matches is much larger than would be expected by chance alone. This requires a chance model; a simple one stipulates that the nucleotide at each site of fragment 1 occurs randomly with probabilities p_{A1}, p_{G1}, p_{C1}, p_{T1}, and that the second fragment is similarly composed with probabilities $p_{A2}, \ldots, p_{T2}$. What is the chance that the fragments match at a particular site if in fact the identity of the nucleotide on fragment 1 is independent of that on fragment 2? The match probability can be calculated using the law of total probability:

$$
\begin{aligned}
P(\text{match}) &= P(\text{match}|\text{A on fragment 1})P(\text{A on fragment 1}) + \\
&\qquad \ldots + P(\text{match}|\text{T on fragment 1})P(\text{T on fragment 1}) \\
&= p_{A2}p_{A1} + p_{G2}p_{G1} + p_{C2}p_{C1} + p_{T2}p_{T1}
\end{aligned}
$$

The problem of determining the probability that they match at k out of a total of n sites is discussed later. ■

1.7 Concluding Remarks

This chapter provides a simple axiomatic development of the mathematical theory of probability. Some subtle issues that arise in a careful analysis of infinite sample spaces have been neglected. Such issues are typically addressed in graduate-level courses in measure theory and probability theory. Certain philosophical questions have also been avoided. One might ask what is meant by the statement "The probability that this coin will land heads up is $\frac{1}{2}$." Two commonly advocated views are the **frequentist approach** and the **Bayesian approach.** According to the frequentist approach, the statement means that if the experiment were repeated many times, the long-run average number of heads would tend to $\frac{1}{2}$. According to the Bayesian approach, the statement is a quantification of the speaker's uncertainty about the outcome of the experiment and thus is a personal or subjective notion; the probability that the coin will land heads up may be different for different speakers, depending on their experience and knowledge of the situation. There has been vigorous and occasionally acrimonious debate among proponents of various versions of these points of view.

In this and ensuing chapters, there are many examples of the use of probability as a model for various phenomena. In any such modeling endeavor, an idealized mathematical theory is hoped to provide an adequate match to characteristics of the phenomenon under study. The standard of adequacy is relative to the field of study and the modeler's goals.

1.8 Problems

1. A coin is tossed three times and the sequence of heads and tails is recorded.

a. List the sample space.

b. List the elements that make up the following events: (1) A = at least two heads, (2) B = the first two tosses are heads, (3) C = the last toss is a tail.

c. List the elements of the following events: (1) A^c, (2) $A \cap B$, (3) $A \cup C$.

2. Two six-sided dice are thrown sequentially, and the face values that come up are recorded.

a. List the sample space.
b. List the elements that make up the following events: (1) $A =$ the sum of the two values is at least 5, (2) $B =$ the value of the first die is higher than the value of the second, (3) $C =$ the first value is 4.
c. List the elements of the following events: (1) $A \cap C$, (2) $B \cup C$, (3) $A \cap (B \cup C)$.

3. An urn contains three red balls, two green balls, and one white ball. Three balls are drawn without replacement from the urn, and the colors are noted in sequence. List the sample space. Define events A, B, and C as you wish and find their unions and intersections.

4. Draw Venn diagrams to illustrate De Morgan's laws:

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

5. Let A and B be arbitrary events. Let C be the event that either A occurs or B occurs, but not both. Express C in terms of A and B using any of the basic operations of union, intersection, and complement.

6. Verify the following extension of the addition rule (a) by an appropriate Venn diagram and (b) by a formal argument using the axioms of probability and the propositions in this chapter.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$
$$- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

7. Prove Bonferroni's inequality:

$$P(A \cap B) \geq P(A) + P(B) - 1$$

8. Prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

9. The weather forecaster says that the probability of rain on Saturday is 25% and that the probability of rain on Sunday is 25%. Is the probability of rain during the weekend 50%? Why or why not?

