Numerical Methods Lesson 4

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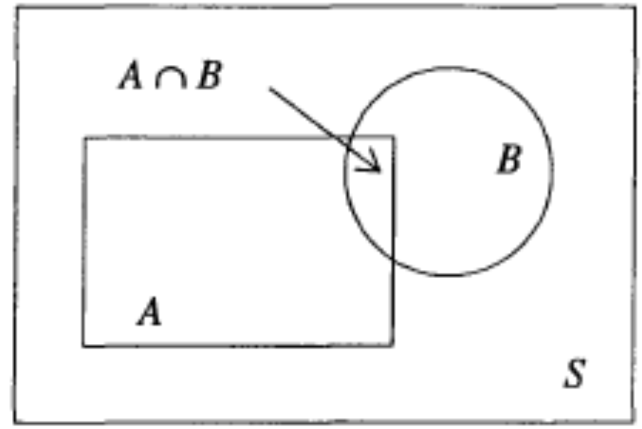
Goals for this Lesson

- Chapter 1 of the book: Statistical Data Analysis *by Glen Cowan*

Probability and Random variables

- A characteristic of a system is said to be **random** when it is not known or cannot be predicted with complete certainty.
- The degree of randomness can be quantified with the concept of **probability**.
- Consider a set S called the **sample space** consisting of a certain number of elements
- A variable that takes on a specific value for each element of the set S is called a **random variable**.
- **conditional probability:** $$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$
- Two subsets A and B are said to be **independent** if

$$P(A \cap B) = P(A)\,P(B).$$



$P(\overline{A}) = 1 - P(A)$ where $\overline{A}$ is the complement of A
$P(A \cup \overline{A}) = 1$
$0 \leq P(A) \leq 1$
$P(\emptyset) = 0$
if $A \subset B$, then $P(A) \leq P(B)$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Example:

- The classical experiment of rolling a six face die with numbers 1-6.
- In this example the random variable is the value obtained (n).
- The probability distribution is uniform: P(n) = 1/6
- For the two subsets A and B defined here:
  - A n B = {} , the empty set
  - A u B = S , the total set
  - P(A) = ½ = P(B)
  - P(A n B) = 0, prob of not obtaining any value is 0.
  - P(A u B) = 1, prob of obtaining any value is 1.
  - P (A | B) = P(A n B) / P(B) = 0 / 0.5 = 0 , prob of A given B is 0 because a value cannot be odd and even at the same time.

S: {1,2,3,4,5,6}

Consider subsets:

A: {1,3,5}

B: {2,4,6}

And probabilities:

P(n) = 1/6 , n ∈ S

Prob. density functions

- Continuous variables

probability to observe x in the interval $[x, x + dx] = f(x)dx$.

