LECTURE 7 SUBSTITUTION

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① $T(n) = 2T(n/2) + \theta(n) = \theta(n \log n)$

BY DEF, $0 \leq d\, n \cdot \log n \leq \underline{2T(n/2) + \theta(n) \leq c \cdot n \log n}$

UPPER BOUND

$\underline{0 \leq d\, n \cdot \log n \leq 2T(n/2)}$

LOWER BOUND $\longrightarrow$ Do them separately

* Find constants and imply if it holds true or not

* Can introduce lower order terms.

② $T(n) = 2T(n/2) + \theta(n) = O(n^2)$

BY DEF, $2T(n/2) + cn \leq dn^2$

$$2\left[d\left(n/2\right)^2\right] + cn \leq d \cdot n^2$$

$$\frac{dn^2}{2} + cn \leq d \cdot n^2$$

$$cn \leq \frac{1}{2} dn^2$$

$$c \leq \frac{dn}{2}$$

$n_0 = 1$

Can find c and that satisfy the equation

$c = 1,\ d = 10$ (safe)

$\therefore$ The recurrence holds true

③ $T(n) = 2T(n/2) + \theta n = \Omega(n)^2$

$0 \leq dn^2 \leq 2T(n/2) + cn$

$2\left[\frac{dn^2}{4}\right] + cn \geq dn^2$

$\frac{dn^2}{2} + cn \geq dn^2$

$cn \geq \frac{1}{2} dn^2$

$c \geq \frac{1}{2} dn$ cannot bound c by n (can get very large)

$\therefore$ The equation doesn't hold true.

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④ $T(n) = 2T(n/2) + \theta n = O(n)$

$0 \leq 2T(n/2) + cn \leq dn$

$2\left[d\frac{n}{2}\right] + cn \leq dn$

$dn + cn \leq dn$

$cn \leq 0$

c and n cannot be < 0

$\therefore$ The equation doesn't hold true.

⑤ $T(n) = T(n-1) + c = O(n)$

$0 \le T(n-1) + c \le dn$

$d(n-1) + c \le dn$

$c = 1$

$d = 2$

$dn - d + c \le dn$

n not there, can pick any n (independent)

$c - d \le 0$

$\therefore$ The equation holds true.

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⑤ $T(n) = T(n-1) + c = O(\log n)$

$0 \le T(n-1) + c \le d \log n$

at inf this becomes 0

$d \log(n-1) + c \le d \log n$

$c \le d(\log n - \log(n-1))$

BUT

$c = 1$

$\times$ $d = 10$ It works for these values

$n_0 = 4$

$c \le 0$ (not possible $\Rightarrow c > 0$)

$\therefore$ The equation doesn't hold true.

NOTE:

After proving upper bound, prove lower bound (approach other side) to know the right bound (the tight bound)

⑥ $T(n) = 8\,T(n/2) + cn^2 = O(n^3)$

$8T(n/2) + cn^2 \leq dn^3 \; (-en^2)$ invisible lower order term to make it work

↳ should be (−)

$$8\left[d\left(\frac{n}{2}\right)^3 + e\left(\frac{n}{2}\right)^2\right] + cn^2 \leq dn^3 - en^2$$

$$-2en^2 + dn^3 + cn^2 \leq dn^3 - en^2$$

$-en^2 + cn^2 \leq 0$ ∵ $c > 0$, the above equation fails, but it should

It failed because of lower order term ⇒ include that

LEARNING:

$$-en^2 + cn^2 \leq 0$$

$$cn^2 \leq en^2$$

If it doesn't hold but looks correct,

you can sub a magic lower order term.

↳ works because $O(g(n))$ is a function with lower order terms.

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⑦ $T(n) = 7T(n/2) + cn^2 = O(n^{\log_2 7})$

$$7T(n/2) + cn^2 \leq d\,n^{\log_2 7} - en^2$$

$$7\left[d\left(n/2\right)^{\log_2 7} - e\left(\frac{n}{2}\right)^2\right] + cn^2 \leq d\,n^{\log_2 7} - en^2$$

$$-\frac{7}{4}en^2 + \cancel{d\,n^{\log_2 7}} + cn^2 \leq \cancel{d\,n^{\log_2 7}} - en^2$$

$$-\frac{3}{4}en^2 + cn^2 \leq 0$$

$$n^2\left(c - \frac{3}{4}e\right) \leq 0 \quad \left\{ \begin{array}{l} n_0 = 1 \\ c = 1 \\ e = 4 \end{array} \right.$$

∴ The equation holds true.

