{Working Title}

by
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A thesis submitted in partial fulfillment
of the requirements for the
Degree of Bachelor of Arts with Honors
in Chemistry

Williams College
Williamstown, Massachusetts

December 18, 2018

Acknowledgements

I would like to express my gratitude to my advisor, Professor Enrique Peacock-López for supporting me throughout my thesis.

Abstract

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Contents

1. Introduction

The study of dynamical systems is traditionally thought to have begun with the publication of "New methods of Celestial Mechanics" by Poincaré and expanded with the work of Lyapunov into a theory of the stability of dynamical systems. It was not until the 1960s however that the use of chaos and stability theory exploded across disciplines.[1]

Dynamics are typically an unnecessary tool when studying the paths of chemical reactions. Their value became apparent however, when Belousov and later Zhabotinsky released published their work on an oscillatory reaction, a reaction that would later come to be known as the BZ reaction.[2] Cycles were long known to exist in the biochemical realm with many famous pathways in organisms such as the Krebs cycle and Calvin cycle; however the BZ reaction was developed to create an inorganic analogue to the Krebs cycle. The development of this cycle allowed for a relatively easily replicable cyclic reaction with with easily measurable indicators of the progress of the reaction.

The study of dynamical chemical systems has since expanded to a variety of other mechanisms such as self-replicating molecules in addition to studying fractal patterns and dimensionality involved in electrochemical deposits and flame patterns. Despite the fairly wide range of background information required to set up these different models, the underlying mathematical theory used to study these models is identical which has contributed to the wide range of interdisciplinary work performed by theorists in the field.

1.1 Background on Dynamical Systems

Traditional dynamical systems are modelled in continuous time as a system of ordinary differential equations. These systems typically treat time as the singular independent variable and solve for the evolution of one or more variables in terms of time. A classic continuous time dynamical system is the simple pendulum which allows one to model the movement of a pendulum in space in terms of time. This model uses a variety of simplifying assumptions in order to reduce the problem of the pendulum into a single variable function, holding the length of the pendulum and acceleration due to gravity constant.

$$\frac{d^2}{dt^2} + \frac{g}{L}\sin\theta = 0 \tag{1.1}$$

Most attempts at modelling real-world systems however require the use of multiple dependent variables in order to effectively model. However, as the amount of variables increases, the complexity of the model increases. Modelling and n-body system acting on each other gravitationally is of obvious interest in astronomy; however, it was quickly found that although a 1 and 2 body system were relatively simple to solve for, the introduction of a third body caused significant complications. The 3-body system is in fact what Poincaré studied in order to develop a theory of chaotic deterministic systems.[3]

That is not to say that these systems cannot arrive at ordered solutions. Although continuous time systems require a minimum of 3 variables in order for chaos to arise, there are typically windows of order in chaotic regimes that allow for stable, oscillatory behavior. The BZ-reaction is still being actively studied and is known to be actually highly complex but is often reduced into 7 primary sub-reactions.[4] A great deal of the work involved in studying dynamical systems is actually on finding ways to simplify models in order to arrive at more mathematically tractable systems. The BZ-reaction has been simplified to a 3-variable system that still provides the complex periodicity and chaotic behavior characteristic of the model.[5]

1.2 Discrete Time Dynamical Systems

Although many processes in the real-world are more intuitively interpreted as operating as a continuous time function, there are many occasions where it is possible and in fact beneficial to think in a discrete-time sense. The prevalence of this type of system varies depending on the exact field of study; however, it is important to note that when computationally modelling continuous time systems, it is impossible for computers to truly operate with continuous variables and thus even these systems are reduced to technically discrete models.

Population dynamics are frequently analyzed as a discrete time system as opposed to continuous time. It is often of more practical use to interpret $t = 0, 1, 2, 3, ...$ as the change in population per year or per season as opposed to determining the change in population over infinitesimally small changes in time. In terms of technique, many of the mathematical principles used in analyzing dynamic systems in continuous time apply to discrete time systems; however, it would be a mistake to assume the two were identical. An important distinction between the two is the nature by which chaos can occur. As described previously, a continuous time system requires 3 or more dependent variables in order for chaos to occur. A discrete time system only requires 1 variable in order to display the same type of chaotic behavior.

The systems discussed throughout this paper will be of the discrete variety due to their nature. Laws pertaining to the physical world are scalable to the infinitesimal degree which allows for their use in continuous models. Economic models do not have a basis in physical laws. It is also important to note that, due to the complexity involved in the human behavior that economic models are trying predict, the exact numerical values of the model are typically of minor concern. The general behavior of the model is significantly more valuable in order to determine the effects of an economic assumption.

