Pep 2. ED y MN 17/8/21

Benjamin Jorquera F-1.

P1) a. $t - 2f(t) = \int_0^t (e^{\tau} - e^{-\tau}) f(t-\tau)\,d\tau$ (1)

Expresamos $\int_0^t e^{\tau} \cdot f(t-\tau)\,d\tau - \int_0^t e^{-\tau} f(t-\tau)\,d\tau$

Por Teorema de la Convolución:

i) $\int_0^t e^{\tau} f(t-\tau)\,d\tau = e^{t} * f(t)$

ii) $\int_0^t e^{-\tau} f(t-\tau)\,d\tau = e^{-t} * f(t)$

Aplicando transformada de Laplace a (1)

$$\mathcal{L}\{t\} - 2\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{t} * f(t)\} - \mathcal{L}\{e^{-t} * f(t)\}$$

$$\Rightarrow \frac{1}{s^2} - 2F(s) = \frac{1}{(s-1)} \cdot F(s) - \frac{1}{(s+1)} \cdot F(s)$$

$$\frac{1}{s^2} = F(s)\left(\underbrace{\frac{1}{(s-1)} - \frac{1}{(s+1)} + 2}_{(s+1)-(s-1)+2(s^2-1)}\right)$$

$$\frac{1}{s^2} = \frac{F(s)\left(2s^2 \cancel{-2} \cancel{+2}\right)}{(s-1)(s+1)}$$

$$F(s) = \frac{(s^2-1)}{2s^4} = \frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{s^4}\right)$$

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Aplicando T.L. Inversa.

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^4}\right\}$$

$$= \frac{1}{2}\left(t - \mathcal{L}^{-1}\left\{\frac{1}{6}\cdot\frac{6}{s^4}\right\}\right)$$

$$= \frac{1}{2}\left(t - \frac{1}{6}\cdot t^3\right)$$

$$\therefore \quad f(t) = \frac{t}{2} - \frac{t^3}{12}.$$

b. $\begin{cases} y''(t) + 4y(t) = f(t) \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$ $\quad f(t) = \begin{cases} 0, & 0 \leq t < 2\pi \\ \operatorname{sen}(t), & t \geq 2\pi \end{cases}$

i) Expresar $f(t) = \operatorname{sen}(t)\,u(t-2\pi)$

ii) Aplicar Transformada de Laplace

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{\operatorname{sen}(t)\,u(t-2\pi)\}$$

- Utilizando Transformadas de Derivadas.

$$s^2 y(s) - s\overset{1}{\cancel{y(0)}} - \cancel{y'(0)} + 4y(s) \qquad \text{con} \quad y(0)=1$$

$$y(s)(s^2+4) - s. \qquad\qquad y'(0)=0$$

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• Utilizando 2= Teorema $\mathcal{L}$ Traslación

$$\mathcal{L}\{sen(t)\,u(t-2\pi)\} = e^{-2\pi s}\,\mathcal{L}\{sen(t+2\pi)\}$$

$$\mathcal{L}\{sen(t+2\pi)\} = \mathcal{L}\{sen(t)\overset{1}{\cancel{\cos(2\pi)}} + \cos(t)\overset{0}{\cancel{sen(2\pi)}}\}$$

$$= \mathcal{L}\{sen(t)\}$$

$$= \frac{1}{s^2+1}\cdot e^{-2\pi s}$$

• Resolviendo la ecuación:

$$Y(s) = \frac{e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s}{(s^2+4)}$$

iii) Aplicando T.L Inversa

① $$\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right\} = u(t-2\pi)\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$$

Realizando una descomposición por fracciones parciales.

