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EXPRESSION THEORY AND THE MEASUREMENT OF APPARENTLY LABILE VALUES*

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Abstract

Systematic inconsistencies in judgments and choices have been attributed to either the encoding of information and/or its evaluation. However, anomalies in decision making can also result from the way people express their internal experiences on different overt response scales. A model that relates underlying preferences for gambles to a variety of observable judgments and choices (e.g. binary choices, judgments of minimum selling price, ratings of attractiveness) is developed. The model, called Expression Theory, explains how stable and consistent underlying preferences can result in inconsistent overt responses, including "preference reversal" phenomena. Thus, one can choose alternative A over B, but rate B as more attractive than A. An axiomatic foundation is developed for Expression Theory that gives insight into the structure of the model and suggests further empirical work. The relative importance of the axioms to the substantive theory is discussed, along with issues relevant to the measurement of the numerical scales.

1. Introduction

Systematic inconsistencies in people's decision-making behavior have provided the focus of much recent research. Such anomalies as intransitive choices (Tversky [25]), violations of dominance (Tversky and Kahneman [26]), and contradictory judgments and choices (Lichtenstein and Slovic [17]) are now well established. Such inconsistencies have suggested to some (Fischhoff et al. [4]) that stable preferences may not exist. Given the sensitivity of judgments and choices to context and task variables, it is important to achieve a theoretical reconciliation of the inconsistencies. Researchers interested in the rationality of decision making have responded by examining generaliza-

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tions of utility theory, to see whether the anomalous behavior might be encompassed by compelling normative principles (see Weber and Camerer [31] for a review of expected utility generalizations). Other investigators have focused on the psychological processes of decision making, quite apart from the possibility of reconciling the behavior with rational principles. Among the latter, some researchers (Tversky and Kahneman [26]) attribute inconsistent behavior to the way decision alternatives and their components – acts, contingencies, and outcomes – are framed or encoded. Others (e.g. Payne [22]) emphasize that people have a repertoire of strategies for evaluating and integrating information, and that contextual factors affect how and when these strategies are used in particular circumstances.

A third possibility provides the focus for this paper. While encoding and evaluation stratgies can undoubtedly result in inconsistent decisions, the lability of preference is not thereby proved. Since no one has a direct and unerring view of people's underlying preferences (assuming they exist at all), it is only the observed decisions, the *expressed* preferences, that can be said to be anomalous. At least some decision anomalies may be due to the manner in which people convert their internal experiences and impressions into overt responses. Indeed, when one considers the wide range of responses that have been used to measure "preference", e.g. binary choices, multi-alternative choices, rank orderings, minimum selling prices, maximum buying prices, ratings, certainty equivalents, lottery equivalents, and so on, it seems plausible that the conversion of preferences to different response scales may involve distinct psychological processes worthy of study in themselves (cf. Upshaw [29]).

The variety of approaches can be illustrated with a particularly intriguing decision anomaly, the preference reversal phenomenon (Lichtenstein and Slovic [17]). This anomaly refers to the fact that people who choose A over B often ask for *less* money to sell A than B, where A and B are monetary gambles. This phenomenon is so striking, counterintuitive, and contrary to standard choice theory, that it naturally has attracted the scrutiny of many researchers. In one line of research, investigators considered whether preference reversals might result from experimental artifact or inadequate controls (e.g. Grether and Plott [9]; see Slovic and Lichtenstein [24] for a review). However, attempts to eliminate the reversals by increasing incentives, checking subjects' understanding of the task, etc., have proved remarkably unsuccessful. Although some experimental manipulations have managed to reduce the frequency of reversals, the phenomenon has stubbornly refused to disappear.

In a second line of research, generalizations of expected utility theory have been brought to bear on preference reversals. Loomes and Sugden [19,20] and Fishburn [5,6] independently developed models that attribute reversals to an intransitive choice process. Karni and Safra [14] (see also Holt [10]) pointed out that widely-used experimental procedures for determining subjects' actual opportunities to sell or play their gambles (Becker, DeGroot and Marschak [1]) do not necessarily encourage subjects to announce prices that equal their certainty equivalents for the gambles. Subjects satisfying Karni and Safra's [14] model of expected utility with rank-dependent probabilities are predicted to produce preference reversals. Unfortunately for these explanations based on expected utility generalizations, Tversky et al. [28] observed preference reversals in the absence of intransitive choice, while using elicitation procedures that are not open to the Karni–Safra criticism of the Becker–DeGroot–Marschak method.

A third line of research has attempted to understand the preference reversal phenomenon in terms of psychological processes, regardless of its implications for normative models. Lichtenstein and Slovic's [17,18] original explanation hinged on subjects' use of an anchoring-and-adjustment heuristic to set minimum selling prices, while using another decision strategy to determine choice. A different way in which the evaluation process might depend on the response mode has recently been proposed by Tversky, Sattath, and Slovic [27]. These authors suggest that subjects give more weight to stimulus dimensions which are compatible with the response scale. Thus, the monetary outcomes of a gamble receive more weight when the response is a monetary amount than when the response is a choice. Goldstein and Einhorn [7] focused on the expression process by which subjects convert their subjective impressions into overt responses, rather than the evaluation process by which they form their impressions. They noted that previous research on preference reversals had confounded the response method (judgment of individual stimuli versus choice among two or more stimuli) with the scale used to express preferences (monetary amounts versus degree of attractiveness). By crossing judgment and choice with attractiveness and minimum selling price (2 x 2), it was shown that there are at least six preference reversal *phenomena,* one for each pair of the four tasks considered. Goldstein and Einhorn [7] then developed a quantitative model of judgment and choice that accounts for both the magnitude and direction of the various reversals. The model attributes the different "preference reversals" to the manner in which people translate their subjective evaluations into numerical judgments and choices.

The present paper seeks to extend the work of Goldstein and Einhorn [7] by putting their model on a firm axiomatic foundation. The axioms are not intended to be prescriptive principles, but to provide insight into the structure of the (descriptive) model, to give guidance regarding further empirical work, and to help in considering whether the model can be generalized fruitfully to other phenomena of psychological and economic interest. The Goldstein–Einhorn model is of interest only partly because it offers a possible explanation of preference reversal phenomena. In view of the robustness of the phenomena despite efforts to eliminate them (Slovic and Lichtenstein [24]), it seems likely that the phenomena have multiple causes and that no single explanation will be complete. Nonetheless, the model is of interest because it provides a framework for considering a set of psychological processes that are an integral part of decision making and which have not received the attention they deserve. Before launching into an axiomatization, we briefly review the model as it applies to preferential binary choice, and judgments of attractiveness and minimum selling price.

2. Expression theory

2.1. TRIPARTITE MODEL OF PREFERENCE

A multiplicity of psychological processes are involved between the time a person observes a stimulus and emits a response. We find it useful to distinguish three subsets of processes that are functionally distinct. Specifically, the theory rests on a conception of the decision-making process as consisting of the following three interrelated functions: (1) alternatives are perceived and *encoded*; (2) encoded alternatives are *evaluated*; and (3) evaluations are *expressed* on a response scale of some type (e.g. price). Encoding, evaluation, and expression are each referred to as a "process" as a shorthand term for the collection of psychological processes that serve the corresponding function. The encoding, evaluation, and expression processes need not operate in a strict sequential manner. For example, the partial evaluation of some items of information might create expectations that influence the encoding of additional information. Although the third process, expression, has not typically been considered, this process helps to explain preference reversals and other decision anomalies while providing a theoretical account that is both general and falsifiable. To underscore that the theory is primarily concerned with the expression of preference rather than with encoding or evaluation, the model described below is referred to as Expression Theory. Because the development of the model was motivated by the preference reversal phenomenon, and research on this phenomenon has concentrated on simple two-outcome monetary gambles, expression theory has been developed for that domain.

