{Working Title}

by
Benjamin Bui

Professor Peacock-Lopez, Advisor

A thesis submitted in partial fulfillment
of the requirements for the
Degree of Bachelor of Arts with Honors
in Chemistry

Williams College
Williamstown, Massachusetts

March 5, 2019

Acknowledgements

I would like to express my gratitude to my advisor, Professor Enrique Peacock-López for supporting me throughout my thesis.

Abstract

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum. Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Contents

1. Introduction

The study of dynamical systems is traditionally thought to have begun with the publication of "New methods of Celestial Mechanics" by Poincaré and expanded with the work of Lyapunov into a theory of the stability of dynamical systems. It was not until the 1960s however that the use of chaos and stability theory exploded across disciplines.[1]

Dynamics are typically an unnecessary tool when studying the paths of chemical reactions. Their value became apparent however, when Belousov and later Zhabotinsky released published their work on an oscillatory reaction, a reaction that would later come to be known as the BZ reaction.[2] Cycles were also known to exist in the biochemical realm with many famous pathways in organisms such as the Krebs cycle and Calvin cycle; however the BZ reaction was developed to create an inorganic analogue to the Krebs cycle.[3] The development of this cycle allowed for a relatively easily replicable cyclic reaction with with easily measurable indicators of the progress of the reaction.

The study of dynamical chemical systems has since expanded to a variety of other mechanisms such as self-replicating molecules[4] in addition to studying fractal patterns and dimensionality involved in electrochemical deposits[5] and flame patterns.[6] Despite the fairly wide range of background information required to set up these different models, the underlying mathematical theory used to study these models is identical which has contributed to the wide range of interdisciplinary work performed by theorists in the field.

1.1 Background on Dynamical Systems

Traditional dynamical systems are modelled in continuous time as a system of ordinary differential equations. These systems typically treat time as the singular independent variable and solve for the evolution of one or more variables in terms of time. A classic continuous time dynamical system is the simple pendulum which allows one to model the movement of a pendulum in space in terms of time. This model uses a variety of simplifying assumptions in order to reduce the problem of the pendulum into a single variable function, holding the length of the pendulum and acceleration due to gravity constant.

$$\frac{d^2}{dt^2} + \frac{g}{L}\sin\theta = 0 \tag{1.1}$$

Most attempts at modelling real-world systems however require the use of multiple dependent variables in order to effectively model. However, as the amount of variables increases, the complexity of the model increases. Modelling and n-body system acting on each other gravitationally is of obvious interest in astronomy; however, it was quickly found that although a 1 and 2 body system were relatively simple to solve for, the introduction of a third body caused significant complications. The 3-body system is in fact what Poincaré studied in order to develop a theory of chaotic deterministic systems.[7]

That is not to say that these systems cannot arrive at ordered solutions. Although continuous time systems require a minimum of 3 variables in order for chaos to arise, there are typically windows of order in chaotic regimes that allow for stable, oscillatory behavior. The BZ-reaction is still being actively studied and is known to be actually highly complex but is often reduced into 7 primary sub-reactions.[3] A great deal of the work involved in studying dynamical systems is actually on finding ways to simplify models in order to arrive at more mathematically tractable systems. The BZ-reaction has been simplified to a 3-variable system that still provides the complex periodicity and chaotic behavior characteristic of the model.[8]

Although many processes in the real-world are more intuitively interpreted as operating as a continuous time function, there are many occasions where it is possible and in fact beneficial to think in a discrete-time sense. The prevalence of this type of system varies depending on the exact field of study; however, it is important to note that when computationally modelling continuous time systems, it is impossible for computers to truly operate with continuous variables and thus even these systems are reduced to technically discrete models.

Population dynamics are frequently analyzed as a discrete time system as opposed to continuous time. It is often of more practical use to interpret $t = 0, 1, 2, 3, ...$ as the change in population per year or per season as opposed to determining the change in population over infinitesimally small changes in time. In terms of technique, many of the mathematical principles used in analyzing dynamic systems in continuous time apply to discrete time systems; however, it would be a mistake to assume the two were identical. An important distinction between the two is the nature by which chaos can occur. As described previously, a continuous time system requires 3 or more dependent variables in order for chaos to occur. A discrete time system only requires 1 variable in order to display the same type of chaotic behavior.