1.3 The Logistic Map, A Mathematical Introduction

The logistic map is regarded as the prototypical chaotic discrete time mapping. The logistic function, which the logistic map is based off of, was developed to study population dynamics but actually garnered widespread use in other disciplines such as the study of autocatalytic reactions, computer science, statistics, and economics.[6]

$$\frac{d}{dx}f(x) = f(x)(1 - f(x)) \tag{1.2}$$

The logistic function has 2 equilibria or points where the derivative of the function is 0. $f(x) = 0$ is an unstable equilibrium but $f(x) = 1$ is a stable equilibrium point which means that other points on the function will tend towards this equilibrium overtime. This can be realized by solving for the derivative of the function at points when $f(x) \in (0, 1)$ which is universally positive and $f(x) \in (1, \infty)$ which is negative. Integrating the differential equation gives the general form equation:

$$f(x) = \frac{e^x}{e^x + C} \tag{1.3}$$

This function gained prominence due to its rapid, exponential growth when $f(x)$ is low and its slow, linear decaying to non-existent growth as population increases. Used by notable mathematicians operating in the field of population dynamics such as Verhulst, Pearl, and Lotka, the model continues to be widely used today and is often the basis upon which other modifications are applied.[7]

$$x_{t=1} = \mu x_t(1 - x_t) \tag{1.4}$$

The logistic map is a difference equation model popularized by Robert May as a discrete time analogue to the logistic function.[8] When interpreted in the biological context, x_t refers to the ratio of the population at time t compared to the maximal population, thus the mapping is bounded between 0 and 1. Here we see intuitively the major difference between difference equation and differential equation based systems. Differential equations solve for the derivative of a variable with respect to time in terms of the variable, difference equations solve for the actual state of the variable in the successive state provided we know the state of the variable in the previous time period. Much like how differential equations can be of higher order with the introduction higher order derivatives, a difference equation can also be of higher order by including more time periods in the function for the state of the variable which is valuable in a variety of the models discussed later.

Much like how the equilibrium points were solved for in the differential equation, difference equations also have equilibrium points where $x_{t+1} = x_t$. Interestingly, unlike the logistic function, there does not exist a fixed point at $x_t = 1$ as this would result in $x_{t+1} = 0$. Solving for the fixed points, we have:

$$x_{t+1} = x_t = 0, \frac{\mu - 1}{\mu} \tag{1.5}$$

The stability of a fixed point is again dependent on the derivative of the function; however, there are differences in the details of our analysis. Treating $f(x) = x_{t+1}$, we see the derivative of the logistic map is:

$$f'(x) = \mu(1 - 2x) \tag{1.6}$$

For reasons that will become clear later, the stability of a point on the map requires that $|f'(x)| < 1$. Solving for when this is true for our two fixed points, we see that $\mu < 1$ provides stability for the origin fixed point and $1 < \mu < 3$ gives stability for the non-zero fixed point. Thus, provided the parameter μ satisfies either of the conditions set previously, it will converge to one of the fixed points in a relatively small, finite amount of iterations.

This behavior can be visualized using a cobweb diagram. This diagram consists of 3 primary elements: a plot of the mapping, a 45° line, and a plot of the variable's trajectory. An example of a cobweb diagram can be seen in Figure 1.1. This diagram shows the trajectory of x starting at a value of 0.1 when there is a stable, non-trivial fixed point.

The 45° line is defined as the line where $x_t = x_{t+1}$ which is useful for determining the result of successive iterations. Beginning from the point x_0, we can then determine

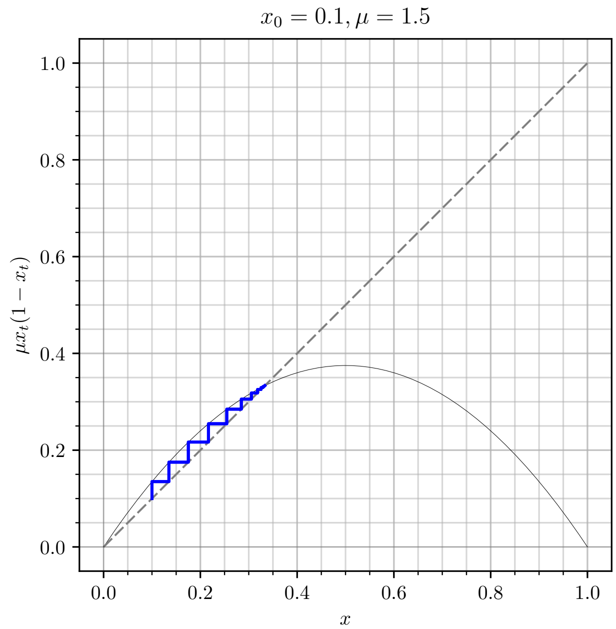

Figure 1.1: Cobweb plot of the logistic map setting $\mu = 1.5$ and $x_0 = 0.1$. The trajectory asymptotically approaches the equilibrium point of $\frac{1}{3}$.

what point x_1 wll be via the mapping. We can then look horizontally to the 45° line until we intersect with it. The x-coordinate of this intersection point is equivalent to the result of the mapping of the previous iteration, thus using this new point will allow us to determine the result of the next iteration of the function. This process can be repeated ad infinitum; however, the result will soon prove uninteresting for stable points and orbits as the trajectory will converge and repeat its behavior.

