ASYMPTOTIC NOTATION

Thursday, September 20, 2018 5:11 PM

- We were able to derive EXACT formulas for FOR LOOPS
  (closed-form formulas involving simple operations)

- OFTEN, WE __AVOID__ deriving exact running time functions

1) Exact formulas can be complicated (sometimes unknown!)

   Insertion sort: # of comparisons is $\frac{n(n-1)}{2}$

   Merge sort: # of comparisons is

   $$\lceil \lg n \rceil n + 2^{\lceil \lg n \rceil} - 1$$

   Difficult to derive

   Confusing to interpret

2) Exact formulas don't tell the whole story

   __Ex__: $2n$ vs $5n$ actual speed depends on hardware details (recursion, cache memory)

__CONVENTION__: for first analysis, we lump running times into classes
In other words, we ignore "small" differences

__Ex__: $n\lceil \lg n \rceil + 2^{\lceil \lg n \rceil} - 1 \sim n \lg n$

negligible

WHICH DETAILS ARE "SMALL"?

1) Additive constants are negligible (when $n \to \infty$)

   $2n$ "same" $2n + 10^6$

2) Multiplicative constants are negligible (when $n \to \infty$)

   $2n$ "same" $10^6 n$

These two considerations lead to the following definition of asymptotic classes.

__Def__: Let $f(n)$, $g(n)$ be running time functions.
We say f and g "grow at the same rate", and write $f \in \Theta(g)$
or equivalently, $g \in \Theta(f)$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \text{nonzero constant}$$

__Ex__: $f(n) = n$
$g(n) = n + 10^6$

$$\lim \frac{f(n)}{[illegible]} = \lim \frac{n}{[illegible]} = \lim \frac{n/n}{[illegible]} = \lim \frac{1}{[illegible]\,10^6} = \frac{1}{1+0} = 1 \neq 0$$

$g(n) = n + 10^6$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n}{n+10^6} = \lim_{n\to\infty} \frac{n/n}{\frac{n}{n} + \frac{10^6}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{10^6}{n}} = \frac{1}{1+0} = 1 \neq 0$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n+10^6}{n} = \lim_{n\to\infty} \frac{n}{n} + \frac{10^6}{n} = \lim_{n\to\infty} 1+0 = 1 \neq 0$$

Ex: $f(n) = 2n$
$g(n) = n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{2n}{n} = 2 \neq 0$$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n}{2n} = 1/2 \neq 0$$

Homework: $n^4 - 6n^3 + 7n^2 + 8n + 10^6 \in \Theta(n^4)$

Def: We say $f(n)$ "grows faster than" $g(n)$, and write $f \in \omega(g)$ [little-omega] if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

Ex: $f(n) = n^2$ $\quad n^2 \in \omega(n)$ because $\lim_{n\to\infty} \frac{n^2}{n} = \lim_{n\to\infty} n = \infty$
$g(n) = n$

Def: We say $f(n)$ "grows slower than" $g(n)$, and write $f \in o(g)$, [little-oh] if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Ex: $n \in o(n^2)$ because $\lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$

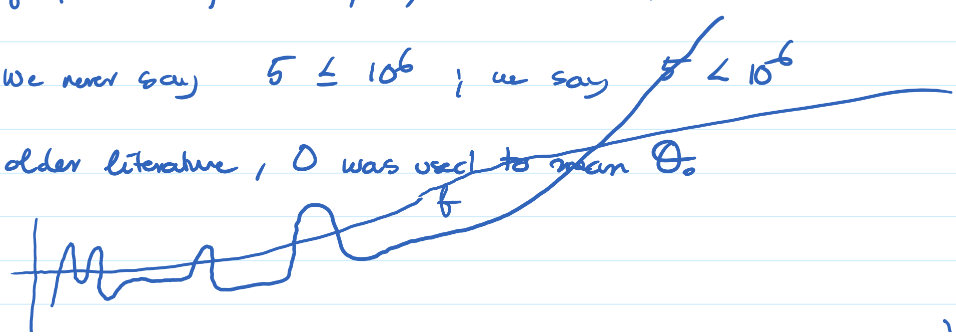
$\Theta \longrightarrow =$
$\omega \longrightarrow >$
$o \longrightarrow <$