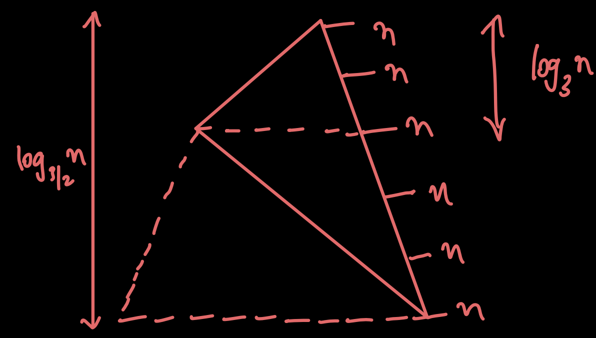
$T(n) = T(n/3) + T\left(\frac{2n}{3}\right) + n = \ ?$

RECURSION TREE

GUESS FROM Δ^{le}:

$h = \min\left(\log_3 n, \log_{3/2} n\right)$

- $\Omega(n \log_3 n)$
- $\Theta(n \log_{3/2} n)$



$T(n/3) + T\left(\frac{2n}{3}\right) + n \le cn \log_{3/2} n$

$\le cn \log n$

(to simplify)

$\Theta\left(\frac{n \log n}{\log 3/2}\right) = \Theta(n \log n)$

$$\left[c\frac{n}{3}\log\left(\frac{n}{3}\right)\right] + \left[c\frac{2n}{3}\log\left(\frac{2n}{3}\right)\right] + n \le cn\log n$$

$$\left[c\frac{n}{3}\left(\cancel{\log} n - \log 3\right)\right] + \left[\frac{c2n}{3}\left(\cancel{\log} n + \log 2 - \log 3\right)\right] + n \le cn \cancel{\log n}$$

$$-cn\log 3 + \frac{2}{3}cn\log 2 + n \le 0$$

$$-c\log 3 + \frac{2}{3}c\log 2 + 1 \le 0$$

$$c\left(\frac{2}{3}\log 2 - \log 3\right) + 1 \le 0$$

$c = 100, n_0 = 1$

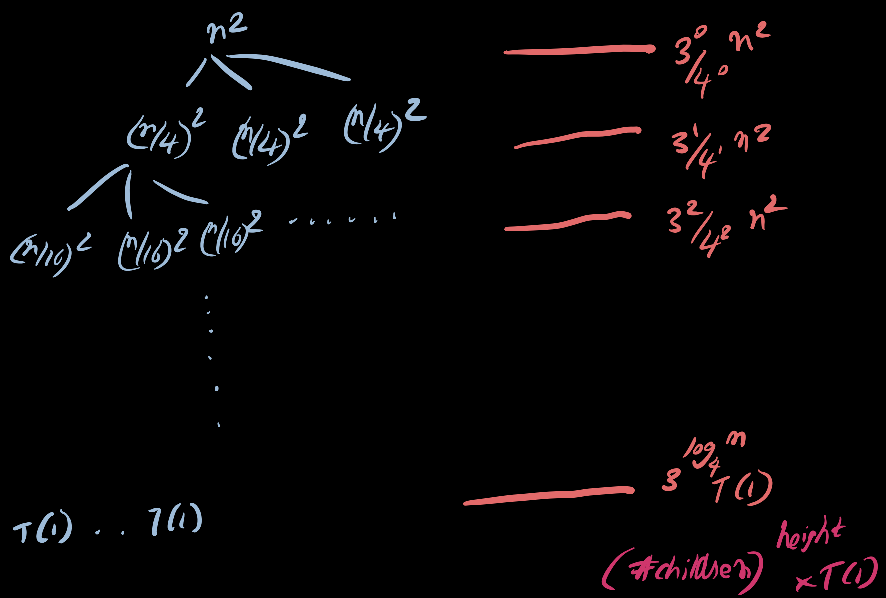
The eq. holds.

$||^{ly}$, it holds for $\Theta(n\log n)$ [try on your own]

$\therefore T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n = \Theta(n\log n)$

LECTURE 8 MASTER THEOREM

① $T(n) = 3\,T(n/4) + n^2$



$$T(n) = \sum_{i=0}^{(\log_4 n)-1} 3^i \left(\frac{n}{4^i}\right)^2 + 3^{\log_4 n}$$

$$= n^2 \underbrace{\sum_{i=0}^{(\log_4 n)-1} \left(\frac{3}{16}\right)^i}_{\text{constant}} + c \cdot n^{\log_4 3}$$

$= O(n^2)$

$= \Omega(n^2)$ → because this least is constant not 0.