10. Make up another example of Simpson's paradox by changing the numbers in Example B of Section 1.4.

11. The first three digits of a university telephone exchange are 452. If all the sequences of the remaining four digits are equally likely, what is the probability that a randomly selected university phone number contains seven distinct digits?

12. In a game of poker, five players are each dealt 5 cards from a 52-card deck. How many ways are there to deal the cards?

13. In a game of poker, what is the probability that a five-card hand will contain (a) a straight (five cards in unbroken numerical sequence), (b) four of a kind, and (c) a full house (three cards of one value and two cards of another value)?

14. The four players in a bridge game are each dealt 13 cards. How many ways are there to do this?

15. How many different meals can be made from four kinds of meat, six vegetables, and three starches if a meal consists of one selection from each group?

16. How many different letter arrangements can be obtained from the letters of the word *statistically*, using all the letters?

17. In acceptance sampling, a purchaser samples 4 items from a lot of 100 and rejects the lot if 1 or more are defective. Graph the probability that the lot is accepted as a function of the percentage of defective items in the lot.

18. A lot of n items contains k defectives, and m are selected randomly and inspected. How should the value of m be chosen so that the probability that at least one defective item turns up is .90? Apply your answer to (a) $n = 1000$, $k = 10$, and (b) $n = 10{,}000$, $k = 100$.

19. A committee consists of five Chicanos, two Asians, three African Americans, and two Caucasians.

a. A subcommittee of four is chosen at random. What is the probability that all the ethnic groups are represented on the subcommittee?
b. Answer the question for part (a) if a subcommittee of five is chosen.

20. A deck of 52 cards is shuffled thoroughly. What is the probability that the four aces are all next to each other?

21. A fair coin is tossed five times. What is the probability of getting a sequence of three heads?

22. A standard deck of 52 cards is shuffled thoroughly, and n cards are turned up. What is the probability that a face card turns up? For what value of n is this probability about .5?

23. How many ways are there to place n indistinguishable balls in n urns so that exactly one urn is empty?

24. If n balls are distributed randomly into k urns, what is the probability that the last urn contains j balls?

25. A woman getting dressed up for a night out is asked by her significant other to wear a red dress, high-heeled sneakers, and a wig. In how many orders can she put on these objects?

26. The game of Mastermind starts in the following way: One player selects four pegs, each peg having six possible colors, and places them in a line. The second player then tries to guess the sequence of colors. What is the probability of guessing correctly?

27. If a five-letter word is formed at random (meaning that all sequences of five letters are equally likely), what is the probability that no letter occurs more than once?

28. How many ways are there to encode the 26-letter English alphabet into 8-bit binary words (sequences of eight 0s and 1s)?

29. A poker player is dealt three spades and two hearts. He discards the two hearts and draws two more cards. What is the probability that he draws two more spades?

30. A group of 60 second graders is to be randomly assigned to two classes of 30 each. (The random assignment is ordered by the school district to ensure against any bias.) Five of the second graders, Marcelle, Sarah, Michelle, Katy, and Camerin, are close friends. What is the probability that they will all be in the same class? What is the probability that exactly four of them will be? What is the probability that Marcelle will be in one class and her friends in the other?

31. Six male and six female dancers perform the Virginia reel. This dance requires that they form a line consisting of six male/female pairs. How many such arrangements are there?

32. A wine taster claims that she can distinguish four vintages of a particular Cabernet. What is the probability that she can do this by merely guessing? (She is confronted with four unlabeled glasses.)

33. An elevator containing five people can stop at any of seven floors. What is the probability that no two people get off at the same floor? Assume that the occupants act independently and that all floors are equally likely for each occupant.

34. Prove the following identity:

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m-n}{n-k} = \binom{m}{n}$$

(*Hint*: How can each of the summands be interpreted?)

