Application of Mathematical Methods in the Study of Chemical Dynamical Systems to an Inventory Multiplier-Accelerator Business Cycle Model

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Motivation

- The study of stochastic processes was deeply influential in explaining macroscale themodynamic properties
- Although first applied by French mathematician Louis Bachelier,[1] Einstein popularized the application of stochastic processes which helped its spread into the fields of finance and economics

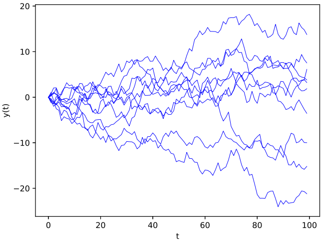

Figure 1: Discrete brownian motion of a 10 independent particles over 100 iterations with a mean of 0 and a standard deviation of 1.

Motivation

- Fractals and chaotic processes are a more recent addition to the chemist's toolkit[2,3]
- Could chaos play a role in economics as well?

Figure 2: Still image of the BZ-reaction

Background

- Lloyd Metzler developed an inventory cycle model in 1941 in order to explore the effect of a Lundberg lag[4]
- Wegener, Westerhoff, and Zaklan expanded on this model in 2009[5]

Consumption and Investment

- Investment is held as endogenous
- Consumption is a proportion of income

$$I_t = \bar{I}$$
$$C_t = bY_t$$

Inventory

- Desired inventory, $\hat{Q}$ is a proportion of expected consumption
- Stock output, S is made to match desired inventory
- Actual inventory, Q depends on the accuracy of the expectation
-

$$\hat{Q}_t = kU_t$$
$$S_t = \hat{Q}_t - Q_{t-1}$$
$$Q_t = \hat{Q}_t - (C_t - U_t)$$

Predicting Consumption

- U^E: "Extrapolaters" believe that consumption will diverge from the steady-state
- U^R: "Regressers" believe that consumption will return to the steady-state
- Firms will choose from the types depending on the consumption level

$$U_t^E = C_{t-1} + c(C_{t-1} - \bar{C})$$
$$U_t^R = C_{t-1} + f(\bar{C} - C_{t-1})$$
$$U_t = w_t U_t^E + (1 - w_t) U_t^R$$
$$w_t = \frac{1}{1 + d(\bar{C} - C_{t-1})^2}$$

Aggregate Output

- Output, Y, is the sum of these production factors
- Can solve for a single fixed point

$$Y_t = I_t + U_t + S_t$$

$$\bar{Y} = \frac{1}{1-b}\bar{I}$$

A Possible Trajectory

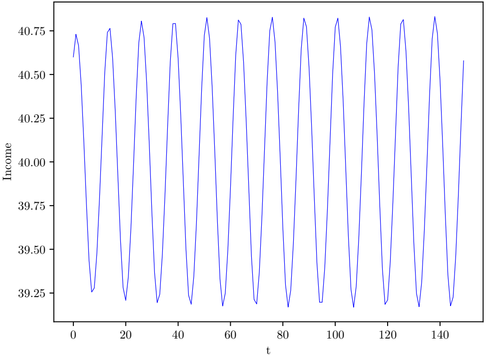

Figure 3: Timeseries with conditions: $Y_0 = 40.6,\ U_0 = 30.3,\ \bar{I} = 10,\ b = 0.75,\ c = 0.3,\ d = 1.0,\ f = 0.1,\ k = 0.1$

Framework

- Consumption, C, has a Robertson lag
- Investment, I, is cubic relative to income change
- There is no Lundberg lag

$$C_t = (1-s)Y_{t-1} + sY_{t-2}$$
$$I_t = v(Y_{t-1} - Y_{t-2}) - v(Y_{t-1} - Y_{t-2})^3$$

The Growth Model

Change in income is normalized[6] to be within the range (-1, 1)

$$\mu \equiv v - s$$
$$\dot{Y}_t = \mu \dot{Y}_{t-1} - (\mu + 1)\dot{Y}_{t-1}^3$$