The reason the logistic map is so frequently studied is because of its ability to exhibit complex behavior beyond a stable equilibrium solution. Once $\mu > 3$, the mapping enters a cyclic region. Much like how fixed points could be solved for by identifying where $x_t = x_{t+1}$, stable oscillatory points can be found by solving for the equilibrium points of higher iterations of the function. A 2-cycle will be such that $x_t = x_{t+2} \neq x_{t+1}$ for example and the stability of a such a cycle can be found using the same methodology as described previously. The logistic map also provides a mechanism to more quantitatively describe what it means for a system to be chaotic. The Lyapunov exponent, named after one of major driving forces in the development of stability analysis, is used to measure the effect of small perturbations in initial conditions on the trajectory of the variable.[9] Conceptually, the logistic map and the systems discussed in this paper are determinisitic. However, chaotic systems have highly divergent trajectories with even small changes in

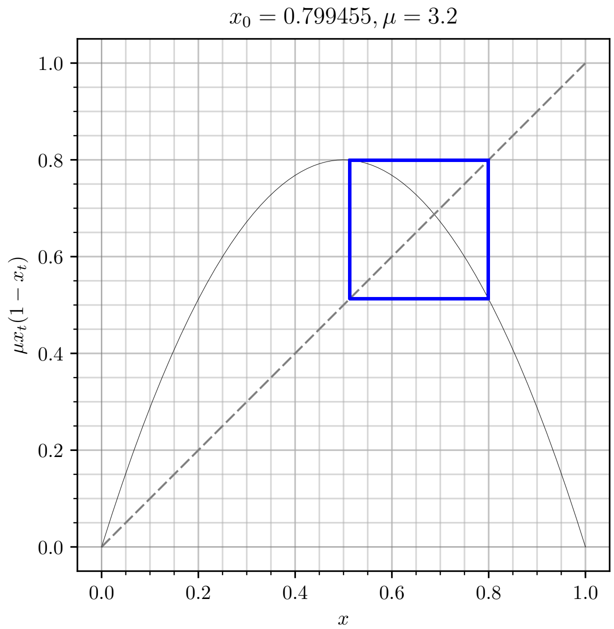

Figure 1.2: 2-period cycle of the logistic map showing only the cyclic behavior.

their initial conditions; thus knowing approximately what the initial conditions are does not provide approximate information on the trajectory of the variable.

In order to quantify this, we begin by taking the absolute value of the derivative of the function as this allows us to effectively magnify the effect of an infinitesimal change in the intial conditions. We then take the natural logarithm of this derivative in order to measure the exponential rate of separation of trajectories. Finally, we take the average separation over an arbitrarily high number of iterations n as exponential separation is not necessitated over all phase space. This gives us a the equation:

$$\lambda_n(x_0) = \frac{1}{n} \sum_{t=1}^{t=n} \ln|f'(x_{t-1})| \tag{1.7}$$

where $\lambda_n(x_0)$ is the lyapunov exponent for a given intial point when allowed to run for n iterations. The true value of the lyapunov exponent is the limit of the infinite series as $n \to \infty$ divided by n; however, the complexity of these maps often makes it practically impossible to analytically solve for the limit. By choosing arbitrarily high values of n though, it is possible to achieve better approximations at the expense of computational time. It is also important to note that, although the lyapunov exponent is a function of the initial condition, as long as the initial state is not in some stable fixed point or cycle, the trajectories will follow that of the chaotic attractor, thus the value of the lyapunov

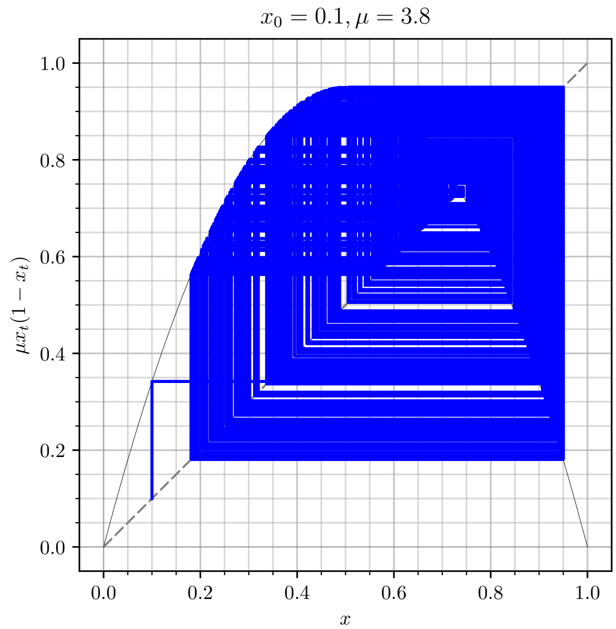

Figure 1.3: Chaotic behavior in the logistic map.

exponent should be mostly consistent regardless of the choice of initial conditions.

Another way visually see the behavior is with a bifurcation diagram. This diagram shows the longterm behavior of the variable for a given variable. Figure 1.5 qualitatively shows the behavior described previously. For parameter values between 0 and 1, we see the origin fixed point is stable. For parameter values between 1 and 3, their is still a single fixed point that is monotonically increasing; however, we can also clearly see the beginning of the 2-period cycle once the parameter exceeds 3. It is difficult however, to determine when predictable higher-order cyclic behavior ends and chaotic behavior begins via qualitative observation of the bifurcation diagram. The benefit of the bifurcation diagram is that it allows us to see both where the bifurcation points are and what behavior the bifurcation points signify. Bifurcation points are where where infinitesimally small quantitative changes in the parameter induce significant qualitative or topological change in the behavior of the mapping such as the transition from a stable fixed point to a 2-period cycle.

