CSE 101 – Oct 25, 2019 (Week 4)

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Graphs

$G = (V, E)$

$x, y \in V$

$e \in E$

Notation: $e = xy = \{x, y\}$

- e joins x to y
- x, y ends of e
- x, y are adjacent
- x is incident with e

Example

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{12, 14, 23, 24, 25, 26, 35, 36, 45, 56\}$

Def An $x - y$ path in G is a sequence of vertices

(k = length, # of edges)

$x = v_0, v_1, v_2, ..., v_k = y$

in which each consecutive pair are adjacent for $i = 1$ to k

$\{v_{i-1}, v_i\} \in E$

and in which no vertex is repeated (except possibly $x = y$)

- If $x = y$, the path is **closed**
- If length of sequence is 1, the path is trivial
- A non-trivial closed path is called a cycle

Example

$1 - 6$ path $1, 2, 6$ (length 2)

$1 - 6$ path $1, 4, 2, 3, 5, 6$ (length 5)

$2 - 2$ cycle $2, 3, 6, 5, 2$ (length 4)

Def

$G = (V, E)$ is **connected** iff for all $x, y \in V$, G contains an $x - y$ path. Otherwise G is called **disconnected**

Example

non-connected graph

$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$E = \{12, 15, 25, 26, 56, 37, 38, 49, 78\}$

Def

A subgraph H of G is a graph in which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

Example

- $\{1, 2, 5, 6\}$, $\{12, 25, 26, 56\}$
  - Connected
- $(\{3, 7, 8, 4\}, \{37, 38\})$
  - Disconnected
- $(\{4\}, \{49\})$ not a graph

Def

A subgraph H of G is a **connected component** iff:

1. H is connected
2. H is maximal with respect to (1)

Def

G is acyclic iff it contains no cycles (non-trivial closed path)

Example

$n = 18$, $m = 15$

acyclic graph

Def

A **tree** is a graph that is both acyclic and connected

Also called a **forest**

Theorem

Let T be a tree on n vertices. Then T has $n-1$ edges

Proof (strong induction on n)

Base case

Note – $P(n)$ is: if T is a tree with n vertices, then T has $n-1$ edges

Goal: Show $\forall n \geq 1 : P(n)$

I. $P(1)$ says "if T is a tree with 1 vertex, then T has 0 edges." Observe there is only one graph with 1 vertex. This is the only tree on 1 vertex.

$P(1)$ is true in this case.

IId.

$\forall n > 1 : P(1) \wedge ... \wedge P(n-1) \Rightarrow P(n)$

Let $n > 1$ be arbitrary. Assume for all k in the range $1 \leq k < n$ that if T' is a tree on k vertices, then T' has $k-1$ edges. We must show that if T is a tree on n vertices, then T has $n-1$ edges.

So let T be a tree on n vertices.

Let $e \in E(T)$ be arbitrary. Remove e from T. This results in two subtrees.

T_1 T_2 each with fewer than n vertices. Let T_i has n_i vertices and m_i edges. $(i = 1, 2)$. Since $n_1 < n$ and $n_2 < n$, by induction hypothesis.

We have $m_1 = n_1 - 1$ and $m_2 = n_2 - 1$

Note: $n_1 + n_2 = n$ (no vertices were removed)

Thus

(# edges in T) $= m_1 + m_2 + 1$

$= (n_1 - 1) + (n_2 - 1) + 1$ (by I.H.)

$= (n_1 + n_2) - 1$

$= n - 1$

$\therefore T$ has $n-1$ edges