size of the particle is $Y_1 = X_1 y_0$; after the second impact, the size is $Y_2 = X_2 X_1 y_0$; and after the nth impact, the size is

$$Y_n = X_n X_{n-1} \cdots X_2 X_1 y_0$$

Then

$$\log Y_n = \log y_0 + \sum_{i=1}^{n} \log X_i$$

and the central limit theorem applies to $\log Y_n$. ■

A similar construction is relevant to the theory of finance. Suppose that an initial investment of value v_0 is made and that returns occur in discrete time, for example, daily. If the return on the first day is R_1, then the value becomes $V_1 = R_1 v_0$. After day two the value is $V_2 = R_2 R_1 v_0$, and after day n the value is

$$V_n = R_n R_{n-1} \cdots R_1 v_0$$

The log value is thus

$$\log V_n = \log v_0 + \sum_{i=1}^{n} \log R_i$$

If the returns are independent random variables with the same distribution, then the distribution of $\log V_n$ is approximately normally distributed.

5.4 Problems

1. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma_i^2$. Show that if $n^{-2} \sum_{i=1}^{n} \sigma_i^2 \to 0$, then $\overline{X} \to \mu$ in probability.

2. Let X_i be as in Problem 1 but with $E(X_i) = \mu_i$ and $n^{-1} \sum_{i=1}^{n} \mu_i \to \mu$. Show that $\overline{X} \to \mu$ in probability.

3. Suppose that the number of insurance claims, N, filed in a year is Poisson distributed with $E(N) = 10{,}000$. Use the normal approximation to the Poisson to approximate $P(N > 10{,}200)$.

4. Suppose that the number of traffic accidents, N, in a given period of time is distributed as a Poisson random variable with $E(N) = 100$. Use the normal approximation to the Poisson to find Δ such that $P(100 - \Delta < N < 100 + \Delta) \approx .9$.

5. Using moment-generating functions, show that as $n \to \infty$, $p \to 0$, and $np \to \lambda$, the binomial distribution with parameters n and p tends to the Poisson distribution.

6. Using moment-generating functions, show that as $\alpha \to \infty$ the gamma distribution with parameters α and λ, properly standardized, tends to the standard normal distribution.

7. Show that if $X_n \to c$ in probability and if g is a continuous function, then $g(X_n) \to g(c)$ in probability.

8. Compare the Poisson cdf and the normal approximation for (a) $\lambda = 10$, (b) $\lambda = 20$, and (c) $\lambda = 40$.

9. Compare the binomial cdf and the normal approximation for (a) $n = 20$ and $p = .2$, and (b) $n = 40$ and $p = .5$.

10. A six-sided die is rolled 100 times. Using the normal approximation, find the probability that the face showing a six turns up between 15 and 20 times. Find the probability that the sum of the face values of the 100 trials is less than 300.

11. A skeptic gives the following argument to show that there must be a flaw in the central limit theorem: "We know that the sum of independent Poisson random variables follows a Poisson distribution with a parameter that is the sum of the parameters of the summands. In particular, if n independent Poisson random variables, each with parameter n^{-1}, are summed, the sum has a Poisson distribution with parameter 1. The central limit theorem says that as n approaches infinity, the distribution of the sum tends to a normal distribution, but the Poisson with parameter 1 is not the normal." What do you think of this argument?

12. The central limit theorem can be used to analyze round-off error. Suppose that the round-off error is represented as a uniform random variable on $[-\frac{1}{2}, \frac{1}{2}]$. If 100 numbers are added, approximate the probability that the round-off error exceeds (a) 1, (b) 2, and (c) 5.

13. A drunkard executes a "random walk" in the following way: Each minute he takes a step north or south, with probability $\frac{1}{2}$ each, and his successive step directions are independent. His step length is 50 cm. Use the central limit theorem to approximate the probability distribution of his location after 1 h. Where is he most likely to be?

14. Answer Problem 13 under the assumption that the drunkard has some idea of where he wants to go so that he steps north with probability $\frac{2}{3}$ and south with probability $\frac{1}{3}$.

15. Suppose that you bet \$5 on each of a sequence of 50 independent fair games. Use the central limit theorem to approximate the probability that you will lose more than \$75.

16. Suppose that $X_1, \ldots, X_{20}$ are independent random variables with density functions

$$f(x) = 2x, \qquad 0 \le x \le 1$$

Let $S = X_1 + \cdots + X_{20}$. Use the central limit theorem to approximate $P(S \le 10)$.

17. Suppose that a measurement has mean μ and variance $\sigma^2 = 25$. Let $\overline{X}$ be the average of n such independent measurements. How large should n be so that $P(|\overline{X} - \mu| < 1) = .95$?

