CSE 101 – Oct 16, 2019 (Week 3)

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Recall

Show if $a, b \in \mathbb{R}$, $b > 0$, then

$$(n+a)^b = \Theta(n^b)$$

$\lim_{n\to\infty}(\frac{(n+a)^b}{n^b}) = \lim_{n\to\infty}((\frac{n+a}{n})^b) = \lim_{n\to\infty}((1+\frac{a}{n})^b)$

$= \lim_{n\to\infty}((1+\frac{a}{n}))^b = 1^b = 1 \in (0, \infty)$

$\therefore (n+a)^b = \Theta(n^b)$

Exercise 9-C

$P(n)$ is polynomial, with degree $k \geq 0$, then $P(n) = \Theta(n^k)$

We have:

$P(n) = a_k n^k + a_{k_1} n^k - 1 + ... + a_1 * n + a_0$

So

$\frac{P(n)}{n^k} = a_k + \frac{a_{k-1}}{n} + ... + \frac{a_1}{n^{k-1}} + \frac{a_0}{n^k}$

Since $0 < a_k < \infty$, $\therefore$ we have $P(n) = \Theta(n^k)$

Exercise 9-D

If $\alpha, \beta \in \mathbb{R}$, then

$$n^\alpha = \begin{cases} o(n^\beta) & \text{if } \alpha < \beta \\ \Theta(n^\beta) & \text{if } \alpha = \beta \\ \omega(n^\beta) & \text{if } \alpha > \beta \end{cases}$$

$\frac{n^\alpha}{n^\beta} = n^{\alpha-\beta} \Rightarrow \begin{cases} 0 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha = \beta \\ \infty & \text{if } \alpha > \beta \end{cases}$

Exercise 9-B

If $a, b \in \mathbb{R}^+$, then

$$a^n = \begin{cases} o(b^n) & \text{if } a < b \\ \Theta(b^n) & \text{if } a = b \\ \omega(b^n) & \text{if } a > b \end{cases}$$

Why?

$\frac{a^n}{b^n} = (\frac{a}{b})^n \Rightarrow$ (as $n \to \infty$) $\begin{cases} 0 & \text{if } a < b \\ 1 & \text{if } a = b \\ \infty & \text{if } \alpha > \beta \end{cases}$

Example (from last night's HW)

$$f(n) + o(f(n)) = \Theta(f(n))$$

Proof Let $h(n) = o(f(n))$. Then $\lim_{n\to\infty}(\frac{h(n)}{f(n)}) = 0$

So

$\lim_{n\to\infty}(\frac{f(n)+h(n)}{f(n)}) = lim_{n\to\infty}(1 + \frac{h(n)}{f(n)}) = 1 \in (0, \infty)$

$\therefore f(n) + h(n) = \Theta(f(n))$

Handout on common functions

- Read floor and ceiling functions

Logarithms: Let $a, b > 1$

$\log_a x$ is inverse of $\exp_a(x) = a^x$

i.e. $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$

$\therefore x = a^{\log_a(x)} = (b^{\log_b(a)})^{\log_a(x)} = b^{log_b(a) * log_a(x)}$

$\therefore log_b(x) = log_b(a) * log_a(x)$

$\therefore \log_b(n = const * log_a(n))$

$\Rightarrow \therefore \log_b(n) = \Theta(\log_a(n)) \Leftarrow$

Also

$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

Also

$a^{\log_b(x)} = a^{\log_a(x)*\log_b(a)} = (a^{\log_a(x)})^{\log_b(a)}$

$\therefore a^{\log_b(x)} = x^{\log_b(a)}$

Stirling's Formula

Let $n \in \mathbb{Z}^+$. then

$$n! = \sqrt{2\pi n} * (\frac{n}{e})^n * (1 + \Theta(\frac{1}{n}))$$

Corollary

1. $n! = o(n^n)$
2. $\log(n!) = \Theta(n \log n)$

Induction handout

Goal: Prove statements of the form

$$\forall n \geq n_0 : P(n)$$

where $P(n)$ is a propositional function of n

Domino analogy

$$| \quad | \quad | \quad |$$

$$n_0 \quad n_0 + 1 \quad n \quad n + 1$$