2.2. BASIC ASSUMPTIONS ABOUT ENCODING AND EVALUATION

To focus attention on the expression process, the encoding and evaluation processes are deliberately left vague. Let $g = (W, p, L)$ denote a gamble in which W is received with probability p, and L is received with probability $1 - p$, where $0 \leqslant p \leqslant 1$ and $W \geqslant L$. Let $u(g)$ denote the subjective worth or basic evaluation of the gamble g, which we also call the "utility" of g. Then we can write

$$u(g) = u(W) - \Delta(g)\,[u(W) - u(L)]\ , \tag{1}$$

where $u(W)$ is the utility of W, $u(L)$ is the utility of L, and Δ is a weight between 0 and 1 that depends on the gamble g. Equation (1) is compatible with many theories of the evaluation process. The only assumption implicit in eq. (1) is that the utility of g is between utilities of W and L. The weight $\Delta(g)$ plays a crucial role in expression theory. Solving eq. (1) for $\Delta(g)$, we have

$$\Delta(g) = [u(W) - u(g)]\,/\,[u(W) - u(L)]\ . \tag{2}$$

One can interpret $\Delta(g)$ as the gamble's proportional reduction in utility due to uncertainty and other situational factors.

2.3. EXPRESSION

Expressing opinions about alternatives requires that some correspondence be established between one's subjective experience of the alternatives and overt responses of the form requested. To provide an account of how people establish a mapping from subjective impressions to overt responses, expression theory distinguishes between what the person must *do*, e.g. judge or choose, and the *scale* the person must do it with, e.g. minimum selling price or attractiveness. Thus, we discuss in turn: (a) choice with respect to attractiveness, (b) judgments of attractiveness (i.e. numerical attractiveness ratings), and (c) judgments of minimum selling price.

Choice/attractiveness. It is assumed that people choose between easily discriminable gambles by comparing the basic evaluations of the gambles. That is, gamble 1 (g_1) will be chosen over gamble 2 (g_2) when $u(g_1) > u(g_2)$. This assumption underlies a great many theories of decision making under uncertainty and will not be discussed further.

Judgment/attractiveness. The difficulty in establishing a direct correspondence between subjective evaluations and overt responses is that the two are often incommensurable. In particular, rating points do not themselves give rise to a sense of attractiveness that can be matched with the attractiveness of the gamble to be rated. The conceptual framework underlying expression theory holds that people overcome the incommensurability between subjective impressions and overt responses by finding a way to normalize both, turning impressions and responses into standardized or even unitless *relative* impressions and *relative* responses that can be matched directly. Evidence in support of this general approach comes from theoretical work in the psychophysics of cross-modality matching (Krantz [15], Shepard [23]).

Specifically, it is proposed that people judge an alternative by considering it in the context of a remembered or imagined collection of other alternatives that have or might have been made available (cf. Kahneman and Miller [13]). This context enables people to normalize their subjective impressions by considering *relations* between the alternatives and its contextual distribution, analogous to standardized z-scores or percentile ranks. The pertinent relation between the alternative and its context may depend on the nature of the response scale, and the ease with which special values of the relative impressions (e.g. the extremes) can be mapped into responses to provide a benchmark for mapping others. The recalled or imagined context may depend on the details of the experimental setting.

Most experimental studies of gambling preferences try to control for strategic portfolio effects with instructions and incentives for subjects to consider each gamble as a separate matter, distinct from the other gambles to be considered.

We assume that subjects comply with these instructions by refraining from using the set of experimental gambles as the context for judging any particular one of them. Instead, subjects imagine an appropriate context for each new gamble as it comes up. Specifically, we assume that subjects consider each gamble in an imagined context of gambles with the same approximate outcomes but different probabilities. This is consistent with the greater familiarity most subjects have with monetary amounts, and it provides the subject with a particularly useful *bounded* context. No gamble in this contextual set can be (much) more attractive than the gamble's largest potential outcome, nor (much) less attractive than its smallest potential outcome. Therefore, a convenient way to relativize one's subjective evaluation of the gamble is to compare its attractiveness with the attractiveness of these bounds, i.e. to consider the value of $\Delta(g)$ as shown in eq. (2).

Many numerical response scales are also bounded. In particular, attractiveness ratings must fall in a range specified by the experimenter. According to expression theory, people translate their subjective evaluations into overt ratings in a manner that capitalizes on the boundedness of both the imagined context and the response scale. To set a rating for gamble g, $R(g)$, on a 20-point scale, say, the person must find a point on the rating scale between 20 and 0 that corresponds to $u(g)$, which is between $u(W)$ and $u(L)$. People are hypothesized to choose a response, $R(g)$, whose relationship with the bounds of the response scale, 20 and 0, emulates the relationship between the subjective evaluation, $u(g)$, and the contextual bounds that limit subjective evaluations, $u(W)$ and $u(L)$. That is, people try to match the *relative* values of utility and ratings. Thus, a rating $R(g)$ is sought for which the relative reduction from 20,

$$\Delta''(g) = [20 - R(g)] / [20 - 0] \;, \tag{3}$$

matches the relative utility reduction $\Delta(g)$. It is assumed that people can establish a monotonic relation between these relative values. Thus, $\Delta'' = h(\Delta)$, for some increasing function h satisfying $h(0) = 0$ and $h(1) = 1$.[1] Because of this reliance on bounded evaluations and response scales, we refer to the process as one of "subjective interpolation".

Judgment/minimum selling price. Setting minimum selling prices also calls for mapping subjective evaluations to overt responses. It may appear that the pricing task does permit a direct correspondence to be established, because amounts of money do give rise to a subjective sense of attractiveness, as do the gambles themselves. However, because our experimental subjects report difficulty and lack of confidence in setting minimum selling prices, we assume that the uncertainty involved in the gamble causes a difference to remain in the quality of one's subjective impressions of gambles and prices. Thus, there is still an incommensurability to be overcome. Moreover, the range of possible prices for a gamble is bounded by the gamble's extreme outcomes, just as the possible ratings are bounded by the extremes of the rating scale.

As a result, the expression theory model treats the pricing task as analogous to the rating task. To set a minimum selling price for gamble g, $MS(g)$, the person must find a point between W and L on the monetary scale that corresponds to $u(g)$, which is between $u(W)$ and $u(L)$. Again, people are hypothesized to use subjective interpolation in choosing a response, $MS(g)$, whose relationship with the bounds of the response scale, W and L, emulates the relationship between the subjective evaluation, $u(g)$, and the contextual bounds that limit subjective evaluations, $u(W)$ and $u(L)$. That is, people try to match the *relative* values of utility and price. Thus, a price $MS(g)$ is sought for which the relative price reduction from W,

$$\Delta'(g) = [W - MS(g)] / [W - L] \ , \tag{4}$$

matches the relative utility reduction, $\Delta(g)$. As before, it is assumed that people can establish a monotonic relation between these relative values. Thus, $\Delta' = f(\Delta)$ for an increasing function f satisfying $f(0) = 0$ and $f(1) = 1$.

2.4. PREFERENCE REVERSALS

Although subjective interpolation is a useful heuristic for solving the problem of matching incommensurables, it is similar to other heuristics in that its thoughtless application can lead to anomalous and inconsistent behavior. Consider, for example, an almost-sure thing $g_1 = (\$4, 0.97, \$0)$, and a relative long-shot $g_2 = (\$16, 0.33, \$0)$, and suppose that a person's basic evaluations satisfy expected utility with $u(x) = \sqrt{x}$, for $x \geqslant 0$. Then, $u(g_1) = 1.94$ and $u(g_2) = 1.32$, so g_1 is chosen over g_2. According to expression theory, the minimum selling prices are formed by subjective interpolation. For g_1, $\Delta_1 = [\sqrt{4} - 1.94] / [\sqrt{4} - \sqrt{0}] = 0.03$. If, for ease of illustration, we assume that f is linear, then we also have $0.03 = \Delta'_1 = [4 - MS(g_1)] / [4 - 0]$, which gives $MS(g_1) = \$3.88$. For g_2, $\Delta_2 = [\sqrt{16} - 1.32] / [\sqrt{16} - \sqrt{0}] = 0.67$. This gives $0.67 = \Delta'_2 = [16 - MS(g_2)] / [16 - 0]$, which yields $MS(g_2) = \$5.28$. Thus, the typical preference reversal is obtained: the sure-thing is chosen, but the long-shot receives the higher price.

