Practical 324

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Regression analysis

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Simple regression

The simple regression analysis is a supervised machine learning approach to creating a model able to predict the value of one outcome variable Y based on one predictor variable X_1, by estimating the intercept b_0 and coefficient (slope) b_1, and accounting for a reasonable amount of error ϵ.

$$Y_i = (b_0 + b_1 * X_{i1}) + \epsilon_i$$

Least squares is the most commonly used approach to generate a regression model. This model fits a line to minimise the squared values of the **residuals** (errors), which are calculated as the squared difference between observed values the values predicted by the model.

$$redidual = \sum(observed - model)^2$$

A model is considered **robust** if the residuals do not show particular trends, which would indicate that *"something"* is interfering with the model. In particular, the assumption of the regression model are:

- **linearity:** the relationship is actually linear;
- **normality** of residuals: standard residuals are normally distributed with mean `0`;
- **homoscedasticity** of residuals: at each level of the predictor variable(s) the variance of the standard residuals should be the same (*homo-scedasticity*) rather than different (*hetero-scedasticity*);
- **independence** of residuals: adjacent standard residuals are not correlated.

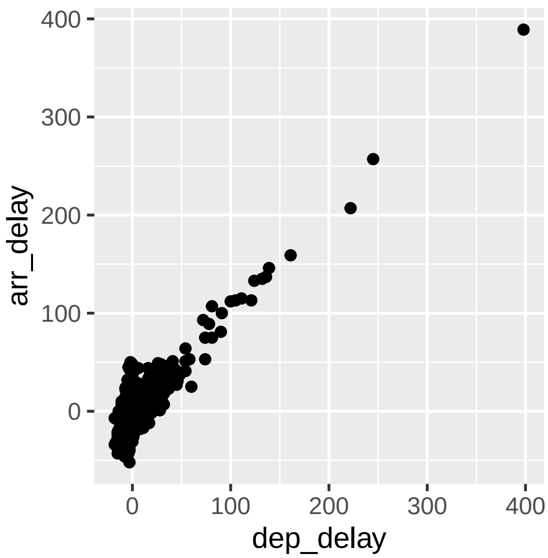
Example

The example that we have seen in the lecture illustrated how simple regression can be used to create a model to predict the arrival delay based on the departure delay of a flight, based on the data available in the `nycflights13` dataset for the flight on November 20th, 2013. The scatterplot below seems to indicate that the relationship is indeed linear.

$$arr\_delay_i = (Intercept + Coefficient_{dep\_delay} * dep\_delay_{i1}) + \epsilon_i$$

```
# Load the library
library(nycflights13)
```

```
# November 20th, 2013
flights_nov_20 <- nycflights13::flights %>%
  filter(!is.na(dep_delay), !is.na(arr_delay), month == 11, day ==20)
```



The code below generates the model using the function `lm`, and the function `summary` to obtain the summary of the results of the test. The model and summary are saved in the variables `delay_model` and `delay_model_summary`, respectively, for further use below. The variable `delay_model_summary` can then be called directly to visualise the result of the test.

```
# Classic R coding version
# delay_model <- lm(arr_delay ~ dep_delay, data = flights_nov_20)
# delay_model_summary <- summary(delay_model)

delay_model <- flights_nov_20 %$%
  lm(arr_delay ~ dep_delay)

delay_model_summary <- delay_model %>%
  summary()

delay_model_summary
```

```
##
## Call:
## lm(formula = arr_delay ~ dep_delay)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -43.906  -9.022  -1.758   8.678  57.052
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) -4.96717    0.43748  -11.35   <2e-16 ***
## dep_delay    1.04229    0.01788   58.28   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 13.62 on 972 degrees of freedom
```

```
## Multiple R-squared:  0.7775, Adjusted R-squared:  0.7773
## F-statistic:  3397 on 1 and 972 DF,  p-value: < 2.2e-16
```

