CSE 101 – Oct 18, 2019 (Week 3)

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Class Notes

- PA2 extended 3 days
- Midterm practice problems posted

Variations on induction step: II

IIa. $\forall n \geq n_0 : P(n) \rightarrow P(n+1)$ *[induction hypothesis]*

Let $n \geq n_0$ be chosen arbitrarily

Assume $P(n)$ is true.

Show as a consequence that $P(n+1)$ is true

IIb. $\forall n > n_0 : P(n-1) \rightarrow P(n)$ *[induction hypothesis]*

Let $n > n_0$ be arbitrary

Assume $P(n-1)$

Show $P(n)$

Note: $n - 1 \geq n_0 \iff n \geq n_0 + 1 \iff n > n_0$

IIa & IIb are *weak induction* or 1st principle of mathematical induction (PMI)

IIc. $\forall n \geq n_0 : P(n_0) \wedge P(n_9 + 1) \wedge ... \wedge P(n) \rightarrow P(n+1)$ *[induction hypothesis]*

Let $n \geq n_0$ be arbitrary

Assume $P(n_0) \wedge ... \wedge P(n)$ true

Show $P(n+1)$

IId. $\forall n > n_0 : P(n_0) \wedge ... \wedge P(n-1) \rightarrow P(n)$

Let $n > n_0$ be arbitrary

Assume $P(n_0) \wedge ... \wedge P(n-1)$

Show $P(n)$

Example

Show: $\forall n \geq 1 : \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Proof **I.** $P(1)$ says $1^2 = \frac{1(1+1)(2*1+1)}{6}$

$1 = 1$, which is true

IIa. Let $n \geq 1$ be chosen arbitrarily.

Assume $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, for this n

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

So

$$\sum_{k=1}^{n+1} k^2 = (\sum_{k=1}^{n} k^2) + (n+1)^2$$

$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2$ *[by the induction hypothesis]*

Note: Label the I.H, I.C. and where the I.H is used (3 things)

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)(n(2n+1) + 6(n+1))}{6}$$

$$= \frac{(n+1)(2n^2 + n + 6n + 6)}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

So the 1st P.M.I

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 1$

Exercise: Do it again using IIb

Example 3 (in handout)

Define $T(n)$ by

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ T(\lfloor \frac{n}{2} \rfloor) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove: $\forall n \geq 1 : T(n) \leq \lg(n)$

(hence $T(n) = O(\log n)$)

Proof $P(1)$ says: $T(1) \leq \lg(1)$, i.e. $0 \leq 0$, which is true

IId. $\forall n > 1 : P(1) \wedge ... \wedge P(n-1) \rightarrow P(n)$

Let $n > 1$ be arbitrary

Assume (for this n) for all k in the range $1 \leq k < n$ that

$$T(k) \leq \lg(k)$$

We must show $T(n) \leq \lg(n)$

So

$T(n) = T(\lfloor \frac{n}{2} \rfloor) + 1$ [by recurrence for $T(n)$]

$\leq \lg(\lfloor \frac{n}{2} \rfloor) + 1$ [by induction hypothesis, with $k = \lfloor \frac{n}{2} \rfloor$ we have $T(\lfloor \frac{n}{2}) \leq \lg(\lfloor \frac{n}{2} \rfloor)$]

$\leq \lg(\frac{n}{2}) + 1$ [since $\lfloor x \rfloor \leq x$ and lg() is increasing]

$= \lg n - \lg 2 + 1$

$= \lg n$

Results follows for all $n \geq 1$ by 2nd P.M.I

Exercise

Define $S(n)$ by

$$S(n) = \begin{cases} 0 & \text{if } n = 1 \\ S(\lfloor \frac{n}{2} \rfloor) + 1 & \text{if } n \geq 2 \end{cases}$$

Prove $\forall n \geq 1 : S(n) \geq \lg(n)$, hence $S(n) = \Omega(\log n)$

Note:

$\lfloor x \rfloor \leq x$ and $x - 1 < \lceil x \rceil$

$\lceil x \rceil \geq x$ and $x + 1 > \lfloor x \rfloor$