Branch and Bound Techniques for Solving Integer Linear Programming Problems

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Outline

Integer Linear Programming

Why?

- Many practical problems need variables to take integer values

How to solve it?

- Exhaustive enumeration is only useful when dealing with few combinations of values
- In general, we deal with a combinatorial problem
- Solving techniques are based on divide and conquer approach

1. Integer Linear Programming
   - Branch and Bound for General IP
   - Branch and Bound for Binary IP

Branch and Bound for General IP

- Solve linear programming relaxations to bound the objective function
- Create branches by adding constraints that eliminate non-integer values

Example

Considering the following Integer Programming problem:

$$Max \;\; z = 11x_1 + 14x_2$$

Subject to

$$\begin{aligned}
1x_1 + 1x_2 &\leq 17 \\
3x_1 + 7x_2 &\leq 63 \\
3x_1 + 5x_2 &\leq 48 \\
3x_1 + 1x_2 &\leq 30 \\
x_1, x_2 &\geq 0 \\
x_1, x_2 &\in \mathbb{Z}
\end{aligned}$$

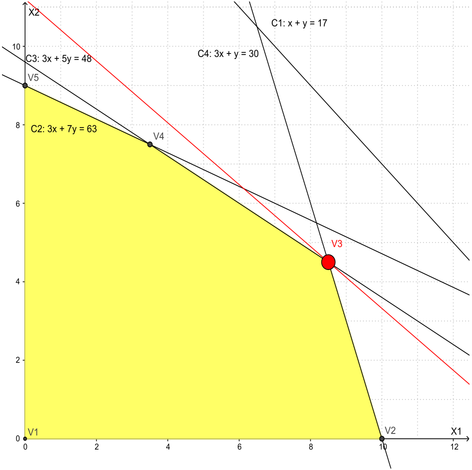
Solving the Linear Programming Relaxation

$$Max \;\; z = 11x_1 + 14x_2$$

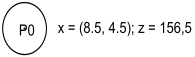
Subject to

$$\begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1, x_2 &\geq 0 \quad x^* = (8.5, 4.5) \;\; z^* = 156.5 \end{aligned}$$

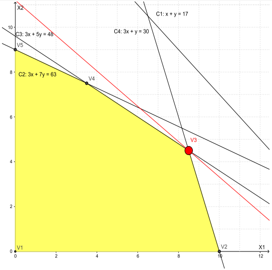
Graphically



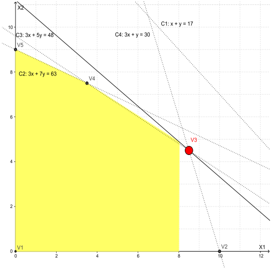
Enumeration tree

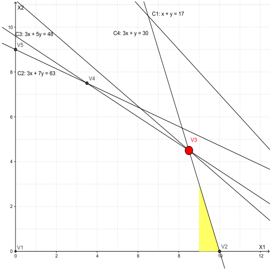


Branching



Branching on x_1:





Solving the Linear Programming Relaxation

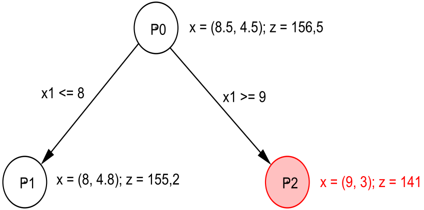
$$Max \;\; z = 11x_1 + 14x_2$$

$$\text{Subject to} \quad \begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1 &\leq 8 \\ x_1, x_2 &\geq 0 \quad x^* = (8, 4.8) \;\; z^* = 155.2 \end{aligned}$$

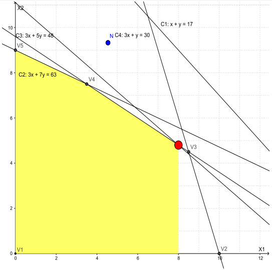
$$Max \;\; z = 11x_1 + 14x_2$$

$$\text{Subject to} \quad \begin{aligned} 1x_1 + 1x_2 &\leq 17 \\ 3x_1 + 7x_2 &\leq 63 \\ 3x_1 + 5x_2 &\leq 48 \\ 3x_1 + 1x_2 &\leq 30 \\ x_1 &\geq 9 \\ x_1, x_2 &\geq 0 \quad x^* = (9, 3) \;\; z^* = 141 \end{aligned}$$