$$\int_S f(x)dx = 1, \qquad F(x) = \int_{-\infty}^{x} f(x')dx',$$

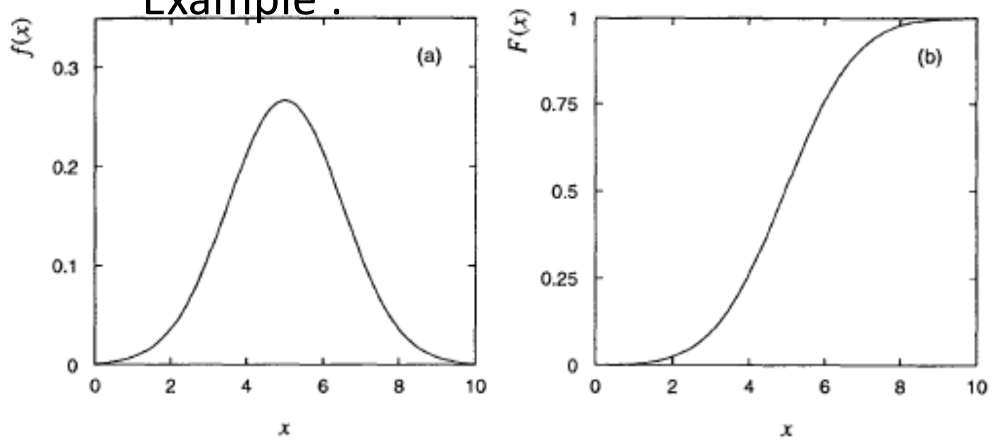
- Discrete variables

probability to observe value $x_i = P(x_i) = f_i$,

$$\sum_{i=1}^{N} f_i = 1. \qquad F(x) = \sum_{x_i \le x} P(x_i).$$

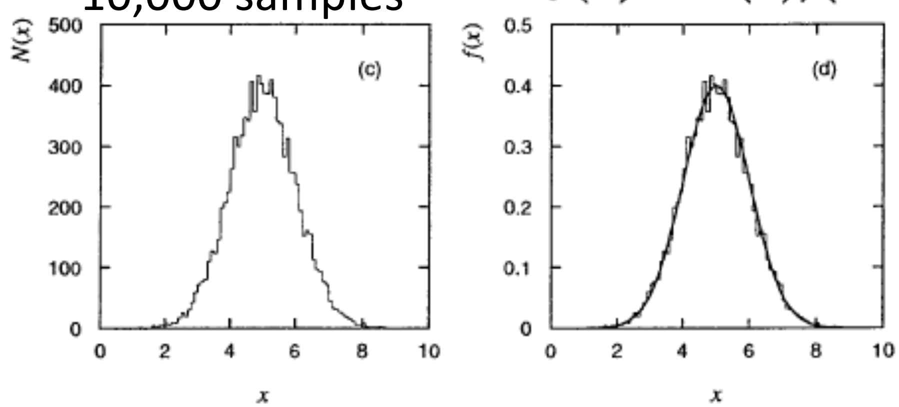
Example :

Cumulative distribution of f(x)

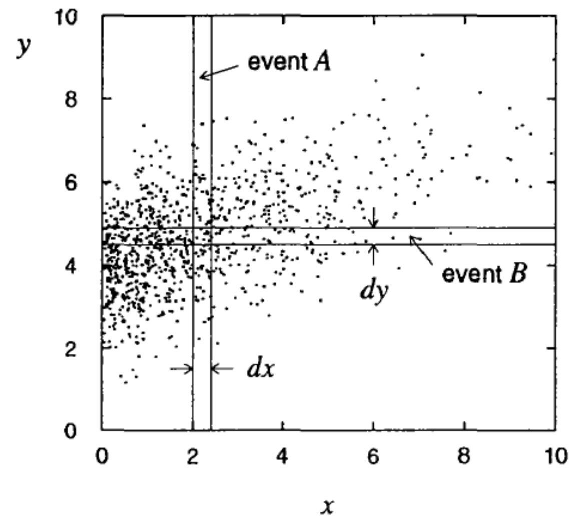


Experiment with 10,000 samples

$$f(x) = N(x)/(n\Delta x).$$



2 dimensional p.d.f.



- **Multidimensional** prob distributions are needed to describe experiments with events described by multiple properties.