Research on the logistic map and other iterated maps has shown the existence of what is called Feigenbaum's constant. This constant can be found by observing the behavior of the periodic cycles of the map. The interval of stability decreases and the ratio of subsequence intervals actually approaches a limit $\delta \approx 4.6692$.[9] All other topologically similar maps with a single local maximum share this Feigenbaum constant. Once the

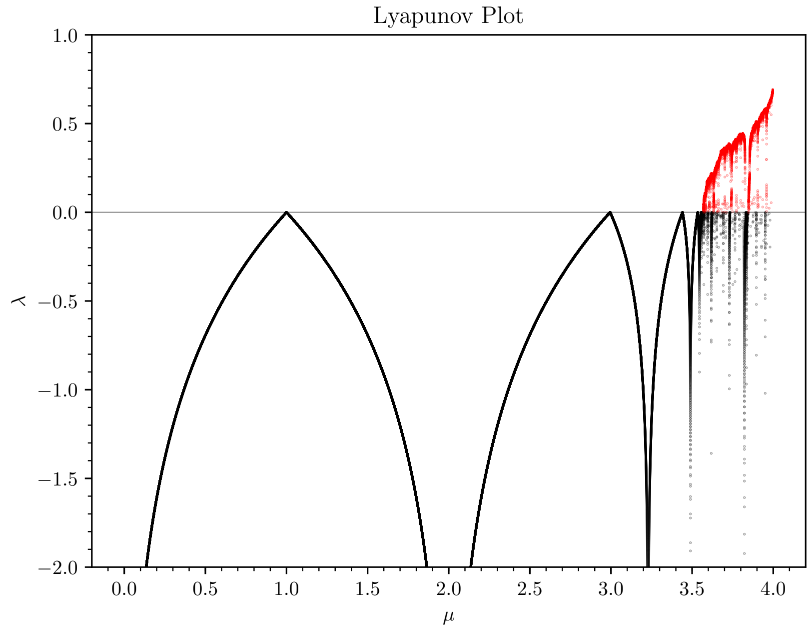

Figure 1.4: Lyapunov exponent plotted against μ for the logistic map. Initial value of 0.1 is used. Red denotes regions where $\lambda \geq 0$, black denotes regions where $\lambda < 0$.

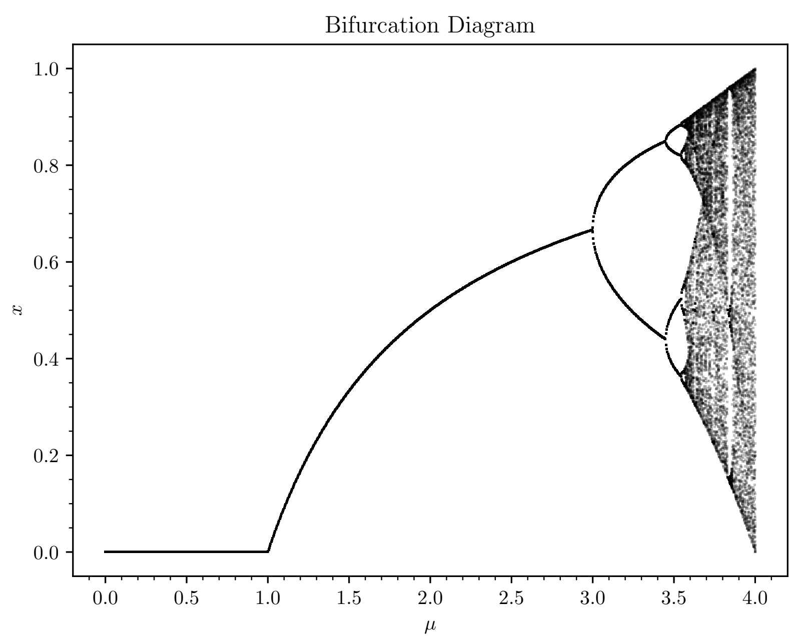

Figure 1.5: Bifurcation diagram plotting x against μ for the logistic map.

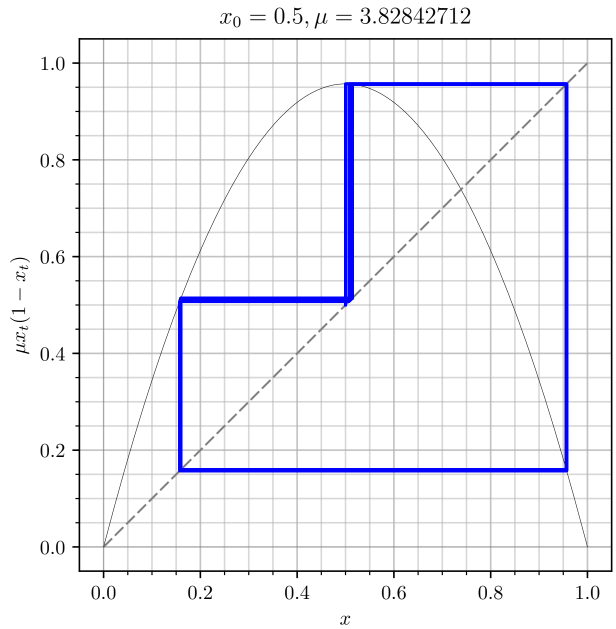

Figure 1.6: 3-period cycle of the logistic map showing only the cyclic behavior.

mapping exceeds this constant, chaos occurs which allows for another method to determine precisely where chaotic behavior occurs.

