude that all outcomes are equally likely? (c) Under this assumption, compute the probability that all Swedes are the winners.
11) A group of 18 Scandinavians consists of 5 Norwegians, 6 Swedes, and 7 Finns. They are seated at random around a table. Compute the following probabilities: (a) that all the Norwegians sit together, (b) that all the Norwegians and all the Swedes sit together, and (c) that all theNorwegians, all the Swedes, and all the Finns sit together.
12) In a lottery, 6 numbers are drawn out of 45. You hit a jackpot if you guessall 6 numbers correctly, and get \$400 if you guess 5 numbers out of 6. Whatare the probabilities of each of those events?
13) There are 21 Bachelor of Science programs at New Mexico Tech. Given 21areas from which to choose, in how many ways can a student select:
    a) A major area and a minor area?
    b) A major area and first and second minor?
14) From a box containing 5 chocolates and 4 hard candies, a child takes ahandful of 4 (at random). What is the probability that exactly 3 of the 4 arechocolates?
15) If a group consist of 8 men and 6 women, in how many ways can a committeeof 5 be selected if:
    a) The committee is to consist of 3 men and 3 women.
    b) There are no restrictions on the number of men and women on thecommittee.
    c) There must at least one man.
    d) There must be at least one of each sex.
16) Suppose we have a lot of 40 transistors of which 8 are defective. If we samplewithout replacement, what is the probability that we get 4 good transistorsin the first 5 draws?
17) A housewife is asked to rank four brands A, B, C, and D of household cleaneraccording to her preference, number one being the one she prefers most, etc.she really has no preference among the four brands. Hence, any ordering isequally likely to occur.
    a) Find the probability that brand A is ranked number one.
    b) Find the probability that brand C is number one D is number 2 in therankings.
    c) Find the probability that brand A is ranked number one or number 2.
18) How many ways can one arrange the letters of the word ADVANTAGE so
19) that the three As are adjacent to each other?
20) Eight tires of different brands are ranked 1 to 8 (best to worst) accordingto mileage performance. If four of these tires are chosen at random by acustomer, find the probability that the best tire among the four selected bythe customer is actually ranked third among the original eight.

5.7Conditional Probability and Independence

Humans often have to act based on incomplete information. If your bosshas looked at you gloomily, you might conclude that something's wrong withyour job performance. However, if you know that she just suffered some lossesin the stock market, this extra information may change your assessment ofthe situation. Conditional probability is a tool for dealing with additionalinformation like this.
Conditional probability is the probability of an event occurring given theknowledge that another event has occurred. The conditional probability ofevent A occurring, given that event B has occurred is denoted by P(A/B)and is read "probability of A given B" and is given by

$$P(A/B)=\frac{P(AB)}{P(B)} \text{ provided } P(B)>0 \text{ Similarly}$$

$$P(B/A)=\frac{P(AB)}{P(A)} \text{ provided } P(A)>0 \Rightarrow P(AB) = P(A/B) \times P(B) = P(B/A) \times P(A)$$

Remark: Another way to express independence is to say that the knowledge of B occurring does not change our assessment of P(A). This means that if A and B are independent then $P(A/B) = P(A)$ and $P(B/A) = P(B)$

ExampleIn a large metropolitan area, the probability of a family owning a colour T.V ,a computer or both 0.86, 0.35 and 0.29 respectively. What is the probability that a family chosen at random during a survey will own a colour T.V and/or a computer?Given that the family chosen at random during a survey owns a colour T.V, what is the probability that it will own a computer?
Solution
Let T and C be the event of owning a colour T.V and a computer respectively. Then
$P(T \cup C) = P(T) + P(C) - P(TC) = 0.86 + 0.35 - 0.29 = 0.92$

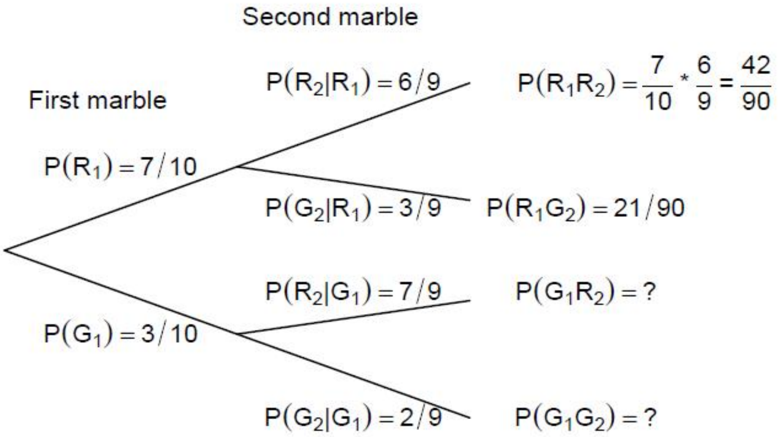
$$P(C/T) - \frac{P(TC)}{P(T)} = \frac{0.29}{0.86} \approx 0.337209$$

