CSE 101 – Dec 4, 2019 (Week 10)

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Priority Queue Operations

- Insert(S, x)
- Max(S): Returns but does not delete element with maximum key
- ExtractMax(S): Returns and deletes element with maximum key
- IncreaseKey(S, x, k)

Chapter 23 – Minimum Weight Spanning Trees (MWST)

Definition

Let $G = (V, E)$ be a graph. A subgraph $H = (V', E')$ is said to **span** G iff

$$V' = V$$

If $w : E \to \mathbb{R}$ is a weight function on G, then

$$w(H) = \sum_{e \in E(H)} w(e)$$

Definition

A spanning subgraph that is also a tree is called a **spanning tree**.

A minimum weight spanning tree in a weighted graph is a spanning tree T such that

$$w(T) \leq w(S)$$

for all spanning trees S in G.

MWST Problem

Given a weighted graph G, determine a MWST in G.

Theorem

G contains a spanning tree $\iff$ G is connected

Proof $\Rightarrow$ Obvious

$\Leftarrow$ **:** Let $n = |V(G)|, \quad m = |E(G)|$. Since G is connected, we have $m \geq n - 1$ (Lemma 3).

Do the following:

- While G contains a cycle
  - Pick an edge on that cycle and remove it

The graph left is acyclic. We haven't removed any vertices, so it's a a spanning subgraph. It's connected since we've only removed cycle edges. The resulting subgraph is therefore a spanning tree. $\square$

Two Famous Algorithms

- Kruskal (and dual)
- Prim

Kruskal

Uses a min priority queue

- Records are edges
- Keys are edge weights

Kruskal(G)

1. $Q = E(G)$ // Keys are edge weights
2. $F = \emptyset$
3. while $|F| < n - 1$ // $n = |V(G)|$
    4. $e = \text{ExtractMin}(Q)$
    5. if $F \cup \{e\}$ is acyclic
        6. $F = F \cup \{e\}$

Theorem

When Kruskal is complete

$$T = (V, F)$$

is a minimum weight spanning tree in G

Dual-Kruskal (Pat's homemade algorithm)

Uses a max priority queue

Dual-Kruskal(G) 1. $Q = E(G)$ // Keys are edge weights

2. $F = E(G)$
3. while $|F| > n - 1$
   4. $e = \text{ExtractMax}(Q)$
   5. if $F - \{e\}$ is connected
      6. $F = F - \{e\}$

Theorem

When complete, $T = (V, F)$ is a minimum weight spanning tree in G

Read: Prim in Ch. 23

Example (P. 632)