For example, if $n = 100$ and $\alpha = .95$, this probability is .96. In words, this means that the probability is .96 that the range of 100 independent random variables covers 95% or more of the probability mass, or, with probability .96, 95% of all further observations from the same distribution will fall between the minimum and maximum. This statement does not depend on the actual form of the distribution. ■

3.8 Problems

1. The joint frequency function of two discrete random variables, X and Y, is given in the following table:

	x			
y	1	2	3	4
1	.10	.05	.02	.02
2	.05	.20	.05	.02
3	.02	.05	.20	.04
4	.02	.02	.04	.10

a. Find the marginal frequency functions of X and Y.
b. Find the conditional frequency function of X given $Y = 1$ and of Y given $X = 1$.

2. An urn contains p black balls, q white balls, and r red balls; and n balls are chosen without replacement.

a. Find the joint distribution of the numbers of black, white, and red balls in the sample.
b. Find the joint distribution of the numbers of black and white balls in the sample.
c. Find the marginal distribution of the number of white balls in the sample.

3. Three players play 10 independent rounds of a game, and each player has probability $\frac{1}{3}$ of winning each round. Find the joint distribution of the numbers of games won by each of the three players.

4. A sieve is made of a square mesh of wires. Each wire has diameter d, and the holes in the mesh are squares whose side length is w. A spherical particle of radius r is dropped on the mesh. What is the probability that it passes through? What is the probability that it fails to pass through if it is dropped n times? (Calculations such as these are relevant to the theory of sieving for analyzing the size distribution of particulate matter.)

5. (Buffon's Needle Problem) A needle of length L is dropped randomly on a plane ruled with parallel lines that are a distance D apart, where $D \geq L$. Show that the probability that the needle comes to rest crossing a line is $2L/(\pi D)$. Explain how this gives a mechanical means of estimating the value of π.

6. A point is chosen randomly in the interior of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Find the marginal densities of the x and y coordinates of the point.

7. Find the joint and marginal densities corresponding to the cdf

$$F(x, y) = (1-e^{-\alpha x})(1-e^{-\beta y}), \qquad x \geq 0, \qquad y \geq 0, \qquad \alpha > 0, \qquad \beta > 0$$

8. Let X and Y have the joint density

$$f(x, y) = \frac{6}{7}(x + y)^2, \qquad 0 \leq x \leq 1, \qquad 0 \leq y \leq 1$$

a. By integrating over the appropriate regions, find (i) $P(X > Y)$, (ii) $P(X + Y \leq 1)$, (iii) $P(X \leq \frac{1}{2})$.
b. Find the marginal densities of X and Y.
c. Find the two conditional densities.

9. Suppose that (X, Y) is uniformly distributed over the region defined by $0 \leq y \leq 1 - x^2$ and $-1 \leq x \leq 1$.

a. Find the marginal densities of X and Y.
b. Find the two conditional densities.

10. A point is uniformly distributed in a unit sphere in three dimensions.

a. Find the marginal densities of the x, y, and z coordinates.
b. Find the joint density of the x and y coordinates.
c. Find the density of the xy coordinates conditional on $Z = 0$.

11. Let U_1, U_2, and U_3 be independent random variables uniform on [0, 1]. Find the probability that the roots of the quadratic $U_1 x^2 + U_2 x + U_3$ are real.

12. Let

$$f(x, y) = c(x^2 - y^2)e^{-x}, \qquad 0 \leq x < \infty, \qquad -x \leq y < x$$

a. Find c.
b. Find the marginal densities.
c. Find the conditional densities.

13. A fair coin is thrown once; if it lands heads up, it is thrown a second time. Find the frequency function of the total number of heads.

14. Suppose that

$$f(x, y) = xe^{-x(y+1)}, \qquad 0 \leq x < \infty, \qquad 0 \leq y < \infty$$

a. Find the marginal densities of X and Y. Are X and Y independent?
b. Find the conditional densities of X and Y.

