ANALYSIS OF RECURSIVE ALGORITHMS

Tuesday, September 25, 2018 5:08 PM

$n! = \underbrace{1 \times 2 \times 3 \times \cdots \times n-1}_{(n-1)!} \times n$

```
fact (n):
  if (n==0)
     return 1;
  return fact(n-1)*n;
```

METHODOLOGY: 2-step process

1) transcribe recursive alg into a RECURRENCE
2) solve recurrence

A RECURRENCE is a system of 2 equations: base, general

Define: $M(n)$ = # of multiplications performed by fact(n)

Recurrence: $\boxed{\begin{array}{l} M(0) = 0 \\ M(n) = 1 + M(n-1), \ n > 0. \end{array}}$

We can use these 2 equations to compute $M(n)$ for any $n \geq 0$.

$M(0) = 0$

$M(1) = 1 + M(0) = 1 + 0 = 1$

$M(2) = 1 + M(1) = 1 + 1 = 2$

$M(3) = 1 + M(2) = 1 + 2 = 3$

$M(4) = 1 + M(3) = 1 + 3 = 4$

Guess: $M(n) = n$ for $n \geq 0$.

Check: a) $M(0) = 0$ ✓

b) $M(n) \stackrel{?}{=} 1 + M(n-1), \ n > 0$

$n \stackrel{?}{=} \cancel{1} + n - \cancel{1}, \ n > 0$

$n = n$ ✓

PRACTICE:

```
f(n)
  if (n==0)
     return 1*2*3;
  return f(n-1) * n * n+1;
```

Find $M(n)$ = # of *'s performed by $f(n)$, $n \geq 0$.

a) $M(0) = 2$

$M(n) = 1 + 1 + M(n-1) = \boxed{2 + M(n-1)}, \ n > 0$

b) $M(0) = 2$

$M(1) = 2 + M(0) = 2+2 = 4$
$M(2) = 2 + M(1) = 2+4 = 6$
$M(3) = 2 + M(2) = 2+6 = 8$

GUESS: $\boxed{M(n) = 2n+2, \ n \geq 0}$

CHECK: $M(0) \stackrel{?}{=} 2$
$2(0)+2 \stackrel{?}{=} 2$ ✓

$M(n) \stackrel{?}{=} 2 + M(n-1), \quad n > 0$
$2n+2 \stackrel{?}{=} 2 + 2*(n-1) + 2$
$2 + 2n - \cancel{2} + \cancel{2}$
$= 2 + 2n$ ✓

Ex:

```
g(n)
  if (n==0)
     return 1*2;
  return g(n-1) * g(n-1);
```

$M(n)$ = # of *'s performed by $g(n)$, $n \geq 0$:

a) $M(0) = 1$
$M(n) = \boxed{1 + 2M(n-1)}, \ n > 0.$

b) $M(0) = 1$
$M(1) = 1 + 2M(0) = 1 + 2.1 = 3$
$M(2) = 1 + 2M(1) = 1 + 2.3 = 7$
$M(3) = 1 + 2M(2) = 1 + 2.7 = 15$

GUESS $\boxed{M(n) = 2^{n+1} - 1, \ n \geq 0}$

CHECK:
$M(0) = 2^{0+1} - 1 = 2^1 - 1 = 1$ ✓
$M(n) \stackrel{?}{=} 1 + 2M(n-1)$
$2^{n+1} - 1 \stackrel{?}{=} 1 + 2(2^{n-1+1} - 1)$
$= 1 + 2(2^n - 1)$
$= 1 + 2^{n+1} - 2$
$= 2^{n+1} - 1$ ✓

$M(n) \in \Theta(2^n)$

GENERALIZATIONS:

$$M(0) = m_0$$
$$M(n) = b + M(n-1), \quad n > 0.$$

a)
$$M(0) = m_0$$
$$M(1) = b + M(0) = b + m_0$$
$$M(2) = b + M(1) = b + b + m_0 = 2b + m_0$$
$$M(3) = b + M(2) = b + 2b + m_0 = 3b + m_0$$

GUESS: $\boxed{M(n) = bn + m_0, \quad n \geq 0}$

CHECK: $M(0) = b.0 + m_0 = m_0$ ✓
$$b + M(n-1) = b + b(n-1) + m_0$$
$$= bn + m_0$$
$$= M(n)$$

Ex1. $m_0 = 0,\ b = 1$ $\qquad M(n) = 1.n + 0 = n$ ✓
$m_0 = 2,\ b = 2$ $\qquad M(n) = 2.n + 2$ ✓