Qualitatively Different Trajectories Possible

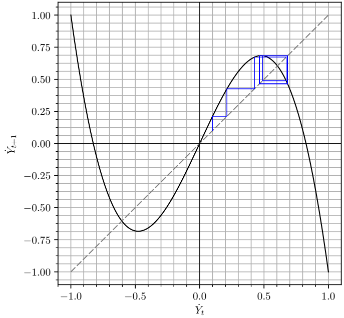

Figure 4: Cobweb plot with conditions $\mu = 2.15$ and $\dot{Y}_0 = 0.1$

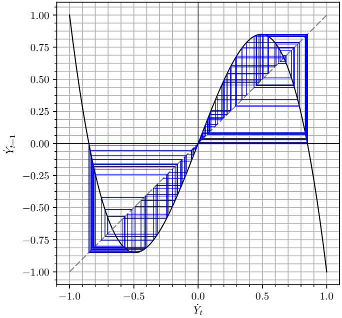

Figure 5: Cobweb plot with conditions $\mu = 2.6$ and $\dot{Y}_0 = 0.1$

Goal

- To incorporate a Robertson and Lundberg lag in the same model
- To provide a reasonable basis for the boundedly rational behavior of firms

Consumption and Investment

- Consumption, C, is identical to the multiplier-accelerator model
- The investment function, I, is qualitatively similar but asymptotically approaches 0

$$C_t = (1-s)Y_{t-1} + sY_{t-2}$$
$$I_t = \frac{\frac{Y_{t-1}-Y_{t-2}}{v}}{\frac{Y_{t-1}-Y_{t-2}}{v} + q}$$

Investment Curve

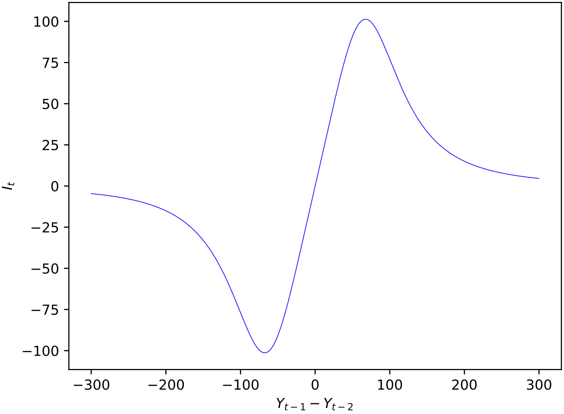

Figure 6: Investment curve such that $v = 500$ and $q = 0.001$

Maximal/Minimal Investment

$$Y_{t-1} - Y_{t-2} = \pm \frac{q^{1/4} v}{3^{1/4}}$$

$$I_t = \pm \frac{3^{3/4}}{3q^{3/4}}$$

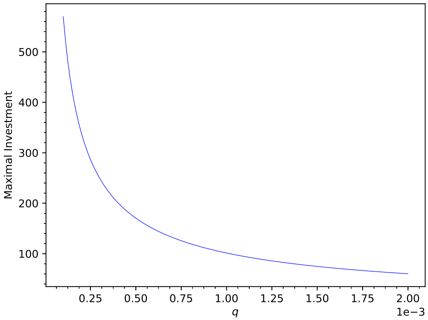

Figure 7: Plot of maximal investment relative to q.

Investment FWHM

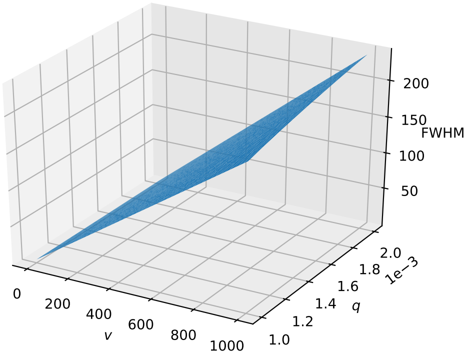

Figure 8: Triangulated plot investment FWHM relative to v and q.

