Evaluating these derivatives at (μ_X, μ_Y) and using the preceding result, we find, if $\mu_X \neq 0$,

$$
\begin{aligned}
E(Z) &\approx \frac{\mu_Y}{\mu_X} + \sigma_X^2 \frac{\mu_Y}{\mu_X^3} - \frac{\sigma_{XY}}{\mu_X^2} \\
&= \frac{\mu_Y}{\mu_X} + \frac{1}{\mu_X^2}\left(\sigma_X^2 \frac{\mu_Y}{\mu_X} - \rho\sigma_X\sigma_Y\right)
\end{aligned}
$$

From this equation, we see that the difference between $E(Z)$ and μ_Y/μ_X depends on several factors. If σ_X and σ_Y are small—that is, if the two concentrations are measured quite accurately—the difference is small. If μ_X is small, the difference is relatively large. Finally, correlation between X and Y affects the difference.

We now consider the variance. Again using the preceding result and evaluating the partial derivatives at (μ_X, μ_Y), we find

$$
\begin{aligned}
\mathrm{Var}(Z) &\approx \sigma_X^2 \frac{\mu_Y^2}{\mu_X^4} + \frac{\sigma_Y^2}{\mu_X^2} - 2\sigma_{XY}\frac{\mu_Y}{\mu_X^3} \\
&= \frac{1}{\mu_X^2}\left(\sigma_X^2 \frac{\mu_Y^2}{\mu_X^2} + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y \frac{\mu_Y}{\mu_X}\right)
\end{aligned}
$$

From this equation, we see that the ratio is quite variable when μ_X is small, paralleling the results in Example A, and that correlation between X and Y, if of the same sign as μ_Y/μ_X, decreases $\mathrm{Var}(Z)$. ■

4.7 Problems

1. Show that if a random variable is bounded—that is, $|X| < M < \infty$—then $E(X)$ exists.

2. If X is a discrete uniform random variable—that is, $P(X = k) = 1/n$ for $k = 1, 2, \ldots, n$—find $E(X)$ and $\mathrm{Var}(X)$.

3. Find $E(X)$ and $\mathrm{Var}(X)$ for Problem 3 in Chapter 2.

4. Let X have the cdf $F(x) = 1 - x^{-\alpha}$, $x \geq 1$.
   a. Find $E(X)$ for those values of α for which $E(X)$ exists.
   b. Find $\mathrm{Var}(X)$ for those values of α for which it exists.

5. Let X have the density

$$
f(x) = \frac{1+\alpha x}{2}, \qquad -1 \leq x \leq 1, \qquad -1 \leq \alpha \leq 1
$$

   Find $E(X)$ and $\mathrm{Var}(X)$.

6. Let X be a continuous random variable with probability density function $f(x) = 2x, 0 \leq x \leq 1$.

a. Find $E(X)$.
b. Let $Y = X^2$. Find the probability mass function of Y and use it to find $E(Y)$.
c. Use Theorem A in Section 4.1.1 to find $E(X^2)$ and compare to your answer in part (b).
d. Find Var(X) according to the definition of variance given in Section 4.2. Also find Var(X) by using Theorem B of Section 4.2.

7. Let X be a discrete random variable that takes on values $0, 1, 2$ with probabilities $\frac{1}{2}, \frac{3}{8}, \frac{1}{8}$, respectively.

a. Find $E(X)$.
b. Let $Y = X^2$. Find the probability mass function of Y and use it to find $E(Y)$.
c. Use Theorem A of Section 4.1.1 to find $E(X^2)$ and compare to your answer in part (b).
d. Find Var(X) according to the definition of variance given in Section 4.2. Also find Var(X) by using Theorem B in Section 4.2.

8. Show that if X is a discrete random variable, taking values on the positive integers, then $E(X) = \sum_{k=1}^{\infty} P(X \geq k)$. Apply this result to find the expected value of a geometric random variable.