The image below highlights the important values in the output: the adjusted R^2 value; the model significance value `p-value` and the related F-statistic information `F-statistic`; the intercept and `dep_delay` coefficient estimates in the `Estimate` column and the related significance values of in the column `Pr(>|t|)`.

```
Call:
lm(formula = arr_delay ~ dep_delay)

Residuals:
    Min      1Q  Median      3Q     Max
-43.906  -9.022  -1.758   8.678  57.052

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -4.96717    0.43748  -11.35   <2e-16 ***
dep_delay    1.04229    0.01788   58.28   <2e-16 ***
---
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Multiple R-squared:  0.7775,    Adjusted R-squared:  0.7773
F-statistic:  3397 on 1 and 972 DF,  p-value: < 2.2e-16
```
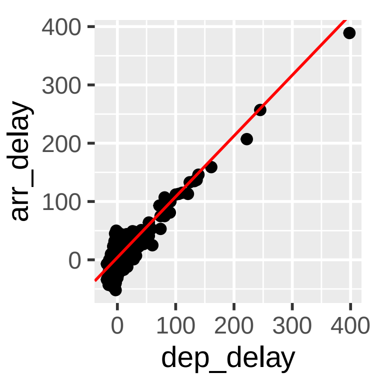

The output indicates:

- **p-value: < 2.2e-16**: $p < .001$ the model is significant;
  - derived by comparing the calulated **F-statistic** value to F distribution 3396.74 having specified degrees of freedom (1, 972);
  - Report as: $F(1, 972) = 3396.74$
- **Adjusted R-squared: 0.7773**: the departure delay can account for 77.73% of the arrival delay;
- **Coefficients**:
  - Intercept estimate -4.9672 is significant;
  - `dep_delay` coefficient (slope) estimate 1.0423 is significant.

```r
flights_nov_20 %>%
  ggplot(aes(x = dep_delay, y = arr_delay)) +
  geom_point() + coord_fixed(ratio = 1) +
  geom_abline(intercept = 4.0943, slope = 1.04229, color="red")
```

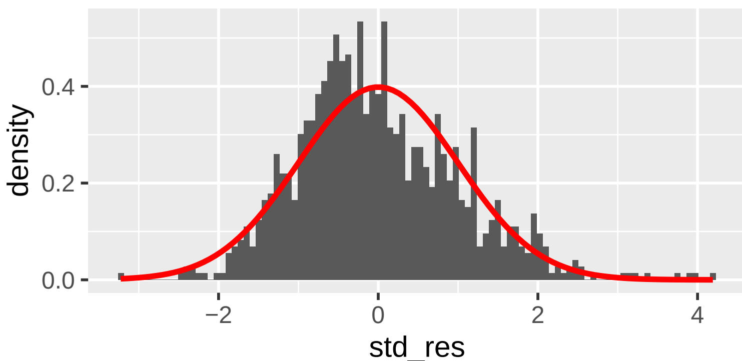


Checking assumptions

Normality The Shapiro-Wilk test can be used to check for the normality of standard residuals. The test should be not significant for robust models. In the example below, the standard residuals are *not* normally distributed. However, the plot further below does show that the distribution of the residuals is not far away from a normal distribution.

```
delay_model %>%
  rstandard() %>%
  shapiro.test()
```

```
##
##  Shapiro-Wilk normality test
##
## data:  .
## W = 0.98231, p-value = 1.73e-09
```



Homoscedasticity The Breusch-Pagan test can be used to check for the homoscedasticity of standard residuals. The test should be not significant for robust models. In the example below, the standard residuals are homoscedastic.

```
library(lmtest)

delay_model %>%
  bptest()
```

```
##
##  studentized Breusch-Pagan test
##
## data:  .
## BP = 0.017316, df = 1, p-value = 0.8953
```

Independence The Durbin-Watson test can be used to check for the independence of residuals. The test should be statistic should be close to 2 (between 1 and 3) and not significant for robust models. In the example below, the standard residuals might not be completely independent. Note, however, that the result depends on the order of the data.