Reduced sample space approach
In case when all the outcomes are equally likely, it is sometimes easier to find conditional probabilities directly, without having to apply the above equation. If we already know that B has happened, we need only to consider outcomes in B, thus reducing our sample space to B.
Then, $P(A/B) = \frac{\text{Number of outcomes in AB}}{\text{Number of outcomes in B}}$
For example, $P(\text{adieis3/adieisodd}) = \frac{1}{3}$ and $P(\text{a die is 4 / a die is odd}) = 0$

Example
Let A = {a family has two boys} and B = {a family of two has at least one boy} Find $P(A/B)$
Solution
The event B contains the following outcomes: $B = \{(B, B), (B, G), (G; B)\}$ and. Only one of these is in A. Thus, $P(A/B) = \frac{1}{3}$. However, if I know that the family has two children, and I see one of the children and it's a boy, then the probability suddenly changes to 1/2. There is a subtle difference in the language and this changes the conditional probability

5.7.1 Tree Diagrams in conditional probability
Suppose we are drawing marbles from a bag that initially contains 7 red and 3 green marbles. The drawing is without replacement that is after we draw the first marble, we do not put it back. Let's denote the events

$R_1 = \{\text{the first marble is red}\}$ $R_2 = \{\text{the second marble is red}\}$ $G_1 = \{\text{tthe first marble is green}\}$ and so on. Let's fill out the tree representing the consecutive choices.



The conditional probability $P(R_2/R_1)$ can be obtained directly from reasoning that after we took the first red marble, there remain 6 red and 3 green marbles. On the other hand, we could use the formula to get $P(R_2/R_1) = \frac{P(R_1R_2)}{P(R_1)} = \frac{42/90}{7/10} = \frac{2}{3}$ where the probability $P(R_2 R_1)$ {same as $P(R_1R_2)$} can be obtained fromcounting the outcomes $P(R_1R_2) = \frac{{}_7C_2}{{}_{10}C_2} = \frac{7}{15}$

Question: Find $P(R_2)$ and $P(R_1/R_2)$.

Example 2 Suppose that of all individuals buying a certain digital camera, 60% include an optional memory card in their purchase, 40% include a set of batteries, and 30% include both a card and batteries. Consider randomly selecting a buyer and let $A = \{\text{memory card purchased}\}$ and $B = \{\text{battery purchased}\}$. Then find P(A/B) and P(B/A).

Solution

From given information, we have $P(A) = 0.60$, $P(B) = 0.40$

and $P(\text{both purchased}) = P(AB) = 0.30$

Given that the selected individual purchased an extra battery, the probability that an optional card was also purchased is $P(A/B) = \frac{P(AB)}{P(B)} = \frac{0.30}{0.40} = 0.75$

That is, of all those purchasing an extra battery, 75% purchased an optional memory card.

Similarly $P(\text{batteryj memorycard}) = P(B/A) = \frac{P(AB)}{P(A)} = \frac{0.30}{0.60} = 0.5$

Notice that $P(A/B) \neq P(A)$ and $P(B/A) \neq P(B)$, that is, the events A and B are dependent.

Remark: The tree diagram may become tedious especially when the tree grows beyond 4 stages. In such a case we can make use of binomial formula which is applicable when:

i) The experiment's outcome can be classified into 2 categories success and failure with probabilities p and 1-p respectively
ii) The experiment is to be repeated n independent times
iii) Our interest is the number of successes

The probability of observing x successes out of n trials is given by:-

$$P(x) = {}_nC_x \times p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, 2, \ldots, n$$

ExampleA fair coin is tossed 10 times, what is the probability of observing exactly 8 heads?

Solution

$n = 10 \quad p = 0.5$ 5 and x = 8 successes Therefore

$$P(X=8) = {}_{10}C_8 \times 0.5^8 (1-0.5)^{10-8} = {}_{10}C_8 \times 0.5^{10} \approx 0.044$$

Exercises

1) A pair of fair dice is rolled once. If the sum of the scores showing up is 6 , find the probability that one of the dice shows a 2.
2) A consumer research organisation has studied the services and warranty provided by 50 new car dealers in a certain city. It's findings are as follows

In Business for	**Good services and a warranty**	**Poor services and a warranty**
At least 10 yeras	16	4
Less than 10 years	10	20