Moreover, it can be shown that if f is linear and $u(0) = 0$, expression theory makes a very strong prediction about people with concave utility functions (whether or not thay satisfy expected utility): the opposite pattern in which g_2 is chosen and g_1 receives the higher price is *impossible*. Some other predictions require fewer assumptions. Irrespective of the utility function or the shape of h, the model forbids a subject from choosing g_1 but rating g_2 as more attractive. Thus, expression theory is highly falsifiable. Further details about expression theory's predictions are given by Goldstein and Einhorn [7].

2.5. MEASUREMENT ISSUES

Utility. The concept of utility, applied both to gambles and to the outcomes of gambles, plays a crucial role in expression theory. However, the term "utility" is ambiguous, and the interpretation of "$u(g)$" is not on a firm footing. The "utility" of gamble g, $u(g)$, was defined earlier as "the subjective worth or basic evaluation" of the gamble g. This phrase suggests a Bernoullian notion along the lines of "hedonic tone", and this interpretation does guide our intuitive picture of a person considering the attractiveness of g. Nevertheless, when it comes to using expression theory to *measure* $u(g)$, "utility" derives its meaning from its relations with other variables in the theory. Thus, to paraphrase von Neumann and Morgenstern [30], utility is what makes the theory work, if anything can. To explicate the conditions under which *some* interpretation of "utility" *can* make the theory work, a representation theorem is needed. An associated uniqueness theorem will then determine what statements about utility are meaningful.

From the above description of expression theory, it is apparent that a desirable uniqueness theorem would show utility to be an interval scale. If u turns out to be an interval scale, then the assumptions used to derive predictions from expression theory would be on firm ground. Specifically, the assumption that $u(0) = 0$ would be formally permissible, and it would be meaningful to speak of the utility function as concave or convex.[2] Most important, utility must be at least as strong as an interval scale in order for the proportional adjustment Δ to be meaningful. Since Δ plays such a central role in expression theory, the need for this is clear. Moreover, if u is an interval scale, then $\Delta(g) = [u(W) - u(g)] / [u(W) - u(L)]$ can be measured as a derived quantity, and it is possible to separate the effects of Δ from the effects of the functions f and h.

Simultaneous representations. It is interesting to note that the present case implicitly involves three representation theorems, one each for choice, minimum selling price, and ratings. However, the three representations are interlocked, in that the *same* utility and delta functions appear in two or more of the representations. Treating the three representation problems separately would not guarantee that these functions exhibit the proper invariance. A representation theorem is needed that uses a single utility function to relate the choice ordering, minimum selling prices, and ratings of two-outcome monetary gambles.

To formalize this, let A be a set of monetary amounts, and let $G = A \times [0, 1] \times A$ be a set of two-outcome gambles. Any $g = (x, p, y) \in G$ is to be interpreted as a gamble in which x is received with probability p and y is received with probability $1 - p$. Let $D = \{(x, p, y) \in G: x = y, \text{ or } p = 0, \text{ or } p = 1\}$ be the set of degernate gambles. Let $\geqslant^*$ be a binary relation on G, interpreted as the choice relation. Let $MS(x, p, y)$ and $R(x, p, y)$ denote the minimum selling price and attractiveness rating, respectively, for the gamble (x, p, y). For concreteness, we will assume that the rating scale runs from 1 to 100.

Our goal is to find conditions on the structure $\langle G, MS, R, \geqslant^{\star}\rangle$ which are sufficient to ensure the existence of functions $u : A \to Re$, $\Delta : G \to [0, 1]$, and strictly increasing functions f and h from $\Delta(G)$ and $\Delta(G - D)$, respectively, into $[0, 1]$, such that for any $(x, p, y), (z, q, w) \in G$,

$$(x, p, y) \geqslant^{\star} (z, q, w)$$
$$\text{iff} \tag{5}$$
$$u(x) - \Delta(x,p,y)[u(x) - u(y)] \geqslant u(z) - \Delta(z, q, w)[u(z) - u(w)],$$

$$MS(x, p, y) = x - f[\Delta(x, p, y)](x - y), \tag{6}$$

and for any nondegenerate $(x, p, y) \in G - D$,

$$R(x, p, y) = 100 - h[\Delta(x, p, y)](100 - 1). \tag{7}$$

3. Representation and uniqueness theorem for expression theory

The models for choice and minimum selling price will be axiomatized simultaneously. Afterwards, the model for ratings will be handled with an axiom that asserts relative ratings to be monotone with relative prices. To axiomatize the model for choice, we exploit a similarity between the desired representation and subjective expected utility (SEU) theory. By rewriting eq. (5) as

$$(x, p, y) \geqslant^{\star} (z, q, w)$$
$$\text{iff} \tag{8}$$
$$u(x)[1 - \Delta(x, p, y)] + u(y)\Delta(x, p, y) \geqslant u(z)[1 - \Delta(z, q, w)] + u(w)\Delta(z, q, w),$$

it can be seen that the Δ function replaces SEU's subjective probability function. In fact, if $\Delta(x, p, y)$ depended only on p, then expression theory's model for choice would reduce to subjectively weighted utility theory. Consequently, it is a necessary property of expression theory that every subset of gambles with fixed Δ have an additive representation. This provides a way of reducing the problem of axiomatizing expression theory to a sequence of more familiar problems.

3.1. BACKGROUND ASSUMPTIONS

Several "innocuous" background assumptions will be used throughout the discussion. These assumptions have less to do with expression theory, per se, than with a general presumption that people's preferences will be "reasonable" if people are given sufficient time to consider easily discriminable alternatives.

DEFINITION 1

Let R be a weak order on $G = A x [0, 1] x A$, and define the relation E by $g_1 \, E \, g_2$ iff $g_1 \, R \, g_2$ and $g_2 \, R \, g_1$. Then R is said to be a *reasonable ordering of G* if the following five conditions are satisfied.

(i) For all $x, y \in A$ and $p \in [0, 1]$, $(x, p, y) \, E \, (y, 1-p, x)$.

(ii) For all $x, y \in A$ and $p \in [0, 1]$, $(x, 1, y) \, E \, (x, p, x)$.

(iii) For all $x, y \in A$, $(x, 1, x) \, R \, (y, 1, y)$ iff $x \geqslant y$.

(iv) For all $x, y \in A$ where $x > y$, $(x, p, y) \, R \, (x, q, y)$ iff $p \geqslant q$.

As background assumptions, we adopt the following three axioms:

(1) $\geqslant^*$ is a reasonable ordering of G.

(2) The order induced by MS is a reasonable ordering of G.

(3) For all $x, y \in A$, $MS(x, 1, y) = x$.

3.2. PARTITIONING G

Minimum selling prices are used to partition the set of gambles G into subsets G'_r, in which all the gambles have the same value of Δ. Although subsequent sections focus mostly on the representation for choice, this initial use of minimum selling price is pivotal to assure that the same delta function appears in the representations for choices, prices, and ratings.

Define $\Delta' : G \to Re$ by

$$\Delta'(x, p, y) = [x - MS(x, p, y)] / [x - y] \qquad \text{if } x \neq y \tag{9a}$$

and

$$\Delta'(x, p, y) = 0 \qquad \text{if } x = y. \tag{9b}$$

Unlike $\Delta(x, p, y)$, the quantity $\Delta'(x, p, y)$ can be computed directly from the subject's minimum selling price. Moreover, if gambles (x, p, y) and (z, q, w) both produce the same value of Δ', then a necessary consequence of expression theory is that $\Delta(x, p, y) = \Delta(z, q, w)$. To see this, note that by eqs. (6) and (9), $\Delta'(x, p, y) = f[\Delta(x, p, y)]$, where f is required to be strictly increasing. The importance of this observation comes from its use in partitioning the gambles in G. Let G'_r denote the subset of G which contains all the gambles (x, p, y) for which $\Delta'(x, p, y) = r$. If expression theory is to be satisfied, all of the gambles in G'_r must be assigned the same (as yet unknown) value of Δ.

3.3. REPRESENTATION OF CHOICE

Within any subset of gambles G'_r, where Δ is a fixed but unknown constant, the desired representation for choice is only slightly stronger than additivity. Therefore, we proceed in three steps. First, we must make sure that the subsets meet certain preconditions for the use of a standard set of additivity axioms. Second, the axioms of additive conjoint measurement (Krantz et al. [16]) are imposed on each subset. Third, axioms are adopted to guarantee that a common utility function can be extracted from each additive representation.