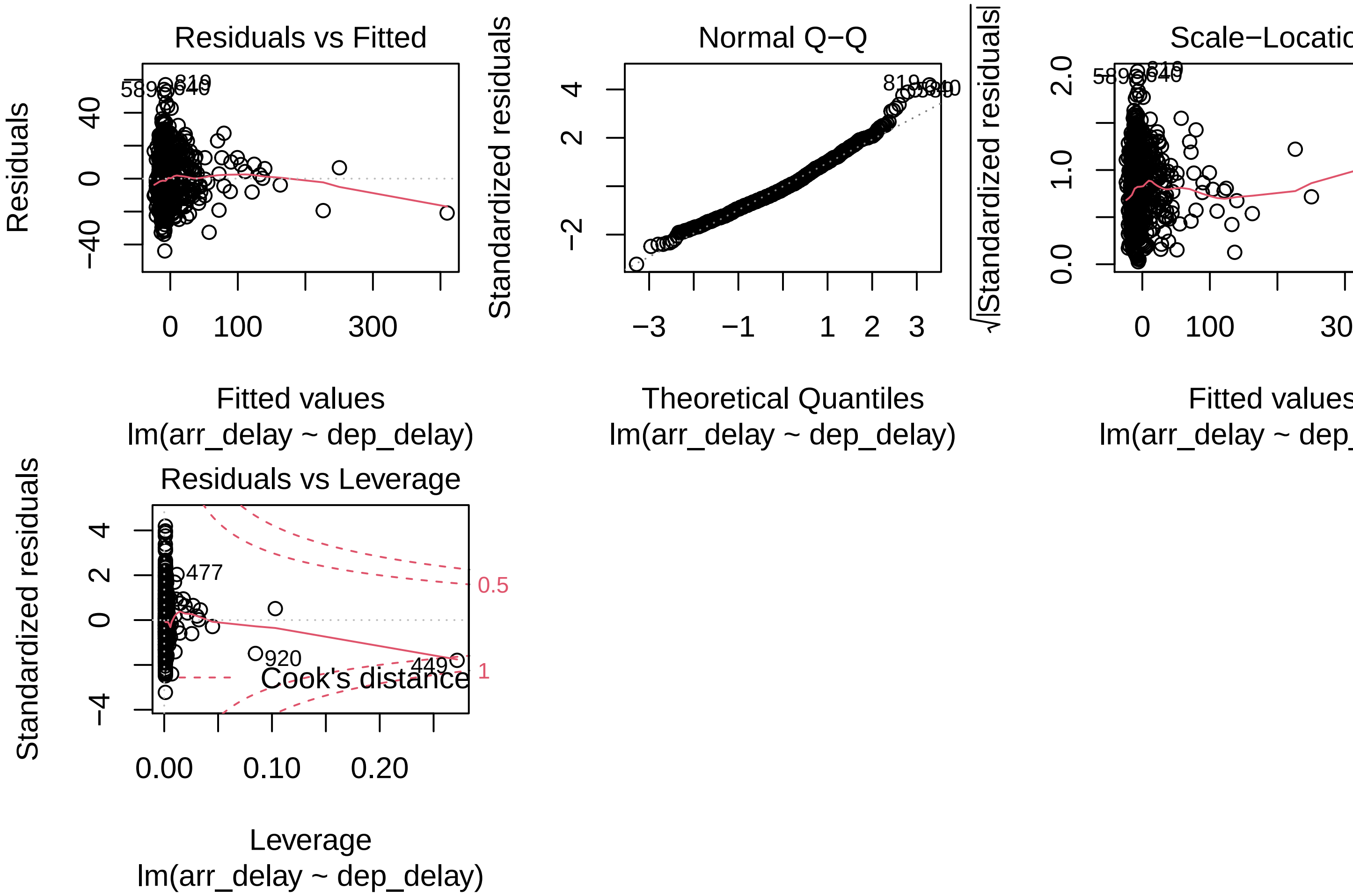
```
# Also part of the library lmtest
delay_model %>%
  dwtest()
```

```
##
##  Durbin-Watson test
##
## data:  .
```

```
## DW = 1.8731, p-value = 0.02358
## alternative hypothesis: true autocorrelation is greater than 0
```

Plots The `plot.lm` function can be used to further explore the residuals visuallly. Usage is illustrated below. The *Residuals vs Fitted* and *Scale-Location* plot provide an insight into the homoscedasticity of the residuals, the *Normal Q-Q* plot provides an illustration of the normality of the residuals, and the *Residuals vs Leverage* can be useful to identify exceptional cases (e.g., Cook's distance greater than 1).

```r
delay_model %>%
  plot()
```



How to report

Overall, we can say that the delay model computed above is fit ($F(1, 972) = 3396.74$, $p < .001$), indicating that the departure delay might account for 77.73% of the arrival delay. However the model is only partially robust. The residuals satisfy the homoscedasticity assumption (Breusch-Pagan test, $BP = 0.02$, $p = 0.9$), and the independence assumption (Durbin-Watson test, $DW = 1.87$, $p = 0.02$), but they are not normally distributed (Shapiro-Wilk test, $W = 0.98$, $p < .001$).