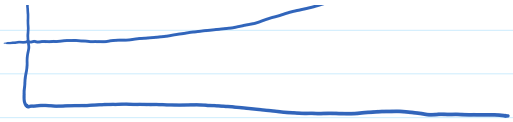
$\Omega \longrightarrow >=$ $\quad f \in \Omega(g)$ if $f \in \Theta(g)$ or $f \in \omega(g)$
$O \longrightarrow <=$ $\quad f \in O(g)$ if $f \in \Theta(g)$ or $f \in o(g)$

We use Ω and O when the exact relationship is not known.
If it is known, use Θ, ω, or o instead.

We never say $5 \leq 10^6$; we say $5 < 10^6$

In older literature, O was used to mean Θ.

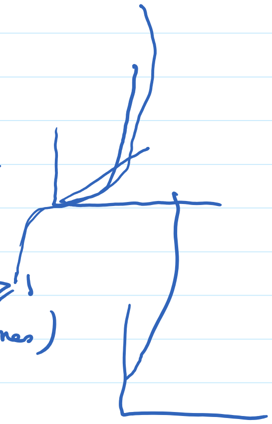




WELL-KNOWN FAMILIES OF RUNNING TIME FUNCTIONS

1) POLYNOMIAL FUNCTIONS : $1, n, n^2, n^3, \cdots, n^c, \cdots, \quad c \geq 0.$

$n^c = n \times n \times \cdots n$ (c times)



2) EXPONENTIAL FUNCTIONS: $2^n, 3^n, \pi^n, \cdots, c^n, \quad c \geq 1$

$c^n = c \times c \times c \cdots \times c$ (n times)

Exponentials grow very fast.

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$\vdots$

$2^{10} = 1024$

$2^{20} \sim 10^6$

$2^{30} \sim 10^9$

$\vdots$

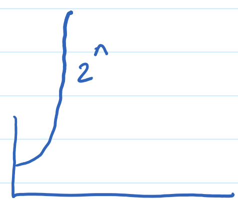
2^{100}

3) LOGARITHMIC FUNCTIONS $\log_2 n, \log_3 n, \cdots, \log_c n \qquad c > 1$

$\log_2 64$: $\quad \frac{64}{2} = 32$ ✓ $\quad \frac{8}{2} = 4$ ✓

$\frac{32}{2} = 16$ ✓ $\quad \frac{4}{2} = 2$ ✓

$\frac{16}{2} = 8$ ✓ $\quad \frac{2}{2} = 1$ ✓



$\boxed{\log_c n \text{ is the INVERSE of } c^n}$

$n \longrightarrow \log_c n \rightarrow c^{(\log_c n)} \longrightarrow n$

$n \longrightarrow c^n \longrightarrow \log_c(c^n) \longrightarrow n$

$\log_c n$ grows very slowly

CONVENTION : $\log_2 n \rightarrow \lg n$

$\log n \rightarrow \log_e n \rightarrow \ln n \qquad e = 2.71828\ldots$

Properties of $\log_c n$

1) $\log_c(xy) = \log_c x + \log_c y$

2) $\log_c(x^n) = n \log_c x$

3) $\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$ $\longrightarrow$ base change formula

$\leftarrow$ constant

3) $\boxed{\log_b x = \dfrac{\log_a x}{\log_a b}} \leftarrow$ constant

$\log_2 16 = 4$ $\quad\quad \log_{10} 16 = \dfrac{\log_2 16}{\log_2 10} = \dfrac{4}{3.\dots}$

Proof : $x = a^{\log_a x}$

$\log_b x = \log_b \left(a^{\log_a x}\right)$

$\log_b x = \log_a x \log_b a$

$\dfrac{\log_b x}{\log_b a} = \log_a x$

THEOREM 1 : $n^a \in \omega(n^b)$ if $a > b \geq 0$.