It is also beneficial to point out that another mechanism exists for cyclic behavior to exist. The previously described method involved taking a mapping and solving for the stability of its double iteration. This allows for $2n$-period stability cycles to exists. However, there does exist odd-ordered cycles such as the 3 cycle; however, it occurs as a window of order in the region of chaos. These windows can be seen in Figure 1.4 where the Lyapunov exponent dips into the negative region past the chaotic bifurcation point. These bifurcation points are known as tangent bifurcations This also allows us to use Sharkovsky's Theorem, which states that any continuous mapping with a 3-period cycle must also have every n-period cycle for every $n \in \mathbb{Z}$.[9] A variety of other mathematical techniques exist to study the dynamics of difference equation mappings but these will be covered more specifically when used for the specific case.

2. The Multiplier-Accelerator Model

2.1 Background

John Maynard Keynes work revolutionized economic thought; however, he never formalized any of his theories into a mathematical theory. This was done over several decades in a process known as the Neo-classical synthesis which sought to connect traditional, classical models with newer Keynesian ideas.[10] In 1939, Paul Samuelson developed a popular model that could display endogenous, business-cycle behavior which married Keynes' investment multiplier theory with Clarke's accelerator principle.[9,11]

The model is often criticized for the amount of simplifying assumptions it makes for the sake of mathematical simplicity. It is for this reason that later models feature significantly more mechanisms at play; however, this model is still useful for studying how cyclic behavior can be derived endogenously purely through economic fundamentals.

2.1.1 Accelerator Theory and Investment

Key to the idea of the accelerator effect is that growth has a positive effect on the level of investment. Economic growth features an overall increase in business profits and business confidence, thereby resulting in increases in fixed investments to grow business further. However, a recession decreases business profits and confidence and damages their ability and willingness to invest for the future.

Earlier implementations of this theory used a simple, linear function to represent the relationship between investment and income. This however, was questioned on its applicability to real world behavior as it implied that business actively destroyed preexisting investments if income declined faster than the natural rate of capital depreciation. In the 1950s, Hicks suggested a linear, piecewise function with upper and lower bounds.[9] The lower bound represented the natural rate of capital depreciation thereby giving a minimal level of investment loss. The upper bound was rationalized to be a result of decreasing marginal gains to productivity from investment as land, labor, and raw materials became limiting factors to production.

Richard Goodwin found a different solution in the form of a hyperbolic tangent function that asymptotically approached the limits of the piecewise function. This function was differentiable at all points which allows us to derive the form of the function used for our multiplier-accelerator model. Our investment function is a linear-cubic taylor series expansion of the hyperbolic tangent function which introduces a fundamental change in the function by causing it to backbend to 0 instead of asymptotically approaching a non-zero limit. This can be rationalized by introducing counter-cyclic government economic policy. During a recession, governments will increase spending in an attempt to counteract market behavior and stimulate the economy. Likewise, governments will take advantage of expansionary periods by increasing taxes and reducing government projects relative to recessionary periods allowing the government to possess the necessary funds to sustain investments come the next recessionary period. Our investment function can thus be thought of as being representative of both public and private investments.[9]

Investments are thus treated as a function of the change in income in the past with lag introduced, making it a second-order difference equation of the linear-cubic form:

$$I_t = \mu(Y_{t-1} - Y_{t-2}) - \mu(Y_{t-1} - Y_{t-2})^3 \tag{2.1}$$

2.1.2 The Keynesian Multiplier and Consumption

The Keynesian multiplier allows us to derive a relationship between income and the level of consumption. Much like investment, the model incorporates a lag into the reaction of consumption to income; however, this is now rationalized as a result of consumer's propensity to save and consume. This model holds propensity as a fixed parameter s and sets propensity to consume as $1 - s$. However, a key simplifying assumption made is by only allowing savings to last for a time period before being consumed. This is accomplished by the 2nd order difference equation:

$$C_t = (1 - s)Y_{t-1} + sY_{t-2} \tag{2.2}$$

We can thus see that consumption is composed of two contributions, a 1-period delayed contribution due to the propensity to consume and a 2-period delayed contribution due to the propensity to save.

2.2 Stability and Chaos in Income Dynamics

This economy does not store any income in an unproductive manner. Moreover, by effectively wrapping public investments in the cubic investment function and closing the economy from exports and imports, we reduce the outlets of income into two possible paths: consumption and investment.

$$Y_t = C_t + I_t \tag{2.3}$$

Incorporating our definitions of consumption and investment, we are able solve for future change in income as a function of previous change in income:

$$Y_t - Y_{t-1} = (\nu - s)(Y_{t-1} - Y_{t-2}) - \nu(Y_{t-1} - Y_{t-2})^3$$

As we are most directly concerned with change in income, we can define a new variable:

$$Y_t - Y_{t-1} = Z_{t-1}$$

Thereby allowing us to simplify change in income to a first order difference equation:

$$Z_t = \mu Z_{t-1} - (\mu + 1)Z_{t-1}^3 | \mu = \nu - s \tag{2.4}$$

This is achieved due to a key economic feature of our parameter ν. The value of ν is dependent on the choice of measure of income, it is thus linearly scalable which allows the model to be rescaled to this new parameter μ as this rescaling has no effect on the linear term.