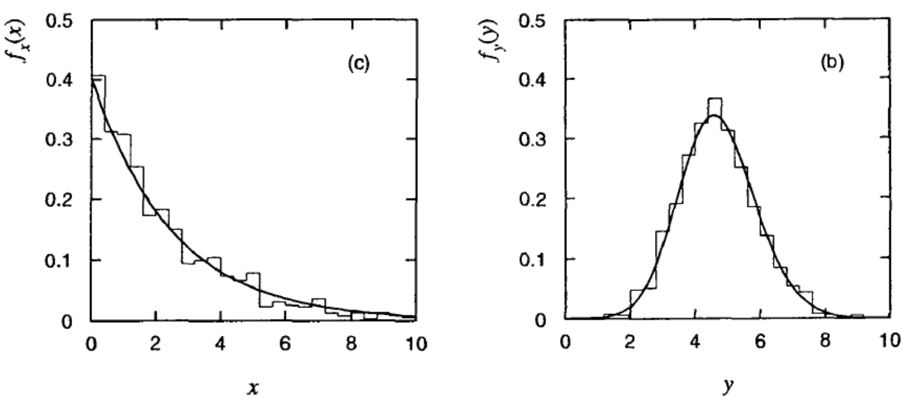
$$\int \int_S f(x,y)dxdy = 1.$$

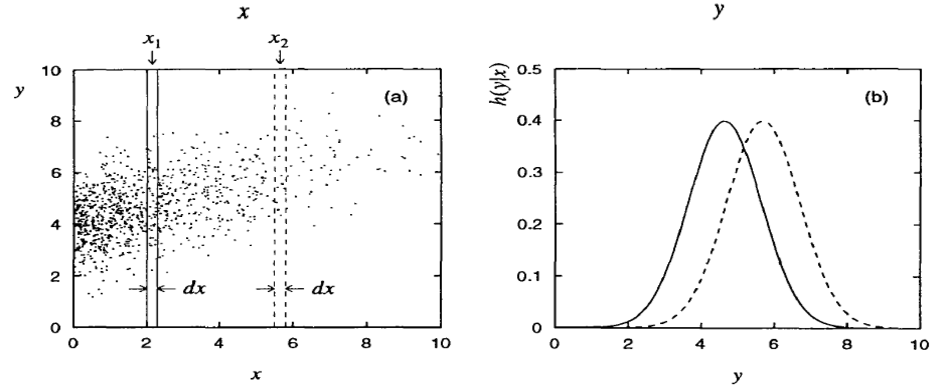
- **Marginal/projection** p.d.f.:

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)dy.$$

- **Conditional** p.d.f.:

$$h(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{f(x,y)}{\int f(x,y')dy'}.$$





Functions of random variables

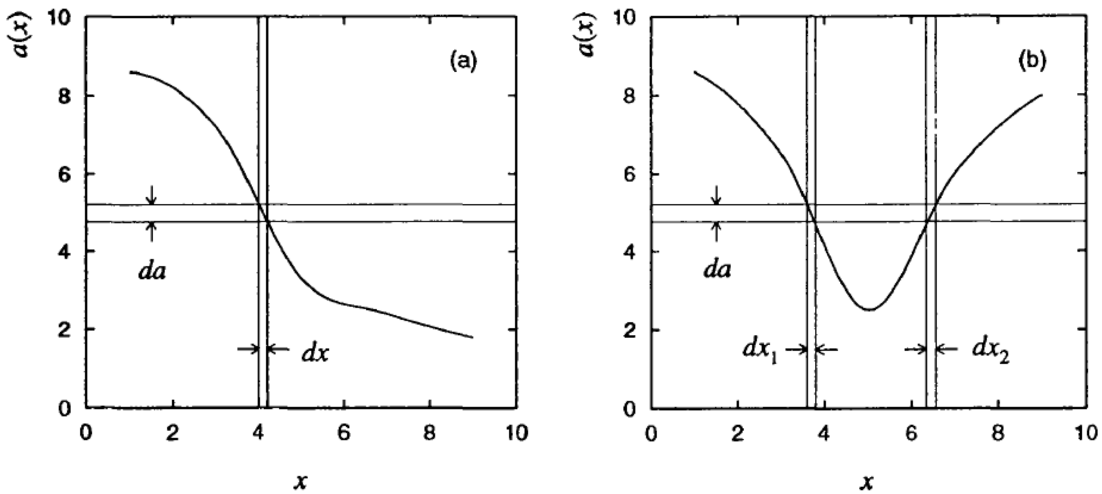
- Functions of random variables are themselves random variables. Suppose a(x) is a continuous function of a continuous random variable x, where x is distributed according to the p.d.f. *f*(x).
- In general:

$$g(a')da' = \int_{dS} f(x)dx,$$

- If a(x) is invertible then:

$$g(a) = f(x(a))\left|\frac{dx}{da}\right|.$$

1-1 map x→a(x) Not 1-1 map x→a(x)



- Generalized multidimensional formula, n random variables with m mappings :
  $a_1(x_1,..,x_n)$,
  …,
  $a_m(x_1,…,x_n)$.