Proof :

$$\lim_{n\to\infty} \frac{n^a}{n^b} = \lim_{n\to\infty} n^{a-b>0} = \infty$$

Ex : $n^{1.1} \in \omega(n)$

THEOREM 2 : $a^n \in \omega(b^n)$ if $a > b > 1$

Proof :

$$\lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} \left(\frac{a}{b}\right)^n = \infty \quad \text{because } \frac{a}{b} > 1$$

Ex : $3^n \in \omega(2^n)$

THEOREM 3 : $\log_a n \in \Theta(\log_b n)$ for $a, b > 1$.

Proof :

$$\lim_{n\to\infty} \frac{\log_a n}{\log_b n} = \frac{\log_b n / \log_b a}{\log_b n} = \lim_{n\to\infty} \frac{1}{\log_b a} \neq 0$$

THEOREM 4 : $n^a \in \omega(\log_b n)$ for $a > 0$, $b > 1$.

Proof :

$$\lim_{n\to\infty} \frac{n^a}{\log_b n} \overset{\text{L'Hopital's rule}}{=} \lim_{n\to\infty} \frac{(n^a)'}{(\log_b n)'} = \lim_{n\to\infty} \frac{a n^{a-1}}{\frac{1}{n \log_e b}} = \lim_{n\to\infty} a n^a \log_e b = \infty$$

Ex : $n^{0.000000001} \in \omega(\log_2 n)$

THEOREM 5 : $a^n \in \omega(n^b)$ $\quad a > 1$, $b \geq 0$

Proof :

$$\lim_{n\to\infty} \frac{a^n}{n^b} = \lim_{n\to\infty} \frac{(a^n)'}{(n^b)'} = \lim_{n\to\infty} \frac{a^n \ln a}{b n^{b-1}} = \dots\dots = \lim_{n\to\infty} \frac{a^n (\ln a)^b}{b!} = \infty$$

Ex : $(1.000001)^n \in \omega(n^{10^6})$

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Fast

$n!$

$\vdots$

3^n

2^n

$\vdots$

n^3

$n^2 \longrightarrow n\lg n$

n

slow

$\log_2 n, \log_3 n, \cdots, \log_c n$

$1, 2, \cdots$, constant

$n^n = n \times n \times n \times \cdots \times n$

$n! = 1 \times 2 \times 3 \times \cdots \times n$

$2^n = 2 \times 2 \times 2 \times \quad \times 2$

$3^n = 3 \times 3 \times 3 \times \quad \times 3$

$\log(n!) \in \Theta(n\lg n)$

$n^{1.5}$ vs $n\lg n$

$$\lim_{n\to\infty} \frac{n^{1.5}}{n\lg n} = \lim_{n\to\infty} \frac{n^{0.5}}{\lg n} = \infty$$

$\in \Theta(n\lg n)$

$\frac{n}{2}\log(\frac{n}{2}) \le \log(n!) \le n\lg n$

$(\frac{n}{2})^{n/2} \le n! \le n^n$

$n! = 1 \times 2 \times \cdots \frac{n}{2} \times (\frac{n}{2}+1) \times (\frac{n}{2}+2) \cdots \times n$

$(\frac{n}{2})^{\frac{n}{2}}$ $\quad \frac{n}{2} \times \frac{n}{2} \times \cdots \frac{n}{2}$

CAUTION: If $\log f(n) \in \Theta(\log g(n))$ ~~then~~ $f(n) \in \Theta(g(n))$ FALSE

Counter example: $f(n) = n!$ $\quad \log(n!) \in \Theta(n\lg n)$

$g(n) = n^n$ $\quad \log(n^n) \in \Theta(n\lg n)$