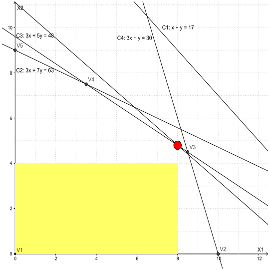
Enumeration tree

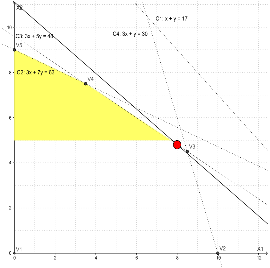


Branching



Branching on x_2:





Solving the Linear Programming Relaxation

$$Max \quad z = 11x_1 + 14x_2$$

Subject to

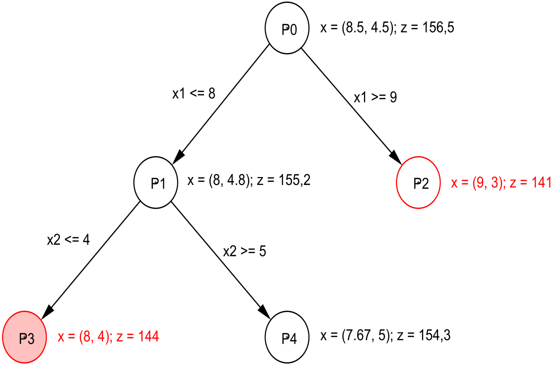
$$\begin{aligned}
1x_1 + 1x_2 &\leq 17 \\
3x_1 + 7x_2 &\leq 63 \\
3x_1 + 5x_2 &\leq 48 \\
3x_1 + 1x_2 &\leq 30 \\
x_1 \leq 8 \quad &\& \quad x_2 \leq 4 \\
x_1, x_2 &\geq 0 \quad x^* = (8, 4) \quad z^* = 144
\end{aligned}$$

$$Max \quad z = 11x_1 + 14x_2$$

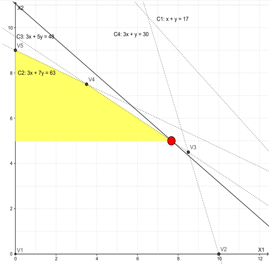
Subject to

$$\begin{aligned}
1x_1 + 1x_2 &\leq 17 \\
3x_1 + 7x_2 &\leq 63 \\
3x_1 + 5x_2 &\leq 48 \\
3x_1 + 1x_2 &\leq 30 \\
x_1 \leq 8 \quad &\& \quad x_2 \geq 5 \\
x_1, x_2 &\geq 0 \quad x^* = (7.67, 5) \quad z^* = 154.3
\end{aligned}$$

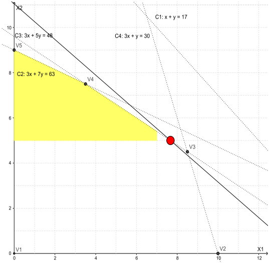
Enumeration tree

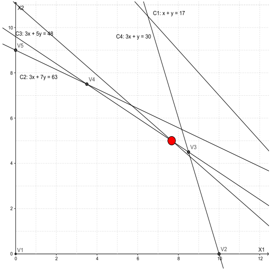


Branching



Branching again on x_1:





Solving the Linear Programming Relaxation

$$Max \quad z = 11x_1 + 14x_2$$

Subject to

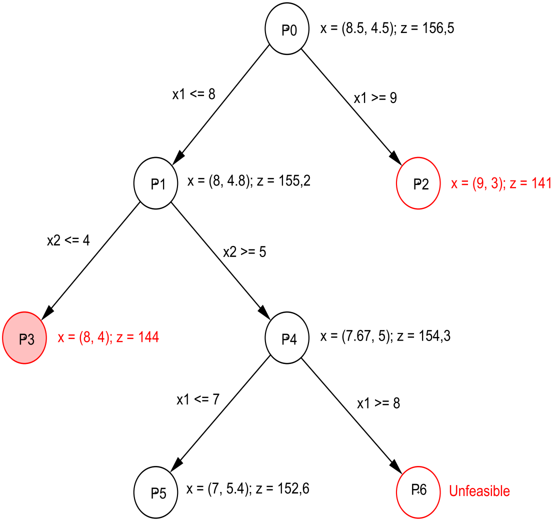
$$\begin{aligned}
1x_1 + 1x_2 &\leq 17 \\
3x_1 + 7x_2 &\leq 63 \\
3x_1 + 5x_2 &\leq 48 \\
3x_1 + 1x_2 &\leq 30 \\
x_1 \leq 8 \ \& \ x_2 &\geq 5 \ \& \ x_1 \leq 7 \\
x_1, x_2 &\geq 0 \quad x^* = (7, 5.4) \quad z^* = 152.6
\end{aligned}$$