Inventory

- Predicted consumption, U, is an average of lagged consumption
- Evolution of stock output, S, and actual inventory, Q, behaves identically to the Wegener model

$$U_t = \frac{C_{t-1} + C_{t-2} + C_{t-2}}{3}$$

$$S_t = kU_t - Q_{t-1}$$

$$Q_t = Q_{t-1} + S_t + (U_t - C_t)$$

Output Growth

Endogenous investment allows for long-run growth or decay

$$Y_t = I_t + S_t + U_t$$
$$\dot{Y}_t = Y_t - Y_{t-1}$$

Output Growth

$$\dot{Y}_t = \frac{\frac{\dot{Y}_{t-1}}{v}}{\left(\frac{\dot{Y}_{t-1}}{v}\right)^4 + q} - \frac{\frac{\dot{Y}_{t-2}}{v}}{\left(\frac{\dot{Y}_{t-2}}{v}\right)^4 + q} + \frac{k+1}{3}\left[(1-s)(\dot{Y}_{t-2} - Y_{t-5}) + s(\dot{Y}_{t-3} - Y_{t-6})\right] + (1-s)\dot{Y}_{t-2} + s\dot{Y}_{t-3}$$

Possible Trajectories

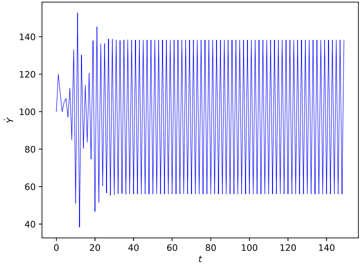

Figure 9: Timeseries plot with parameters: $s = 0.6$, $k = 0.3$, $v = 500$, $q = 0.001$. Initial values of $\dot{Y}$ are: 100, 120, 110, 100, 105, 107

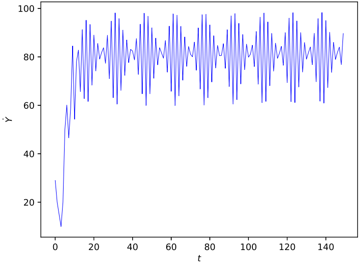

Figure 10: Timeseries plot of income growth rate over 150 iterations. Parameters identical to Fig. 9. Initial values of $\dot{Y}$ are: 29, 20 , 15 , 10, 20, 50

Determining Chaos

$$
\begin{aligned}
\dot{Y}_t &= f(\dot{Y}_{t-1}, \dot{A}_{t-1}, \dot{B}_{t-1}, \dot{C}_{t-1}, \dot{D}_{t-1}, \dot{G}_{t-1}) \\
\dot{A}_t &= g(\dot{Y}_{t-1}, \dot{A}_{t-1}, \dot{B}_{t-1}, \dot{C}_{t-1}, \dot{D}_{t-1}, \dot{G}_{t-1}) \\
\dot{B}_t &= h(\dot{Y}_{t-1}, \dot{A}_{t-1}, \dot{B}_{t-1}, \dot{C}_{t-1}, \dot{D}_{t-1}, \dot{G}_{t-1}) \\
\dot{C}_t &= j(\dot{Y}_{t-1}, \dot{A}_{t-1}, \dot{B}_{t-1}, \dot{C}_{t-1}, \dot{D}_{t-1}, \dot{G}_{t-1}) \\
\dot{D}_t &= l(\dot{Y}_{t-1}, \dot{A}_{t-1}, \dot{B}_{t-1}, \dot{C}_{t-1}, \dot{D}_{t-1}, \dot{G}_{t-1}) \\
\dot{G}_t &= m(\dot{Y}_{t-1}, \dot{A}_{t-1}, \dot{B}_{t-1}, \dot{C}_{t-1}, \dot{D}_{t-1}, \dot{G}_{t-1})
\end{aligned}
$$