The `stargazer` function of the `stargazer` library can be applied to the model `delay_model` to generate a nicer output in RMarkdown PDF documents by including `results = "asis"` in the R snippet option.

```r
# Install stargazer if not yet installed
# install.packages("stargazer")
```

```r
library(stargazer)

# Not rendered in bookdown
stargazer(delay_model)
```

% Table created by stargazer v.5.2.2 by Marek Hlavac, Harvard University. E-mail: hlavac at fas.harvard.edu
% Date and time: Sun, Nov 29, 2020 - 11:44:24 PM

Table 1:

	Dependent variable:
	arr_delay
dep_delay	1.042^{***}
	(0.018)
Constant	-4.967^{***}
	(0.437)
Observations	974
R^2	0.778
Adjusted R^2	0.777
Residual Std. Error	13.618 (df = 972)
F Statistic	$3,396.742^{***}$ (df = 1; 972)
Note:	*p<0.1; **p<0.05; ***p<0.01

Multiple regression

The multiple regression analysis is a supervised machine learning approach to creating a model able to predict the value of one outcome variable Y based on two or more predictor variables $X_1 \dots X_M$, by estimating the intercept b_0 and the coefficients (slopes) $b_1 \dots b_M$, and accounting for a reasonable amount of error ϵ.

$$Y_i = (b_0 + b_1 * X_{i1} + b_2 * X_{i2} + \cdots + b_M * X_{iM}) + \epsilon_i$$

The assumptions are the same as the simple regression, plus the assumption of **no multicollinearity**: if two or more predictor variables are used in the model, each pair of variables not correlated. This assumption can be tested by checking the variance inflation factor (VIF). If the largest VIF value is greater than 10 or the average VIF is substantially greater than 1, there might be an issue of multicollinearity.

Example

The example below explores whether a regression model can be created to estimate the number of people in Leicester commuting to work using public transport (`u120`) in Leicester, using the number of people in different occupations as predictors.

For instance, occupations such as skilled traders usually require to travel some distances with equipment, thus the related variable `u163` is not included in the model, whereas professional and administrative occupations might be more likely to use public transportation to commute to work.

A multiple regression model can be specified in a similar way as a simple regression model, using the same `lm` function, but adding the additional predictor variables using a `+` operator.

```
leicester_2011OAC <- read_csv("2011_OAC_Raw_uVariables_Leicester.csv")
```

```
# u120: Method of Travel to Work, Public Transport
# u159: Employment, Managers, directors and senior officials
# u160: Employment, Professional occupations
# u161: Employment, Associate professional and technical occupations
# u162: Employment, Administrative and secretarial occupations
# u163: Employment, Skilled trades occupations
# u164: Employment, Caring, leisure and other service occupations
# u165: Employment, Sales and customer service occupations
# u166: Employment, Process, plant and machine operatives
# u167: Employment, Elementary occupations
public_transp_model <- leicester_2011OAC %$%
  lm(u120 ~ u160 + u162 + u164 + u165 + u167)

public_transp_model %>%
  summary()
```

```
##
## Call:
## lm(formula = u120 ~ u160 + u162 + u164 + u165 + u167)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -16.8606  -4.0247  -0.1084   3.7912  24.6359
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  3.19593    0.75048   4.258 2.26e-05 ***
## u160         0.06912    0.01416   4.881 1.24e-06 ***
## u162         0.17000    0.03328   5.108 3.93e-07 ***
## u164         0.28641    0.03589   7.979 4.17e-15 ***
## u165         0.21311    0.03107   6.858 1.25e-11 ***
## u167         0.32008    0.02156  14.846  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 5.977 on 963 degrees of freedom
## Multiple R-squared:  0.436,  Adjusted R-squared:  0.4331
## F-statistic: 148.9 on 5 and 963 DF,  p-value: < 2.2e-16
```

```
# Not rendered in bookdown
stargazer(public_transp_model)
```

