HW-4

Given the cost function $J(w_0, w_1) = \frac{1}{2m}\sum_{i=1}^{m}(w_0 + w_1 x^{(i)} - y_i)^2$ determine the definiteness of its Hessian matrix and the convexity of the function. Assume 1-D dataset $m = 1$. Show your work.

$\rightarrow$ $$J(w_0, w_1) = \frac{1}{2m}\sum_{i=1}^{m}(w_0 + w_1 x^{(i)} - y_i)^2$$

Assume $m = 1$, 1-D dataset

$$\frac{dJ(w)}{dw_0} = \frac{1}{m}\sum_{i=1}^{m}(w_0 + w_1 x^{(i)} - y_i)$$

when $m = 1 \Rightarrow \frac{dJ(w)}{dw_0} = w_0 + w_1 x - y_1$

$$\frac{dJ(w)}{dJ_1} = \frac{1}{m}\sum_{i=1}^{m} x^i(w_0 + w_1 x^{(i)} - y_i)$$

when $m = 1 \Rightarrow \frac{dJ(w)}{dw_1} = x(w_0 + w_1 x - y_1)$

$$\frac{dJ(w)}{dw_1} = \frac{1}{m}\sum_{i=1}^{m} x^{(i)}(w_0 + w_1 x^{(i)} - y_i)$$

when $m = 1$, $\frac{dJ(w)}{dw_1} = x(w_0 + w_1 x - y_1)$

$$\frac{d^2 J(w)}{dw_0^2} = 1 \qquad \frac{d^2 J(w)}{dw_1^2} = x * x = x^2$$

$$\frac{d^2 J(w)}{dw_0 dw_1} = \frac{d^2 J(w)}{dw_1 dw_0} = x$$

Hessian matrix, $H = \begin{bmatrix} \frac{d^2 J(w)}{dw_0^2} & \frac{d^2 J(w)}{dw_0 dw_1} \\ \frac{d^2 J(w)}{dw_1 dw_0} & \frac{d^2 J(w)}{dw_1^2} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix}$

For a square matrix H, $HV = \lambda V$ ..... where

λ = eign value of H

V = eigenvector associated with H

To find the eign value λ,

$|H - \lambda I| = 0$ ..... where I = identity matrix

i.e. $\begin{vmatrix} 1-\lambda & x \\ x & x^2-\lambda \end{vmatrix} = 0$

Calculating the determinant,

$$(1-\lambda)(x^2-\lambda) - x^2 = 0$$

$$\cancel{x^2} - \lambda - x^2\lambda + \lambda^2 - \cancel{x^2} = 0$$

$$-\lambda - \lambda(x^2-\lambda) = 0$$

$$-\lambda\left[1 + (x^2-\lambda)\right] = 0$$

i.e. $\lambda_1 = 0$ or $\lambda_2 = 1 + x^2$

$\because$ $\lambda_1 = 0$ and $\lambda_2 > 0$

$\therefore$ given cost function is positive semidefinite (PSD) $\Rightarrow$ convex.