CSE 101 – Oct 11, 2019 (Week 2)

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PA1

- Run unit tests from Piazza
- Last extension of PA1

Example

Show that $\sqrt{n+10} = \Theta(\sqrt{n})$

Proof: **Goal**: Find positive c_1, c_2, n_0 such that

$$0 \le c_1\sqrt{n} \le \sqrt{n+10} \le c_2\sqrt{n}$$

for all $n \ge n_0$. Let

$$c_1 = 1, c_2 = \sqrt{2}, n_0 = 10$$

Then if $n \ge n_0$, we have

$-10 \le 0$ and $10 \ge n$

$\therefore -10 \le (1-1)n$ and $10 \le (2-1)n$

$\therefore -10 \le (1-c_1^2)n$ and $10 \le (c_2^2-1)n$

$\therefore c_1^2 n \le n+10$ and $n+10 \le c_2^2 n$

$\therefore 0 \le c_1^2 n \le n+10 \le c_2^2 n$

$\therefore 0 \le c_1\sqrt{n} \le \sqrt{n+10} \le c_2\sqrt{n}$

$\therefore \sqrt{n+10} = \Theta(\sqrt{n})$

Exercise

Let $a, b \in \mathbb{R}$ and $b > 0$. Show

$$(n+a)^b = \Theta(n^b)$$

Theorem

If $h(n) = O(g(n))$ and $f(n) \leq h(n)$ for all sufficiently large n, then

$$f(n) = O(g(n))$$

Proof By hypothesis, we have pos c_1, n_1 such that

(1) $\forall n \geq n_1,\ 0 \leq h(n) \leq c_1 g(n)$

Also there exists pos n_2 such that

(2) $\forall \geq n_2 : 0 \leq f(n) \leq h(n)$

We must show there exist positive c, n_0 such that

(3) $\forall n \geq n_0 : 0 \leq f(n) \leq cg(n)$

Let $c = c_1$ and $n_0 = \max(n_1, n_2)$

Then if $n \geq n_0$, we have both **(1)** and **(2)** true, therefore **(3)** is also true.

$$\therefore f(n) = O(g(n))$$

~ *Think of Ω as the "greater than" function*

Exercise

If $h(n) = \Omega(g(n))$ and $f(n) \geq h(n)$ for sufficiently large n, then

$$f(n) = \Omega(g(n))$$

Exercise

If $h_1(n) = \Omega(g(n))$ and $h_2(n) = O(g(n))$ and $h_1(n) \leq f(n) \leq h_2(n)$ for sufficiently large n, then

$$f(n) = \Theta(g(n))$$

Example

Let $k \geq 0$. Then

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

Proof $\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n * n^k = n^{k+1} = O(n^{k+1})$

Also

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k$$

$$\geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \rceil \frac{n}{2} \rceil^2$$

$= (n - \lceil \frac{n}{2} \rceil + 1) \lceil \frac{n}{2} \rceil^k$

$= (\lfloor \frac{n}{2} \rfloor + 1) \lceil \frac{n}{2} \rceil^k$

$\geq ((\frac{n}{2} - 1) + 1)(\frac{n}{2})^k$

$= (\frac{n}{2})^{k+1}$

$= (\frac{1}{2})^{k+1} * n^{k+1}$

$= const * n^{k+1} = \Omega(n^{k+1})$

$$\therefore \sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

Definition

$o(g(n)) = \{f(n) | \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n)\}$

"$g(n)$ is a strict asymptotic upper-bound"

Review $O(g(n)) = \{f(n) | \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n)\}$

Obviously: $o(g(n)) \subseteq O(g(n))$

Lemma

$f(n) = o(g(n)) \iff \lim_{n->\infty}(\frac{f(n)}{g(n)}) = 0$

$f(n) = o(g(n))$ is equivalent to

$\forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq \frac{f(n)}{g(n)} < c$

This is the definition of

$\lim_{n->\infty}(\frac{f(n)}{g(n)}) = 0$