35. Prove the following two identities both algebraically and by interpreting their meaning combinatorially.

a. $\binom{n}{r} = \binom{n}{n-r}$

b. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

36. What is the coefficient of x^3y^4 in the expansion of $(x+y)^7$?

37. What is the coefficient of $x^2y^2z^3$ in the expansion of $(x+y+z)^7$?

38. A child has six blocks, three of which are red and three of which are green. How many patterns can she make by placing them all in a line? If she is given three white blocks, how many total patterns can she make by placing all nine blocks in a line?

39. A monkey at a typewriter types each of the 26 letters of the alphabet exactly once, the order being random.

a. What is the probability that the word *Hamlet* appears somewhere in the string of letters?

b. How many independent monkey typists would you need in order that the probability that the word appears is at least .90?

40. In how many ways can two octopi shake hands? (There are a number of ways to interpret this question—choose one.)

41. A drawer of socks contains seven black socks, eight blue socks, and nine green socks. Two socks are chosen in the dark.

a. What is the probability that they match?
b. What is the probability that a black pair is chosen?

42. How many ways can 11 boys on a soccer team be grouped into 4 forwards, 3 midfielders, 3 defenders, and 1 goalie?

43. A software development company has three jobs to do. Two of the jobs require three programmers, and the other requires four. If the company employs ten programmers, how many different ways are there to assign them to the jobs?

44. In how many ways can 12 people be divided into three groups of 4 for an evening of bridge? In how many ways can this be done if the 12 consist of six pairs of partners?

45. Show that if the conditional probabilities exist, then

$$
\begin{aligned}
&P(A_1 \cap A_2 \cap \cdots \cap A_n) \\
&\quad = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1})
\end{aligned}
$$

46. Urn A has three red balls and two white balls, and urn B has two red balls and five white balls. A fair coin is tossed. If it lands heads up, a ball is drawn from urn A; otherwise, a ball is drawn from urn B.

a. What is the probability that a red ball is drawn?
b. If a red ball is drawn, what is the probability that the coin landed heads up?

47. Urn A has four red, three blue, and two green balls. Urn B has two red, three blue, and four green balls. A ball is drawn from urn A and put into urn B, and then a ball is drawn from urn B.

a. What is the probability that a red ball is drawn from urn B?
b. If a red ball is drawn from urn B, what is the probability that a red ball was drawn from urn A?

48. An urn contains three red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn.

a. What is the probability that the second ball drawn is white?
b. If the second ball drawn is white, what is the probability that the first ball drawn was red?

49. A fair coin is tossed three times.

a. What is the probability of two or more heads given that there was at least one head?
b. What is the probability given that there was at least one tail?

50. Two dice are rolled, and the sum of the face values is six. What is the probability that at least one of the dice came up a three?

51. Answer Problem 50 again given that the sum is less than six.

52. Suppose that 5 cards are dealt from a 52-card deck and the first one is a king. What is the probability of at least one more king?

53. A fire insurance company has high-risk, medium-risk, and low-risk clients, who have, respectively, probabilities .02, .01, and .0025 of filing claims within a given year. The proportions of the numbers of clients in the three categories are .10, .20, and .70, respectively. What proportion of the claims filed each year come from high-risk clients?

54. This problem introduces a simple meteorological model, more complicated versions of which have been proposed in the meteorological literature. Consider a sequence of days and let R_i denote the event that it rains on day i. Suppose that $P(R_i \mid R_{i-1}) = \alpha$ and $P(R_i^c \mid R_{i-1}^c) = \beta$. Suppose further that only today's weather is relevant to predicting tomorrow's; that is, $P(R_i \mid R_{i-1} \cap R_{i-2} \cap \cdots \cap R_0) = P(R_i \mid R_{i-1})$.

a. If the probability of rain today is p, what is the probability of rain tomorrow?
b. What is the probability of rain the day after tomorrow?
c. What is the probability of rain n days from now? What happens as n approaches infinity?