9. This is a simplified inventory problem. Suppose that it costs c dollars to stock an item and that the item sells for s dollars. Suppose that the number of items that will be asked for by customers is a random variable with the frequency function $p(k)$. Find a rule for the number of items that should be stocked in order to maximize the expected income. (*Hint:* Consider the difference of successive terms.)

10. A list of n items is arranged in random order; to find a requested item, they are searched sequentially until the desired item is found. What is the expected number of items that must be searched through, assuming that each item is equally likely to be the one requested? (Questions of this nature arise in the design of computer algorithms.)

11. Referring to Problem 10, suppose that the items are not equally likely to be requested but have known probabilities $p_1, p_2, \ldots, p_n$. Suggest an alternative searching procedure that will decrease the average number of items that must be searched through, and show that in fact it does so.

12. If X is a continuous random variable with a density that is symmetric about some point, ξ, show that $E(X) = \xi$, provided that $E(X)$ exists.

13. If X is a nonnegative continuous random variable, show that

$$E(X) = \int_0^{\infty} [1 - F(x)]\, dx$$

Apply this result to find the mean of the exponential distribution.

14. Let X be a continuous random variable with the density function

$$f(x) = 2x, \qquad 0 \leq x \leq 1$$

a. Find $E(X)$.
b. Find $E(X^2)$ and $\mathrm{Var}(X)$.

15. Suppose that two lotteries each have n possible numbers and the same payoff. In terms of expected gain, is it better to buy two tickets from one of the lotteries or one from each?

16. Suppose that $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. Let $Z = (X - \mu)/\sigma$. Show that $E(Z) = 0$ and $\mathrm{Var}(Z) = 1$.

17. Find (a) the expectation and (b) the variance of the kth-order statistic of a sample of n independent random variables uniform on [0, 1]. The density function is given in Example C in Section 3.7.

18. If $U_1, \ldots, U_n$ are independent uniform random variables, find $E(U_{(n)} - U_{(1)})$.

19. Find $E(U_{(k)} - U_{(k-1)})$, where the $U_{(i)}$ are as in Problem 18.

20. A stick of unit length is broken into two pieces. Find the expected ratio of the length of the longer piece to the length of the shorter piece.

21. A random square has a side length that is a uniform [0, 1] random variable. Find the expected area of the square.

22. A random rectangle has sides the lengths of which are independent uniform random variables. Find the expected area of the rectangle, and compare this result to that of Problem 21.

23. Repeat Problems 21 and 22 assuming that the distribution of the lengths is exponential.

24. Prove Theorem A of Section 4.1.2 for the discrete case.

25. If X_1 and X_2 are independent random variables following a gamma distribution with parameters α and λ, find $E(R^2)$, where $R^2 = X_1^2 + X_2^2$.

26. Referring to Example B in Section 4.1.2, what is the expected number of coupons needed to collect r different types, where $r < n$?

27. If n men throw their hats into a pile and each man takes a hat at random, what is the expected number of matches? (*Hint:* Express the number as a sum.)

28. Suppose that n enemy aircraft are shot at simultaneously by m gunners, that each gunner selects an aircraft to shoot at independently of the other gunners, and that each gunner hits the selected aircraft with probability p. Find the expected number of aircraft hit by the gunners.

29. Prove Corollary A of Section 4.1.1.

30. Find $E[1/(X + 1)]$, where X is a Poisson random variable.

31. Let X be uniformly distributed on the interval $[1, 2]$. Find $E(1/X)$. Is $E(1/X) = 1/E(X)$?

32. Let X have a gamma distribution with parameters α and λ. For those values of α and λ for which it is defined, find $E(1/X)$.

33. Prove Chebyshev's inequality in the discrete case.

34. Let X be uniform on [0, 1], and let $Y = \sqrt{X}$. Find $E(Y)$ by (a) finding the density of Y and then finding the expectation and (b) using Theorem A of Section 4.1.1.