Determining Chaos

$$J = \begin{bmatrix} \frac{\partial f}{\partial Y_{t-1}} & \frac{\partial f}{\partial A_{t-1}} & s + \frac{k+1}{3}s & 0 & \frac{(k+1)(s-1)}{3} & -\frac{(k+1)s}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial \dot{Y}_{t-1}} = \frac{1}{v\left(q + \frac{Y_{t-1}^4}{v^4}\right)} - \frac{4Y_{t-1}^4}{v^5\left(q + \frac{Y_{t-1}^4}{v^4}\right)}$$

$$\frac{\partial f}{\partial \dot{A}_{t-1}} = 1 + \frac{1}{3}(k+1)(1-s) - s + \frac{4A_{t-1}^4}{v^5\left(q + \frac{A_{t-1}^4}{v^4}\right)^2} - \frac{1}{v\left(q + \frac{A_{t-1}^4}{v^4}\right)}$$

Calculating the MLE

$$\lambda = \lim_{j \to \infty} \frac{1}{j} \sum_{j=1}^{t=j} \ln |J_t \cdot v_t|$$

$$v_{t+1} = \frac{J_t \cdot v_t}{|J_t \cdot v_t|}$$

Propensity to Consume: s

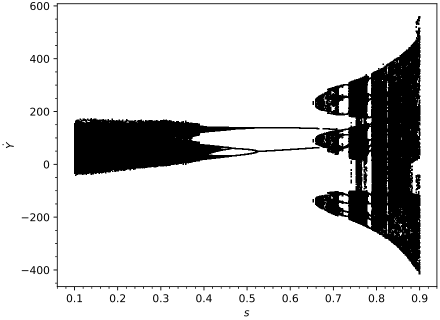

Figure 11: Bifurcation diagram varying s

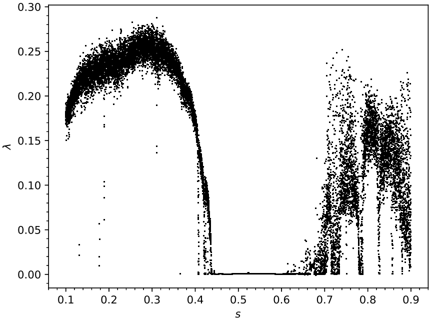

Figure 12: Lyapunov plot varying s

Investment (FWHM): Measured by v

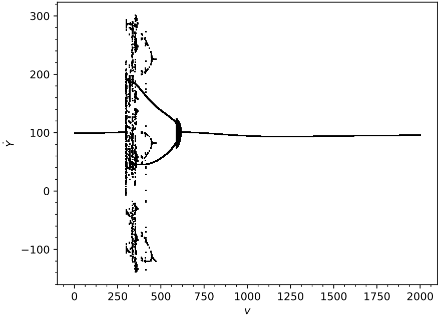

Figure 13: Bifurcation diagram varying v

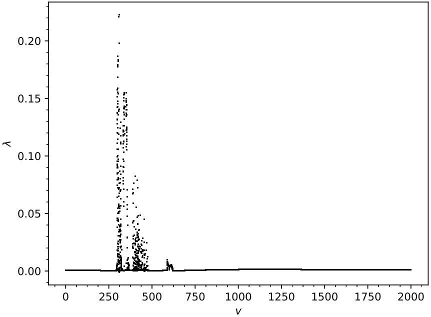

Figure 14: Lyapunov plot varying v

