CSE 101 – Oct 30, 2019 (Week 5)

Notes provided by Ben Sihota bsihota@ucsc.edu

Example

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + n^2 & \text{if } n \geq 2 \end{cases}$$

$T(smile) = (smile)^2 + T(\lfloor \frac{smile}{2} \rfloor)$

$T(n) = n^2 + T(\lfloor \frac{n}{2} \rfloor)$

$= n^2 + \lfloor \frac{n}{2} \rfloor^2 + T(\lfloor \frac{\frac{n}{2}}{2} \rfloor)$

$= n^2 + \lfloor \frac{n}{2} \rfloor^2 + T(\lfloor \frac{n}{2^2} \rfloor)$

$= n^2 + \lfloor \frac{n}{2} \rfloor^2 + \lfloor \frac{n}{2} \rfloor^2 + T(\lfloor \frac{\frac{n}{2^2}}{2} \rfloor)$

$= n^2 + \lfloor \frac{n}{2} \rfloor^2 + \lfloor \frac{n}{2} \rfloor^2 + T(\lfloor \frac{n}{2^3} \rfloor)$

...

$$= \sum_{i=0}^{k-1} \lfloor \frac{n}{2^i} \rfloor^2 + T(\lfloor \frac{n}{2^k} \rfloor)$$

Find first k s.t. $\lfloor \frac{n}{2^k} \rfloor = 1$

i.e. $1 \leq \frac{n}{2^k} < 2$

$2^k \leq n < 2^{k+1}$

$k \leq \lg(n) < k + 1$

$\therefore k = \lfloor \lg(n) \rfloor$

Thus

$$T(n) = \sum_{i=0}^{\lfloor \lg(n) \rfloor - 1} \lfloor \frac{n}{2^i} \rfloor^2 + 1$$

So

$$T(n) = \sum_{i=0}^{k-1} \lfloor \frac{n}{2^i} \rfloor^2 + 1$$

$$\leq \sum_{i=0}^{k-1} (\frac{n}{2^i})^2 + 1 \text{ [since } \lfloor x \rfloor \leq x]$$

$= n^2 * \sum_{i=0}^{k-1} (\frac{1}{4})^i + 1$

$= n^2 * (\frac{1}{1-\frac{1}{4}}) + 1$

$= \frac{1}{3} n^2 + 1 = O(n^2)$

$\therefore T(n) = O(n^2)$

$T(n) = \sum_{i=0}^{k-1} \lfloor \frac{n}{2^i} \rfloor^2 + 1$

$\geq n^2 + 1 = \Omega(n^2)$

$\therefore T(n) = \Theta(n^2)$

Master method

Applies to:

$$T(n) = aT(\frac{n}{b}) + f(n)$$

Master theorem

Let $a \geq 1, b > 1$ and $f(n)$ be an asymptotically positive function

Let $T(n)$ be defined by the recurrence

$$T(n) = aT(\frac{n}{b}) + f(n)$$

Then

Case 1 If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then

$$T(n) = \Theta(n^{\log_b(a)})$$

Case 2 If $f(n) = \Theta(n^{\log_b(a)})$, then

$$T(n) = \Theta(f(n) * \log(n))$$

$$= \Theta(n^{\log_b(a)} * \log(n))$$

Case 3 If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ for some $\epsilon > 0$, and [if $af(\frac{n}{b}) \leq c * f(n)$ for some c in the range $0 < c < 1$, and for all sufficiently large n] $\Leftarrow$ {regularity condition}, then

$$T(n) = \Theta(f(n))$$

Rules

- Compare $f(n)$ to $n^{\log_b(a)}$ in all cases
- Case 2: *?*
- Case 1: $n^{\log_b(a)}$ wins by a polynomial factor: n^ϵ
- Case 3: $f(n)$ wins by a polynomial factor n^ϵ. Also have regularity condition

Example

$T(n) = 8T(\frac{n}{2}) + n^3$

Compare: n^3 to $n^{\log_2 8} = n^3$

By case 2: $T(n) = \Theta(n^3 * \log(n))$

Example

$T(n) = 5T(\frac{n}{4}) + n$

Compare: n to $n^{\log_4 5}$

Note $5 > 4 \rightarrow \log_4 5 > \log_4 4 = 1$

Let $\epsilon = \log_4 5 - 1$, then $\epsilon > 0$

and $\log_4 5 - \epsilon = 1$, so

$n = O(n') = O(n^{\log_4 5 - \epsilon})$

$\therefore$ by case 1: $T(n) = \Theta(n^{\log_4 5})$