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{A}{(s^2+4)} + \frac{B}{(s^2+1)} \quad As^2+A+Bs^2+4B=1$$

$$\left.\begin{array}{l} A+B=0 \\ A+4B=1 \end{array}\right\} \quad \begin{array}{l} A=-1/3 \\ B=1/3 \end{array}$$

$$\Rightarrow u(t-2\pi)\left(\mathcal{L}^{-1}\left\{-\frac{1}{3(s^2+4)}\cdot\frac{2}{2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{3(s^2+1)}\right\}\right)$$

$$\Rightarrow u(t-2\pi)\left(-\frac{1}{6}\cdot sen(2(t-2\pi)) + \frac{1}{3}sen(t-2\pi)\right)$$

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② $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos(2t)$

$\therefore \; y(t) = \left(-\frac{1}{6}\,\text{sen}(2(t-2\pi)) + \frac{1}{3}\,\text{sen}(t-2\pi)\right)u(t-2\pi) + \cos(2t)$

$y(t) = \left(-\frac{1}{6}\,\text{sen}(2t) + \frac{1}{3}\,\text{sen}(t)\right)u(t-2\pi) + \cos(2t)$

$\Rightarrow \quad y(t) = \begin{cases} \cos(2t) & 0 \leq t < 2\pi \\ \cos(2t) + \frac{1}{3}\left(\text{sen}(t) - \frac{1}{2}\,\text{sen}(2t)\right) & t \geq 2\pi. \end{cases}$

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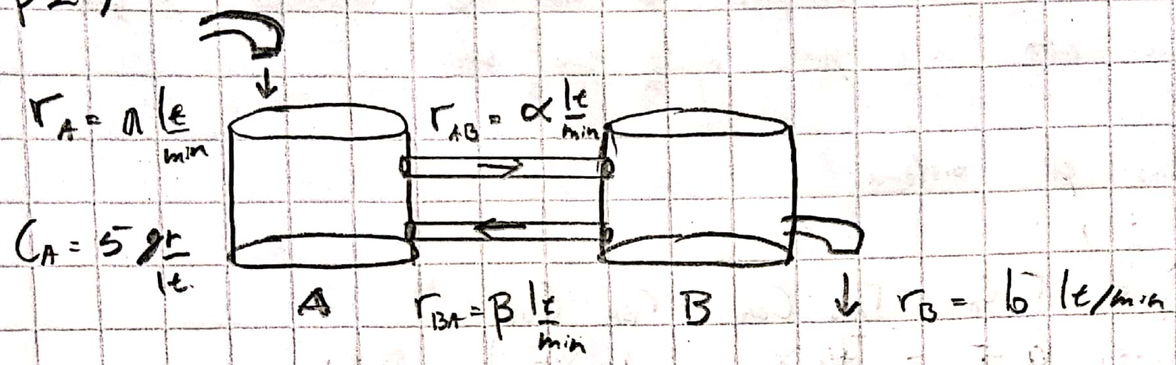
P2)



$V_A = V_B = 1\ lt.$ inicialmente.

a Construimos un sistema de ecuaciones:

$$\left.\begin{array}{l} a+\beta = \alpha \\ \alpha = \beta + b \end{array}\right\} \quad \begin{array}{l} \beta = \alpha - a \\ \alpha = \beta + b \end{array}$$

b. $a = 6\ lt/min$ y $\alpha = 7\beta\ lt/min$

Reemplazando en el sistema anterior

$$\beta = 7\beta - 6 \Rightarrow \beta = 1\ lt/min$$
$$7\beta - \beta = b \Rightarrow b = 6\ lt/min$$

Las concentraciones en los estanques con respecto al tiempo son

$$C_{OA} = \frac{X_A}{V_A} = X_A \quad y \quad C_{OB} = \frac{X_B}{V_B} = X_B.$$

Además se tiene que: $r_{AB} = 7\ lt/min$; $r_{BA} = 1\ lt/min$
$r_A = 6\ lt/min$; $r_B = 6\ lt/min$

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El volumen en los estanques se mantiene constante ya que lo que entra es igual a lo que sale.

