

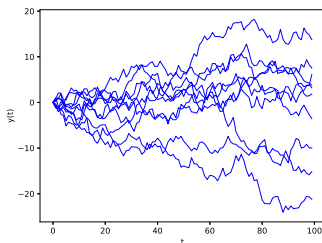
# Application of Mathematical Methods in the Study of Chemical Dynamical Systems to an Inventory Multiplier-Accelerator Business Cycle Model

Benjamin Bui

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## Motivation

- The study of stochastic processes was deeply influential in explaining macroscale thermodynamic properties
- This quickly spread into the fields of economics and finance



**Figure 1:** Discrete brownian motion of a 10 independent particles over 100 iterations with a mean of 0 and a standard deviation of 1.

## Motivation

- Fractals and chaotic processes are a more recent addition to the chemist's toolkit<sup>1,2</sup>
- Could chaos play a role in economics as well?



Figure 2: Still image of the BZ-reaction

## Background

- Lloyd Metzler developed an inventory cycle model in 1941 in order to explore the effect of a Lundberg lag<sup>3</sup>
- Wegener, Westerhoff, and Zaklan expanded on this model in 2009<sup>4</sup>

## Consumption and Investment

- Investment is held as endogenous
- Consumption is a proportion of income

$$I_t = \bar{I}$$
$$C_t = bY_t$$

# Inventory

- Desired inventory is a proportion of expected consumption
- Stock output is made to match desired inventory
- Actual inventory depends on the accuracy of the expectation

$$\hat{Q}_t = kU_t$$

$$S_t = \hat{Q}_t - Q_{t-1}$$

$$Q_t = \hat{Q}_t - (C_t - U_t)$$

## Predicting Consumption

- "Extrapolaters" believe that consumption will diverge from the steady-state
- "Regressers" believe that consumption will return to the steady-state
- Firms will choose from the types depending on the consumption level

$$U_t^E = C_{t-1} + c(C_{t-1} - \bar{C})$$

$$U_t^R = C_{t-1} + f(\bar{C} - C_{t-1})$$

$$U_t = w_t U_t^E + (1 - w_t) U_t^R$$

$$w_t = \frac{1}{1 + d(\bar{C} - C_{t-1})^2}$$

## Aggregate Output

- Output is the sum of these production factors
- Can solve for a single fixed point

$$Y_t = I_t + U_t + S_t$$
$$\bar{Y} = \frac{1}{1-b} \bar{I}$$



## A Possible Trajectory

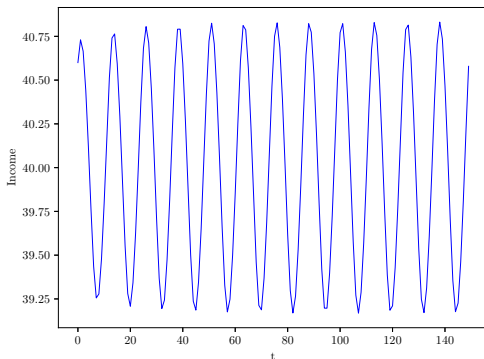


Figure 3: Timeseries with conditions:  $Y_0 = 40.6$ ,  $U_0 = 30.3$ ,  $\bar{I} = 10$ ,  $b = 0.75$ ,  $c = 0.3$ ,  $d = 1.0$ ,  $f = 0.1$ ,  $k = 0.1$

## Framework

- Consumption has a Robertson lag
- Investment is cubic relative to income change
- There is no Lundberg lag

$$C_t = (1 - s)Y_{t-1} + sY_{t-2}$$

$$I_t = v(Y_{t-1} - Y_{t-2}) - v(Y_{t-1} - Y_{t-2})^3$$

## The Growth Model

Change in income is normalized<sup>5</sup>  
to be within the range  $(-1, 1)$

$$\mu \equiv v - s$$

$$\dot{Y}_t = \mu \dot{Y}_{t-1} - (\mu + 1) \dot{Y}_{t-1}^3$$

## Qualitatively Different Trajectories Possible

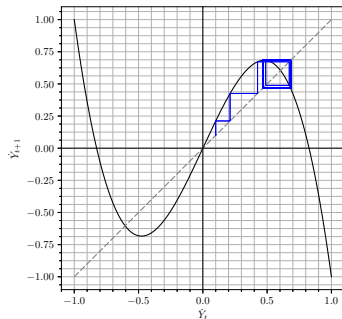


Figure 4: Cobweb plot with conditions  $\mu = 2.15$  and  $\dot{Y}_0 = 0.1$

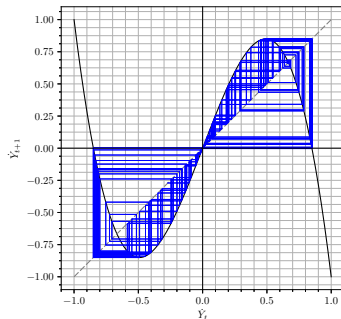


Figure 5: Cobweb plot with conditions  $\mu = 2.6$  and  $\dot{Y}_0 = 0.1$

## Goal

- To incorporate a Robertson and Lundberg lag in the same model
- To provide a reasonable basis for the boundedly rational behavior of firms

