CSE 101 – Nov 8, 2019 (Week 6)

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Lemma 5

Let G be connected with n vertices and m edges. Assume also $m = n - 1$. Then G is acyclic, hence a tree

Proof Assume, to get a contradiction, that G is not acyclic, containing a cycle C.

Let e be any edge on C and remove it. Call the result:

$G - e$

Observe $G - e$ is connected.

By lemma 3:

$$|E(G - e) \geq |V(G - e)| - 1$$

$\therefore m - 1 \geq n - 1$

$\therefore m \geq n$

But also: $m = n - 1$

$\therefore n - 1 \geq n$ (contradiction!)

This contradiction shows our assumption must be false, so no such cycle exists. $\therefore G$ is acyclic.

Lemma 6

Let G be acyclic with n vertices and $medges$ Suppose also that $m = n - 1$. Then G is also connected.

Proof Let k be the # of connected components in G. By lemma 2, we have $m = n - k$. But also, we have $m = n - 1$, so $n - 1 = n - k$. Therefore, $k = 1$, hence G is connected.

Lemma 7

Let G be a connected graph with n vertices and m edges. Suppose also $m = n$. Then G contains exactly 1 cycle. (called **unicyclic**).

Proof Handout:

Consider 3 Properties

1. G is connected
2. G is acyclic
3. $m = n - 1$

Note **Lemma 1**: (i) and (ii) $\rightarrow$ (iii)

Lemma 5: (i) and (iii) $\rightarrow$ (ii)

Lemma 6: (ii) and (iii) $\Rightarrow$ (i)

Logical possibilities

i	ii	iii	Possible?
F	F	F	?
F	F	T	?
F	T	F	?
F	T	T	No, by Lemma 6
T	F	F	?
T	F	T	No, by Lemma 5
T	T	F	No, by Lemma 1
T	T	T	Yes, any tree!

Exercise: Show all ? are true; find smallest example

Theorem (Treeness theorem)

The following are equivalent.

a) G is a tree

b) G contains a **unique** $x - y$ path for all $x, y \in V(G)$

c) G is connected, but if any edge e is removed, the result $G - e$ is disconnected

d) G is connected and $m = n - 1$

e) G is acyclic and $m = n - 1$

f) G is acyclic, but if any edge e is added (joining any 2 non-adjacent vertices), then the result $G + e$ contains a unique cycle

Read proof in book or handout

Directed graphs

Definition

A **directed graph (digraph)** is a pair of sets

$$G = (V, E)$$

where

$$V \neq \emptyset$$

$$E \subseteq V^2 = V * V$$

(x origin, y terminus)

Example

Drawing

$V = \{1, 2, 3, 4\}$

$E = \{(1, 2), (1, 4), (3, 1), (4, 2), (4, 3)\}$

Now allowed: (x, x),

$(x, y) \neq (y, x)$

Representation

- Incidence Matrix
- Adjacency Matrix
- Adjacency List