Investment (Maximum/Minimum): Measured by q

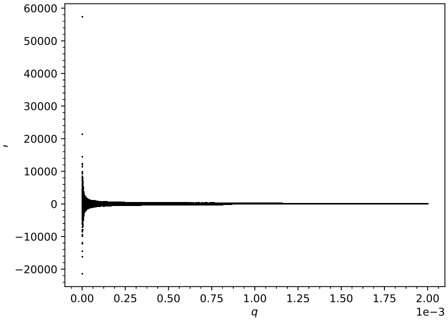

Figure 15: Bifurcation diagram varying q

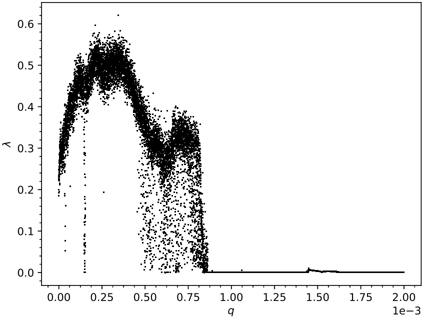

Figure 16: Lyapunov plot varying q

Sensitivity to Initial Conditions

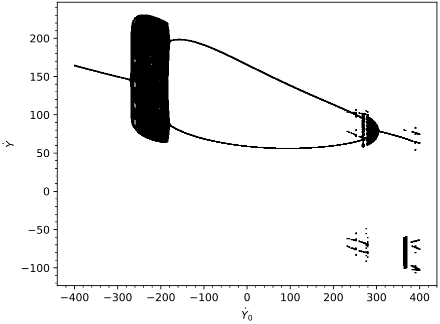

Figure 17: Bifurcation diagram varying $\dot{Y}_0$

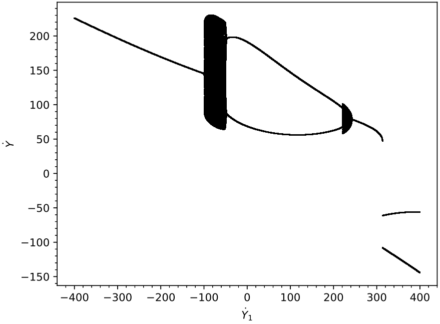

Figure 18: Bifurcation diagram varying $\dot{Y}_1$

Sensitivity to Initial Conditions

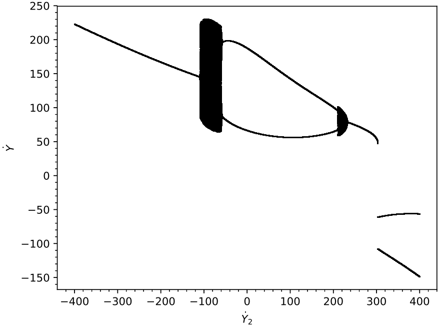

Figure 19: Bifurcation diagram varying $\dot{Y}_2$

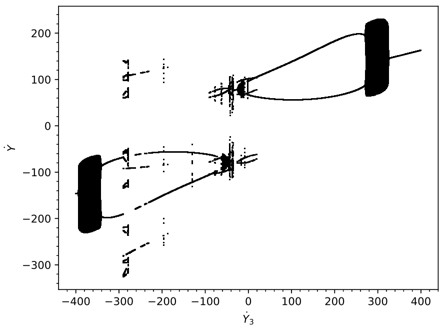

Figure 20: Bifurcation diagram varying $\dot{Y}_3$

Sensitivity to Initial Conditions

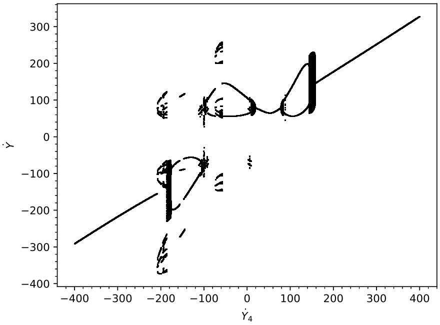

Figure 21: Bifurcation diagram varying $\dot{Y}_4$

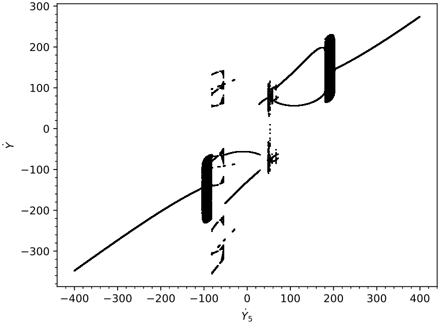

Figure 22: Bifurcation diagram varying $\dot{Y}_5$

Next steps

- Further investigate the fractal structure of the strange attractors
- Open the economy to foreign trade and improve the investment mechanism
- Improve consumption prediction mechanism

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