$$Max \quad z = 11x_1 + 14x_2$$

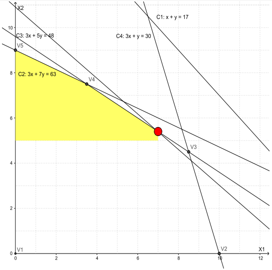
Subject to

$$\begin{aligned}
1x_1 + 1x_2 &\leq 17 \\
3x_1 + 7x_2 &\leq 63 \\
3x_1 + 5x_2 &\leq 48 \\
3x_1 + 1x_2 &\leq 30 \\
x_1 \leq 8 \ \& \ x_2 &\geq 5 \ \& \ x_1 \geq 8 \\
x_1, x_2 &\geq 0 \quad \textit{Unfeasible}
\end{aligned}$$

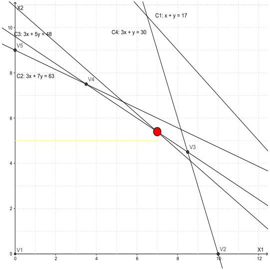
Enumeration tree

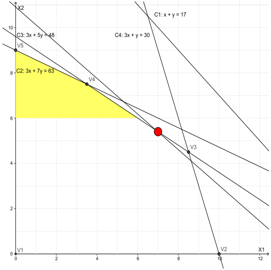


Branching



Branching again on x_2:





Solving the Linear Programming Relaxation

$$Max \quad z = 11x_1 + 14x_2$$

Subject to

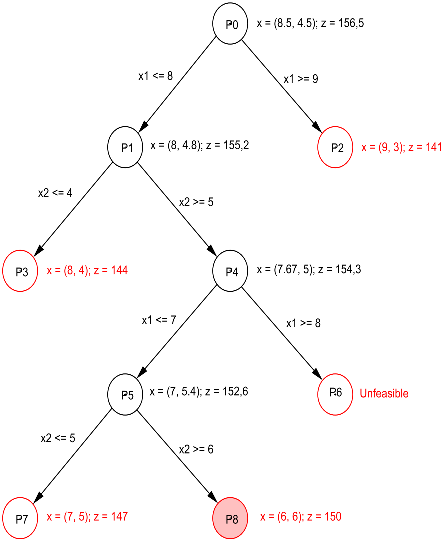
$$\begin{array}{rcl}
1x_1 + 1x_2 & \leq & 17 \\
3x_1 + 7x_2 & \leq & 63 \\
3x_1 + 5x_2 & \leq & 48 \\
3x_1 + 1x_2 & \leq & 30 \\
x_1 \leq 8 \;\; \& \;\; x_2 \geq 5 & \& & x_1 \leq 7 \;\; \& \;\; x_2 \leq 5 \\
x_1, x_2 & \geq & 0 \quad x^* = (7,5) \quad z^* = 147
\end{array}$$

$$Max \quad z = 11x_1 + 14x_2$$

Subject to

$$\begin{array}{rcl}
1x_1 + 1x_2 & \leq & 17 \\
3x_1 + 7x_2 & \leq & 63 \\
3x_1 + 5x_2 & \leq & 48 \\
3x_1 + 1x_2 & \leq & 30 \\
x_1 \leq 8 \;\; \& \;\; x_2 \geq 5 & \& & x_1 \leq 7 \;\; \& \;\; x_2 \geq 6 \\
x_1, x_2 & \geq & 0 \quad x^* = (6,6) \quad z^* = 150
\end{array}$$

Enumeration tree



Branch and Bound for Binary IP

- Solve linear programming relaxations to bound the objective function
- Create branches by adding constraints that fix variables