## Consumption and Investment

- Consumption is identical to the multiplier-accelerator model
- The investment function is qualitatively similar but asymptotically approaches 0

$$C_t = (1 - s)Y_{t-1} + sY_{t-2}$$

$$I_t = \frac{\frac{Y_{t-1} - Y_{t-2}}{v}}{\frac{Y_{t-1} - Y_{t-2}}{v} + q}$$

## Investment Curve

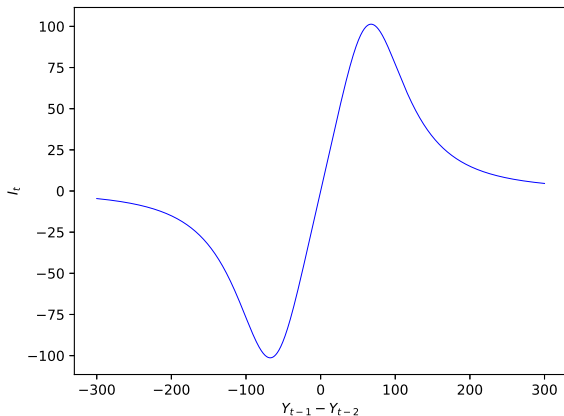


Figure 6: Investment curve such that  $\nu = 500$  and  $q = 0.001$

## Maximal/Minimal Investment

$$Y_{t-1} - Y_{t-2} = \pm \frac{q^{1/4} v}{3^{1/4}}$$

$$I_t = \frac{3^{3/4}}{3q^{3/4}}$$

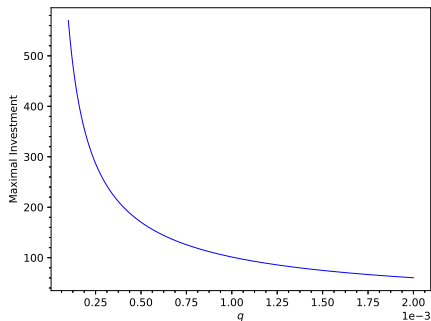


Figure 7: Plot of maximal investment relative to  $q$ .



## Investment FWHM

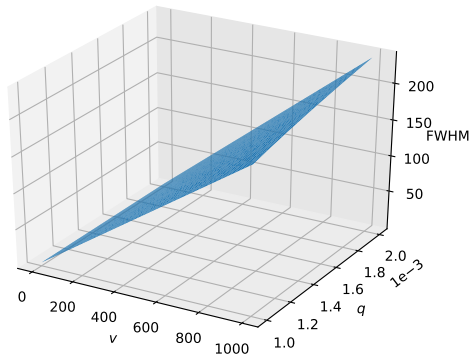


Figure 8: Triangulated plot investment FWHM relative to  $v$  and  $q$ .

# Inventory

- Predicted consumption is an average of lagged consumption
- Inventory production is otherwise identical to Wegener's model

$$U_t = \frac{C_{t-1} + C_{t-2} + C_{t-2}}{3}$$

$$S_t = kU_t - Q_{t-1}$$

$$Q_t = Q_{t-1} + S_t + (U_t - C_t)$$

## Output Growth

Endogenous investment allows for long-run growth or decay

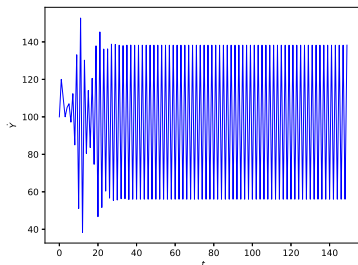
$$Y_t = I_t + S_t + U_t$$

$$\dot{Y}_t = Y_t - Y_{t-1}$$

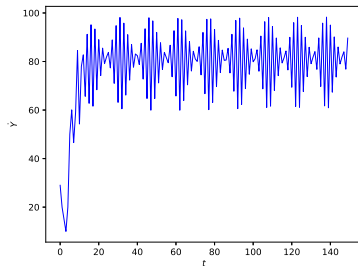
## Output Growth

$$\begin{aligned}\dot{Y}_t = & \frac{\frac{\dot{Y}_{t-1}}{v}}{\left(\frac{\dot{Y}_{t-1}}{v}\right)^4 + q} - \frac{\frac{\dot{Y}_{t-2}}{v}}{\left(\frac{\dot{Y}_{t-2}}{v}\right)^4 + q} + \\ & \frac{k+1}{3} \left[ (1-s)(\dot{Y}_{t-2} - Y_{t-5}) + s(\dot{Y}_{t-3} - Y_{t-6}) \right] \\ & + (1-s)\dot{Y}_{t-2} + s\dot{Y}_{t-3}\end{aligned}$$