35. Find the mean of a negative binomial random variable. (*Hint:* Express the random variable as a sum.)

36. Consider the following scheme for group testing. The original lot of samples is divided into two groups, and each of the subgroups is tested as a whole. If either subgroup tests positive, it is divided in two, and the procedure is repeated. If any of the groups thus obtained tests positive, test every member of that group. Find the expected number of tests performed, and compare it to the number performed with no grouping and with the scheme described in Example C in Section 4.1.2.

37. For what values of p is the group testing of Example C in Section 4.1.2 inferior to testing every individual?

38. This problem continues Example A of Section 4.1.2.

a. What is the probability that a fragment is the leftmost member of a contig?
b. What is the expected number of fragments that are leftmost members of contigs?
c. What is the expected number of contigs?

39. Suppose that a segment of DNA of length 1,000,000 is to be shotgun sequenced with fragments of length 1000.

a. How many fragment would be needed so that the chance of an individual site being covered is greater than 0.99?
b. With this choice, how many sites would you expect to be missed?

40. A child types the letters Q, W, E, R, T, Y, randomly producing 1000 letters in all. What is the expected number of times that the sequence QQQQ appears, counting overlaps?

41. Continuing with the previous problem, how many times would we expect the word "TRY" to appear? Would we be surprised if it occurred 100 times? (*Hint:* Consider Markov's inequality.)

42. Let X be an exponential random variable with standard deviation σ. Find $P(|X - E(X)| > k\sigma)$ for $k = 2, 3, 4$, and compare the results to the bounds from Chebyshev's inequality.

43. Show that $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y)$.

44. If X and Y are independent random variables with equal variances, find $\text{Cov}(X+Y,\ X-Y)$.

45. Find the covariance and the correlation of N_i and N_j, where $N_1,\ N_2,\ \ldots,\ N_r$ are multinomial random variables. (*Hint:* Express them as sums.)

46. If $U = a + bX$ and $V = c + dY$, show that $|\rho_{UV}| = |\rho_{XY}|$.

47. If X and Y are independent random variables and $Z = Y - X$, find expressions for the covariance and the correlation of X and Z in terms of the variances of X and Y.

48. Let U and V be independent random variables with means μ and variances σ^2. Let $Z = \alpha U + V\sqrt{1-\alpha^2}$. Find $E(Z)$ and ρ_{UZ}.

49. Two independent measurements, X and Y, are taken of a quantity μ. $E(X) = E(Y) = \mu$, but σ_X and σ_Y are unequal. The two measurements are combined by means of a weighted average to give

$$Z = \alpha X + (1-\alpha)Y$$

where α is a scalar and $0 \le \alpha \le 1$.

a. Show that $E(Z) = \mu$.
b. Find α in terms of σ_X and σ_Y to minimize $\text{Var}(Z)$.
c. Under what circumstances is it better to use the average $(X+Y)/2$ than either X or Y alone?

50. Suppose that X_i, where $i = 1,\ \ldots,\ n$, are independent random variables with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$. Let $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$. Show that $E(\overline{X}) = \mu$ and $\text{Var}(\overline{X}) = \sigma^2/n$.

51. Continuing Example E in Section 4.3, suppose there are n securities, each with the same expected return, that all the returns have the same standard deviations, and that the returns are uncorrelated. What is the optimal portfolio vector? Plot the risk of the optimal portfolio versus n. How does this risk compare to that incurred by putting all your money in one security?

52. Consider two securities, the first having $\mu_1 = 1$ and $\sigma_1 = 0.1$, and the second having $\mu_2 = 0.8$ and $\sigma_2 = 0.12$. Suppose that they are negatively correlated, with $\rho = -0.8$.

a. If you could only invest in one security, which one would you choose, and why?
b. Suppose you invest 50% of your money in each of the two. What is your expected return and what is your risk?
c. If you invest 80% of your money in security 1 and 20% in security 2, what is your expected return and your risk?
d. Denote the expected return and its standard deviation as functions of π by $\mu(\pi)$ and $\sigma(\pi)$. The pair $(\mu(\pi), \sigma(\pi))$ trace out a curve in the plane as π varies from 0 to 1. Plot this curve.
e. Repeat **b–d** if the correlation is $\rho = 0.1$.