Example

$$Max \;\; z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to

$$\begin{aligned} 6x_1 + 3x_2 + 5x_3 + 2x_4 &\leq 10 \\ x_3 + x_4 &\leq 1 \\ -x_1 + x_3 &\leq 0 \\ -x_2 + x_4 &\leq 0 \\ x_i &\in \{0, 1\} \;\; \forall i = 1, \ldots, 4 \end{aligned}$$

Constraints allowing values

A constraint allowing values for variables:

$$x_i \in \{0, 1\}$$

is equivalent to the following constraints:

$$\begin{aligned} x_i &\geq 0 \\ x_i &\leq 1 \\ x_i &\in \mathbb{Z} \end{aligned}$$

This correspond to an IP problem with upper bounds for all variables.

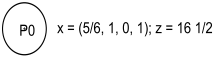
Solving the Linear Programming Relaxation

$$Max \;\; z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to

$$\begin{aligned}
6x_1 + 3x_2 + 5x_3 + 2x_4 &\leq 10 \\
x_3 + x_4 &\leq 1 \\
-x_1 + x_3 &\leq 0 \\
-x_2 + x_4 &\leq 0 \\
x_i &\leq 1 \;\; \forall i = 1, \ldots, 4 \\
x_i &\geq 0 \;\; \forall i = 1, \ldots, 4 \;\; x^* = (\frac{5}{6}, 1, 0, 1); \;\; z^* = 16\frac{1}{2}
\end{aligned}$$

Enumeration tree



Branching

Remember that $x_i \in \{0,1\}$ is equivalent to the following constraints:

$$\begin{aligned} x_i &\geq 0 \\ x_i &\leq 1 \\ x_i &\in \mathbb{Z} \end{aligned}$$

When branching we will add constraints $x_i \leq 0$ and $x_i \geq 1$:

$$\begin{array}{cc} x_i \geq 0 & x_i \geq 0 \\ x_i \leq 1 & x_i \leq 1 \\ x_i \leq 0 & x_i \geq 1 \\ x_i \in \mathbb{Z} & x_i \in \mathbb{Z} \\ \Downarrow & \Downarrow \\ x_i = 0 & x_i = 1 \end{array}$$

Branching

Subproblem 1 (original problem $\wedge$ $x_1 = 0$):

$$Max \;\; z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to

$$\begin{aligned} 6x_1 + 3x_2 + 5x_3 + 2x_4 &\leq 10 \\ x_3 + x_4 &\leq 1 \\ -x_1 + x_3 &\leq 0 \\ -x_2 + x_4 &\leq 0 \\ x_1 &= 0 \\ x_i &\in \{0, 1\} \;\; \forall i = 1, \ldots, 4 \end{aligned}$$

Subproblem 2 (original problem $\wedge$ $x_1 = 1$):

$$Max \;\; z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Subject to

$$\begin{aligned} 6x_1 + 3x_2 + 5x_3 + 2x_4 &\leq 10 \\ x_3 + x_4 &\leq 1 \\ -x_1 + x_3 &\leq 0 \\ -x_2 + x_4 &\leq 0 \\ x_1 &= 1 \\ x_i &\in \{0, 1\} \;\; \forall i = 1, \ldots, 4 \end{aligned}$$

Solving the Linear Programming Relaxation

Subproblem 1: $Max \;\; z = 5x_2 + 6x_3 + 4x_4$

Subject to

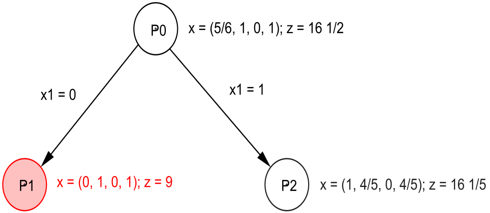
$$
\begin{aligned}
3x_2 + 5x_3 + 2x_4 &\leq 10 \\
x_3 + x_4 &\leq 1 \\
x_3 &\leq 0 \\
-x_2 + x_4 &\leq 0 \\
x_i &\leq 1 \;\; \forall i = 2, \ldots, 4 \\
x_i &\geq 0 \;\; \forall i = 2, \ldots, 4 \;\; x^* = (0, 1, 0, 1); \;\; z^* = 9
\end{aligned}
$$