% Table created by stargazer v.5.2.2 by Marek Hlavac, Harvard University. E-mail: hlavac at fas.harvard.edu
% Date and time: Sun, Nov 29, 2020 - 11:44:25 PM

```
public_transp_model %>%
  rstandard() %>%
  shapiro.test()
```

```
##
##  Shapiro-Wilk normality test
##
## data:  .
## W = 0.9969, p-value = 0.05628
```

Table 2:

	Dependent variable:
	u120
u160	0.069***
	(0.014)
u162	0.170***
	(0.033)
u164	0.286***
	(0.036)
u165	0.213***
	(0.031)
u167	0.320***
	(0.022)
Constant	3.196***
	(0.750)
Observations	969
R^2	0.436
Adjusted R^2	0.433
Residual Std. Error	5.977 (df = 963)
F Statistic	148.884*** (df = 5; 963)
Note:	*p<0.1; **p<0.05; ***p<0.01

```r
public_transp_model %>%
  bptest()
```

```
##
##  studentized Breusch-Pagan test
##
## data:  .
## BP = 45.986, df = 5, p-value = 9.142e-09
```

```r
public_transp_model %>%
  dwtest()
```

```
##
##  Durbin-Watson test
##
## data:  .
## DW = 1.8463, p-value = 0.007967
## alternative hypothesis: true autocorrelation is greater than 0
```

```r
library(car)

public_transp_model %>%
  vif()
```

```
##     u160     u162     u164     u165     u167
## 1.405480 1.486768 1.163760 1.353682 1.428418
```

The output above suggests that the model is fit ($F(5, 963) = 148.88$, $p < .001$), indicating that a model based on the number of people working in the five selected occupations can account for 43.31% of the number of people using public transportation to commute to work. However the model is only partially robust. The residuals are normally distributed (Shapiro-Wilk test, $W = 1$, $p = 0.06$) and there seems to be no multicollinearity with average VIF 1.37, but the residuals don't satisfy the homoscedasticity assumption (Breusch-Pagan test, $BP = 45.99$, $p < .001$), nor the independence assumption (Durbin-Watson test, $DW = 1.85$, $p < .01$).

The coefficient values calculated by the `lm` functions are important to create the model, and provide useful information. For instance, the coefficient for the variable `u165` is `0.21`, which indicates that if the number of people employed in sales and customer service occupations increases by one unit, the number of people using public transportation to commute to work increases by `0.21` units, according to the model. The coefficients also indicate that the number of people in elementary occupations has the biggest impact (in the context of the variables selected for the model) on the number of people using public transportation to commute to wor0, whereas the number of people in professional occupations has the lowest impact.

In this example, all variables use the same unit and are of a similar type, which makes interpretating the model relatively simple. When that is not the case, it can be useful to look at the standardized β, which provide the same information but measured in terms of standard deviation, which make comparisons between variables of different types easier to draw. For instance, the values calculated below using the function `lm.beta` of the library `QuantPsyc` indicate that if the number of people employed in sales and customer service occupations increases by one standard deviation, the number of people using public transportation to commute to work increases by `0.19` standard deviations, according to the model.

```r
# Install lm.beta library if necessary
# install.packages("lm.beta")
library(lm.beta)

lm.beta(public_transp_model)
```

```
##
```

```
## Call:
## lm(formula = u120 ~ u160 + u162 + u164 + u165 + u167)
##
## Standardized Coefficients::
## (Intercept)        u160        u162        u164        u165        u167
##   0.0000000   0.1400270   0.1507236   0.2083107   0.1931035   0.4293988
```

Exercise 9.1

Question 9.1.2: Is the number of people using public transportation to commute to work statistically, linearly related to mean age (`u020`)?

Question 9.1.3: Is the number of people using public transportation to commute to work statistically, linearly related to (a subset of) the age structure categories (`u007` to `u019`)?