  J is the Jacobian determinant for the transformation.

$$g(a_1,\ldots,a_n) = f(x_1,\ldots,x_n)|J|,$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_1}{\partial a_2} & \cdots & \frac{\partial x_1}{\partial a_n} \\ \frac{\partial x_2}{\partial a_1} & \frac{\partial x_2}{\partial a_2} & \cdots & \frac{\partial x_2}{\partial a_n} \\ \vdots & & & \vdots \\ & & \cdots & \frac{\partial x_n}{\partial a_n} \end{vmatrix}.$$

Expectation values

- **Population mean** E[x]

$$E[x] = \int_{-\infty}^{\infty} x f(x) dx = \mu.$$

- For any function a(x):

$$E[a] = \int_{-\infty}^{\infty} a g(a) da = \int_{-\infty}^{\infty} a(x) f(x) dx,$$

- **n_{th} moment :**

$$E[x^n] = \int_{-\infty}^{\infty} x^n f(x) dx = \mu'_n,$$

n_{th} central moment:

$$E[(x - E[x])^n] = \int_{-\infty}^{\infty} (x - \mu)^n f(x) dx = \mu_n,$$

Variance and standard deviation

- The second central moment is called the **variance** V[x]**:**

$$E[(x - E[x])^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \sigma^2 = V[x],$$

$$E[(x - E[x])^2] = E[x^2] - \mu^2.$$

- The square root of the variance is the **standard deviation**:

  σ = sqrt(V[x])

- Note that all distributions have these properties no matter what is the underlying model.

Exercises 1

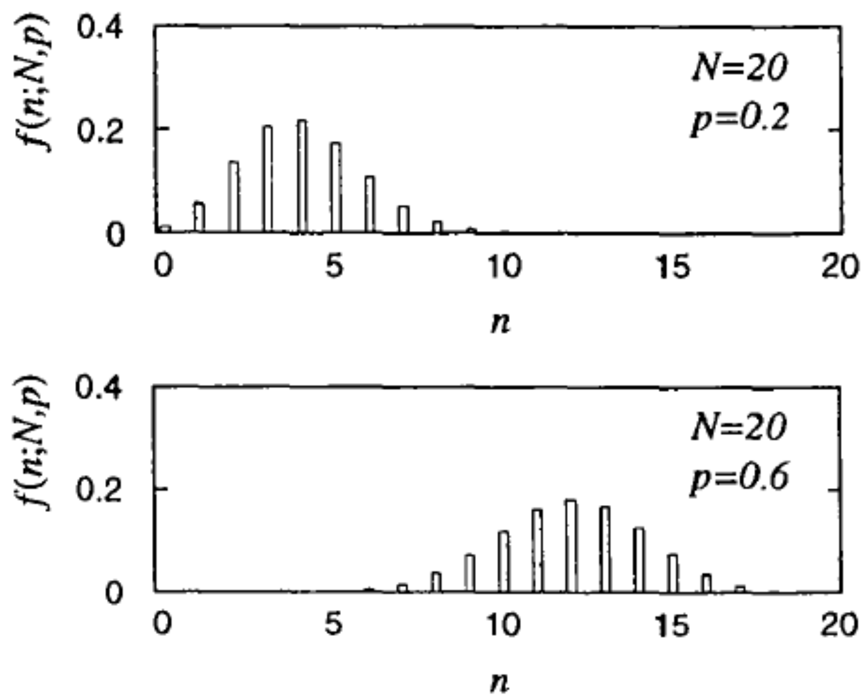
- Revive Lesson 1 code for generating events from a Gaussian distribution and generate a dataset of 100 events.
- Produce the same graphs as before using ROOT including statistics box which shows Mean and RMS
- Run the fit from Lesson 2 to obtain the fitted values for mu and sigma.
- Write your own code to read the dataset and calculate:
  - Expectation value E[x] = mu
  - Variance V[x] = E[(x-mu)^2] and standard deviation
- Rerun with 10,000 events.
- Write a txt file summarizing and comparing the results of the three approaches above. Publish this text file in the Lesseon 4 Results directory.