18. Suppose that a company ships packages that are variable in weight, with an average weight of 15 lb and a standard deviation of 10. Assuming that the packages come from a large number of different customers so that it is reasonable to model their weights as independent random variables, find the probability that 100 packages will have a total weight exceeding 1700 lb.

19. a. Use the Monte Carlo method with $n = 100$ and $n = 1000$ to estimate $\int_0^1 \cos(2\pi x)\, dx$. Compare the estimates to the exact answer.

b. Use Monte Carlo to evaluate $\int_0^1 \cos(2\pi x^2)\, dx$. Can you find the exact answer?

20. What is the variance of the estimate of an integral by the Monte Carlo method (Example A of Section 5.2)? [*Hint:* Find $E(\hat{I}^2(f))$.] Compare the standard deviations of the estimates of part (a) of previous problem to the actual errors you made.

21. This problem introduces a variation on the Monte Carlo integration technique of Example A of Section 5.2. Suppose that we wish to evaluate

$$I(f) = \int_a^b f(x)\, dx$$

Let g be a density function on $[a, b]$. Generate $X_1, \cdots, X_n$ from g and estimate I by

$$\hat{I}(f) = \frac{1}{n}\sum_{i=1}^{n} \frac{f(X_i)}{g(X_i)}$$

a. Show that $E(\hat{I}(f)) = I(f)$.

b. Find an expression for $\mathrm{Var}(\hat{I}(f))$. Give an example for which it is finite and an example for which it is infinite. Note that if it is finite, the law of large numbers implies that $\hat{I}(f) \to I(f)$ as $n \to \infty$.

c. Show that if $a = 0$, $b = 1$, and g is uniform, this is the same Monte Carlo estimate as that of Example A of Section 5.2. Can this estimate be improved by choosing g to be other than uniform? (*Hint*: Compare variances.)

22. Use the central limit theorem to find Δ such that $P(|\hat{I}(f) - I(f)| \leq \Delta) = .05$, where $\hat{I}(f)$ is the Monte Carlo estimate of $\int_0^1 \cos(2\pi x)\, dx$ based on 1000 points.

23. An irregularly shaped object of unknown area A is located in the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$. Consider a random point distributed uniformly over the square; let $Z = 1$ if the point lies inside the object and $Z = 0$ otherwise. Show that $E(Z) = A$. How could A be estimated from a sequence of n independent points uniformly distributed on the square?

24. How could the central limit theorem be used to gauge the probable size of the error of the estimate of the previous problem? Denoting the estimate by $\hat{A}$, if $A = .2$, how large should n be so that $P(|\hat{A} - A| < .01) \approx .99$?

25. Let X be a continuous random variable with density function $f(x) = \frac{3}{2}x^2$, $-1 \leq x \leq 1$. Sketch this density function. Use the central limit theorem to sketch the approximate density function of $S = X_1 + \cdots + X_{50}$, where the X_i are independent random variables with density f. Similarly, sketch the approximate density functions of $S/50$ and $S/\sqrt{50}$. For each sketch, label at least three points on the horizontal axis.

26. Suppose that a basketball player can score on a particular shot with probability .3. Use the central limit theorem to find the approximate distribution of S, the number of successes out of 25 independent shots. Find the approximate probabilities that S is less than or equal to 5, 7, 9, and 11 and compare these to the exact probabilities.

27. Prove that if $a_n \to a$, then $(1 + a_n/n)^n \to e^a$.

28. Let f_n be a sequence of frequency functions with $f_n(x) = \frac{1}{2}$ if $x = \pm(\frac{1}{2})^n$ and $f_n(x) = 0$ otherwise. Show that $\lim f_n(x) = 0$ for all x, which means that the frequency functions do not converge to a frequency function, but that there exists a cdf F such that $\lim F_n(x) = F(x)$.

29. In addition to limit theorems that deal with sums, there are limit theorems that deal with extreme values such as maxima or minima. Here is an example. Let $U_1, \ldots, U_n$ be independent uniform random variables on [0, 1], and let $U_{(n)}$ be the maximum. Find the cdf of $U_{(n)}$ and a standardized $U_{(n)}$, and show that the cdf of the standardized variable tends to a limiting value.

30. Generate a sequence $U_1, U_2, \ldots, U_{1000}$ of independent uniform random variables on a computer. Let $S_n = \sum_{i=1}^{n} U_i$ for $n = 1, 2, \ldots, 1000$. Plot each of the following versus n:

a. S_n
b. S_n/n
c. $S_n - n/2$
d. $(S_n - n/2)/n$
e. $(S_n - n/2)/\sqrt{n}$

Explain the shapes of the resulting graphs using the concepts of this chapter.