53. Show that $\text{Cov}(X,\ Y) \le \sqrt{\text{Var}(X)\text{Var}(Y)}$.

54. Let X, Y, and Z be uncorrelated random variables with variances σ_X^2, σ_Y^2, and σ_Z^2, respectively. Let

$$U = Z + X$$
$$V = Z + Y$$

Find $\mathrm{Cov}(U, V)$ and ρ_{UV}.

55. Let $T = \sum_{k=1}^{n} kX_k$, where the X_k are independent random variables with means μ and variances σ^2. Find $E(T)$ and $\mathrm{Var}(T)$.

56. Let $S = \sum_{k=1}^{n} X_k$, where the X_k are as in Problem 55. Find the covariance and the correlation of S and T.

57. If X and Y are independent random variables, find $\mathrm{Var}(XY)$ in terms of the means and variances of X and Y.

58. A function is measured at two points with some error (for example, the position of an object is measured at two times). Let

$$X_1 = f(x) + \varepsilon_1$$
$$X_2 = f(x + h) + \varepsilon_2$$

where ε_1 and ε_2 are independent random variables with mean zero and variance σ^2. Since the derivative of f is

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

it is estimated by

$$Z = \frac{X_2 - X_1}{h}$$

a. Find $E(Z)$ and $\mathrm{Var}(Z)$. What is the effect of choosing a value of h that is very small, as is suggested by the definition of the derivative?
b. Find an approximation to the mean squared error of Z as an estimate of $f'(x)$ using a Taylor series expansion. Can you find the value of h that minimizes the mean squared error?
c. Suppose that f is measured at three points with some error. How could you construct an estimate of the second derivative of f, and what are the mean and the variance of your estimate?

59. Let (X, Y) be a random point uniformly distributed on a unit disk. Show that $\mathrm{Cov}(X, Y) = 0$, but that X and Y are not independent.

60. Let Y have a density that is symmetric about zero, and let $X = SY$, where S is an independent random variable taking on the values $+1$ and -1 with probability $\frac{1}{2}$ each. Show that $\mathrm{Cov}(X, Y) = 0$, but that X and Y are not independent.

61. In Section 3.7, the joint density of the minimum and maximum of n independent uniform random variables was found. In the case $n = 2$, this amounts to X and Y, the minimum and maximum, respectively, of two independent random variables uniform on [0, 1], having the joint density

$$f(x, y) = 2, \qquad 0 \le x \le y$$

a. Find the covariance and the correlation of X and Y. Does the sign of the correlation make sense intuitively?
b. Find $E(X|Y = y)$ and $E(Y|X = x)$. Do these results make sense intuitively?
c. Find the probability density functions of the random variables $E(X|Y)$ and $E(Y|X)$.
d. What is the linear predictor of Y in terms of X (denoted by $\hat{Y} = a + bX$) that has minimal mean squared error? What is the mean square prediction error?
e. What is the predictor of Y in terms of X [$\hat{Y} = h(X)$] that has minimal mean squared error? What is the mean square prediction error?

62. Let X and Y have the joint distribution given in Problem 1 of Chapter 3.

a. Find the covariance and correlation of X and Y.
b. Find $E(Y|X = x)$ for $x = 1, 2, 3, 4$. Find the probability mass function of the random variable $E(Y|X)$.

63. Let X and Y have the joint distribution given in Problem 8 of Chapter 3.

a. Find the covariance and correlation of X and Y.
b. Find $E(Y|X = x)$ for $0 \le x \le 1$.

64. Let X and Y be jointly distributed random variables with correlation ρ_{XY}; define the *standardized* random variables $\tilde{X}$ and $\tilde{Y}$ as $\tilde{X} = (X - E(X))/\sqrt{\mathrm{Var}(X)}$ and $\tilde{Y} = (Y - E(Y))/\sqrt{\mathrm{Var}(Y)}$. Show that $\mathrm{Cov}(\tilde{X}, \tilde{Y}) = \rho_{XY}$.