## Possible Trajectories



**Figure 9:** Timeseries plot with parameters:  $s = 0.6$ ,  $k = 0.3$ ,  $v = 500$ ,  $q = 0.001$ . Initial values of  $\dot{Y}$  are: 100, 120, 110, 100, 105, 107



**Figure 10:** Timeseries plot of income growth rate over 150 iterations. Parameters identical to Fig. 9. Initial values of  $\dot{Y}$  are: 29, 20, 15, 10, 20, 50

## Determining Chaos

$$J = \begin{bmatrix} \frac{\partial f}{\partial Y_{t-1}} & \frac{\partial f}{\partial A_{t-1}} & s + \frac{k+1}{3}s & 0 & \frac{(k+1)(s-1)}{3} & -\frac{(k+1)s}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial \dot{Y}_{t-1}} = \frac{1}{v \left( q + \frac{Y_{t-1}^4}{v^4} \right)} - \frac{4Y_{t-1}^4}{v^5 \left( q + \frac{Y_{t-1}^4}{v^4} \right)}$$

$$\frac{\partial f}{\partial \dot{A}_{t-1}} = 1 + \frac{1}{3}(k+1)(1-s) - s + \frac{4A_{t-1}^4}{v^5 \left( q + \frac{A_{t-1}^4}{v^4} \right)^2} - \frac{1}{v \left( q + \frac{A_{t-1}^4}{v^4} \right)}$$

## Calculating the MLE

$$\lambda = \lim_{j \rightarrow \infty} \frac{1}{j} \sum_{j=1}^{t=j} \ln |J_t \cdot v_t|$$