55. This problem continues Example D of Section 1.5 and concerns occupational mobility.

a. Find $P(M_1 \mid M_2)$ and $P(L_1 \mid L_2)$.
b. Find the proportions that will be in the three occupational levels in the third generation. To do this, assume that a son's occupational status depends on his father's status, but that given his father's status, it does not depend on his grandfather's.

56. A couple has two children. What is the probability that both are girls given that the oldest is a girl? What is the probability that both are girls given that one of them is a girl?

57. There are three cabinets, A, B, and C, each of which has two drawers. Each drawer contains one coin; A has two gold coins, B has two silver coins, and C has one gold and one silver coin. A cabinet is chosen at random, one drawer is opened, and a silver coin is found. What is the probability that the other drawer in that cabinet contains a silver coin?

58. A teacher tells three boys, Drew, Chris, and Jason, that two of them will have to stay after school to help her clean erasers and that one of them will be able to leave. She further says that she has made the decision as to who will leave and who will stay at random by rolling a special three-sided Dungeons and Dragons die. Drew wants to leave to play soccer and has a clever idea about how to increase his chances of doing so. He figures that one of Jason and Chris will certainly stay and asks the teacher to tell him the name of one of the two who will stay. Drew's idea is that if, for example, Jason is named, then he and Chris are left and they each have a probability .5 of leaving; similarly, if Chris is named, Drew's probability of leaving is still .5. Thus, by merely asking the teacher a question, Drew will increase his probability of leaving from $\frac{1}{3}$ to $\frac{1}{2}$. What do you think of this scheme?

59. A box has three coins. One has two heads, one has two tails, and the other is a fair coin with one head and one tail. A coin is chosen at random, is flipped, and comes up heads.

a. What is the probability that the coin chosen is the two-headed coin?
b. What is the probability that if it is thrown another time it will come up heads?
c. Answer part (a) again, supposing that the coin is thrown a second time and comes up heads again.

60. A factory runs three shifts. In a given day, 1% of the items produced by the first shift are defective, 2% of the second shift's items are defective, and 5% of the third shift's items are defective. If the shifts all have the same productivity, what percentage of the items produced in a day are defective? If an item is defective, what is the probability that it was produced by the third shift?

61. Suppose that chips for an integrated circuit are tested and that the probability that they are detected if they are defective is .95, and the probability that they are declared sound if in fact they are sound is .97. If .5% of the chips are faulty, what is the probability that a chip that is declared faulty is sound?

62. Show that if $P(A \mid E) \geq P(B \mid E)$ and $P(A \mid E^c) \geq P(B \mid E^c)$, then $P(A) \geq P(B)$.

63. Suppose that the probability of living to be older than 70 is .6 and the probability of living to be older than 80 is .2. If a person reaches her 70th birthday, what is the probability that she will celebrate her 80th?

64. If B is an event, with $P(B) > 0$, show that the set function $Q(A) = P(A \mid B)$ satisfies the axioms for a probability measure. Thus, for example,

$$P(A \cup C \mid B) = P(A \mid B) + P(C \mid B) - P(A \cap C \mid B)$$

65. Show that if A and B are independent, then A and B^c as well as A^c and B^c are independent.

66. Show that $\emptyset$ is independent of A for any A.

67. Show that if A and B are independent, then

$$P(A \cup B) = P(A) + P(B) - P(A)P(B)$$

68. If A is independent of B and B is independent of C, then A is independent of C. Prove this statement or give a counterexample if it is false.

69. If A and B are disjoint, can they be independent?

70. If $A \subset B$, can A and B be independent?

71. Show that if A, B, and C are mutually independent, then $A \cap B$ and C are independent and $A \cup B$ and C are independent.

72. Suppose that n components are connected in series. For each unit, there is a backup unit, and the system fails if and only if both a unit and its backup fail. Assuming that all the units are independent and fail with probability p, what is the probability that the system works? For $n = 10$ and $p = .05$, compare these results with those of Example F in Section 1.6.