If a person randomly selects one of these new car dealers ;
   a) What is the probability that he gets one who provides good services and a warranty
   b) Who has been in business for at least 10 years, what is the probability that hehe provides good services and a warranty
   c) What is the probability that one of these new car dealers who has been in business for less than 10 years will provide good services and a warranty?
3) Three machines A, B and C produces 50%, 30% and 20% respectively of the total number of items in a factory. The percentage of defective outputs of these machines are 3%, 4% and 5% respectively. If an item is selected at random:-
   a) Find the probability that it is defective
   b) And found to be defective, what is the probability that it was produced by machine A?
4) A year has 53 Sundays. What is the conditional probability that it is a leap year?
5) The probability that a majority of the stockholders of a company will attend a special meeting is 0.5. If the majority attends, then the probability that an important merger will be approved is 0.9. What is the probability that a majority will attend and the merger will be approved?
6) Let events A, B have positive probabilities. Show that, if P(A/B) = P(A) then also P(B/A) = P(B).
7) The cards numbered 1 through 10 are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, what is the probability that it is ten?
8) In the roll of a fair die, consider the events A = {2, 4, 6} = "even numbers" and B = {4, 5, 6} ="high scores". Find the probability that die showing an even number given that it is a high score.
9) There are two urns. In the first urn there are 3 white and 2 black balls and in the second urn there 1 white and 4 black balls. From a randomly chosen urn, one ball is drawn. What is the probability that the ball is white?
10) The level of college attainment of US population by racial and ethnic group in 1998 is given in the following table

Racial or Ethnic Group	No of Adults (Millions)	%age with Associate's Degree	%age With Bachelor's Degree	%age with Graduate or Professional Degree
Natives	1.1	6.4	6.1	3.3
Blacks	16.8	5.3	7.5	3.8
Asians	4.3	7.7	22.7	13.9
Hispanics	11.2	4.8	5.9	3.3
Whites	132.0	6.3	13.9	7.7

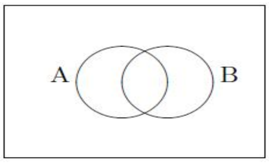
The percentages given in the right three columns are conditional percentages.

a) How many Asians have had a graduate or professional degree in 1998?
b) What percent of all adult Americans has had a Bachelor's degree?
c) Given that the person had an Associate's degree, what is the probability that the person was Hispanic?

11) The dealer's lot contains 40 cars arranged in 5 rows and 8 columns. We pick one car at random. Are the events A = {the car comes from an odd-numbered row} and B = {the car comes from one of the last 4 columns} independent? Prove your point of view.

12) You have sent applications to two colleges. If you are considering your chances to be accepted to either college as 60%, and believe the results are statistically independent, what is the probability that you'll be accepted to at least one?How will your answer change if you applied to 5 colleges?

13) In a high school class, 50% of the students took Spanish, 25% took French and 30% of the students took neither. Let A be the event that a randomly chosen student took Spanish, and B be the event that a student took French. Fill in either the Venn diagram or a 2-way table and answer the questions:



	B	B'	
A			
A'			

a) Describe in words the meaning of the event AB'. Find the probabilityof this event.
b) Are the events A, B independent? Explain with numbers why or why not.
c) If it is known that the student took Spanish, what are the chances that she also took French?

14) One half of all female physicists are married. Among those married, 50% aremarried to other physicists, 29% to scientists other than physicists and 21%to non-scientists’. Among male physicists, 74% are married. Among them, 7%are married to other physicists, 11% to scientists other than physicists and82% to non-scientists What percent of all physicists are female? [Hint: Thisproblem can be solved as is, but if you want to, assume that physicists comprise1% of all population.]

15) Error-correcting codes are designed to withstand errors in data being sentover communication lines. Suppose we are sending a binary signal (consistingof a sequence of 0's and 1's), and during transmission, any bit may get flippedwith probability p, independently of any other bit. However, we might chooseto repeat each bit 3 times. For example, if we want to send a sequence 010,we will code it as 000111000. If one of the three bits flips, say, the receivergets the sequence 001111000, he will still be able to decode it as 010 bymajority voting. That is, reading the first three bits, 001, he will interpretit as an attempt to send 000. However, if two of the three bits are flipped,for example 011, this will be interpreted as an attempt to send 111, and thusdecoded incorrectly.What is the probability of a bit being decoded incorrectly under thisscheme?

5.8 Bayes’ Rule

Events $B_1, B_2, \ldots, B_K$ are said to be a partition of the sample space S if thefollowing two conditions are satisfied.i) $B_i B_J = \phi$ for each pair i, j and ii) $B_1 \cup B_2 \cup \ldots \cup B_K = S$

This situation often arises when the statistics are available in subgroups ofa population. For example, an insurance company might know accident ratesfor each age group B_i. This will give the company conditional probabilities $P(A/B_i)$ (if we denote A = {event of accident}.

Question:If we know the conditional probabilities $P(A/B_i)$, ,how dowe find the unconditional P(A)?

Consider a case when k = 2:
The event A can be written as the union of mutually exclusive events AB_1 and AB_2, that is $A = AB_1 \cup AB_2$ it follows that $P(A) = P(AB_1) + P(AB_2)$
If the conditional probabilities $P(A/B_1)$) and $P(A/B_2)$) are known, that is

$$P(A/B_1) = \frac{P(AB_1)}{P(B_1)} \text{ and } P(A/B_2) = \frac{P(AB_2)}{P(B_2)} \quad \text{then} \quad P(A) = P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2)$$

Suppose we want to find probability of the form $P(B_i/A)$, which can be written as

$$P(B_i/A) = \frac{P(AB_i)}{P(A)} = \frac{P(B_i) \times P(A/B_i)}{P(A)} = \frac{P(B_i) \times P(A/B_i)}{P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2)}$$

This calculation generalizes to k > 2 events as follows.