15. Suppose that X and Y have the joint density function

$$f(x, y) = c\sqrt{1 - x^2 - y^2}, \qquad x^2 + y^2 \leq 1$$

a. Find c.

b. Sketch the joint density.
c. Find $P(X^2 + Y^2) \leq \frac{1}{2}$.
d. Find the marginal densities of X and Y. Are X and Y independent random variables?
e. Find the conditional densities.

16. What is the probability density of the time between the arrival of the two packets of Example E in Section 3.4?

17. Let (X, Y) be a random point chosen uniformly on the region $R = \{(x, y) : |x| + |y| \leq 1\}$.

a. Sketch R.
b. Find the marginal densities of X and Y using your sketch. Be careful of the range of integration.
c. Find the conditional density of Y given X.

18. Let X and Y have the joint density function

$$f(x, y) = k(x - y), \qquad 0 \leq y \leq x \leq 1$$

and 0 elsewhere.

a. Sketch the region over which the density is positive and use it in determining limits of integration to answer the following questions.
b. Find k.
c. Find the marginal densities of X and Y.
d. Find the conditional densities of Y given X and X given Y.

19. Suppose that two components have independent exponentially distributed lifetimes, T_1 and T_2, with parameters α and β, respectively. Find (a) $P(T_1 > T_2)$ and (b) $P(T_1 > 2T_2)$.

20. If X_1 is uniform on [0, 1], and, conditional on X_1, X_2, is uniform on $[0, X_1]$, find the joint and marginal distributions of X_1 and X_2.

21. An instrument is used to measure very small concentrations, X, of a certain chemical in soil samples. Suppose that the values of X in those soils in which the chemical is present is modeled as a random variable with density function $f(x)$. The assay of a soil reports a concentration only if the chemical is first determined to be present. At very low concentrations, however, the chemical may fail to be detected even if it is present. This phenomenon is modeled by assuming that if the concentration is x, the chemical is detected with probability $R(x)$. Let Y denote the concentration of a chemical in a soil in which it has been determined to be present. Show that the density function of Y is

$$g(y) = \frac{R(y)f(y)}{\int_0^\infty R(x)f(x)\,dx}$$

22. Consider a Poisson process on the real line, and denote by $N(t_1, t_2)$ the number of events in the interval (t_1, t_2). If $t_0 < t_1 < t_2$, find the conditional distribution of $N(t_0, t_1)$ given that $N(t_0, t_2) = n$. (*Hint:* Use the fact that the numbers of events in disjoint subsets are independent.)

23. Suppose that, conditional on N, X has a binomial distribution with N trials and probability p of success, and that N is a binomial random variable with m trials and probability r of success. Find the unconditional distribution of X.

24. Let P have a uniform distribution on [0, 1], and, conditional on $P = p$, let X have a Bernoulli distribution with parameter p. Find the conditional distribution of P given X.

25. Let X have the density function f, and let $Y = X$ with probability $\frac{1}{2}$ and $Y = -X$ with probability $\frac{1}{2}$. Show that the density of Y is symmetric about zero—that is, $f_Y(y) = f_Y(-y)$.

26. Spherical particles whose radii have the density function $f_R(r)$ are dropped on a mesh as in Problem 4. Find an expression for the density function of the particles that pass through.

27. Prove that X and Y are independent if and only if $f_{X|Y}(x|y) = f_X(x)$ for all x and y.

28. Show that $C(u, v) = uv$ is a copula. Why is it called "the independence copula"?

29. Use the Farlie-Morgenstern copula to construct a bivariate density whose marginal densities are exponential. Find an expression for the joint density.

30. For $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$, show that $C(u, v) = \min(u^{1-\alpha}v, uv^{1-\beta})$ is a copula (the Marshall-Olkin copula). What is the joint density?

31. Suppose that (X, Y) is uniform on the disk of radius 1 as in Example E of Section 3.3. Without doing any calculations, argue that X and Y are not independent.

32. Continuing Example E of Section 3.5.2, suppose you had to guess a value of θ. One plausible guess would be the value of θ that maximizes the posterior density. Find that value. Does the result make intuitive sense?

33. Suppose that, as in Example E of Section 3.5.2, your prior opinion that the coin will land with heads up is represented by a uniform density on [0, 1]. You now spin the coin repeatedly and record the number of times, N, until a heads comes up. So if heads comes up on the first spin, $N = 1$, etc.

a. Find the posterior density of Θ given N.
b. Do this with a newly minted penny and graph the posterior density.

