# CSE 101 – Nov 1, 2019 (Week 5)

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**Example (Case 3 of Master Theorem)**

$T(n) = 5T(\lfloor \frac{n}{4} \rfloor) + n$

**Compare**: $n^2$ to $n^{\log_4 5}$

**Note**: $5 < 16$ implies that $\log_4 5 < \log_4 16 = 2$

Let $\epsilon = 2 - \log_4 5 > 0$. Then:

$$2 = \log_4 5 + \epsilon$$

$\therefore n^2 = \Omega(n^2) = \Omega(n^{\log_4 5 + \epsilon})$

**Need**: $5(\frac{n}{4})^2 \leq cn^2$ for some $c \in (0, 1)$ and all sufficiently large $n$

i.e. $\frac{5}{16} n^2 \leq cn^2$

$\ldots \ \frac{5}{16} \leq c$

So pick any $c$ in the range

$$\frac{5}{16} \leq c < 1$$

and regularity condition is satisfied.

By Case 3: $T(n) = \Theta(n^2)$

**Example**

$$T(n) = \begin{cases} 80 & n < 100 \\ 8T(\lceil \dfrac{n}{2} \rceil) + 10n^3 - n^{1.5} + n \log n & n \geq 100 \end{cases}$$

Same as: $T(n) = 8T(\frac{n}{2}) + n^3$

This was 1st example

Case 2: $T(n) = \Theta(n^3 \log n)$

**Example**

$T(n) = T(\lfloor \frac{n}{2} \rfloor) + 2T(\lceil \frac{n}{2} \rceil) + \log(n!)$

Simplify: $T(n) = 3T(\frac{n}{2}) + n \log n$

Compare: $n \log n$ to $n^{\log_2(3)}$

Let $\epsilon = (\log_2 3 - 1)$

**Note**: $2 < 3 \to \log_2 2 < \log_2 3 \to 1 < \log_2 3$

$\to \epsilon > 0$

also $2\epsilon = \log_2 3 - 1$

$1 + \epsilon = \log_2 3 - \epsilon$

also

$\dfrac{n \log n}{n^{1+\epsilon}} = \dfrac{n \log n}{n * n^{\epsilon}} \to 0$

$\therefore n \log n = o(n^{1+\epsilon}) \subseteq O(n^{1+\epsilon} = O(n^{\log_2 3 - \epsilon}))$

By case 1: $T(n) = \Theta(n^{\log_2 3})$

**Remark** In this problem:

$$\epsilon = \log_2 3 - 1$$

does **not** work. Why?

$1 = \log_2 3 - \epsilon$

$\frac{n \log n}{n} = \log n \to \infty$

$\therefore n \log n = \omega(n^1) = \omega(n^{\log_2 3 - \epsilon})$

$\therefore n \log n \neq O(n^{\log_2 3 - \epsilon})$

**Example**

$T(n) = 2T(\dfrac{n}{2}) + \dfrac{n}{\log n}$

**Compare**: $\frac{n}{\log n}$ to $n^1$

**Note**: $\dfrac{n}{\log n} = o(n)$ since $\dfrac{\frac{n}{\log n}}{n} = \dfrac{1}{\log n} \to 0$

$\therefore n$ wins, but not by a polynomial factor.

Pick any $\epsilon > 0$. Then

$$\frac{\frac{n}{\log n}}{n^{1-\epsilon}} = \frac{n/\log n}{n/n^2} = \frac{n^\epsilon}{\log n} \to \infty$$

$$\therefore \frac{n}{\log n} = \omega(n^{1-\epsilon})$$

$\therefore$ case 1 does not apply

$\therefore$ master theorem does not apply!

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|---|---|---|---|---|
| Case 1 | *gap* | Case 2 | *gap* | Case 3 |

**Exercise**: Find gap, i.e. a recurrence $T(n) = aT(\frac{n}{b}) + f(n)$ such that $f(n) = \omega(n^{\log_b a})$ but $f(n) \neq \Omega(n^{\log_b a+\epsilon})$ for any $\epsilon > 0$

**Exercise**: Find $T(n) = aT(\frac{n}{b}) + f(n)$ such that:

$\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a+\epsilon})$ but regularity condition fails

**Recall: HW Problem**

Write an algorithm that counts # of inversions in $A[1...n]$ in time $\Theta(n \log n)$