TheoremIf $B_1, B_2, \ldots, B_K$ form a partition of the sample space S such that $P(B_i) \neq 0$ for i = 1, 2, . . . k; then for any event $A \subseteq S$,

$$P(A) = \sum_{i=1}^{k} P(AB_i) = \sum_{i=1}^{k} P(B_i) \times P(A/B_i) \quad \text{Subsequently,} \quad P(B_i/A) = \frac{P(AB_i)}{P(A)} = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{k} P(B_i) \times P(A/B_i)}$$

This last equation is often called Law of Total Probability.

Example 1A rare genetic disease (occuring in 1 out of 1000 people) is diagnosed usinga DNA screening test. The test has false positive rate of 0.5%, meaning that $P(\text{test positive / no disease}) = 0.005$. Given that a person has tested positive,what is the probability that this person actually has the disease?First, guess the answer, then read on.
Solution
Let's reason in terms of actual numbers of people, for a change.
Imagine 1000 people, 1 of them having the disease. How many out of 1000will test positive? One that actually has the disease, and about 5 disease-freepeople who would test false positive. Thus, $P(\text{disease/ test positive}) \approx \frac{1}{6}$.
It is left as an exercise for the reader to write down the formal probabilitycalculation.

Example 2 At a certain assembly plant, three machines make 30%, 45%, and 25%, respectively,of the products. It is known from the past experience that 2%, 3%and 2% of the products made by each machine, respectively, are defective.Now, suppose that a finished product is randomly selected.

a) What is the probability that it is defective?
b) If a product were chosen randomly and found to be defective, what isthe probability that it was made by machine 3?

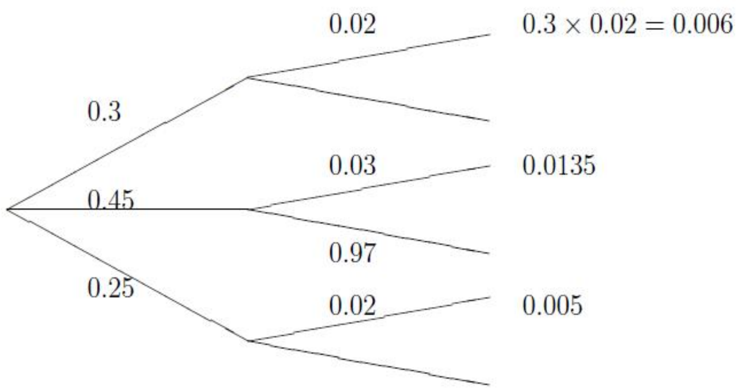
Solution
Consider the following events:
A: the product is defectiveand B_i: the product is made by machine i=1, 2, 3,
Applying additive and multiplicative rules, we can write
(a)

$$P(A) = P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3)$$

$$= (0.3)(0.02) + (0.45)(0.03) + (0.25)(0.02) = 0.006 + 0.0135 + 0.005 = 0.0245$$

(b) Using Bayes' rule $P(B_3/A) = \dfrac{P(B_3) \times P(A/B_3)}{P(A)} = \dfrac{0.005}{0.0245} = 0.2041$

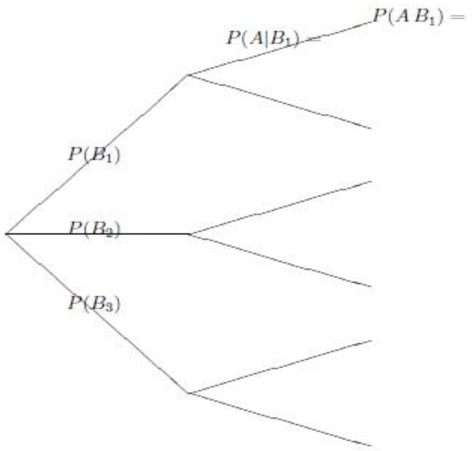
This calculation can also be represented using a tree diagram as follows



Here, the first branching represents probabilities of the events B_1 and the second branching represents conditional probabilities $P(A/B_1)$. The probabilitiesof intersections, given by the products, are on the right. P(A) is their sum.