GENERALIZATION
$$M(0) = m_0$$
$$M(n) = b + \underline{a} M(n-1), \quad a \neq 1$$

$$M(0) = m_0$$
$$M(1) = b + aM(0) = b + am_0$$
$$M(2) = b + aM(1) = b + a[b + am_0] = b + ba + m_0 a^2 = ba^0 + ba^1 + m_0 a^2$$
$$M(3) = b + aM(2) = b + a(ba^0 + ba^1 + m_0 a^2) = b.a^0 + ba^1 + ba^2 + m_0 a^3$$

$$M(n) = (b.a^0 + b.a^1 + ba^2 + \cdots + ba^{n-1}) + (m_0 a^n)$$
$$= b(a^0 + a^1 + a^2 + \cdots + a^{n-1}) + m_0 a^n$$
$$= \left(b \sum_{i=0}^{n-1} a^i\right) + m_0 a^n = \boxed{b\left(\frac{1-a^n}{1-a}\right) + m_0 a^n}$$

GEOMETRIC SUM: $\displaystyle\sum_{i=0}^{n} a^i, \quad a \neq 1 = \frac{1-a^{n+1}}{1-a}$

$$S = a^0 + \cancel{a^1} + \cancel{a^2} + \cdots + \cancel{a^n}$$
$$aS = \cancel{a^1} + \cancel{a^2} + \cdots + \cancel{a^n} + a^{n+1}$$
$$S - aS = a^0 - a^{n+1}$$
$$S(1-a) = a^0 - a^{n+1} = 1 - a^{n+1}$$
$$\boxed{S = \frac{1-a^{n+1}}{1-a}}$$

```
f(n)
{
    if (n==0)
        return 1*2*...*...   (m_0 multiplications)

    return f(n-1) + f(n-1) + ...... + f(n-1)   (a times)
                                              b multiplications
}
```

3rd example : $m_0 = 1$
$b = 1$
$a = 2$

$$M(n) = 1\left(\frac{1-2^n}{1-2}\right) + 1\cdot 2^n$$

$$= 2^n - 1 + 2^n$$

$$= \boxed{2^{n+1} - 1}$$

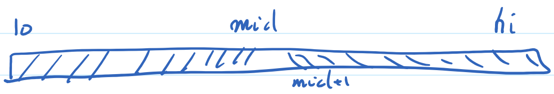
MERGE SORT

$\underbrace{4 \;\; 7 \;\; 1 \;\; 2} \;\|\; \underbrace{3 \;\; 8 \;\; 6 \;\; 5}$

1 2 4 7   3 5 6 8

1 2 3 4 5 6 7 8

MERGE



ANALYSIS OF MERGE

$$Merge(n) = \sum_{i=0}^{\cancel{n-1}\; n-2} (1) = n-\cancel{1}^{\,2}-0+1 = n-1$$

ANALYSIS OF MERGE SORT

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$$\boxed{\begin{aligned} MS(1) &= 0 \\ MS(n) &= MS\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + MS\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + (n-1),\; n>1. \end{aligned}}$$

$MS(1) = 0$

$MS(2) = MS(1) + MS(1) + (2-1) = 0+0+1 = 1$

$MS(3) = MS(1) + MS(2) + (3-1) = 3$

$MS(4) = MS(2) + MS(2) + (4-1) = 1+1+3 = 5$

$MS(5) = MS(2) + MS(3) + (5-1) = 1+3+4 = 8$

Numerical evidence suggests that

$MS(n) \in \Theta(n \lg n)$

TREE METHOD

$MS(1) = 0$

$MS(n) = MS(\lfloor \frac{n}{2} \rfloor) + MS(\lceil \frac{n}{2} \rceil) + (n-1),\ n > 0.$

To get rid of $\lfloor\ \rfloor$ and $\lceil\ \rceil$ we assume $\boxed{n = 2^m}$

$MS(1) = 0$

$MS(2^m) = MS(\lfloor \frac{2^m}{2} \rfloor) + MS(\lceil \frac{2^m}{2} \rceil) + 2^m - 1$

$= 2\,MS(2^{m-1}) + 2^m - 1$

$MS(2^m) = 2\,MS(2^{m-1}) + 2^m - 1$

$= 2[2\,MS(2^{m-2}) + 2^{m-1} - 1] + 2^m - 1 = 2^2 MS(2^{m-2}) + 2^m - 2 + 2^m - 1$

$= 2^2 MS(2^{m-2}) + 2.2^m - 3$

$= 2^2(2\,MS(2^{m-3}) + 2^{m-2} - 1) + 2 \cdot 2^m - 3$

$= 2^3 MS(2^{m-3}) + 2^m - 2^2 + 2^m + 2^m - 1 - 2$

$= 2^3 MS(2^{m-3}) + 3 \cdot 2^m - 1 - 2 - 2^2$

$= \ldots\ldots$

$= 2^m MS(2^0) + m \cdot 2^m - \boxed{1 - 2 - 2^2 - \cdots - 2^{m-1}}$

$= 2^m \cdot 0 + m \cdot 2^m - 2^m + 1$

$= \boxed{m \cdot 2^m - 2^m + 1}$

$MS(2^{10}) = 10.1024 - 1024 + 1 = 9.1024 + 1$

$\begin{array}{r} 10240 \\ 1023 \\ \hline \boxed{9217} \end{array}$