CSE 101 – Oct 14, 2019 (Week 3)

Notes provided by Ben Sihota bsihota@ucsc.edu

Logistics

- HW2 extended by one day
- PA2 may be extended, but not a week
- Samples for PA2 posted

Review

Little-o Lemma

$f(n) = o(g(n)) \iff \lim_{n\to\infty}(\frac{f(n)}{g(n)}) = 0$

Example

$ln(n) = o(n)$ ... why?

$$\lim_{n\to\infty}(\frac{ln(n)}{n}) = \lim_{n\to\infty}(\frac{\frac{1}{n}}{1})$$

$$= \lim_{n\to\infty}(\frac{1}{n}) = 0$$

Example

$ln(n) = o(n^k)$, for any $k > 0$

$$\lim_{n\to\infty}(\frac{ln(n)}{n^k}) = \lim_{n\to\infty}(\frac{\frac{1}{n}}{kn^{k-1}}) = \lim_{n\to\infty}(\frac{1}{kn^k}) = 0$$

Example

$n^k = o(e^n)$ ... why?

If $k > 0$ and $k \in \mathbb{R}$:

$$\lim_{n\to\infty}(\frac{n^k}{e^n}) = \lim_{n\to\infty}(\frac{kn^{k-1}}{e^n}) = \lim_{n\to\infty}(\frac{k(k-1)n^{k-2}}{e^n})$$

...

$= 0$

After $\lceil k \rceil$ applications of L'Hopitals

Exercise

Replace e by any $b > 1$ in last 3 examples

Exercise

Show:

$$o(g(n)) \cap \Omega(g(n)) = \emptyset$$

Hint: Use contradiction – assume $f(n) \in o(g(n)) \cap \Omega(g(n))$

$$\forall c_1 > 0, \exists n_1 > o, \forall n \geq n_1 : 0 \leq f(n) < c_1 g(n)$$

Since $f(n) = \Omega(g(n))$:

$$\forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2 : 0 \leq c_2 g(n) \leq f(n)$$

Definition

"strict asymptotic lower bound"

$\omega(g(n)) = \{f(n) | \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq c * g(n) < f(n)\}$

Recall:

$\Omega(g(n)) = \{f(n) | \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n)\}$

Obviously: $\omega(g(n)) \subseteq \Omega(g(n))$

Exercise

Prove: $f(n) = \omega(g(n)) \iff \lim_{n\to\infty}(\frac{f(n)}{g(n)}) = \infty$

Exercise

Prove: $\omega(g(n)) \cap O(g(n)) = \emptyset$

Picture

Not shown: *Venn diagram*

Theorem

If $\lim\limits_{n\to\infty}(\dfrac{f(n)}{g(n)}) = L$ where $0 \leq L < \infty$, then $f(n) = O(g(n))$

Note: The converse if false

Proof The limit statement says:

$\forall \epsilon > 0, \exists n_0 > 0, \forall n \geq n_0 : |\dfrac{f(n)}{g(n)} - L| < \epsilon$

This holds for all $\epsilon > 0$. In particular, for $\epsilon = 1$, we have

$\exists n_0 > 0, \forall n \geq n_0 : |\dfrac{f(n)}{g(n)} - L| < 1$

$\therefore \exists n_0 > 0, \forall n \geq n_0 : -1 < \dfrac{f(n)}{g(n)} - L < 1$

$\therefore \exists n_0 > 0, \forall n \geq n_0 : \dfrac{f(n)}{g(n)} < 1 + 1$

$\therefore \exists n_0 > 0, \forall n \geq n_0 : f(n) < (1 + L)g(n)$

$\therefore \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq c * g(n)$

$\therefore f(n) = O(g(n))$

Theorem

If $\lim_{n\to\infty}(\dfrac{f(n)}{g(n)}) = L$ where $0 < L \leq \infty$, then: $f(n) = \Omega(g(n))$

Note: Converse is false

Proof We have $\lim_{n\to\infty}(\dfrac{g(n)}{f(n)}) = L'$ where $L' = \dfrac{1}{L}$

So $0 \leq L' < \infty$

By last theorem: $g(n) = O(f(n))$

By previously proved theorem: $f(n) = \Omega(g(n))$

Exercise

Prove if $\lim_{n\to\infty}(\dfrac{f(n)}{g(n)}) = L$ where $0 < L < \infty$, then $f(n) = \Theta(g(n))$

Note: Converse is false

Example

A : $g(n) = n, f(n) = (1 + sin(n))n$

Obviously, $f(n) = O(g(n))$

Also: $f(n) \neq \Omega(g(n))$

and

$$\lim_{n\to\infty}(\frac{f(n)}{g(n)}) = \lim_{n\to\infty}(1 + sin(n)) \text{ D.N.E.}$$

Example

B : $g(n) = n$, $f(n) = (2 + sin(n))n$

$\therefore f(n) = \Omega(g(n))$

But $\dfrac{f(n)}{g(n)} = 2 + sin(n)$, limit D.N.E.

Not shown: *Another Venn diagram*