Exercises

1) Lucy is undecided as to whether to take a Math course or a Chemistry course. She estimates that her probability of receiving an A grade would be 0.5 in amath course, and $\frac{2}{3}$ in a chemistry course. If Lucy decides to base her decisionon the flip of a fair coin, what is the probability that she gets an A?
2) Of the customers at a gas station, 70% use regular gas, and 30% use diesel.Of the customers who use regular gas, 60% will fill the tank completely, andof those who use diesel, 80% will fill the tank completely.
   a) What percent of all customers will fill the tank completely?
   b) If a customer has filled up completely, what is the probability it was acustomer buying diesel?
3) In 2004, 57% of White households directly and/or indirectly owned stocks,compared to 26% of Black households and 19% of Hispanic households. Thedata for Asian households is not given, but let's assume the same rate as forWhites. Additionally, 77% of households are classified as either White orAsian, 12% as African American, and 11% as Hispanic.
   a) What proportion of all families owned stocks?
   b) If a family owned stock, what is the probability it was White/Asian?
4) Drawer one has five pairs of white and three pairs of red socks, while drawertwo has three pairs of white and seven pairs of red socks. One drawer isselected at random a pair of socks is selected at random from that drawer.
   a) What is the probability that it is a white pair of socks?
   b) Suppose a white pair of socks is obtained. What is the probability thatit came from drawer two?
5) For an on-line electronics retailer, 5% of customers who buy Zony digitalcameras will return them, 3% of customers who buy Lucky Star digital cameraswill return them, and 8% of customers who buy any other brand willreturn them. Also, among all digital cameras bought, there are 20% Zony'sand 30% Lucky Stars.Fill in the tree diagram and answer the questions.
   a) What percent of all cameras are returned?
   b) If the camera was just returned, what is the probability it is a LuckyStar?
   c) What percent of all cameras sold were Zony and were not returned?



6) Three newspapers, A, B, and C are published in a certain city. It is estimatedfrom a survey that that of the adult population: 20% read A, 16% read B,14% read C, 8% read both A and B, 5% read both A and C, 4% read both Band C, 2% read all three. What percentage reads at least one of the papers?Of those that read at least one, what percentage reads both A and B?
7) Suppose $P(A/B) = 0.3$, $P(B) = 0.4$ and $P(B/A) = 0.6$. Find $P(A)$ and $P(A \cup B)$
8) This is the famous Monty Hall problemf A contestant on a game show isasked to choose among 3 doors. There is a prize behind one door and nothingbehind the other two. You (the contestant) have chosen one door. Then, thehost is flinging one other door open, and there's nothing behind it. Whatis the best strategy? Should you switch to the remaining door, or just staywith the door you have chosen? What is your probability of success (gettingthe prize) for either strategy?
9) There are two children in a family. We overheard about one of them referredto as a boy.
   a) Find the probability that there are 2 boys in the family.
   b) Suppose that the oldest child is a boy. Again, find the probability that there are 2 boys in the family.g [Why is it different from part (a)?]
10) At a university, two students were doing well for the entire semester butfailed to show up for a final exam. Their excuse was that they travelled outof state and had a at tire. The professor gave them the exam in separaterooms, with one question worth 95 points: \which tire was it?". Find theprobability that both students mentioned the same tire.
11) In firing the company's CEO, the argument was that during the six yearsof her tenure, for the last three years the company's market share was lowerthan for the first three years. The CEO claims bad luck. Find the probabilitythat, given six random numbers, the last three are the lowest among six.

6 DISCRETE PROBABILITY DISTRIBUTIONS

Introduction

In application of probability, we are often interested in a number associated with the outcome of a random experiment. Such a quantity whose value is determined by the outcome of a random experiment is called a **random variable**. It can also be defined as any quantity or attribute whose value varies from one unit of the population to another.

A **discrete** random variable is function whose range is finite and/or countable, Ie it can only assume values in a finite or countably infiniteset of values. A **continuous** random variable is one that can take any value in an interval of real numbers. (There are *unaccountably*many real numbers in an interval of positive length.)

6.1 Discrete Random Variables

A random variable X is said to be discrete if it can take on only a finite or countable number of possible values x. Consider the experiment of flipping a fair coin three times. The number of tails that appear is noted as a discrete random variable. *X*= *"*number of tails that appear in 3 flips of a fair coin".There are 8 possible outcomes of the experiment: namely the sample space consists of

$$S = \{HHH\ , HHT\ , HTH\ , HTT\ , THH\ , THT\ , TTH\ , TTT\ \}$$
$$X = \{\ 0 \quad 1, \quad 1, \quad 2, \quad 1, \quad 2, \quad 2, \quad 3\}$$

are the corresponding values taken by the random variable *X.*

Now, what are the possible values that *X* takes on and what are the probabilities of *X* taking a particular value?

From the above we see that the possible values of *X* are the 4 values

$$X = \{0, 1, 2, 3\}$$

Ie the sample space is a disjoint union of the 4 events $\{X = j\}$ for j=0,1,2,3

Specifically in our example:

$\{X=0\}=\{HHH\}$ $\{X=1\}=\{HHT, HTH, THH\}$

$\{X=2\}=\{TTH, HTT, THT\}$ $\{X=3\}=\{TTT\}$

Since for a fair coin we assume that each element of the sample space is equally likely (with probability $\frac{1}{8}$, we find that the probabilities for the various values of *X,* called the *probability distribution*of*X* or the *probability mass function (pmf)*. can be summarized in the following table listing the possible values beside the probability of that value

x	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Note: The probability that *X* takes on the value *x*, ie $p(X = x)$, is defined as the sum of the probabilities of all points in *S* that are assigned the value *x*.

