Ayudantía 1.

1. Demostraciones:

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx = \sqrt{2\pi} \quad \text{(Integral Gaussiana)}$$

$$I = \int_{-\infty}^{+\infty} e^{-x^2/2}dx \quad ; \quad I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx\right)^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2/2}dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2/2}\,dy\right)$$

→ se convierte en una integral multivariable.

$$I^2 = \iint_{\mathbb{R}^2} e^{-\left(\frac{x^2+y^2}{2}\right)}dx\,dy \quad ; \text{ cambio de variables a polares} \quad \begin{array}{l} x^2+y^2 = r^2 \\ \text{Jacobiano} = r\,dr\,d\theta \\ \mathbb{R}^2 \to \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq \infty \end{array} \end{array}$$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta \quad ; \text{ sustitución } \quad \begin{array}{l} \mu = -r^2/2 \\ d\mu = -r\,dr \end{array}$$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} -e^{\mu}\,d\mu\,d\theta = \int_0^{2\pi}\left[-e^{\mu}\right]d\theta \quad ; \text{ deshacemos sustitución}$$

$$I^2 = \int_0^{2\pi}\left[-e^{-r^2/2}\right]_{r=0}^{r=\infty} d\theta = \int_0^{2\pi}[0+1]\,d\theta = 2\pi \quad ; \quad I^2 = 2\pi \quad ; \quad I = \sqrt{2\pi}$$

QED

$$\int_0^{\infty^+} \frac{\nu^K}{\Gamma(K)} x^{K-1} e^{-\nu x}\,dx = 1 \qquad \text{donde: función gama} = \Gamma(K) = \int_0^{\infty^+} \mu^{K-1}e^{-\mu}\,d\mu$$

$$\int_0^{\infty} \frac{\nu^K}{\Gamma(K)} x^{K-1}e^{-\nu x}\,dx = \frac{\nu^K}{\Gamma(K)}\int_0^{\infty} x^{K-1}e^{-\nu x}\,dx \quad ; \text{ cambio de variable} \quad \begin{array}{l} y = \nu x \\ dy = \nu dx \end{array}$$

$$= \frac{\nu^K}{\Gamma(K)}\int_0^{\infty}\left(\frac{y}{\nu}\right)^{K-1} e^{-y}\frac{1}{\nu}dy = \frac{1}{\Gamma(K)}\underbrace{\int_0^{\infty} y^{K-1}e^{-y}dy}_{\Gamma(K)} = \frac{1}{\Gamma(K)}\cdot\Gamma(K) = 1 \quad \text{QED}$$

$$\sum_{K=0}^{n}\binom{n}{K} = 2^n$$

$$(a+b)^n = \sum_{K=0}^{n}\binom{n}{K}a^K b^{n-K} \quad \text{si } a \text{ y } b \text{ son } 1,$$

$$(1+1)^n = \sum_{K=0}^{n}\binom{n}{K}1^K 1^{n-K} \longleftrightarrow (2)^n = \sum_{K=0}^{n}\binom{n}{K} \quad \text{QED}$$

$$\sum_{i=0}^{K}\frac{e^{-\alpha}\cdot\alpha^{K-i}}{(K-i)!}\cdot\frac{e^{-\beta}\cdot\beta^i}{i!} = \frac{e^{-(\alpha+\beta)}(\alpha+\beta)^K}{K!} \qquad \text{donde } \alpha,\beta > 0 \;;\; K \in \mathbb{Z}^+$$

$$e^{-\alpha}\cdot e^{-\beta}\sum_{i=0}^{K}\frac{\alpha^{K-i}}{(K-i)!}\cdot\frac{\beta^i}{i!}\left(\frac{(K!)}{(K!)}\right) = e^{-(\alpha+\beta)}\sum_{i=0}^{K}\frac{\alpha^{K-i}\beta^i}{K!}\binom{K}{i} = \frac{e^{-(\alpha+\beta)}}{K!}\underbrace{\sum_{i=0}^{K}\binom{K}{i}\beta^i\alpha^{K-i}}_{(\beta+\alpha)^K}$$