Part 2: 1D distributions

Binomial distribution

- Experiment with only two possible outcomes:
  - Prob of outcome #1 : p
  - Prob of outcome #2 : (1-p)
- Example: a coin flip
  - Heads prob p = 0.5
  - Tails prob (1-p)= 0.5
- Suppose we run this experiment N times.
  - What is the probability of obtaining n successes (e.g. Heads)?
  - ANSWER: the probability is given by the **binomial** distribution.

$$f(n; N, p) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n},$$

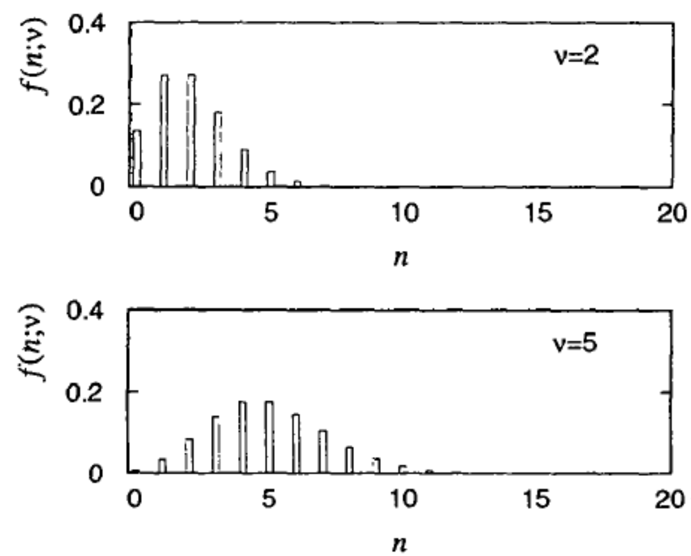


Poisson distribution

- The Poisson distribution is derived from the Binomial distribution in following limits:
  - N → infinity
  - p → 0

  But pN → $\boldsymbol{\nu}$ is constant
- The formula at the right can be derived mathematically by applying this limits
- This situation actually occurs frequently in real experiments where we make a large number of trials (e.g. particle collisions) and we are looking for a very rare signal.

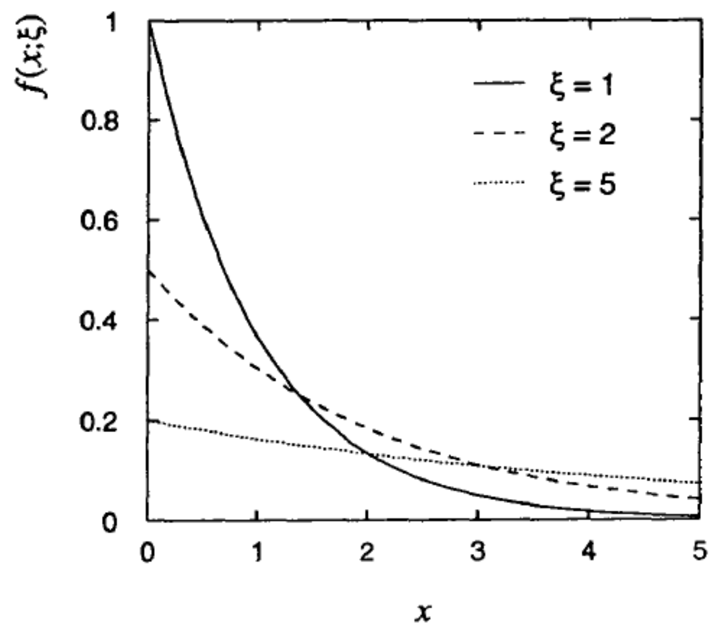
$$f(n;\nu) = \frac{\nu^n}{n!} e^{-\nu},$$



Exponential distribution

- The **Exponential** distribution has the formula the right.
- Example: A population of particles/nuclei with certain lifetime
- In the formula at the right if x = time then **ξ** is the lifetime of the particle. It is the time it takes the population to decrease by 1/e.