We can say that this pmf places mass $\frac{3}{8}$on the value $X=2$.

The"masses" (or probabilities) for a pmf should be between 0 and 1.

The total mass (i.e. total probability) must add up to 1.

Definition: The **probability mass function** of a discrete variable is a, table, formula or graph that specifies the proportion (or probabilities) associated with each possible value the random variable can take. The mass function $P(X = x)$ (or simply p(x) has the following properties:

$$0 \le p(x) \le 1 \text{ and} \sum_{\text{all}\,x} p(x) = 1$$

More generally, let X have the following properties

i) It is a discrete variable that can only assume values $x_1, x_2, \ldots. x_n$

ii) The probabilities associated with these values are

$$P(X = x_1) = p_1 \;,\; P(X = x_2) = p_2 \ldots\ldots P(X = x_n) = p_n$$

Then X is a discrete random variable if $0 \le p_i \le 1$ and $\sum_{i=1}^{n} p_i = 1$

Remark: We denote random variables with capital letters while realized or particular values are denoted by lower case letters.

Example 1 Two tetrahedral dice are rolled together once and the sum of the scores facing down was noted. Find the pmf of the random variable 'the sum of the scores facing down.'

Solution

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

$$X = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

Therefore the pmf is given by the table below

x	2	3	4	5	6	7	8
P(X=x)	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

This can also be written as a function $P(X = x) = \begin{cases} \frac{x-1}{16} \text{ for} x = 2, 3, 4, 5 \\ \frac{9-x}{16} \text{ for} x = 6, 7, 8 \end{cases}$

Example 2
The pmf of a discrete random variable W is given by the table below

w	-3	-2	-1	0	1
P(W=w)	0.1	0.25	0.3	0.15	d

Find the value of the constant d, $P(-3 \le w < 0)$, $P(w > -1)$ and $P(-1 < w < 1)$

Solution

$$\sum_{\text{all w}} p(W = w) = 1 \quad \Rightarrow \quad 0.1 + 0.25 + 0.3 + 0.15 + d = 1 \quad \Rightarrow \quad d = 0.2$$

$P(-3 \le w < 0) = P(W = -3) + P(W = -2) + P(W = -1) = 0.65$

$P(w > -1) = P(w = 0) + P(w = 1) = 0.15 + 0.2 = 0.35$

$P(-1 < w < 1) = P(W = 0) = 0.15$

Example 3A discrete random variable Y has a pmf given by the table below

y	0	1	2	3	4
P(Y=y)	c	2c	5c	10c	17c

Find the value of the constant c hence computes $P(1 \le Y < 3)$

Solution

$$\sum_{\text{all y}} p(Y = y) = 1 \quad \Rightarrow \quad c(1 + 2 + 5 + 10 + 17) = 1 \quad \Rightarrow \quad c = \tfrac{1}{35}$$

$$P(1 \le Y < 3) = P(Y = 1) + P(Y = 2) = \tfrac{2}{35} + \tfrac{5}{35} = \tfrac{1}{5}$$

Exercise

1. A die is loaded such that the probability of a face showing up is proportional to the face number. Determine the probability of each sample point.
2. Roll a fair die and let X be the square of the score that show up. Write down the probability distribution of X hence compute $P(X < 15)$ and $P(3 \le X < 30)$
3. Let X be the random variable the number of fours observed when two dice are rolled together once. Show that X is a discrete random variable.
4. The pmf of a discrete random variable X is given by $P(X = x) = kx$ for $x = 1, 2, 3, 4, 5, 6$
   Find the value of the constant k, $P(X < 4)$ and $P(3 \le X < 6)$
5. A fair coin is flip until a head appears. Let N represent the number of tosses required to realize a head. Find the pmf of N , $P(N < 2)$ and $P(N \ge 2)$
6. A discrete random variable Y has a pmf given by $P(Y = y) = c\left(\tfrac{3}{4}\right)^y$ for $y = 0, 1, 2, ...$ Find the value of the constant c , $P(X < 3)$ and $P(X \ge 3)$
7. Verify that $f(y) = \dfrac{2y}{k(k+1)}$ for $y = 0, 1, 2, ...k$ can serve as a pmf of a random variable X.
8. For each of the following determine c so that the function can serve as a pmf of a random variable X.
   a) $f(x) = c$ for $x = 1, 2, 3, 4, 5$
   b) $f(x) = cx$ for $x = 1, 2, 3, 4, 5$
   c) $f(x) = cx^2$ for $x = 0, 1, 2, ...k$
   d) $f(x) = \tfrac{c}{2}$ for $x = -1, 0, 1, 2$
   e) $f(x) = \tfrac{(x-2)}{c}$ for $x = 1, 2, 3, 4, 5$
   f) $f(x) = \tfrac{(x^2 - x + 1)}{c}$ for $x = 1, 2, 3, 4, 5$
   g) $f(x) = c(x^2 + 1)$ for $x = 0, 1, 2, 3$
   h) g) $f(x) = cx\left({}_3C_x\right)$ for $x = 1, 2, 3$
   i) $f(x) = c\left(\tfrac{1}{6}\right)^x$ for $x = 0, 1, 2, 3.....$
   j) $f(x) = c2^{-x}$ for $x =$ for $x = 0, 1, 2, ...$
9. A coin is loaded so that heads is three times as likely as the tails.