Subproblem 2: $Max \;\; z = 9 + 5x_2 + 6x_3 + 4x_4$

Subject to

$$
\begin{aligned}
3x_2 + 5x_3 + 2x_4 &\leq 4 \\
x_3 + x_4 &\leq 1 \\
x_3 &\leq 1 \\
-x_2 + x_4 &\leq 0 \\
x_i &\leq 1 \;\; \forall i = 2, \ldots, 4 \\
x_i &\geq 0 \;\; \forall i = 2, \ldots, 4 \;\; x^* = (1, \frac{4}{5}, 0, \frac{4}{5}); \;\; z^* = 16\frac{1}{5}
\end{aligned}
$$

Enumeration tree



Branching

Subproblem 3 (subproblem 2 $\wedge$ $x_2 = 0$):

Subject to

$$
\begin{aligned}
& Max \;\; z = 9 + 5x_2 + 6x_3 + 4x_4 \\
3x_2 + 5x_3 + 2x_4 &\leq 4 \\
x_3 + x_4 &\leq 1 \\
x_3 &\leq 1 \\
-x_2 + x_4 &\leq 0 \\
x_2 &= 0 \\
x_i &\in \{0, 1\} \;\; \forall i = 2, \ldots, 4
\end{aligned}
$$

Subproblem 4 (subproblem 2 $\wedge$ $x_2 = 1$):

Subject to

$$
\begin{aligned}
& Max \;\; z = 9 + 5x_2 + 6x_3 + 4x_4 \\
3x_2 + 5x_3 + 2x_4 &\leq 4 \\
x_3 + x_4 &\leq 1 \\
x_3 &\leq 1 \\
-x_2 + x_4 &\leq 0 \\
x_2 &= 1 \\
x_i &\in \{0, 1\} \;\; \forall i = 2, \ldots, 4
\end{aligned}
$$

Solving the Linear Programming Relaxation

Subproblem 3:
Subject to

$$Max \;\; z = 9 + 6x_3 + 4x_4$$

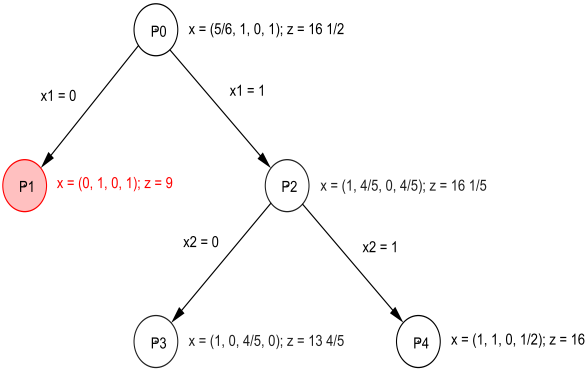
$$\begin{aligned}
5x_3 + 2x_4 &\leq 4\\
x_3 + x_4 &\leq 1\\
x_3 &\leq 1\\
x_4 &\leq 0\\
x_i &\leq 1 \;\; \forall i = 3,4\\
x_i &\geq 0 \;\; \forall i = 3,4 \;\; x^* = (1, 0, \frac{4}{5}, 0); \;\; z = 13\frac{4}{5}
\end{aligned}$$

Subproblem 4:
Subject to

$$Max \;\; z = 14 + 6x_3 + 4x_4$$

$$\begin{aligned}
5x_3 + 2x_4 &\leq 1\\
x_3 + x_4 &\leq 1\\
x_3 &\leq 1\\
x_4 &\leq 1\\
x_i &\leq 1 \;\; \forall i = 3,4\\
x_i &\geq 0 \;\; \forall i = 3,4 \;\; x^* = (1, 1, 0, \frac{1}{2}); \;\; z^* = 16
\end{aligned}$$

Enumeration tree



Branching

Subproblem 5 (subproblem 4 $\wedge$ $x_4 = 0$):

Subject to

$$Max \;\; z = 14 + 6x_3 + 4x_4$$

$$\begin{aligned}
5x_3 + 2x_4 &\leq 1 \\
x_3 + x_4 &\leq 1 \\
x_3 &\leq 1 \\
x_4 &\leq 1 \\
x_4 &= 0 \\
x_i &\in \{0,1\} \;\; \forall i = 3,4
\end{aligned}$$

Subproblem 6 (subproblem 4 $\wedge$ $x_4 = 1$):