This rescaling allows us to significantly simplify analysis of the model as forces the model to pass through the points (0,0), (-1,1), and (1,-1).

Solving for fixed points, we have the trivial point $Z = 0$; however, there also exists two non-origin points:

$$Z = \pm\sqrt{\frac{\mu - 1}{\mu + 1}}$$

Both of these points are stable in the region $0 < \mu < 2$ such that a positive initial value will tend towards the positive the positive fixed point while a negative initial value will tend towards the negative fixed point. This can be interpreted as having the economy face steady growth or decay ad infinitum which does not describe typical business cycle behavior.

Setting $\mu > 2$, the fixed points lose stability and we then see the onset of stable cycles.

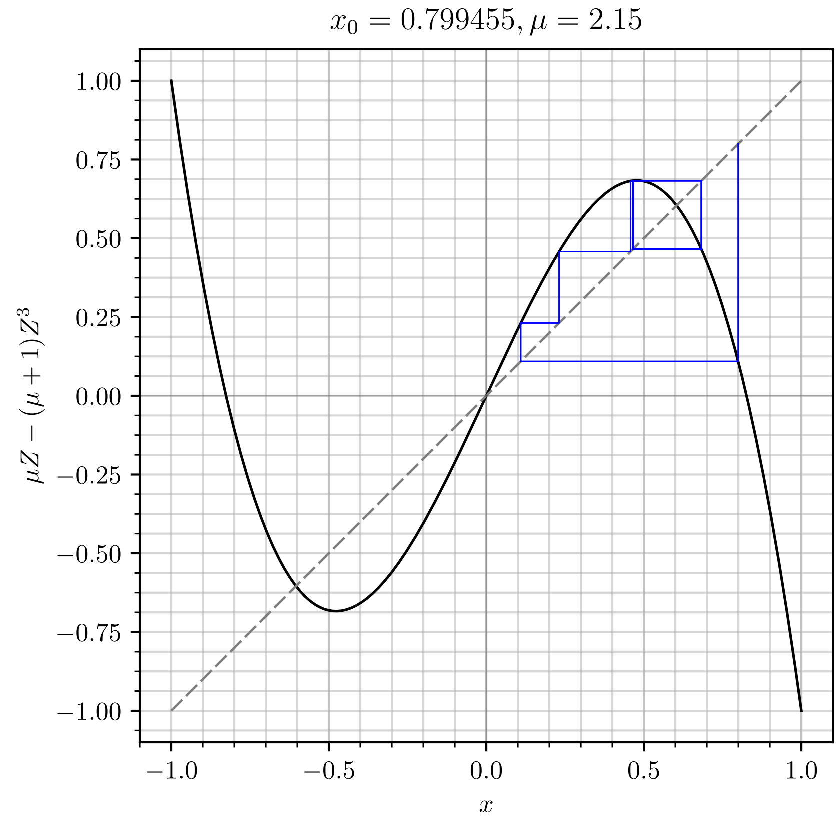

Figure 2.1: Cyclic growth behavior in the multiplier-accelerator model. As $Z > 0$ for all iterations, the economy is perpetually growing albeit at an unsteady rate.

However, it is important to note that these cycles feature oscillating levels of growth in the case of the positive domain or oscillating levels of decay in the negative domain. As the cycles do not oscillate between the positive and negative domains, the economy will still always either grow or decay depending on its initial state and parameter

The feigenbaum point of this map occurs when $\mu \sim 2.402$ which indicates the boundary point where chaos dominates the mapping. However, the chaotic behavior is still bounded in in one quadrant over successive iterations. Solving for the parameter value such that the maximal of the mapping is also the zero of the function allows us to define the boundary where the mapping's behavior is no longer bounded over successive iterations. The maximal of this mapping occurs at:

$$Z = \pm\frac{1}{\sqrt{3}}\sqrt{\frac{mu}{\mu+1}}$$

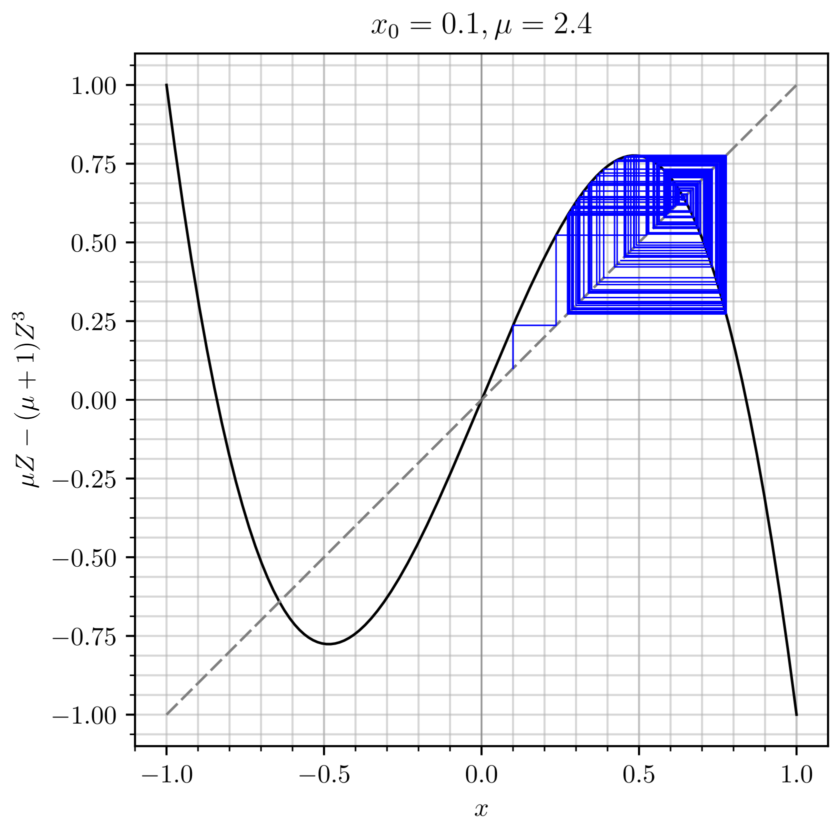

Figure 2.2: Chaotic behavior in the multiplier-accelerator model contained in the positive quadrant.

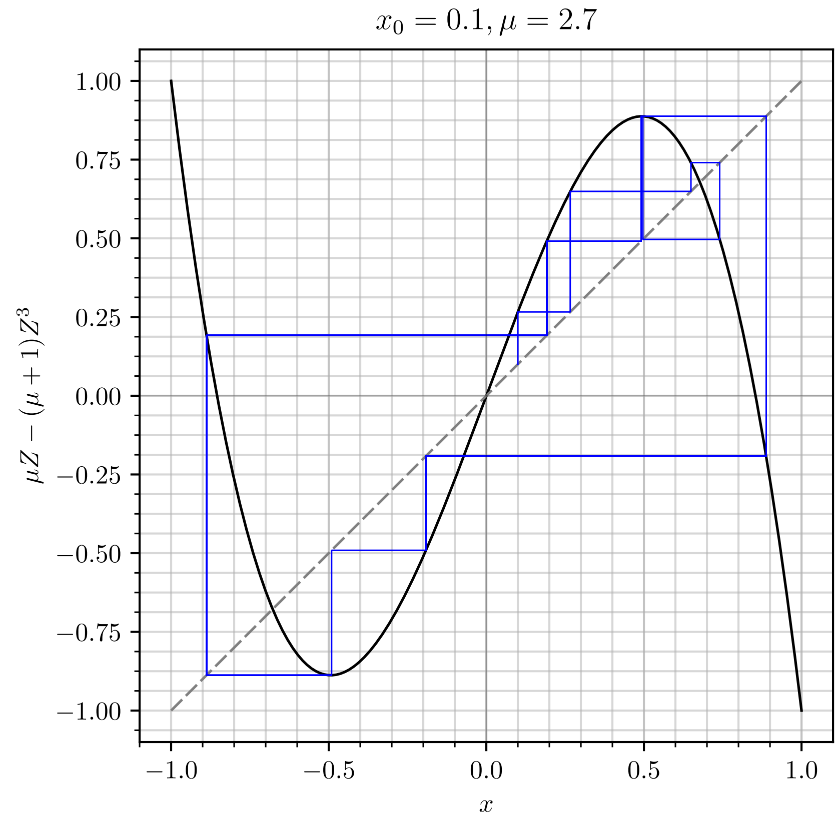

Figure 2.3: Stable 6-period cycle in the multiplier-accelerator model featuring growth and decay.

This is equivalent to the zero of the mapping when

$$\mu = \frac{3\sqrt{3}}{2} \sim 2.5981$$

This point marks when the economy can begin actually facing cyclic growth and decay; however, as the behavior of the mapping is chaotic for most of this region, it does not feature regular shifts between growth and decay.

However, chaotic regions often features windows of order as is the case of this particular mapping when $\mu = 2.7$ which features a stable, attractive cycle of 6 periodicity that is still able to feature growth and decay. Setting $\mu > 3$ sees the model explode as the maximum exceeds the box set. This features unstable levels of extremely high magnitude growth and decay after a few relatively few iterations and is thus seen of little use in terms of applying to real-world business cycle dynamics.