73. A system has n independent units, each of which fails with probability p. The system fails only if k or more of the units fail. What is the probability that the system fails?

74. What is the probability that the following system works if each unit fails independently with probability p (see Figure 1.5)?

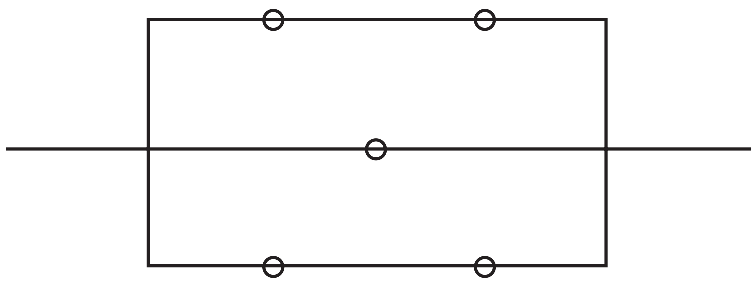

FIGURE **1.5**

75. This problem deals with an elementary aspect of a simple branching process. A population starts with one member; at time $t = 1$, it either divides with probability p or dies with probability $1 - p$. If it divides, then both of its children behave independently with the same two alternatives at time $t = 2$. What is the probability that there are no members in the third generation? For what value of p is this probability equal to .5?

76. Here is a simple model of a queue. The queue runs in discrete time ($t = 0, 1, 2, \ldots$), and at each unit of time the first person in the queue is served with probability p and, independently, a new person arrives with probability q. At time $t = 0$, there is one person in the queue. Find the probabilities that there are 0, 1, 2, 3 people in line at time $t = 2$.

77. A player throws darts at a target. On each trial, independently of the other trials, he hits the bull's-eye with probability .05. How many times should he throw so that his probability of hitting the bull's-eye at least once is .5?

78. This problem introduces some aspects of a simple genetic model. Assume that genes in an organism occur in pairs and that each member of the pair can be either of the types a or A. The possible genotypes of an organism are then AA, Aa, and aa (Aa and aA are equivalent). When two organisms mate, each independently contributes one of its two genes; either one of the pair is transmitted with probability .5.

a. Suppose that the genotypes of the parents are AA and Aa. Find the possible genotypes of their offspring and the corresponding probabilities.

b. Suppose that the probabilities of the genotypes AA, Aa, and aa are p, $2q$, and r, respectively, in the first generation. Find the probabilities in the second and third generations, and show that these are the same. This result is called the Hardy-Weinberg Law.

c. Compute the probabilities for the second and third generations as in part (b) but under the additional assumption that the probabilities that an individual of type AA, Aa, or aa survives to mate are u, v, and w, respectively.

79. Many human diseases are genetically transmitted (for example, hemophilia or Tay-Sachs disease). Here is a simple model for such a disease. The genotype aa is diseased and dies before it mates. The genotype Aa is a carrier but is not diseased. The genotype AA is not a carrier and is not diseased.

a. If two carriers mate, what are the probabilities that their offspring are of each of the three genotypes?

b. If the male offspring of two carriers is not diseased, what is the probability that he is a carrier?

c. Suppose that the nondiseased offspring of part (b) mates with a member of the population for whom no family history is available and who is thus assumed to have probability p of being a carrier (p is a very small number). What are the probabilities that their first offspring has the genotypes AA, Aa, and aa?

d. Suppose that the first offspring of part (c) is not diseased. What is the probability that the father is a carrier in light of this evidence?

80. If a parent has genotype Aa, he transmits either A or a to an offspring (each with a $\frac{1}{2}$ chance). The gene he transmits to one offspring is independent of the one he transmits to another. Consider a parent with three children and the following events: $A = \{\text{children 1 and 2 have the same gene}\}$, $B = \{\text{children 1 and 3 have the same gene}\}$, $C = \{\text{children 2 and 3 have the same gene}\}$. Show that these events are pairwise independent but not mutually independent.