Construimos el sistema:

$$x'_A = r_A C_A - r_{AB} \cdot C_{OA} + r_{BA} \cdot C_{OB}$$
$$= 6 \cdot 5 - 7 \cdot X_A + 1 \cdot X_B$$
$$= -7X_A + X_B + 30$$

$$X'_B = r_{AB} \cdot C_{OA} - r_{BA} \cdot C_{OB} - r_B \cdot C_{OB}$$
$$= 7X_A - 1 \cdot X_B - 6 \cdot X_B$$
$$= 7X_A - 7X_B$$

$$\Rightarrow \quad X' = \begin{pmatrix} -7 & 1 \\ 7 & -7 \end{pmatrix} X + \begin{pmatrix} 30 \\ 0 \end{pmatrix}$$

c) Resolvemos el sistema homogéneo mediante vectores y valores propios.

$$|A - \lambda I| = \begin{vmatrix} -7-\lambda & 1 \\ 7 & -7-\lambda \end{vmatrix} = (-7-\lambda)^2 - 7$$
$$= \lambda^2 + 14\lambda + 49 - 7$$
$$= \lambda^2 + 14\lambda + 42$$

Raíces de $p(\lambda)$: $\lambda_1 = -7 - \sqrt{7}$
$\lambda_2 = \sqrt{7} - 7$

$$\Delta = \frac{-14 \pm \sqrt{14^2 - 4 \cdot 42}}{2} = \frac{-7 \pm \sqrt{28}}{2} = \frac{-7 \pm \cancel{2}\sqrt{7}}{\cancel{2}}$$

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i) caso $\lambda = -7-\sqrt{7}$

$$\begin{pmatrix} \cancel{-7}\cancel{+7}+\sqrt{7} & 1 \\ 7 & \sqrt{7} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} \sqrt{7}a + b = 0 \\ 7a + \sqrt{7}b = 0 \end{matrix}\right\} \quad b = -\sqrt{7}a.$$

$$\Rightarrow \quad V_1 = \begin{pmatrix} a \\ -\sqrt{7}a \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{7} \end{pmatrix}$$

ii) caso $\lambda = \sqrt{7} - 7.$

$$\begin{pmatrix} \cancel{-7} - \sqrt{7} \cancel{+7} & 1 \\ 7 & -\sqrt{7} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} -\sqrt{7}a + b = 0 \\ 7a - \sqrt{7}b = 0 \end{matrix}\right\} \quad b = \sqrt{7}a$$

$$\Rightarrow \quad V_2 = \begin{pmatrix} a \\ \sqrt{7}a \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{7} \end{pmatrix}$$

$\therefore$ La solución homogenea es

$$X_h(t) = C_1 \begin{pmatrix} 1 \\ -\sqrt{7} \end{pmatrix} e^{(-7-\sqrt{7})t} + C_2 \begin{pmatrix} 1 \\ \sqrt{7} \end{pmatrix} e^{(\sqrt{7}-7)}$$

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para la solución particular utilizamos el Método de coeficientes indeterminados.

$$X_p = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad X'_p = 0 \qquad F(t) = \begin{pmatrix} 30 \\ 0 \end{pmatrix}$$

Reemplazando: $$0 = \begin{pmatrix} -7 & 1 \\ 7 & -7 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 30 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -7a_1 + a_2 &= -30 \\ 7a_1 - 7a_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} -6a_2 &= -30 \\ a_2 &= 5 \\ 7a_1 &= 35 \\ a_1 &= 5 \end{aligned}$$

$$\therefore X_p(t) = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

Finalmente la solución general es:

$$X(t) = C_1 \begin{pmatrix} 1 \\ -\sqrt{7} \end{pmatrix} e^{(-7-\sqrt{7})t} + C_2 \begin{pmatrix} 1 \\ \sqrt{7} \end{pmatrix} e^{(\sqrt{7}-7)t} + \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

Por componentes:

$$X_A(t) = C_1 e^{(-7-\sqrt{7})t} + C_2 e^{(\sqrt{7}-7)t} + 5$$

$$X_B(t) = -\sqrt{7} C_1 e^{(-7-\sqrt{7})t} + \sqrt{7} C_2 e^{(\sqrt{7}-7)t} + 5.$$