a. For 3 independent tosses of the coin find the pmf of the total number of heads realized and the probability of realizing at most 2 heads.
b. A game is played such that you earn 2 points for a head and loss 5 points for a tail. Write down the probability distribution of the total scores after 4 independent tosses of the coin

10. For an on-line electronics retailer, X = "the number of Zony digital cameras returned per day" follows the distribution given by

x	0	1	2	3	4	5
P(X=x)	0.05	0.1	t	0.2	0.25	0.1

Find the value of t and P(X > 3)

11. Out of 5 components, 3 are domestic and 2 are imported. 3 components are selected at random (without replacement). Obtain the PMF of X ="number of domestic components picked" (make a table).

6.2 Expectation and Variance of a Random Variable

6.2.2 Expected Values

One of the most important things we'd like to know about a random variable is: what value does it take on average? What is the average price of a computer? What is the average value of a number that rolls on a die? The value is found as the average of all possible values, weighted by how often they occur (i.e. probability)

Definition: Let X be a discrete r.v. with probability function p(x). Then the **expected value** of X, denoted $E(X)$ or μ, is given by $\mathrm{E(x)} = \mu = \sum_{x=-\infty}^{\infty} xp(X = x).$

Theorem: Let Xbe a discrete r.v. with probability function $p(X=x)$ and let $g(x)$ be a real-valued function of X. ie$g$: ℝ → ℝ, then the expected value of $g(x)$ is given by

$$\mathrm{E}[\mathrm{g(x)}] = \sum_{x=-\infty}^{\infty} g(x)p(X = x).$$

Theorem: Let X be a discrete r.v. with probability function p(x). Then

(i) $E(c) = c$, where c is any real constant;

(ii) $\mathrm{E}[\mathrm{ax} + \mathrm{b}] == a\mu + b$ where a and b are constants

(iii) $\mathrm{E}[\mathrm{kg(x)}] = k\mathrm{E}[\mathrm{g(x)}]$ where $g(x)$ is a real-valued function of X

(iv) $\mathrm{E}[\mathrm{ag}_1(\mathrm{x}) \pm b\mathrm{g}_2(\mathrm{x})] = a\mathrm{E}[\mathrm{g}_1(\mathrm{x})] \pm b\mathrm{E}[\mathrm{g}_2(\mathrm{x})]$ and in general

$$\mathrm{E}\left[\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{c_i g_i(x)}\right] = \sum_{\mathrm{i=1}}^{n} c_i \mathrm{E}[\mathrm{g_i(x)}] \text{where } \mathrm{g_i's}(\mathrm{x}) \text{are real-valued functions of } X.$$

This property of expectation is called *linearity property*

Proof

(i) $\mathrm{E}[\mathrm{c}] = \sum_{all\ x} cP(X = x) = c\sum_{all\ x} P(X = x) = c(1) = c$

(ii) $\mathrm{E}[\mathrm{ax} + \mathrm{b}] = \sum_{all\ x} (\mathrm{ax} + \mathrm{b})P(x) = \sum_{all\ x} \mathrm{ax}P(x) + \sum_{all\ x} \mathrm{b}P(x) = a\sum_{all\ x} xP(x) + b\sum_{all\ x} P(x) = a\mu + b$

(iii) $\mathrm{E}[\mathrm{kg(x)}] = \sum_{all\ x} kg(x)P(X = x) = k\sum_{all\ x} g(x)P(X = x) = k\mathrm{E}[\mathrm{g(x)}]$

(iv) $\mathrm{E}[\mathrm{ag}_1(\mathrm{x}) \pm b\mathrm{g}_2(\mathrm{x})] = \mathrm{E}[\mathrm{ag}_1(\mathrm{x})] \pm E[b\mathrm{g}_2(\mathrm{x})] = a\mathrm{E}[\mathrm{g}_1(\mathrm{x})] \pm b\mathrm{E}[\mathrm{g}_2(\mathrm{x})]$ from part iii

6.2.3 Variance and Standard Deviation

Definition: Let X be ar.v with mean $E(X)=\mu$, the **variance** of X, denoted σ^2 orVar(X),is given by $Var(X)=\sigma^2=E(X-\mu)^2$.The units for variance are square units. The quantity that has the correct units is **standard deviation**, denoted σ . It'sactually the positive square root ofVar(X) .