Preconditions for additivity axioms. Unfortunately, G'_r fails to meet a precondition of the axioms of additive conjoint measurement, viz., that the subset G'_r contain a gamble for each $(x, y) \in AxA$. Therefore, we augment G'_r with degenerate gambles of the form $(x, 1, x)$. Let G_r denote the enriched set, i.e. $G_r = G'_r \cup \{(x, 1, x) : x \in A\}$. Next, we adopt axiom 4 which, in addition to axioms 1–3, implies that the sets G_r do meet this precondition of the additivity axioms.

(4) For any $x, y \in A$ where $x > y$, and for any $a, b, c \in Re$ where $a \geqslant b \geqslant c$, if there are probabilities p_a and p_c so that $MS(x, p_a, y) = a$ and $MS(x, p_c, y) = c$, then there is a probability p_b such that $MS(x, p_b, y) = b$.

It is proven in the appendix that if axioms 1–4 are satisfied, then for each $(x, y) \in AxA$ and for any $r \in [0, 1]$ there is a gamble $(x, p, y) \in G_r$. Moreover, if $r \neq 0$ or if $x \neq y$, then (x, p, y) and (x, q, y) are both members of G_r iff $p = q$.

Additivity axioms. Now consider the additivity of gambles in G_r. The following notation will simplify matters. For any distinct $x, y \in A$, and any $r \in [0, 1]$, let $[x, r, y]$ denote the unique gamble (x, p, y) which is a member of G_r. Also, let $[x, r, x]$ be an alternate notation for $(x, 1, x)$. For any r, $0 \leqslant r \leqslant 1$, define the binary relation $\geqslant_r$ on AxA by

$$(x, y) \geqslant_r (z, w) \text{ iff } [x, r, y] \geqslant^* [z, r, w]. \tag{10}$$

To assure additivity of the gambles in G_r, we adopt the following as an axiom of the representation theorem:

(5) For each value $r \in (0, 1)$, the triple $\langle A, A, \geqslant_r\rangle$ is a symmetric additive conjoint structure.

To be an additive conjoint structure, each triple $\langle A, A, \geqslant_r\rangle$ must satisfy six properties: (1) weak ordering; (2) independence; (3) Thomsen condition; (4) restricted solvability; (5) archimedean property; and (6) each component is essential. To be symmetric, the structure must also satisfy the following condition: (7) for every $x, y \in A$, there exist $a, b, c, d \in A$ such that $[x, r, a] =^* [y, r, b]$ and $[c, r, x] =^* [d, r, y]$.[3] (See Krantz et al. [16], pp. 256–257, for fuller definitions and the representation theorem for additivity.)

It follows that there exist real-valued functions ϕ_r and ψ_r, defined on A, such that for all $(x, p, y), (z, q, w) \in G_r$,

$$(x, p, y) \geqslant^* (z, q, w) \text{ iff } \phi_r(x) + \psi_r(y) \geqslant \phi_r(z) + \psi_r(w). \tag{11}$$

Moreover, ϕ_r and ψ_r are interval scales with a common unit.

Utility function. Having established eq. (11) for each G_r, the next issue is that eq. (8) requires the representation to be slightly stronger than additivity. Specifically, there must be a utility function such that the additive representation is a weighted average of the values assigned by the utility function to the outcomes. Krantz et al. [16] show that if $\langle A, A, \geqslant_r \rangle$ is an additive conjoint structure, then one further condition is necessary and sufficient for there to exist an interval scale u_r, defined on A, and nonzero constants α_r, β_r, such that for all $(x, p, y), (z, q, w) \in G_r$,

$$(x, p, y) \geqslant^* (z, q, w) \text{ iff } \alpha_r u_r(x) + \beta_r u_r(y) \geqslant \alpha_r u_r(z) + \beta_r u_r(w). \tag{12}$$

The condition is that x, y, and z form a standard sequence on the first component (i.e. x, y, and z are equally spaced under ϕ_r) if and only if x, y, and z form a standard sequence on the second component (i.e. x, y, and z are equally spaced under ψ_r).

To assure this equal-spacing condition, we adopt the following axiom.

(6) For all $x, y, z, w, a, b, c, d \in A$, and all $r, s \in (0, 1)$

(i) if $[x, r, a] \geqslant^* [y, r, b]$ and $[w, r, b] \geqslant^* [z, r, a]$, then $[c, s, z] \geqslant^* [d, s, w]$ implies $[c, s, x] \geqslant^* [d, s, y]$,

(ii) if $[a, r, x] \geqslant^* [b, r, y]$ and $[b, r, w] \geqslant^* [a, r, z]$, then $[z, s, c] \geqslant^* [w, s, d]$ implies $[x, s, c] \geqslant^* [y, t, d]$.

Axiom 6(i) can be interpreted via eq. (11) as saying

$$\begin{aligned}&\text{if } \phi_r(x) - \phi_r(y) \geqslant \psi_r(b) - \psi_r(a) \geqslant \phi_r(z) - \phi_r(w),\\ &\text{then } \psi_s(z) - \psi_s(w) \geqslant \phi_s(d) - \phi_s(c)\\ &\quad \text{implies } \psi_s(x) - \psi_s(y) \geqslant \phi_s(d) - \phi_s(c).\end{aligned}$$

To see how the axiom implies the equal-spacing condition, suppose that x, y, and z form a standard sequence on the first component. That is, there exist $a, b \in A$ such that

$$[x, r, a] =^* [y, r, b] \quad \text{and} \quad [y, r, a] =^* [z, r, b],$$

and therefore by eq. (11),

$$\phi_r(x) - \phi_r(y) = \psi_r(b) - \psi_r(a) = \phi_r(y) - \phi_r(z).$$

Then we have both

$$[x, r, a] \geqslant^\star [y, r, b] \quad \text{and} \quad [z, r, b] \geqslant^\star [y, r, a],$$

and

$$[y, r, a] \geqslant^\star [z, r, b] \quad \text{and} \quad [y, r, b] \geqslant^\star [x, r, a].$$

Applying axiom 6(i) twice, it follows that for any $s \in (0, 1)$ and for all $c, d \in A$,

$$[c, s, x] \geqslant^\star [d, s, y] \quad \text{iff} \quad [c, s, y] \geqslant^\star [d, s, z].$$

By symmetry (axiom 5), there exist $c, d \in A$ such that $[c, s, x] =^\star [d, s, y]$, and it follows that $[c, s, y] =^\star [d, s, z]$. Thus,

$$\psi_s(x) - \psi_s(y) = \phi_s(d) - \phi_s(c) = \psi_s(y) - \psi_s(z),$$

and x, y, and z are equally spaced under ψ_s, for all s, including $s = r$. Axiom 6(ii) is used similarly, establishing eq. (12).

Interpreting axiom 6 in terms of eq. (12), we can obtain a stronger result. Because axiom 6 must hold for distinct r and s, it implies that if x, y, and z are equally spaced under u_r, then they are also equally spaced under u_s. It follows that there exist constants $a_{rs} > 0$ and b_{rs}, such that $u_r = a_{rs}u_s + b_{rs}$. Therefore, we can choose some particular s and write all other utility functions u_r in terms of u_s. Rewriting eq. (12), the constants a_{rs} and b_{rs} cancel and we suppress the subscript on u_s to obtain for all $(x, p, y), (z, q, w) \in G_r$, $r \in (0, 1)$,

$$(x, p, y) \geqslant^\star (z, q, w) \quad \text{iff} \quad \alpha_r u(x) + \beta_r u(y) \geqslant \alpha_r u(z) + \beta_r u(w). \tag{13}$$

Once eq. (13) is established, we can divide through by $(\alpha_r + \beta_r)$ and define $\Delta_r = \beta_r/(\alpha_r + \beta_r)$, to obtain for all $(x, p, y), (z, q, w) \in G_r$, $r \in (0, 1)$,

$$\begin{gathered}(x, p, y) \geqslant^\star (z, q, w) \\ \text{iff} \\ u(x)(1 - \Delta_r) + u(y)\Delta_r \geqslant u(z)(1 - \Delta_r) + u(w)\Delta_r,\end{gathered} \tag{14}$$

as desired. Moreover, by axiom 1(i, ii, and iv), for any $(x, p, y) \in G_r$, where $x > y$, $(x, 1, x) \geqslant^\star (x, p, y) \geqslant^\star (y, 1, y)$. By eq. (14), this implies that $u(x) \geqslant u(x)(1 - \Delta_r) + u(y)\Delta_r \geqslant u(y)$, which yields $1 \geqslant \Delta_r \geqslant 0$.