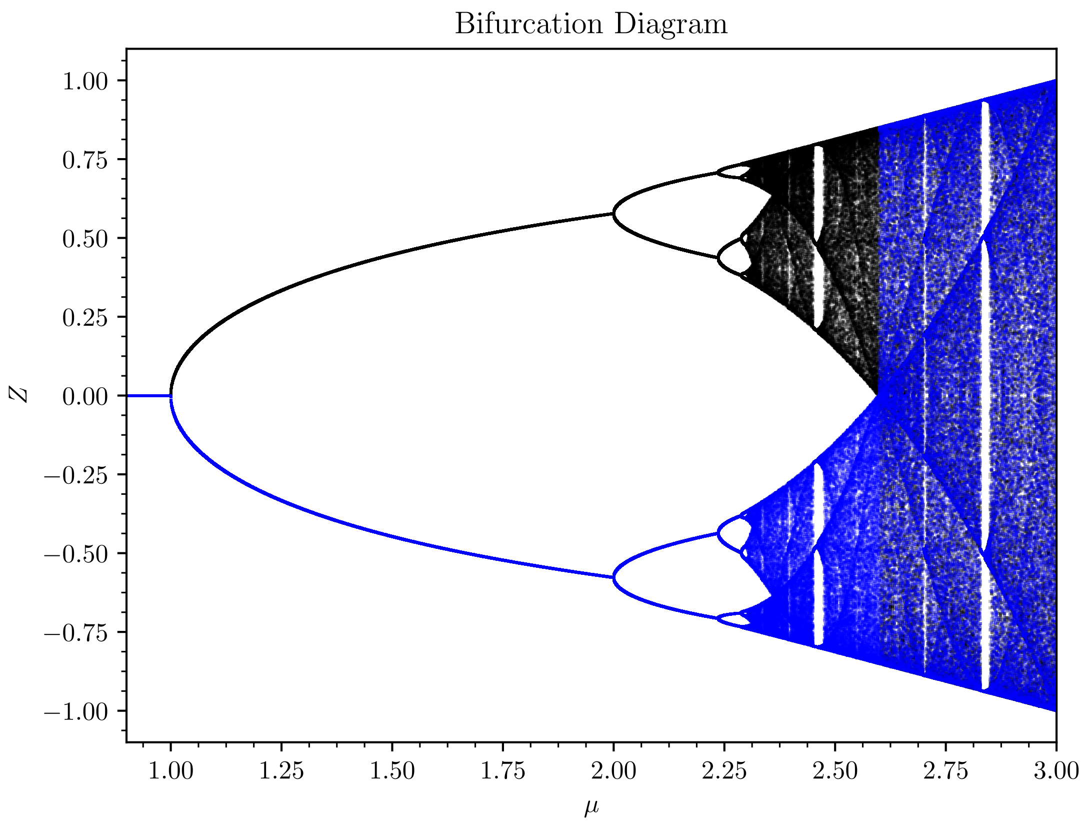

Figure 2.4: Bifurcation diagram of multiplier accelerator model. Black denotes plot with an initial value of 0.1, blue denotes plot with an initial value of -0.1. Convergence point denotes parameter value such that mapping can cross between quadrants.

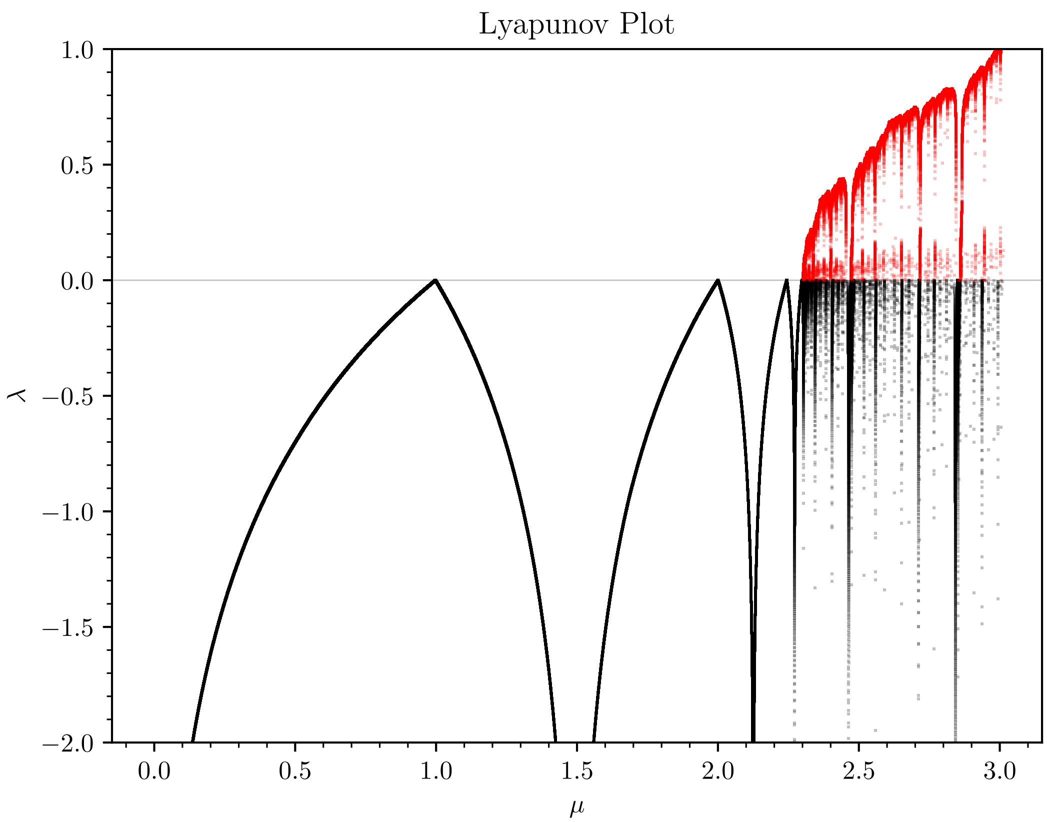

Figure 2.5: Lyapunov exponent plotted against μ for the samuelson-hicks. Initial value set to 0.1. Red denotes regions where $\lambda \geq 0$, black denotes regions where $\lambda < 0$

3. Conclusion

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