Subject to

$$Max \;\; z = 14 + 6x_3 + 4x_4$$

$$\begin{aligned}
5x_3 + 2x_4 &\leq 1 \\
x_3 + x_4 &\leq 1 \\
x_3 &\leq 1 \\
x_4 &\leq 1 \\
x_4 &= 1 \\
x_i &\in \{0,1\} \;\; \forall i = 3,4
\end{aligned}$$

Solving the Linear Programming Relaxation

Subproblem 5:

$$Max \;\; z = 14 + 6x_3$$

Subject to

$$\begin{aligned} 5x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\geq 0 \;\; x^* = (1, 1, \tfrac{1}{5}, 0); \;\; z^* = 15\tfrac{1}{5} \end{aligned}$$

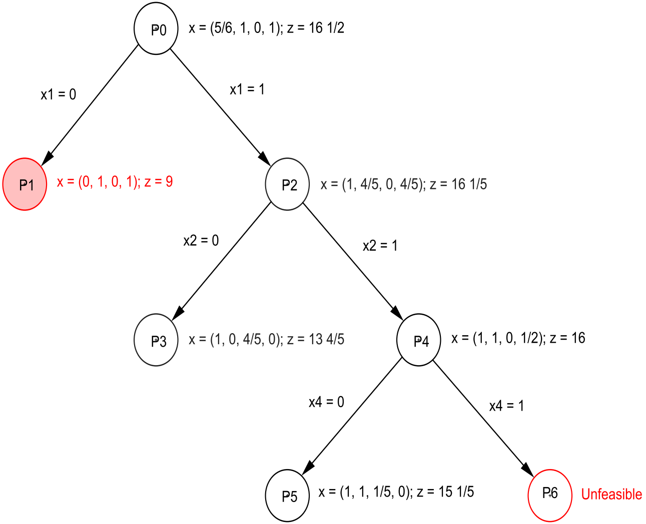
Subproblem 6:

$$Max \;\; z = 18 + 6x_3$$

Subject to

$$\begin{aligned} 5x_3 &\leq -1 \\ x_3 &\leq 0 \\ x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\geq 0 \;\; \textit{Unfeasible} \end{aligned}$$

Enumeration tree



Branching

Subproblem 7 (subproblem 5 $\wedge$ $x_3 = 0$):

Subject to

$$Max \quad z = 14 + 6x_3$$

$$\begin{aligned} 5x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\geq 0 \\ x_3 &= 0 \\ x_3 &\in \{0, 1\} \end{aligned}$$

Subproblem 8 (subproblem 5 $\wedge$ $x_3 = 1$):

Subject to

$$Max \quad z = 14 + 6x_3$$

$$\begin{aligned} 5x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\leq 1 \\ x_3 &\geq 0 \\ x_3 &= 1 \\ x_3 &\in \{0, 1\} \end{aligned}$$

Solving the Linear Programming Relaxation

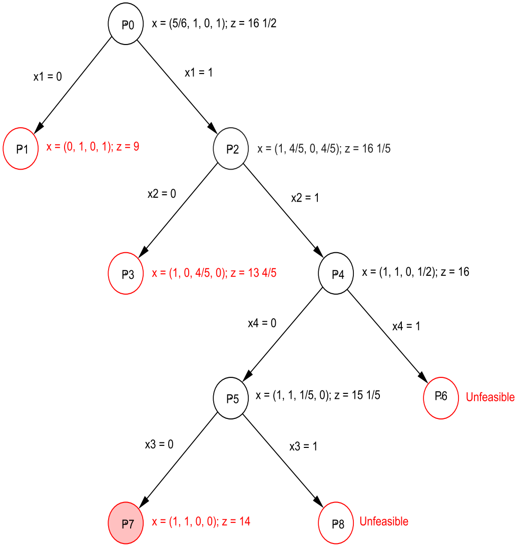
Subproblem 7:

- Variable enumeration: $x = (1, 1, 0, 0)$
- Feasible solution
- $z = 14$

Subproblem 8:

- Variable enumeration: $x = (1, 1, 1, 0)$
- Unfeasible solution

Enumeration tree



Solving Mixed Integer Programming Problems

Mixed Integer Programming Problems involve real, integer, and binary variables:

- Real variables: Simplex Method
- Integer variables: Simplex Method + constraints $\leq$ and $\geq$
- Binary variables: Simplex Method + constraints $=$