If $(x, p, y) \in G_0$, it follows from eq. (9) and from axioms 2(iv) and 3 that either $x = y$ or $p = 1$. In either case, it follows from axiom 1(ii) that $(x, p, y) =^\star (x, 1, x) \in G_s$. Thus, for any $(x, p, y), (z, q, w) \in G_0$,

$$(x, p, y) \geqslant^\star (z, q, w) \text{ iff } (x, 1, x) \geqslant^\star (z, 1, z)$$
$$\text{iff } u(x) \geqslant u(z) \text{ by eq. (14).}$$

Therefore, eq. (14) can be extended to include the case that $r = 0$ by defining $\Delta_0 = 0$.

If $(x, p, y) \in G_1$, it follows from eq. (9) and axioms 2(i and iv) and 3 that $x \neq y$ and $p = 0$. It follows from axiom 1(i and ii) that $(x, p, y) =^\star (y, 1, y) \in G_s$. Therefore, eq. (14) can be extended to include the case that $r = 1$ by defining $\Delta_1 = 1$. For any $(x, p, y) \in G_r'$, $r \in [0, 1]$, let $\Delta(x, p, y) = \Delta_r$.

Bridging the subsets. The representation established in eq. (14) is restricted to choices among the gambles in subset G_r. It remains to establish the analogous representation for choices between gambles in different subsets. To do this, we adopt axiom 7, that each gamble $(x, p, y) \in G$ has a "certainty choice-equivalent" z.[4]

(7) For each $(x, p, y) \in G$, there is a $z \in A$ such that $(z, 1, z) =^\star (x, p, y)$.

It follows from axiom 1(iii) that the certainty choice-equivalent of any gamble is unique. Axioms 1 and 7 are used as follows. Let $(x, p, y), (z, q, w) \in G$. Then, for some values r and t, $(x, p, y) \in G_r'$ and $(z, q, w) \in G_t'$. By the assumption of certainty choice-equivalents (axiom 7), there are values $a, b \in A$ such that $(a, 1, a) =^\star (x, p, y)$ and $(b, 1, b) =^\star (z, q, w)$. Since $\geqslant^\star$ is assumed to be a weak order (axiom 1), it follows that

$$(x, p, y) \geqslant^\star (z, q, w) \text{ iff } (a, 1, a) \geqslant^\star (b, 1, b).$$

Since by definition both $(a, 1, a)$ and $(b, 1, b)$ are members of G_s, we can apply eq. (14) to find that

$$(a, 1, a) \geqslant^\star (b, 1, b) \text{ iff } u(a) \geqslant u(b).$$

Since $(a, 1, a) \in G_r$ and $(b, 1, b) \in G_t$, we can apply eq. (14) to find that

$$u(a) = u(x)(1 - \Delta_r) + u(y)\Delta_r,$$

and

$$u(b) = u(z)(1 - \Delta_t) + u(w)\Delta_t.$$

Therefore, we obtain the desired representation,

$$(x, p, y) \geqslant^{\star} (z, q, w)$$

$$\text{iff}$$

$$u(x)(1 - \Delta_r) + u(y)\Delta_r \geqslant u(z)(1 - \Delta_t) + u(w)\Delta_t.$$

3.4. REPRESENTATION THEOREM

As already shown, axioms 1–7 are sufficient to obtain the desired representation for choice. Axioms 8 and 9 are used to obtain representations for minimum selling prices and ratings. The proof of theorem 1 is given in the appendix.

THEOREM 1

Let $G'_r = \{(x, p, y) \in G : \Delta'(x, p, y) = r\}$ and let $G_r = G'_r \cup \{(x, 1, x) : x \in A\}$. Suppose that axioms 1–7 are satisfied, and in addition that the following axioms are satisfied.

(8) For any $r, s \in [0, 1]$, $s \geqslant r$ iff there exist $m \in (0, 1)$ and $x, y, z, w, a, b \in A$, satisfying the following properties:

(i) $[x, m, a] \geqslant^{\star} [y, m, b]$ and $[w, m, b] \geqslant^{\star} [z, m, a]$ with at least one of the relations strict, and

(ii) $[x, r, y] \geqslant^{\star} [z, r, w]$ and $[z, s, w] \geqslant^{\star} [x, s, y]$.

(9) Let (x, p, y), $(z, q, w) \in G - D$ be nondegenerate gambles where $(x, p, y) \in G'_r$ and $(z, q, w) \in G'_s$. Then

$$r \geqslant s \quad \text{iff} \quad R(x, p, y) \leqslant R(z, q, w).$$

Then there are functions $u : A \to Re$, $\Delta : G \to [0, 1]$, and strictly increasing functions f and h from $\Delta[G]$ and $\Delta[G - D]$, respectively, into $[0, 1]$ such that for any $(x, p, y), (z, q, w) \in G$,

$$(x, p, y) \geqslant^{\star} (z, q, w)$$

$$\text{iff} \tag{15}$$

$$u(x) - \Delta(x, p, y)[u(x) - u(y)] \geqslant u(z) - \Delta(z, q, w)[u(z) - u(w)],$$

$$MS(x, p, y) = x - f[\Delta(x, p, y)](x - y), \tag{16}$$

and for any nondegenerate $(x, p, y) \in G - D$,

$$R(x, p, y) = 100 - h[\Delta(x, p, y)]\,(100 - 1). \tag{17}$$

Moreover, u is an interval scale, and Δ, f, and h are absolute scales.

4. Testing expression theory

Tests of axiomatic theories can be conducted in rather different forms, depending on one's purpose, perspective, and the desired level of generality. In this section, we discuss issues regarding the empirical validity of expression theory as a general model of the relationships among choices, prices, and ratings. In the next section, some issues are considered that arise in the more specific use of expression theory to account for particular decision anomalies, such as the preference reversal phenomena. At the more general level, it is desirable to test predictions that are: (1) necessary consequences of the theory, and (2) as general as possible, i.e. involving no or few ancillary assumptions regarding the shapes of the utility, f, or h functions, etc. Prime candidates for these predictions are the axioms of the representation theorem. Therefore, it is important to consider the necessity, testability, and relative importance of different axioms.

4.1. RELATIVE IMPORTANCE OF THE AXIOMS

The axioms of theorem 1 are sufficient for the expression theory model, but there are several reasons why the axioms are not all equally important or substantively interesting. First, it seems likely that some of the axioms will be present in virtually any acceptable theory of decision making, and thus do not help to discriminate expression theory from any other model. The first three axioms, i.e. the "background assumptions", and possibly axiom 7, are of this sort. In a sense, axioms of this kind help to define the intended domain of the theory. If a subject were to violate these assumptions, we would regard it not so much as a violation of expression theory, but rather as an indication that the subject was inattentive, or rushed, or misunderstood the task.

Second, some of the axioms should be regarded as idealizing approximations of what subjects are capable of doing. Axioms 4 and 7, if taken literally and in the presence of the background assumptions, would require subjects to be capable of making infinitesimal distinctions. Clearly, these axioms are only approximately true. However, replacing these axioms with others that are more literally true would greatly complicate the mathematics, and lead us to consider perceptual issues that would take us far from our main interests.

Third, some of the axioms are statements that bear not only on subjects' decision processes, but also on the richness of the stimulus set. For example, the symmetry assumption (axiom 5) asserts that for every $x, y \in A$, there exist $a, b \in A$ such that $[x, r, a] =^* [y, r, b]$. Thus, $[u(x) - u(y)] [1 - \Delta_r] = [u(b) - u(a)] \Delta_r$. That is, every "$x - y$ utility interval" (weighted by $1 - \Delta_r$) can be matched by a "$b - a$ utility interval" (weighted by Δ_r). Surely this assumption depends as much on the richness of the set A of monetary amounts as it does on subjects' psychological processes.