34. This problem continues Example E of Section 3.5.2. In that example, the prior opinion for the value of Θ was represented by the uniform density. Suppose that the prior density had been a beta density with parameters $a = b = 3$, reflecting a stronger prior belief that the chance of a 1 was near $\frac{1}{2}$. Graph this prior density. Following the reasoning of the example, find the posterior density, plot it, and compare it to the posterior density shown in the example.

35. Find a newly minted penny. Place it on its edge and spin it 20 times. Following Example E of Section 3.5.2, calculate and graph the posterior distribution. Spin another 20 times, and calculate and graph the posterior based on all 40 spins. What happens as you increase the number of spins?

36. Let $f(x) = (1 + \alpha x)/2$, for $-1 \leq x \leq 1$ and $-1 \leq \alpha \leq 1$.

a. Describe an algorithm to generate random variables from this density using the rejection method.

b. Write a computer program to do so, and test it out.

37. Let $f(x) = 6x^2(1 - x)^2$, for $-1 \leq x \leq 1$.

a. Describe an algorithm to generate random variables from this density using the rejection method. In what proportion of the trials will the acceptance step be taken?

b. Write a computer program to do so, and test it out.

38. Show that the number of iterations necessary to generate a random variable using the rejection method is a geometric random variable, and evaluate the parameter of the geometric frequency function. Show that in order to keep the number of iterations small, $M(x)$ should be chosen to be close to $f(x)$.

39. Show that the following method of generating discrete random variables works (D. R. Fredkin). Suppose, for concreteness, that X takes on values $0, 1, 2, \ldots$ with probabilities $p_0, p_1, p_2, \ldots$. Let U be a uniform random variable. If $U < p_0$, return $X = 0$. If not, replace U by $U - p_0$, and if the new U is less than p_1, return $X = 1$. If not, decrement U by p_1, compare U to p_2, etc.

40. Suppose that X and Y are discrete random variables with a joint probability mass function $p_{XY}(x, y)$. Show that the following procedure generates a random variable $X \sim p_{X|Y}(x|y)$.

a. Generate $X \sim p_X(x)$.

b. Accept X with probability $p(y|X)$.

c. If X is accepted, terminate and return X. Otherwise go to Step a.

Now suppose that X is uniformly distributed on the integers $1, 2, \ldots, 100$ and that given $X = x$, Y is uniform on the integers $1, 2, \ldots, x$. You observe $Y = 44$. What does this tell you about X? Simulate the distribution of X, given $Y = 44$, 1000 times and make a histogram of the value obtained. How would you estimate $E(X|Y = 44)$?

41. How could you extend the procedure of the previous problem in the case that X and Y are continuous random variables?

42. **a.** Let T be an exponential random variable with parameter λ; let W be a random variable independent of T, which is ± 1 with probability $\frac{1}{2}$ each; and let $X = WT$. Show that the density of X is

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda |x|}$$

which is called the **double exponential density.**

b. Show that for some constant c,

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq c e^{-|x|}$$

Use this result and that of part (a) to show how to use the rejection method to generate random variables from a standard normal density.

43. Let U_1 and U_2 be independent and uniform on [0, 1]. Find and sketch the density function of $S = U_1 + U_2$.

44. Let N_1 and N_2 be independent random variables following Poisson distributions with parameters λ_1 and λ_2. Show that the distribution of $N = N_1 + N_2$ is Poisson with parameter $\lambda_1 + \lambda_2$.

45. For a Poisson distribution, suppose that events are independently labeled A and B with probabilities $p_A + p_B = 1$. If the parameter of the Poisson distribution is λ, show that the number of events labeled A follows a Poisson distribution with parameter $p_A\lambda$.

46. Let X and Y be jointly continuous random variables. Find an expression for the density of $Z = X - Y$.

47. Let X and Y be independent standard normal random variables. Find the density of $Z = X + Y$, and show that Z is normally distributed as well. (*Hint:* Use the technique of completing the square to help in evaluating the integral.)