65. How has the assumption that N and the X_i are independent been used in Example D of Section 4.4.1?

66. A building contains two elevators, one fast and one slow. The average waiting time for the slow elevator is 3 min. and the average waiting time of the fast elevator is 1 min. If a passenger chooses the fast elevator with probability $\frac{2}{3}$ and the slow elevator with probability $\frac{1}{3}$, what is the expected waiting time? (Use the law of total expectation, Theorem A of Section 4.4.1, defining appropriate random variables X and Y.)

67. A random rectangle is formed in the following way: The base, X, is chosen to be a uniform [0, 1] random variable and after having generated the base, the height is chosen to be uniform on [0, X]. Use the law of total expectation, Theorem A of Section 4.4.1, to find the expected circumference and area of the rectangle.

68. Show that $E[\mathrm{Var}(Y|X)] \le \mathrm{Var}(Y)$.

69. Suppose that a bivariate normal distribution has $\mu_X = \mu_Y = 0$ and $\sigma_X = \sigma_Y = 1$. Sketch the contours of the density and the lines $E(Y|X = x)$ and $E(X|Y = y)$ for $\rho = 0, .5$, and $.9$.

70. If X and Y are independent, show that $E(X|Y = y) = E(X)$.

71. Let X be a binomial random variable representing the number of successes in n independent Bernoulli trials. Let Y be the number of successes in the first m trials, where $m < n$. Find the conditional frequency function of Y given $X = x$ and the conditional mean.

72. An item is present in a list of n items with probability p; if it is present, its position in the list is uniformly distributed. A computer program searches through the list sequentially. Find the expected number of items searched through before the program terminates.

73. A fair coin is tossed n times, and the number of heads, N, is counted. The coin is then tossed N more times. Find the expected total number of heads generated by this process.

74. The number of offspring of an organism is a discrete random variable with mean μ and variance σ^2. Each of its offspring reproduces in the same manner. Find the expected number of offspring in the third generation and its variance.

75. Let T be an exponential random variable, and conditional on T, let U be uniform on $[0, T]$. Find the unconditional mean and variance of U.

76. Let the point (X, Y) be uniformly distributed over the half disk $x^2 + y^2 \le 1$, where $y \ge 0$. If you observe X, what is the best prediction for Y? If you observe Y, what is the best prediction for X? For both questions, "best" means having the minimum mean squared error.

77. Let X and Y have the joint density

$$f(x, y) = e^{-y}, \qquad 0 \le x \le y$$

a. Find $\mathrm{Cov}(X, Y)$ and the correlation of X and Y.
b. Find $E(X|Y = y)$ and $E(Y|X = x)$.
c. Find the density functions of the random variables $E(X|Y)$ and $E(Y|X)$.

78. Show that if a density is symmetric about zero, its skewness is zero.

79. Let X be a discrete random variable that takes on values 0, 1, 2 with probabilities $\frac{1}{2}$, $\frac{3}{8}$, $\frac{1}{8}$, respectively. Find the moment-generating function of X, $M(t)$, and verify that $E(X) = M'(0)$ and that $E(X^2) = M''(0)$.

80. Let X be a continuous random variable with density function $f(x) = 2x$, $0 \le x \le 1$. Find the moment-generating function of X, $M(t)$, and verify that $E(X) = M'(0)$ and that $E(X^2) = M''(0)$.

81. Find the moment-generating function of a Bernoulli random variable, and use it to find the mean, variance, and third moment.

82. Use the result of Problem 81 to find the mgf of a binomial random variable and its mean and variance.

83. Show that if X_i follows a binomial distribution with n_i trials and probability of success $p_i = p$, where $i = 1, \ldots, n$ and the X_i are independent, then $\sum_{i=1}^{n} X_i$ follows a binomial distribution.