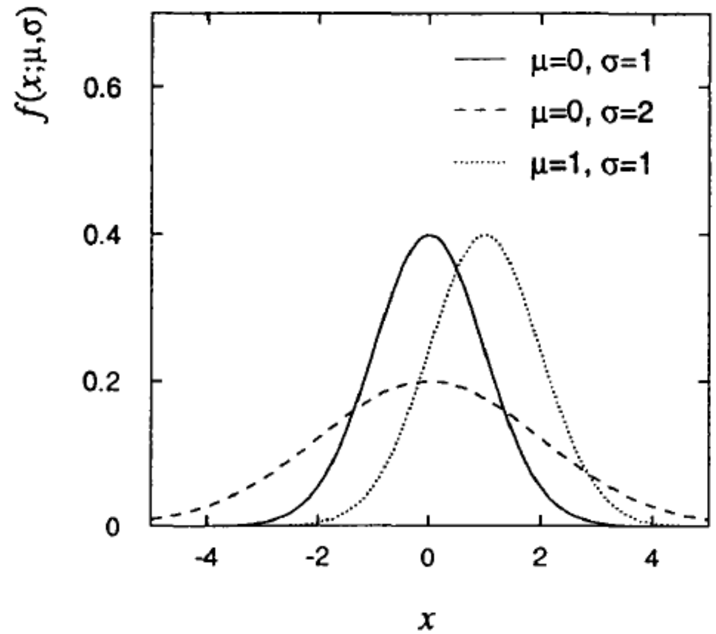
$$f(x;\xi) = \frac{1}{\xi} e^{-x/\xi}.$$



Gaussian distribution

- Very common distribution for many types of measurements.
- For example, where there is common value expected ($\boldsymbol{\mu}$) but there is a measurement uncertainty ($\boldsymbol{\sigma}$).

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right),$$



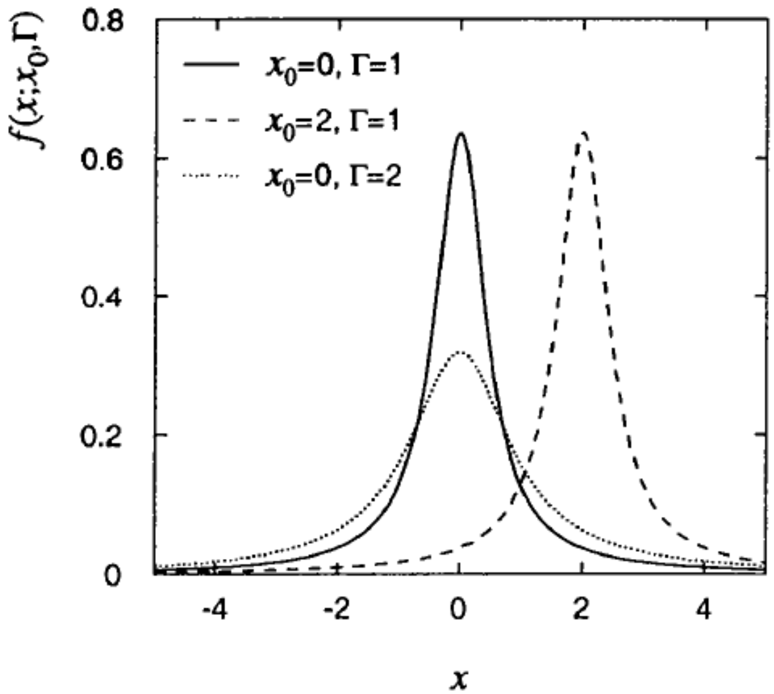
INTERESING:

- NOTE 1 : the Poisson distribution in the limit of large parameter $\boldsymbol{\nu}$ can be approximated by a Gaussian distribution with parameters: $\boldsymbol{\mu} = \boldsymbol{\nu}$ and $\boldsymbol{\sigma}^2 = \boldsymbol{\nu}$
- NOTE 2 (**Central Limit Theorem**) : the Gaussian distribution describes the distribution of the sum of many random variables with arbitrary distributions.