48. Let T_1 and T_2 be independent exponentials with parameters λ_1 and λ_2. Find the density function of $T_1 + T_2$.

49. Find the density function of $X + Y$, where X and Y have a joint density as given in Example D in Section 3.3.

50. Suppose that X and Y are independent discrete random variables and each assumes the values 0, 1, and 2 with probability $\frac{1}{3}$ each. Find the frequency function of $X + Y$.

51. Let X and Y have the joint density function $f(x, y)$, and let $Z = XY$. Show that the density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f\left(y, \frac{z}{y}\right) \frac{1}{|y|}\, dy$$

52. Find the density of the quotient of two independent uniform random variables.

53. Consider forming a random rectangle in two ways. Let U_1, U_2, and U_3 be independent random variables uniform on [0, 1]. One rectangle has sides U_1 and U_2, and the other is a square with sides U_3. Find the probability that the area of the square is greater than the area of the other rectangle.

54. Let X, Y, and Z be independent $N(0, \sigma^2)$. Let Θ, Φ, and R be the corresponding random variables that are the spherical coordinates of (X, Y, Z):

$$\begin{aligned} x &= r \sin\phi \cos\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\phi \end{aligned}$$
$$0 \le \phi \le \pi, \qquad 0 \le \theta \le 2\pi$$

Find the joint and marginal densities of Θ, Φ, and R. (*Hint:* $dx\,dy\,dz = r^2 \sin\phi\; dr\,d\theta\,d\phi$.)

55. A point is generated on a unit disk in the following way: The radius, R, is uniform on $[0, 1]$, and the angle Θ is uniform on $[0, 2\pi]$ and is independent of R.

a. Find the joint density of $X = R\cos\Theta$ and $Y = R\sin\Theta$.
b. Find the marginal densities of X and Y.
c. Is the density uniform over the disk? If not, modify the method to produce a uniform density.

56. If X and Y are independent exponential random variables, find the joint density of the polar coordinates R and Θ of the point (X, Y). Are R and Θ independent?

57. Suppose that Y_1 and Y_2 follow a bivariate normal distribution with parameters $\mu_{Y_1} = \mu_{Y_2} = 0$, $\sigma^2_{Y_1} = 1$, $\sigma^2_{Y_2} = 2$, and $\rho = 1/\sqrt{2}$. Find a linear transformation $x_1 = a_{11}y_1 + a_{12}y_2$, $x_2 = a_{21}y_1 + a_{22}y_2$ such that x_1 and x_2 are independent standard normal random variables. (*Hint:* See Example C of Section 3.6.2.)

58. Show that if the joint distribution of X_1 and X_2 is bivariate normal, then the joint distribution of $Y_1 = a_1X_1 + b_1$ and $Y_2 = a_2X_2 + b_2$ is bivariate normal.

59. Let X_1 and X_2 be independent standard normal random variables. Show that the joint distribution of

$$
\begin{aligned}
Y_1 &= a_{11}X_1 + a_{12}X_2 + b_1 \\
Y_2 &= a_{21}X_1 + a_{22}X_2 + b_2
\end{aligned}
$$

is bivariate normal.

60. Using the results of the previous problem, describe a method for generating pseudorandom variables that have a bivariate normal distribution from independent pseudorandom uniform variables.

61. Let X and Y be jointly continuous random variables. Find an expression for the joint density of $U = a + bX$ and $V = c + dY$.

62. If X and Y are independent standard normal random variables, find $P(X^2 + Y^2 \leq 1)$.

63. Let X and Y be jointly continuous random variables.

a. Develop an expression for the joint density of $X + Y$ and $X - Y$.
b. Develop an expression for the joint density of XY and Y/X.
c. Specialize the expressions from parts (a) and (b) to the case where X and Y are independent.

64. Find the joint density of $X + Y$ and X/Y, where X and Y are independent exponential random variables with parameter λ. Show that $X + Y$ and X/Y are independent.

65. Suppose that a system's components are connected in series and have lifetimes that are independent exponential random variables with parameters λ_i. Show that the lifetime of the system is exponential with parameter $\sum \lambda_i$.