$$\sigma=\sqrt{Var(X)}=\sqrt{E(X-\mu)^2}.$$

Theorem: $Var(X)=E(X-\mu)^2=E(X)^2-\mu^2$

Proof:

$Var(X)=E(X-\mu)^2=E(X^2-2X\mu+\mu^2)=E(X)^2-2\mu E(X)+\mu^2=E(X)^2-\mu^2$ Since $E(X)=\mu$

Theorem: $Var(aX+b)=a^2\ \mathrm{var}(X)$

Proof:

Recall that $E[aX+b]=a\mu+b$ therefore

$$Var(aX+b)=E[(aX+b)-(a\mu+b)]^2=E[a(X-\mu)]^2==E[a^2(X-\mu)^2]=a^2E[(X-\mu)^2]=a^2\ \mathrm{var}(X)$$

Remarks

(i) The expected value of X always lies between the smallest and largest values of X.

(ii) In computations, bear in mind that variance cannot be negative!

Example 1

Given a probability distribution of X as below, find the mean and standard deviation of X.

x	0	1	2	3
P(X=x)	1/8	1/4	3/8	1/4

Solution

x	0	1	2	3	total
$p(X=x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	1
$xp(X=x)$	0	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{7}{4}$
$x^2p(X=x)$	0	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{9}{4}$	4

$E(X)=\mu=\sum_{x=0}^{3}xp(X=x)=1.75$ and

standard deviation

$\sigma=\sqrt{E(X^2)-\mu^2}\ =\sqrt{4-1.75^2}\approx 0.968246$

Example 2 The probability distribution of a r.v X is as shown below, find the mean and standard deviation of; a)X b) $Y=12X+6$.

x	0	1	2
P(X=x)	1/6	1/2	1/3

Solution

x	0	1	2	total
$p(X=x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$	1
$xp(X=x)$	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{7}{6}$
$x^2p(X=x)$	0	$\frac{1}{2}$	$\frac{4}{3}$	$\frac{11}{6}$

$E(X)=\mu=\sum_{x=0}^{2}xp(X=x)=\frac{7}{6}$ and

$E(X^2)=\sum_{x=0}^{2}x^2p(X=x)=\frac{11}{6}$

Standard deviation $\sigma=\sqrt{E(X^2)-\mu^2}\ =\sqrt{\frac{11}{6}-(\frac{7}{6})^2}\ =\sqrt{\frac{17}{6}}\approx 1.6833$

Now $E(Y)=12E(X)+6=12(\frac{7}{6})+6=20$

$Var(Y)=Var(12X+6)=12^2\times Var(X)=144\times\sqrt{\frac{17}{6}}\approx 242.38812$

Exercise

1. Suppose X has a probability mass function given by the table below

x	2	3	4	5	6
P(X=x)	0.01	0.25	0.4	0.3	0.04

Find the mean and variance of; X

2. Suppose X has a probability mass function given by the table below

x	11	12	13	14	15
P(X=x)	0.4	0.2	0.2	0.1	0.1

Find the mean and variance of; X

3. Let X be a random variable with $P(X = 1) = 0.2$, $P(X = 2) = 0.3$, and $P(X = 3) = 0.5$. What is the expected value and standard deviation of; a) X b) $Y = 5X - 10$?
4. A random variable W has the probability distribution shown below,

w	0	1	2	3
P(W=w)	2d	0.3	d	0.1

Find the values of the constant d hence determine the mean and variance of W. Also find the mean and variance of $Y = 10X + 25$

5. A random variable X has the probability distribution shown below,

x	1	2	3	4	5
P(X=x)	7c	5c	4c	3c	c

Find the values of the constant c hence determine the mean and variance of X.

6. The random variable Z has the probability distribution shown below,

z	2	3	5	7	11
P(Z=z)	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	x	y

If $E(Z) = 4\frac{2}{3}$,find the values of x and y hence determine the variance of Z

7. A discrete random variable M has the probability distribution $f(m) = \begin{cases} \frac{m}{36}, & m = 1,2,3,...,8 \\ 0, & elsewhere \end{cases}$, find the mean and variance of M
8. For a discrete random variable Y the probability distribution is $f(y) = \begin{cases} \frac{5-y}{10}, & y = 1,2,3,4 \\ 0, & elsewhere \end{cases}$, calculate $E(Y)$ and var(Y)
9. Suppose X has a pmf given by $f(x) = \begin{cases} kx & \text{for } x = 1,2,3,4 \\ 0, & elsewhere \end{cases}$, find the value of the constant k hence obtain the mean and variance of X
10. A team of 3 is to be chosen from 4 girl and 6 boys. If X is the number of girls in the team, find the probability distribution of X hence determine the mean and variance of X
11. A fair six sided die has; '1' on one face, '2' on two of it's faces and '3' on the remaining three faces. The die is rolled twice. If T is the total score write down the probability distribution of T hence determine;
    a) the probability that T is more than 4
    b) the mean and variance of T