# CSE 101 – Nov 4, 2019 (Week 6)

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### Notes

- PA3 extended 2 days

### Graphs

**Recall**: A graph is a pair $(V, E)$ of **sets**.

$V \neq \emptyset$

$E \subseteq V^{(2)} = \{$ un-ordered pairs of $V\}$

### ***Not*** allowed

- Self-loops: $\{x, x\}$ (not an edge [a singleton])
- Parallel edges: edge set $(E)$ cannot contain the unordered pair $xy$ twice

### Definition

Let $x \in V(G)$. The **degree of** $x$ is

$\deg(x) = \#$ of edges incident with $x = \#$ of vertices adjacent to $x4

**Note**: $\deg_{G_1}(x) = \deg_{G_2}(\phi(x))$

if $\phi : V(G_1) \to V(G_2)$ is an isomorphism

### Definition

The **degree sequence** of $G$ is the sequence of vertex degrees in sorted order (increasing).

### Example

| Degree | $x$ |
|---|---|
| 2 | 1 |
| 5 | 2 |
| 3 | 3 |
| 3 | 4 |
| 4 | 5 |
| 3 | 6 |

Degree sequence: $(2, 3, 3, 3, 4, 5)$

**Lemma (Handshake Lemma)**

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|$$

**Exercise**: Show that # of vertices with odd degree must be even, in any graph

**Lemma 1**

If $T$ is a tree with $n$ vertices and $m$ edges, then $m = n - 1$

**Proof** **I.** if $m = 0$, then since $T$ is connected, must have $n = 1$

**IId.** Let $m > 0$ be arbitrary. Assume that if $T'$ is a tree with fewer than $m$ edges, then

$|E(T')| = |V(T')| - 1$

We must show that

$|E(T)| = |V(T)| - 1$

i.e. $m = n - 1$

Let $T$ be any tree with $m$ edges and $n$ vertices. Then pick any $e \in E(T)$ and remove it. _*this results in two subtrees, $T_1, T_2$ each with fewer than $m$ edges (proof in handout)_

Let $m_i = |E(T_i)|$ and $n_i = |V(T_i)|$

Then, by the induction hypothesis, we have:

$m_i = n_i - 1$, $(i = 1, 2)$

Observe $n = n_1 + n_2$ since no vertices were removed. Thus,

$m = m_1 + m_2 + 1$

$= (n_1 - 1) + (n_2 - 1) + 1$

$= n - 1$

**Lemma 2**

If $G$ is an acyclic graph with $n$ vertices, $m$ edges and $k$ connected components, then

$$m = n - k$$

(not an induction proof, you may use Lemma 1 without proof)

**Proof** Let $T_1, T_2, ..., T_k$ be the connected components of $G$ (which are necessarily trees). Let $m_i$, $n_i$ be # of edges, vertices (resp.) of $T_i (1 \leq i \leq k)$

By Lemma 1

$m_i = n_i - 1 (1 \leq i \leq k)$

So

$m = \sum_{i=1}^{k} m_i$

$= \sum_{i=1}^{k} (n_i - 1)$

$= \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$

$= n - k$

**Lemma 3**

If G is a connected graph with $n$ vertices and $m$ edges, then $m \geq n - 1$

**Proof** **I.** If $m = 0$, then $G$, being connected, can have only 1 vertex, i.e. $n = 1$. Thus $m \geq n - 1$ becomes $0 \geq 0$, which is true.

**IId.** Let $m > 0$ be arbitrary. Assume that if $G'$ is a connected graph with fewer than $m$ edges, then

$$|E(G')| \geq |V(G')| - 1$$

We must show that if $G$ is a connected graph with $m$ edges, then

$$|E(G)| \geq |V(G)| - 1$$

i.e. $m \geq n - 1$

Pick any edge $e \in E(G)$ and remove it. Let $G - e$ be the result. We have 2 cases

**Case 1**: $G - e$ is connected

Since $G - e$ has fewer than $m$ edges, the induction hypothesis gives

$$|E(G - e)| \geq |V(G - e)| - 1$$

$\therefore m - 1 \geq n - 1$

$\therefore m \geq n > n - 1$

Thus $m \geq n - 1$

**Case 2**: $G - e$ is disconnected

***** $G - e$ consists of exactly 2 connected components, $H_1$ and $H_2$ (proof in handout). Since both $H_1, H_2$ have fewer than $m$ edges, we have by induction hypothesis

$|E(H_i)| \geq |V(H_i)| - 1$, $(i = 1, 2)$

**Note** $n = |V(H_1)| + |V(H_2)|$, since no vertices were removed

Thus:

$m = |E(H_1)| + |E(H_2)| + 1$

$\geq (|V(H_1) - 1) + (|V(H_2)| - 1) + 1$

$= (|V(H_1)| + |V(H_2)|) - 1$

$= n - 1$

$\therefore m \geq n - 1$

**Lemma 4**

If $G$ is a graph with $n$ vertices, $m$ edges and $k$ connected components, then

$$m \geq n - k$$