PROOF:

$3\,T(n/4) + n^2 \le c n^2$

$3\left[c\left(\frac{n}{4}\right)^2\right] + n^2 \le c n^2$

$\frac{3}{16} c n^2 + n^2 \le c n^2$

$1 \le \frac{13}{16} c$ $\quad n_0 = 1$, $c = 16$

The equation holds true

can prove lower bound (Ω) as well by flipping ($\ge$)

$n_0 = 1$, $c = 1$

$\therefore \Theta(n^2)$

② $T(n) = a\,T(n/b) + f(n)$

$$f(n) \downarrow a f(n/b) \downarrow a^2 f(n/b^2) \downarrow \cdots\; T(1)\cdot a^{\log_a n}$$

$\frac{n}{b^x} = 1$

$x = \log_b n$

$$= \sum_{i=0}^{\log_b n - 1} a^i f(n/b^i) + n^{\log_b a}\cdot T(1)$$

$$= f(n) + a f(n/b) + \cdots$$

$$= c\cdot n^{\log_b a}$$

$$= O(\quad) \rightarrow 3 \text{ cases}$$

Base Behind Master Theorem

MASTER THEOREM

$T(n) = a\,T(n/b) + f(n)$

$a \geq 1$

$b > 1$

$f(n) > 0$

① $f(n) = O(n^{\log_b a - \varepsilon})$ $\quad \varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

② $f(n) = \Theta(n^{\log_b a} \log^k n)$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

③ $f(n) = \Omega(n^{\log_b a + \varepsilon})$ & $a f(n/b) \leq c\, f(n)$, $c < 1$, then $T(n) = \Theta(f(n))$

① $f(n) \leq c \cdot n^{\log_b a - \varepsilon}$

$$\leq c \frac{n^{\log_b a}}{n^{\varepsilon}}$$ (polynomially smaller)

② $f(n) \geq c \cdot n^{\log_b a + \varepsilon}$

$$\geq c\, n^{\log_b a} \times n^{\varepsilon}$$ (polynomially larger)

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① $T(n) = 3\,T(n/4) + n^2$

$a = 3$
$b = 4$
$f(n) = n^2$

3

$n^2 = \Omega\left(n^{\log_4 3 + \varepsilon}\right)$

$2 = \log_4 3 + \varepsilon$

$\varepsilon = 2 - 0.2$

$= 1.4 > 0$

$T(n) = \Theta(n^2)$

$a f(n/b) < c f(n/b)$

$3\left(\frac{n}{4}\right)^2 < c n^2$

$\frac{3}{16} < c$

(PROVED)

② $T(n) = 8T(n/2) + dn^2$

$dn^2 = O(n^{3-\varepsilon})$

$\varepsilon = 1$ ①

$\therefore T(n) = \theta(n^3)$

③ $T(n) = 2T(n/2) + n$

$a = 2$

$b = 2$ $n^{\log_2 2} = n$ ②

$f(n) = n$

(k=0, use k as log power)

$\therefore T(n) = \theta(n \log n)$

④ $T(n) = 2T(n/2) + n \log n$

$n \log n = \Omega(n^{\log_2 2 + \varepsilon})$ [REMEMBER REFLEXIVITY]

$n \log n \geq c \cdot n \cdot n^{\varepsilon}$

$\log n \geq c \cdot n^{\varepsilon}$

cannot bound a logarithmic function

by polynomial.

Trying case ②

$n \log n = \theta(n^{\log_2 2} \log^k n)$

$k = 1$

$\therefore T(n) = \theta(n \log^2 n)$

⑤ $T(n) = T(n/2) + 1$

$a = 1 \geq 1$

$b = 2 > 1$

$1 > 0$

$1 = \theta(n^{\log_2 1} \log^k n)$

$k = 0$

$\therefore$ Ans $= \theta(1 \cdot \log^{k+1} n) \Rightarrow k =$

$T(n) = \theta(\log n)$

⑥ $T(n) = 27\, T(n/3) + n^3 \log n$

CASE 3 will fail

Try case 2

$n^3 \log n = \theta(n^3 \log^k n)$

$k = 1 \geq 0$

$\therefore T(n) = \theta(n^3 \log^2 n)$

[when there is $\log n$ is $f(n)$ always go for general case ②]

⑦ $T(n) = 27\, T(n/3) + n^3 / \log n$

Fails case ② $\Rightarrow$ cannot find ε

$\frac{n^3}{\log n} = \theta(n^3 \log^k n)$

$k = -1 \ \ ! \ngeq 0$

$\therefore$ None of the cases can be used

8 $T(n) = T(n/8) + T(n/4) + T(n/3) + n$

$T(n/8)$ → goes down fast → Ω

$T(n/3)$ → goes down slow → O

Cannot use M.T directly but use it for guessing

approx. from 3 calls above

3 $T(n/8) + n = \Omega(n)$

3 $T(n/3) + n = O(n \log n)$ → This will fail when you try to prove.

$\therefore T(n) = \Theta(n)$

CHAP 8 FROM HERE