84. Referring to Problem 83, show that if the p_i are unequal, the sum does not follow a binomial distribution.

85. Find the mgf of a geometric random variable, and use it to find the mean and the variance.

86. Use the result of Problem 85 to find the mgf of a negative binomial random variable and its mean and variance.

87. Under what conditions is the sum of independent negative binomial random variables also negative binomial?

88. Let X and Y be independent random variables, and let α and β be scalars. Find an expression for the mgf of $Z = \alpha X + \beta Y$ in terms of the mgf's of X and Y.

89. Let $X_1, X_2, \ldots, X_n$ be independent normal random variables with means μ_i and variances σ_i^2. Show that $Y = \sum_{i=1}^{n} \alpha_i X_i$, where the α_i are scalars, is normally distributed, and find its mean and variance. (*Hint:* Use moment-generating functions.)

90. Assuming that $X \sim N(0, \sigma^2)$, use the mgf to show that the odd moments are zero and the even moments are

$$\mu_{2n} = \frac{(2n)!\sigma^{2n}}{2^n(n!)}$$

91. Use the mgf to show that if X follows an exponential distribution, cX $(c > 0)$ does also.

92. Suppose that Θ is a random variable that follows a gamma distribution with parameters λ and α, where α is an integer, and suppose that, conditional on Θ, X follows a Poisson distribution with parameter Θ. Find the unconditional distribution of $\alpha + X$. (*Hint:* Find the mgf by using iterated conditional expectations.)

93. Find the distribution of a geometric sum of exponential random variables by using moment-generating functions.

94. If X is a nonnegative integer-valued random variable, the **probability-generating function** of X is defined to be

$$G(s) = \sum_{k=0}^{\infty} s^k p_k$$

where $p_k = P(X = k)$.

a. Show that

$$p_k = \frac{1}{k!} \frac{d^k}{ds^k} G(s) \bigg|_{s=0}$$

b. Show that

$$\left.\frac{dG}{ds}\right|_{s=1} = E(X)$$

$$\left.\frac{d^2G}{ds^2}\right|_{s=1} = E[X(X-1)]$$

c. Express the probability-generating function in terms of moment-generating function.

d. Find the probability-generating function of the Poisson distribution.

95. Show that if X and Y are independent, their joint moment-generating function factors.

96. Show how to find $E(XY)$ from the joint moment-generating function of X and Y.

97. Use moment-generating functions to show that if X and Y are independent, then

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$$

98. Find the mean and variance of the compound Poisson distribution (Example H in Section 4.5).

99. Find expressions for the approximate mean and variance of $Y = g(X)$ for (a) $g(x) = \sqrt{x}$, (b) $g(x) = \log x$, and (c) $g(x) = \sin^{-1} x$.

100. If X is uniform on [10, 20], find the approximate and exact mean and variance of $Y = 1/X$, and compare them.

101. Find the approximate mean and variance of $Y = \sqrt{X}$, where X is a random variable following a Poisson distribution.

102. Two sides, x_0 and y_0, of a right triangle are independently measured as X and Y, where $E(X) = x_0$ and $E(Y) = y_0$ and $\text{Var}(X) = \text{Var}(Y) = \sigma^2$. The angle between the two sides is then determined as

$$\Theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

Find the approximate mean and variance of Θ.

103. The volume of a bubble is estimated by measuring its diameter and using the relationship

$$V = \frac{\pi}{6}D^3$$

Suppose that the true diameter is 2 mm and that the standard deviation of the measurement of the diameter is .01 mm. What is the approximate standard deviation of the estimated volume?

104. The position of an aircraft relative to an observer on the ground is estimated by measuring its distance r from the observer and the angle θ that the line of sight from the observer to the aircraft makes with the horizontal. Suppose that the measurements, denoted by R and Θ, are subject to random errors and are independent of each other. The altitude of the aircraft is then estimated to be $Y = R \sin \Theta$.

a. Find an approximate expression for the variance of Y.
b. For given r, at what value of θ is the estimated altitude most variable?