Breit-Wigner

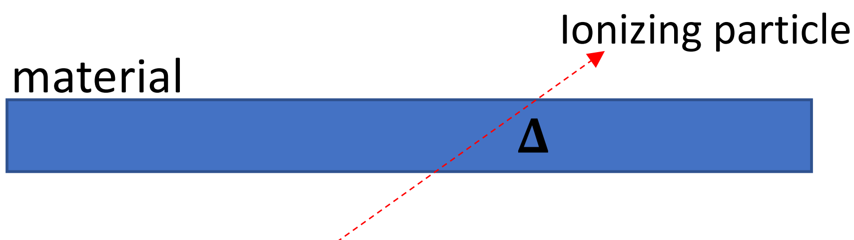
- Describes the mass distribution for an unstable particle (resonance).
- x_0 is the Mass
- $\boldsymbol{\Gamma}$ is the Width, related to the lifetime ($\boldsymbol{\tau}$) of the particle by the relation $\boldsymbol{\Gamma} = 1/\boldsymbol{\tau}$ in natural units.

$$f(x;\Gamma,x_0) = \frac{1}{\pi}\frac{\Gamma/2}{\Gamma^2/4+(x-x_0)^2},$$



Landau

- The **Landau** distribution describes the amount of energy deposited in a material by the passage of an ionizing particle.



- The formulas are complex, derived from first principles in physics. The distribution must solved numerically due to the integral.

$$f(\Delta;\beta) = \frac{1}{\xi}\phi(\lambda), \quad 0 \leq \Delta < \infty,$$

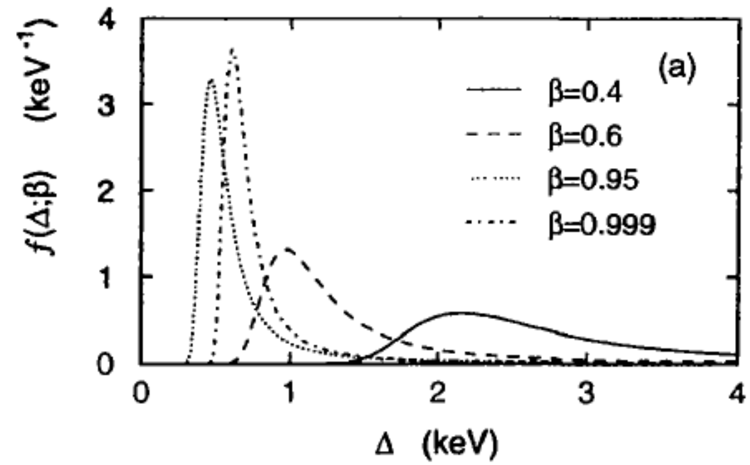
$$\xi = \frac{2\pi N_A e^4 z^2 \rho \sum Z}{m_e c^2 \sum A} \frac{d}{\beta^2},$$

$$\lambda = \frac{1}{\xi}\left[\Delta - \xi\left(\log\frac{\xi}{\epsilon'} + 1 - \gamma_E\right)\right],$$

$$\epsilon' = \frac{I^2 \exp(\beta^2)}{2 m_e c^2 \beta^2 \gamma^2},$$

$$\phi(\lambda) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \exp(u \log u + \lambda u)\, du,$$

$$\Delta_{mp} = \xi\left[\log(\xi/\epsilon') + 0.198\right],$$



Exercises 2

- Write code to implement the formulas and make the following graphs:
  - Binomial (1 graph) with N = 100 , p = 0.5 (coin flip experiment)
  - Poisson (2 graphs): v=1 and v = 10 , use range n: 0-20
  - Exponential (2 graphs) : parameter values 0.5 and 2
  - Breit-Wigner (2 graphs):
    - Mass = 91, Width = 2.5 (the Z boson)
    - Mass = 80, Width = 2.0 (the W boson)
- Use the TGraph object in ROOT
  - Use 1000 points
  - Set points TGraph::SetPoint(n,x,y)
  - Draw in Canvas: TGraph::Draw("alp")
- Publish these graphs including proper labels in the Lesson 4 Results folder.