$$= \frac{e^{-(\alpha+\beta)}}{K!}\cdot(\alpha+\beta)^K \quad \text{QED}$$

$$\sum_{y=x}^{+\infty}\binom{y}{x}p^x(1-p)^{y-x}\,\frac{\nu^y\cdot e^{-\nu}}{y!}=\frac{(\nu p)^x\cdot e^{-\nu p}}{x!}\quad ;\ x\in\mathbb{N}_0,\ \nu>0,\ 0<p<1$$

$$\sum_{y=x}^{+\infty}\frac{y!}{(y-x)!\,x!}p^x(1-p)^{y-x}\cdot\frac{\nu^y\cdot e^{-\nu}}{y!}=\frac{p^x e^{-\nu}}{x!}\sum_{y=x}^{\infty}\frac{(1-p)^{y-x}\cdot\nu^y}{(y-x)!}$$

$\left[* \sum_{k=x}^{\infty}\phi^k=\frac{\phi^x}{1-\phi}\right]$ → Aplicamos $\mu=y-x$ para obtener este formato:

$$=\frac{p^x e^{-\nu}}{x!}\sum_{y=x}^{\infty}\frac{(1-p)^{\mu}}{\mu!}\nu^{\mu+x}=\frac{p^x e^{-\nu}}{x!}\sum_{y=x}^{\infty}\frac{(1-p)^{\mu}}{\mu}\nu^{\mu}\cdot\nu^{x}$$

$$=\frac{p^x e^{-\nu}}{x!}\cdot\nu^x\cdot\sum_{\mu=0}\frac{(1-p)^{\mu}\nu^{\mu}}{\mu!}=\frac{(p\nu)^x e^{-\nu}}{x!}\sum_{\mu=0}^{\infty}\frac{[(1-p)\nu]^{\mu}}{\mu!}=\frac{(p\nu)^x e^{-\nu}}{x!}\cdot e^{\nu-\nu p}$$

$$=\frac{(\nu p)^x e^{-\nu p}}{x!}\quad \text{QED}$$

2. Encontrar máximo/s

$$f(x)=\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}\quad ;\ x\in\mathbb{R},\ \mu\in\mathbb{R},\ \sigma>0$$

$f'(x)=f(x)\cdot\left[\left(\frac{-1}{\sigma}\right)+\exp\left(-\frac{(x-\mu)}{\sigma}\right)\cdot\frac{1}{\sigma}\right]=0$ → Sabemos que $f(x)\neq 0$ ya que buscamos un máximo.

$\frac{-1}{\sigma}+\exp\left(\frac{\mu-x}{\sigma}\right)\cdot\frac{1}{\sigma}=0$ → sólo se cumple si $x=\mu$

→ Debemos comprobar si $x=\mu$ es un máximo.

$f''(x)\Big|_{x=\mu}=0+f(\mu)\left[-\frac{1}{\sigma^2}\right]=-\frac{1}{\sigma^2}\left(\frac{1}{e}\right)<0$, entonces es máximo ✓✓ $f(\mu)=\frac{1}{e}$

3: Demostrar

$\int_0^t f(x)\,dx=1-\exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$ para $t>0$, donde $f(x)=\frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]$, $x>0$

→ usamos la sustitución $\mu=\left(\frac{x}{\eta}\right)^{\beta}$; $d\mu=\frac{\beta x^{\beta-1}}{\eta^{\beta}}dx$

$$\int_0^{(t/\eta)^{\beta}}e^{-\mu}d\mu=\left(-e^{-\mu}\right)\Big|_0^{(t/\eta)^{\beta}}=-e^{-(t/\eta)^{\beta}}-(-1)$$

$$=1-e^{-(t/\eta)^{\beta}}$$

$$=1-\exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right]$$

QED