With the above qualifications in mind, we consider (parts of) axioms 5, 6, 8, and 9 to capture the psychological assumptions of expression theory. Axioms 5, 6, and 8 speak to the interlocking of minimum selling price and preferential choice. The importance of axioms 5 and 6 is not so much that they require choice to be reflected by a weighted sum of utilities for payoffs, but rather *when* they require this, viz., when examining gambles with a fixed relative adjustment in price. Axioms 8 and 9 extend this by asserting that relative adjustments in utility, minimum selling price, and ratings, as opposed to raw values, should be monotone with one another.

4.2. EXPERIMENTAL CONTROL OF THE FACTORS

A problem in testing the axioms of expression theory lies in the inability of experimenters to manipulate all the relevant factors. Experimenters have direct control over the payoffs and probabilities of the gambles evaluated by subjects. However, most of the axioms, including the key axioms 5, 6, 8, and 9, involve choices between gambles with the same relative adjustment in minimum selling price Δ'. Unfortunately, the values of Δ' are not known in advance, let alone determined by the experimenter. Rather, Δ' is observed after the subject sets a minimum selling price for (x, p, y). As a result, it may be very difficult to construct appropriate stimuli in advance of the experiment. One solution to this difficulty is to conduct two-stage tests of the axioms. In the first stage, the subject would be asked to give a sequence of minimum selling prices in response to gambles generated interactively by a computer. The program would suggest different gambles (x, p_i, y) in a search for the gamble satisfying $MS(x, p, y) = K$, for specified values of K. Although such an experiment would be arduous, there seem to be few alternatives at present. Moreover, interactive computer packages for decision research are beginning to appear (Johnson et al. [12]).

5. Examining implications of expression theory

Assuming that expression theory survives tests of the key axioms, there are important and interesting predictions of the theory that depend on knowledge of the shapes of the functions u, Δ, f, and h. For example, it was pointed out that if $\Delta(x, p, y)$ depended only on p, then the expression theory model for choice would reduce to SEU. Also, recall that in using expression theory to account for the preference reversal phenomena, we assumed that utility was concave and that f and h

were linear. In addition, measuring the shapes of u, Δ, f, and h would enable expression theory to be used in accounting for individual differences. In this section, we discuss how these issues can be investigated further.

5.1. SHAPE OF THE UTILITY FUNCTION

The utility function in expression theory need not satisfy the von Neumann–Morgenstern requirement that $u(x, p, y) = pu(x) + (1 - p)u(y)$. Indeed, expression theory says nothing about the functional form of $u(x, p, y)$. Moreover, the shape of the utility function has different implications in the two theories. In expression theory, the shape of the utility curve influences the kinds of preference reversals that are possible, while von Neumann-Morgenstern utility has no such implication. Further, risk aversion is not equivalent to a concave utility function in expression theory. Thus, new procedures are required to assess the shape of the utility curve in expression theory.

The shape of the utility curve can be studied by observing the ordering of utility increments that correspond to equally spaced monetary amounts. Let $x, y, z, w \in A$ be amounts such that $x > z$, and $x - y = z - w > 0$. Then the utility increment $u(x) - u(y)$ will equal the utility increment $u(z) - u(w)$ if the utility curve is linear. If utility is concave, then $u(x) - u(y) < u(z) - u(w)$, and if utility is convex, then $u(x) - u(y) > u(z) - u(w)$. Thus, the problem reduces to being able to order utility increments. Assuming that expression theory is satisfied, the utility increment $u(x) - u(y)$ is less than the increment $u(z) - u(w)$ if it is possible to find $a, b \in A$ and $r \in (0, 1)$, such that

$$[x, r, a] \leqslant^* [y, r, b] \text{ and } [z, r, a] \geqslant^* [w, r, b].$$

Using this observation to study the shape of utility entails the experimental difficulties involved with having only indirect control over the value of Δ'. Nevertheless, it is conceptually clear how one could assess the shape of utility. Let $x_1, x_2, x_3, \ldots$ be a sequence of amounts such that $x_{i+1} > x_i$. One could construct the sequence so that $x_{i+1} - x_i = x_i - x_{i-1}$, and then observe the ordering of the increments $u(x_{i+1}) - u(x_i)$. Alternatively, one could construct the sequence so that $u(x_{i+1}) - u(x_i) = u(x_i) - u(x_{i-1})$, and then observe the ordering of the increments $x_{i+1} - x_i$. To construct the sequence in the latter case, one would select distinct $a, b \in A$ and $r \in (0, 1)$, and then seek amounts $x_1, x_2, x_3, \ldots$ such that $[x_{i+1}, r, a] =^* [x_i, r, b]$.

5.2. FACTORS CONTROLLING DELTA

As already mentioned, the expression theory model for choice would reduce to SEU if $\Delta(x, p, y)$ depended only on p. However, expression theory allows Δ to be affected by the payoffs as well as their probabilities. In view of the many well-known violations of SEU, it seems likely that $\Delta(x, p, y)$ will in fact depend on both probabilities and payoffs. This possibility reintroduces an issue that was investigated in the 1950's and 1960's: whether probabilities and payoffs have independent effects on the attractiveness of gambles (Edwards [3], Irwin [11], Marks [21]; see also Goldstein et al. [8]). To examine whether and how $\Delta(x, p, y)$ depends on payoffs and probabilities, one could manipulate these variables and observe the orderings of (1) values of Δ, and (2) increments in the value of Δ. Studying the ordering of Δ is fairly straightforward. Since expression theory requires Δ to be monotone with Δ', one can infer the ordering of Δ from the ordering of Δ', which can be computed from the subject's minimum selling price Assessing the ordering of Δ increments is more involved.

The problem can be reduced to an easier one as follows. Let $g_i \in G$, $i = 1, \ldots, 4$, and let $\Delta'(g_i) = r_i$ and $\Delta(g_i) = \Delta_i$. In attempting to determine the ordering of $\Delta_4 - \Delta_3$ and $\Delta_2 - \Delta_1$, dealing with gambles that offer fixed payoffs will be easier than considering the original g_i. Select $\alpha, \beta \in A$, where $\alpha \neq \beta$. By definition, the gamble $[\alpha, r_i, \beta]$ has $\Delta'([\alpha, r_i, \beta]) = r_i = \Delta'(g_i)$. Therefore, $\Delta([\alpha, r_i, \beta]) = \Delta(g_i) = \Delta_i$. Now let $C_i \in A$, $i = 1, \ldots, 4$, be the certainty choice-equivalents, respectively, of $[\alpha, r_i, \beta]$, $i = 1, \ldots, 4$. From the expression theory representation for choice, it follows that

$$\Delta_i = [u(\alpha) - u(C_i)] / [u(\alpha) - u(\beta)].$$

Therefore,

$$\Delta_4 - \Delta_3 \geqslant \Delta_2 - \Delta_1 \quad \text{iff} \quad u(C_4) - u(C_3) \leqslant u(C_2) - u(C_1).$$

Thus, the problem of ordering delta increments has been reduced to the problem of ordering utility increments. It follows that $\Delta_4 - \Delta_3 \geqslant \Delta_2 - \Delta_1$ if it is possible to find $\gamma, \delta \in A$ and $k \in (0, 1)$, such that

$$[C_4, k, \gamma] \leqslant^* [C_3, k, \delta] \quad \text{and} \quad [C_2, k, \gamma] \geqslant^* [C_1, k, \delta].$$