$$v_{t+1} = \frac{J_t \cdot v_t}{|J_t \cdot v_t|}$$

## Propensity to Consume

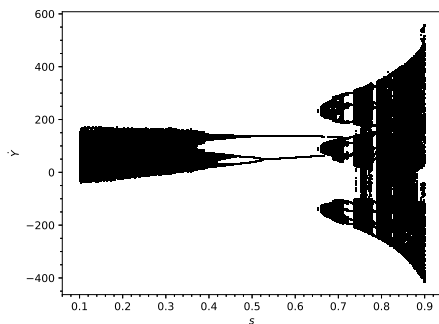


Figure 11: Bifurcation diagram  
varying  $s$

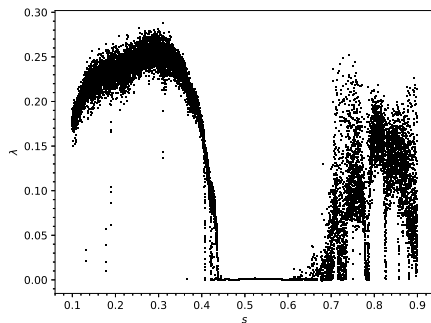


Figure 12: Lyapunov plot varying  $s$



## Investment (FWHM)

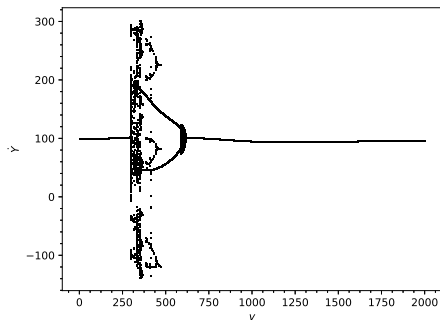


Figure 13: Bifurcation diagram  
varying  $v$

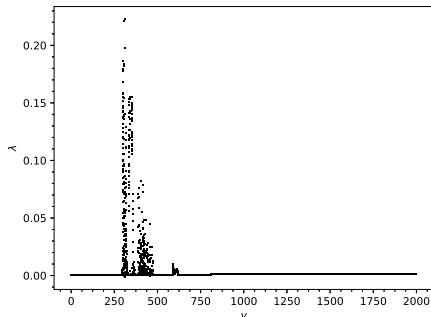


Figure 14: Lyapunov plot varying  $v$

## Investment (Maximum/Minimum)

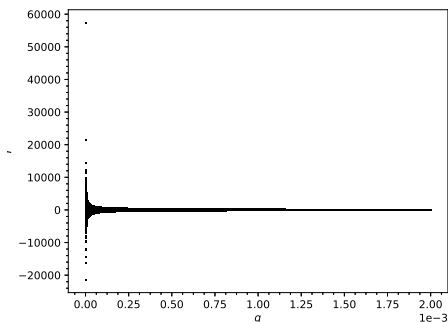


Figure 15: Bifurcation diagram varying  $q$

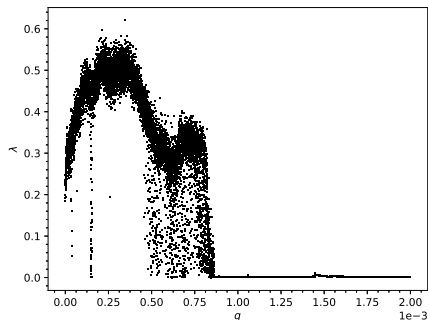


Figure 16: Lyapunov plot varying  $q$

## Sensitivity to Initial Conditions

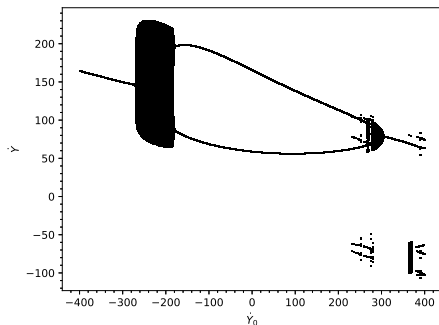


Figure 17: Bifurcation diagram  
varying  $\dot{Y}_0$

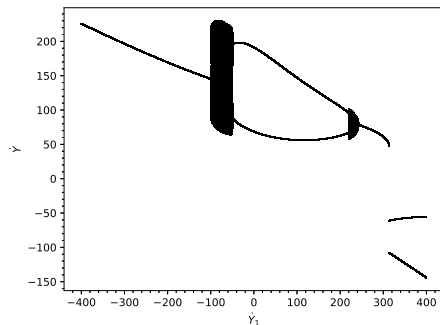


Figure 18: Bifurcation diagram  
varying  $\dot{Y}_1$

## Sensitivity to Initial Conditions

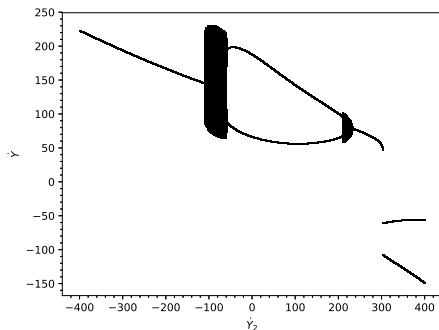


Figure 19: Bifurcation diagram  
varying  $\dot{Y}_2$

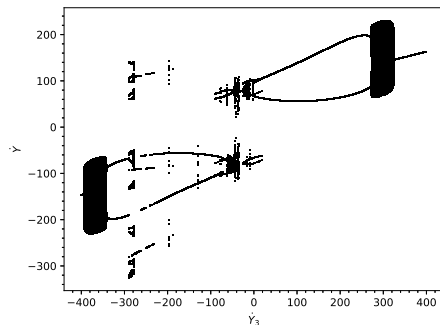


Figure 20: Bifurcation diagram  
varying  $\dot{Y}_3$

## Sensitivity to Initial Conditions

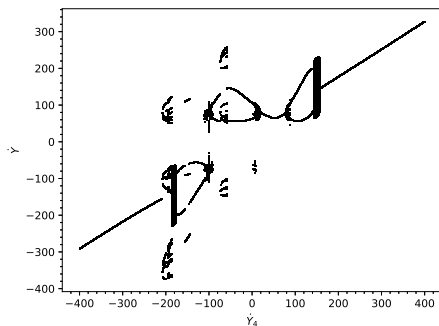


Figure 21: Bifurcation diagram  
varying  $\dot{Y}_4$

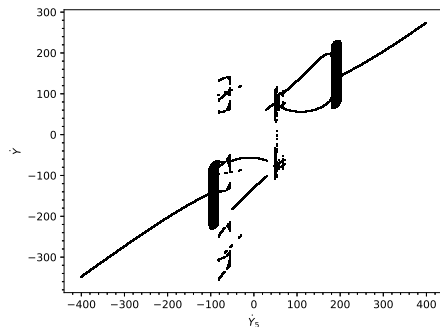


Figure 22: Bifurcation diagram  
varying  $\dot{Y}_5$

## Next steps

- Further investigate the fractal structure of the strange attractors
- Open the economy to foreign trade and improve the investment mechanism
- Improve consumption prediction mechanism

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