66. Each component of the following system (Figure 3.19) has an independent exponentially distributed lifetime with parameter λ. Find the cdf and the density of the system's lifetime.

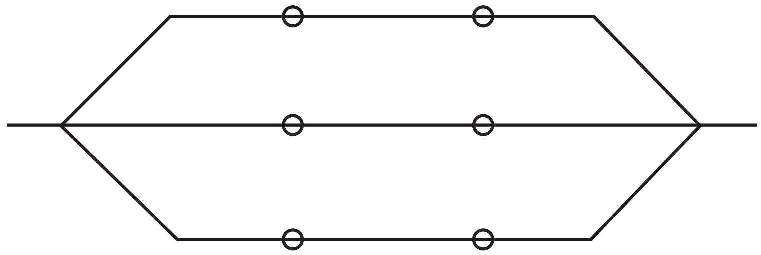

FIGURE **3.19**

67. A card contains n chips and has an error-correcting mechanism such that the card still functions if a single chip fails but does not function if two or more chips fail. If each chip has a lifetime that is an independent exponential with parameter λ, find the density function of the card's lifetime.

68. Suppose that a queue has n servers and that the length of time to complete a job is an exponential random variable. If a job is at the top of the queue and will be handled by the next available server, what is the distribution of the waiting time until service? What is the distribution of the waiting time until service of the next job in the queue?

69. Find the density of the minimum of n independent Weibull random variables, each of which has the density

$$f(t) = \beta\alpha^{-\beta}t^{\beta-1}e^{-(t/\alpha)^{\beta}}, \qquad t \geq 0$$

70. If five numbers are chosen at random in the interval [0, 1], what is the probability that they all lie in the middle half of the interval?

71. Let $X_1, \ldots, X_n$ be independent random variables, each with the density function f. Find an expression for the probability that the interval $(-\infty, X_{(n)}]$ encompasses at least $100\nu\%$ of the probability mass of f.

72. Let $X_1, X_2, \ldots, X_n$ be independent continuous random variables each with cumulative distribution function F. Show that the joint cdf of $X_{(1)}$ and $X_{(n)}$ is

$$F(x, y) = F^n(y) - [F(y) - F(x)]^n, \qquad x \leq y$$

73. If $X_1, \ldots, X_n$ are independent random variables, each with the density function f, show that the joint density of $X_{(1)}, \ldots, X_{(n)}$ is

$$n!f(x_1)f(x_2)\cdots f(x_n), \qquad x_1 < x_2 < \cdots < x_n$$

74. Let U_1, U_2, and U_3 be independent uniform random variables.

a. Find the joint density of $U_{(1)}$, $U_{(2)}$, and $U_{(3)}$.

b. The locations of three gas stations are independently and randomly placed along a mile of highway. What is the probability that no two gas stations are less than $\frac{1}{3}$ mile apart?

75. Use the differential method to find the joint density of $X_{(i)}$ and $X_{(j)}$, where $i < j$.

76. Prove Theorem A of Section 3.7 by finding the cdf of $X_{(k)}$ and differentiating. (*Hint:* $X_{(k)} \leq x$ if and only if k or more of the X_i are less than or equal to x. The number of X_i less than or equal to x is a binomial random variable.)

77. Find the density of $U_{(k)} - U_{(k-1)}$ if the U_i, $i = 1, \ldots, n$ are independent uniform random variables. This is the density of the spacing between adjacent points chosen uniformly in the interval [0, 1].

78. Show that

$$\int_0^1 \int_0^y (y - x)^n \, dx \, dy = \frac{1}{(n+1)(n+2)}$$

79. If T_1 and T_2 are independent exponential random variables, find the density function of $R = T_{(2)} - T_{(1)}$.

80. Let $U_1, \ldots, U_n$ be independent uniform random variables, and let V be uniform and independent of the U_i.

a. Find $P(V \leq U_{(n)})$.
b. Find $P(U_{(1)} < V < U_{(n)})$.

81. Do both parts of Problem 80 again, assuming that the U_i and V have the density function f and the cdf F, with F^{-1} uniquely defined. *Hint:* $F(U_i)$ has a uniform distribution.