One way to consider whether the value of $\Delta(x, p, y)$ depends on payoffs as well as probabilities is to ask whether the ordering of increments $\Delta(x, p, y) - \Delta(x, q, y)$ and $\Delta(x, r, y) - \Delta(x, s, y)$ depends on the payoffs. This would be the case if, for example,

$$\Delta(x, p, y) - \Delta(x, q, y) \geqslant \Delta(x, r, y) - \Delta(x, s, y),$$

but

$$\Delta(z, p, w) - \Delta(z, q, w) \leqslant \Delta(z, r, w) - \Delta(z, s, w).$$

with at least one of the inequalities strict. Let $a, b, c, d \in A$ be the certainty choice-equivalents of (x, p, y), (x, q, y), (x, r, y), and (x, s, y), respectively. Similarly, let $a', b', c', d' \in A$ be the certainty choice-equivalents of (z, p, w), (z, q, w), (z, r, w), and (z, s, w), respectively. By the reasoning in the preceding paragraph,

$$\Delta(x, p, y) - \Delta(x, q, y) \geqslant \Delta(x, r, y) - \Delta(x, s, y) \quad \text{iff} \quad u(b) - u(a) \geqslant u(d) - u(c).$$

Similarly,

$$\Delta(z, p, w) - \Delta(z, q, w) \leqslant \Delta(z, r, w) - \Delta(z, s, w)$$

$$\text{iff}$$

$$u(b') - u(a') \leqslant u(d') - u(c').$$

Therefore, the delta function depends on payoffs as well as probabilities if it is possible to find $\alpha, \beta, \alpha', \beta' \in A$ and $k, k' \in (0, 1)$, such that

$$[b, k, \alpha] \geqslant^{*} [a, k, \beta] \quad \text{and} \quad [d, k, \alpha] \leqslant^{*} [c, k, \beta],$$

while

$$[b', k', \alpha'] \leqslant^{*} [a', k', \beta'] \quad \text{and} \quad [d', k', \alpha'] \geqslant^{*} [c', k', \beta'],$$

with at least one of the relations strict.

5.3. THE PROPORTIONAL MATCHING FUNCTIONS

Expression theory entails the notion that proportional adjustments, i.e. delta's, are matched in forming judgments of different kinds. However, a match has been loosely defined as any monotonic function that relates, for example, Δ to Δ' in the case of minimum selling price. This was done to allow for individual differences in the psychophysics of magnitude, number preferences, tendencies to avoid using the endpoints of response scales, and other "nuisance" factors. However, there are also substantive issues regarding the nature of proportional matching. Consider, for example, judgments of minimum selling price, maximum buying price, and certainty equivalence for gambles. According to expression theory, all three judgments involve translating the same Δ onto a common monetary response scale, but these translations can involve three different proportional matching functions. Thus, expression theory allows for reversals among these judgments, although there should be no reversals among the proportional adjustments.

According to expression theory, the proportional matching functions for minimum selling prices and ratings satisfy

$$f[\Delta(x, p, y)] = [x - MS(x, p, y)] / [x - y],$$

and

$$h[\Delta(x, p, y)] = [100 - R(x, p, y)] / [100 - 1].$$

These values can be computed from subjects' responses. However, these values are the composition of the f and Δ functions and the h and Δ functions, respectively, for minimum selling prices and ratings. To study the proportional matching functions more closely, it is necessary to separate f and h from Δ. The shapes of the f and h functions can be studied in a fashion that is analogous to the procedures outlined above for studying the shape of utility. For example, consider minimum selling prices. Suppose that (x, p, y), (z, q, w), (a, r, b), $(c, s, d) \in G$ are gambles such that $\Delta(x, p, y) > \Delta(a, r, b)$, and $\Delta(x, p, y) - \Delta(z, q, w) = \Delta(a, r, b) - \Delta(c, s, d) > 0$. Then concave f implies that $f[\Delta(x, p, y)] - f[\Delta(z, q, w)] < f[\Delta(a, r, b)] - f[\Delta(c, s, d)]$. Linear f implies that the two f-increments are equal, and convex f implies that $f[\Delta(x, p, y)] - f[\Delta(z, q, w)] > f[\Delta(a, r, b)] - f[\Delta(c, s, d)]$. Since the values of $f[\Delta]$ can be computed from subjects' responses, the problem reduces to assessing the ordering of delta increments. It was shown above that this reduces in turn to determining the ordering of utility increments, which can be handled as discussed earlier.

6. Summary and conclusions

Decision researchers have focused much of their attention on systematically inconsistent behavior. People's decisions often conflict with one another (e.g. preference reversals) or with principles of reasonable behavior (e.g. dominance). These inconsistencies are striking because they appear to reflect defective performance on the part of otherwise competent people. Other researchers have attributed the decision anomalies either to: (1) processes of information encoding that depend on the presentation of information and the perspective of the subject, or (2) the strategic selection of an information evaluation process in response to task and context conditions. There is compelling evidence that each approach does account for some of the inconsistencies (Tversky and Kahneman [26], Payne [22]). It has been suggested here, however, that these explanations are not complete. Studies of preference reversals show that subjects presented with the same two-outcome explicit gambles under identical conditions will express different preference orders depending on the response mode. Attractiveness ratings and minimum selling prices are particularly disparate (Goldstein and Einhorn [7]). Due to the simplicity of the gambles in these studies, the explicitness of their presentation, and the lack of time-pressure or other task manipulations, it seems difficult to attribute the reversals to changes in either the encoding or evaluation of information.

Because psychologists can observe only the overt responses of subjects, it is interesting and important to study how subjects might go about expressing their internal experiences and impressions. Particularly when the experimenter asks the subject for a numerical or graded response, does this seem like a nontrivial task. Because of the incommensurability of internal impressions and overt numerical responses, the subject has difficulty in establishing a direct mapping from stimuli to responses. We suggest that the subject overcomes this difficulty by a process we call "subjective interpolation", in which the relative standing of the stimulus is matched to a relative response. These ideas made specific to two-outcome monetary gambles, along with some presumptions about how values are relativized and what it means to "match" relative impressions and responses, result in the model we call Expression Theory.

Since expression theory has been successful in accounting for a complex pattern of six interlocking preference reversal phenomena (Goldstein and Einhorn [7]), we are encouraged to prod the model further. To this end, an axiomatic analysis of expression theory has been provided. The axioms are prime candidates for further empirical tests of the model although, as discussed, the axioms are not all equally interesting and there are some attendant experimental difficulties. Also discussed were ways of assessing the measurement scales, so that expression theory might be used to account for individual differences and other phenomena, such as the dependent effects of probabilities and outcomes on the attractiveness of gambles (cf. Edwards [3]).

Finally, a larger set of issues surrounds the study of how subjects convert their impressions into overt responses. Most psychologists define their research interests in terms of underlying psychological constructs, e.g. the structure of preferences, perceptions, cognitions, etc. Of course, none of these constructs can be studied except through the overt behavior of subjects. This is certainly not a new insight. Indeed, radical behaviorists early in this century urged that psychologists abandon all reference to the underlying constructs. A more modern position is to recognize that the methods of observation can distort the observer's view of underlying constructs. Attempts are then made to assess the degree to which unwanted "method variance" obscures the substantively more interesting constructs (Campbell and Fiske [2]). We urge a shift in emphasis that follows the recognition that method variance may itself be systematic. To the extent that methods do have systematic effects on observed behavior, this itself is substantively interesting and calls for theoretical explanation. Especially when overt behavior can result in consequences for individuals (e.g. choices, prices), it should be regarded as more than just a conduit to underlying constructs.

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Appendix

LEMMA 1

Let $G'_r = \{(x,p,y) \in G : \Delta'(x,p,y) = r\}$ and let $G_r = G'_r \cup \{(x,1,x) : x \in A\}$. Suppose that axioms 1–4 are satisfied. Then for each $(x,y) \in AxA$ and for any $r \in [0,1]$, there is a gamble $(x,p,y) \in G_r$. Moreover, if $r \neq 0$ or if $x \neq y$, then (x,p,y) and (x,q,y) are both members of G_r iff $p = q$.

Proof of lemma 1

Let $(x,y) \in AxA$. If $x = y$, then by definition of G_r, $(x,1,x) \in G_r$, for any r. Further, if $r \neq 0$, then for $p \neq 1$, (x,p,x) is not in G_r. This is because, by definition of Δ', $\Delta'(x,p,x) = 0$, and inclusion of $(x,p\ x)$ in G_r would contradict the assumption that $r \neq 0$.

Suppose that $x > y$. By axiom 3, $MS(x,1,y) = x$. By axioms 2(i) and 3, $MS(x,0,y) = MS(y,1,x) = y$. Then by axiom 4, for any $r \in [0,1]$, there is a probability p such that $MS(x,p,y) = x - r(x-y)$, which yields $\Delta'(x,p,y) = r$. It follows from axiom 2(iv) that this value of p is unique.

Suppose that $x < y$. By the preceding paragraph, for any $r \in [0,1]$ there is a unique gamble (y,q,x) such that $\Delta'(y,q,x) = 1 - r$. By axiom 2(i), it follows that $\Delta'(x,1-q,y) = r$. QED

Proof of theorem 1

By the discussion in the text, axioms 1–7 are sufficient for there to exist real-valued functions u and Δ, defined on A and G, respectively, so that for any

(x, p, y), $(z, q, w) \in G$, eq. (5) is satisfied. Consider the uniqueness of the functions u and Δ. It has already been observed that u is an interval scale. Suppose that in addition to the functions u and Δ, other functions $u^\star$ and $\Delta^\star$ also provided the desired representation for choice. By axiom 7, for each $(x, p, y) \in G$ there exists a (unique) gamble $(z, 1, z)$ such that $(z, 1, z) =^\star (x, p, y)$. Then

$$u(z) = u(x)[1 - \Delta(x, p, y)] + u(y)\Delta(x, p, y),$$

and

$$u^\star(z) = u^\star(x)[1 - \Delta^\star(x, p, y)] + u^\star(y)\Delta^\star(x, p, y).$$

Thus,

$$\Delta(x, p, y) = [u(x) - u(z)] / [u(x) - u(y)],$$

and

$$\Delta^\star(x, p, y) = [u^\star(x) - u^\star(z)] / [u^\star(x) - u^\star(y)].$$

Since, as observed above, u is an interval scale, there exist constants $a > 0$, and b, such that $u^\star = au + b$. It easily follows that $\Delta^\star(x, p, y) = \Delta(x, p, y)$. Thus, Δ is an absolute scale.

The importance of axiom 8 is that it can be used to establish the property

$$\Delta'(x, p, y) \geqslant \Delta'(z, q, w) \quad \text{iff} \quad \Delta(x, p, y) \geqslant \Delta(z, q, w).$$

To see this, suppose there exist $m \in (0, 1)$ and $x, y, z, w, a, b \in A$, satisfying

(i) $[x, m, a] \geqslant^\star [y, m, b]$ and $[w, m, b] \geqslant^\star [z, m, a]$ with at least one of the relations strict, and

(ii) $[x, r, y] \geqslant^\star [z, r, w]$ and $[z, s, w] \geqslant^\star [x, s, y]$.

Then by eq. (14), it follows from $[x, r, y] \geqslant^\star [z, r, w]$ that

$$u(x) - \Delta_r [u(x) - u(y)] \geqslant u(z) - \Delta_r [u(z) - u(w)],$$

or

$$u(x) - u(z) \geqslant \Delta_r \{[u(x) - u(y)] - [u(z) - u(w)]\}.$$

By the hypothesis that $[x, m, a] \geqslant^\star [y, m, b]$ and $[w, m, b] \geqslant^\star [z, m, a]$, with at least one of the relations strict, it follows that

$$[u(x) - u(y)] > [u(z) - u(w)].$$

Therefore, $[x, r, y] \geqslant^* [z, r, w]$ implies that

$$[u(x) - u(z)] / \{[u(x) - u(y)] - [u(z) - u(w)]\} \geqslant \Delta_r.$$

Similarly, the hypothesis that $[z, s, w] \geqslant^* [x, s, y]$ implies that

$$[u(x) - u(z)] / \{[u(x) - u(y)] - [u(z) - u(w)]\} \leqslant \Delta_s.$$

Thus, it follows that $\Delta_s \geqslant \Delta_r$. Axiom 8 asserts that this is equivalent to $s \geqslant r$, for all $r, s \in [0, 1]$. Thus, axiom 8 implies that

$$\Delta'(x, p, y) \geqslant \Delta'(z, q, w) \quad \text{iff} \quad \Delta(x, p, y) \geqslant \Delta(z, q, w),$$

for all $(x, p, y), (z, q, w) \in G$.

It follows from the above, and the construction that $\Delta_0 = 0$ and $\Delta_1 = 1$, that the function f, defined by $\Delta'(x, p, y) = f[\Delta(x, p, y)]$, is: (1) defined on $\Delta[G]$; (2) well defined; (3) strictly increasing; and (4) satisfies $f(0) = 0$ and $f(1) = 1$. Therefore, if $x \neq y$, we can rewrite Δ' to yield

$$[x - MS(x, p, y)] / [x - y] = f[\Delta(x, p, y],$$

and thus,

$$MS(x, p, y) = x - f[\Delta(x, p, y)] (x - y).$$

If $x = y$, then by axioms 2 and 3, $MS(x, p, x) = MS(x, 1, y) = x$. Since $\Delta'(x, p, x) = \Delta(x, p, x) = 0$, and $f(0) = 0$,

$$MS(x, p, y) = x - f[\Delta(x, p, y)] (x - y) = x,$$

as required. Thus, the desired representation for minimum selling price is established. Moreover, it is clear that any other function f' providing this representation must satisfy $\Delta'(x, p, y) = f'[\Delta(x, p, y)]$, implying that $f' = f$. Thus, f is an absolute scale.

Finally, consider axiom 9. Let $(x, p, y), (z, q, w) \in G - D$ be nondegenerate gambles where $(x, p, y) \in G'_r$ and $(z, q, w) \in G'_s$. Then axiom 9 asserts that

$$r \geqslant s \quad \text{iff} \quad R(x, p, y) \leqslant R(z, q, w),$$

and it easily follows that

$$r \geqslant s \quad \text{iff} \quad [100 - R(x, p, y)] / [100 - 1] \geqslant [100 - R(z, q, w)] / [100 - 1].$$

Moreover, since it was shown above that Δ' is monotone with Δ, we have

$$\Delta(x, p, y) \geqslant \Delta(z, q, w)$$

iff

$$[100 - R(x, p, y)] / [100 - 1] \geqslant [100 - R(z, q, w)] / [100 - 1].$$

Therefore, the function h defined by

$$[100 - R(x, p, y)] / [100 - 1] = h[\Delta(x, p, y)]$$

is: (1) defined on $\Delta[G - D]$; (2) well defined; and (3) strictly increasing. Finally, it is immediate that

$$R(x, p, y) = 100 - h[\Delta(x, p, y)](100 - 1),$$

and that h is an absolute scale. QED

Footnotes

1 For convenience, the proportional adjustment in the ratings Δ'' is based on the entire length of the rating scale, i.e. $\Delta'' = [20 - R(g)] / [20]$. Only minor changes would be needed if it were assumed that subjects reserve some room at the extremes of the rating scale for extraordinary gambles. In this case, the subjective interpolation process would be one in which the person first establishes a correspondence between the endpoints of the utility scale and some nearly extreme ratings, say 15 and 5. Then he or she would attempt to find a rating such that the proportional adjustment in basic evaluation Δ matches the proportional adjustment in ratings, in this case defined as $\Delta'' = [15 - R(g)] / [15 - 5]$.

2 It should be noted that even if u is an interval scale, it cannot be assumed to have the same properties as the von Neumann–Morgenstern utility function, which is derived from expected utility theory instead of expression theory. In particular, a concave utility function in expression theory does not necessarily imply risk aversion.

3 The assumption of symmetry is not required for additivity, but incorporating it into axiom 5 simplifies the development. Also, we will overlook the fact that there is a slight redundancy between axioms 1 and 5. If $\geqslant^*$ is a weak order on G (part of axiom 1), it follows that for any r, $\geqslant_r$ is a weak order on AxA (part of axiom 5).

[4] The term "certainty equivalent" is often used to refer to the judgment given by a subject when asked to produce the value z such that $(z, 1, z) =^* (x, p, y)$. Because expression theory distinguishes sharply between judgments and choices, the term "certainty choice-equivalent" is used to emphasize that the statement $(z, 1, z) =^* (x, p, y)$ is a statement about preferential choice, whether or not z is the value judged by the subject to be the "certainty equivalent".