The systems discussed throughout this paper will be of the discrete variety due to their nature. Laws pertaining to the physical world are scalable to the infinitesimal degree which allows for their use in continuous models. Economic models do not have a basis in physical laws. It is also important to note that, due to the complexity involved in the human behavior that economic models are trying predict, the exact numerical values of the model are typically of minor concern. The general behavior of the model is significantly more valuable in order to determine the effects of an economic assumption.

The logistic map is regarded as the prototypical chaotic discrete time mapping. The logistic function, which the logistic map is based off of, was developed to study population dynamics but actually garnered widespread use in other disciplines such as the study of autocatalytic reactions, computer science, statistics, and economics.[9]

$$\frac{d}{dx}f(x) = f(x)(1 - f(x)) \tag{1.2}$$

The logistic function has 2 equilibria or points where the derivative of the function is 0. $f(x) = 0$ is an unstable equilibrium but $f(x) = 1$ is a stable equilibrium point which means that other points on the function will tend towards this equilibrium overtime. This can be realized by solving for the derivative of the function at points when $f(x) \in (0, 1)$ which is universally positive and $f(x) \in (1, \infty)$ which is negative. Integrating the differential equation gives the general form equation:

$$f(x) = \frac{e^x}{e^x + C} \tag{1.3}$$

This function gained prominence due to its rapid, exponential growth when $f(x)$ is low and its slow, linear decaying to non-existent growth as population increases. Used by notable mathematicians operating in the field of population dynamics such as Verhulst, Pearl, and Lotka, the model continues to be widely used today and is often the basis upon which other modifications are applied.[10]

$$x_{t=1} = \mu x_t(1 - x_t) \tag{1.4}$$

The logistic map is a difference equation model popularized by Robert May as a discrete time analogue to the logistic function.[11] When interpreted in the biological context, x_t refers to the ratio of the population at time t compared to the maximal population, thus the mapping is bounded between 0 and 1. Here we see intuitively the major difference between difference equation and differential equation based systems. Differential equations solve for the derivative of a variable with respect to time in terms of the variable, difference equations solve for the actual state of the variable in the successive state provided we know the state of the variable in the previous time period. Much like how differential equations can be of higher order with the introduction higher order derivatives, a difference equation can also be of higher order by including more time periods in the function for the state of the variable which is valuable in a variety of the models discussed later.

Much like how the equilibrium points were solved for in the differential equation, difference equations also have equilibrium points where $x_{t+1} = x_t$. Interestingly, unlike the logistic function, there does not exist a fixed point at $x_t = 1$ as this would result in $x_{t+1} = 0$. Solving for the fixed points, we have:

$$x_{t+1} = x_t = 0, \frac{\mu - 1}{\mu} \tag{1.5}$$

The stability of a fixed point is again dependent on the derivative of the function; however, there are differences in the details of our analysis. Treating $f(x) = x_{t+1}$, we see the derivative of the logistic map is:

$$f'(x) = \mu(1 - 2x) \tag{1.6}$$

For reasons that will become clear later, the stability of a point on the map requires that $|f'(x)| < 1$. Solving for when this is true for our two fixed points, we see that $\mu < 1$ provides stability for the origin fixed point and $1 < \mu < 3$ gives stability for the non-zero fixed point. Thus, provided the parameter μ satisfies either of the conditions set previously, it will converge to one of the fixed points in a relatively small, finite amount of iterations.

This behavior can be visualized using a cobweb diagram. This diagram consists of 3 primary elements: a plot of the mapping, a 45° line, and a plot of the variable's trajectory. An example of a cobweb diagram can be seen in Figure 1.1. This diagram shows the trajectory of x starting at a value of 0.1 when there is a stable, non-trivial fixed point.

The 45° line is defined as the line where $x_t = x_{t+1}$ which is useful for determining the result of successive iterations. Beginning from the point x_0, we can then determine what point x_1 wll be via the mapping. We can then look horizontally to the 45° line until we intersect with it. The x-coordinate of this intersection point is equivalent to the result of the mapping of the previous iteration, thus using this new point will allow us to determine the result of the next iteration of the function. This process can be repeated ad infinitum; however, the result will soon prove uninteresting for stable points and orbits as the trajectory will converge and repeat its behavior.

The reason the logistic map is so frequently studied is because of its ability to exhibit

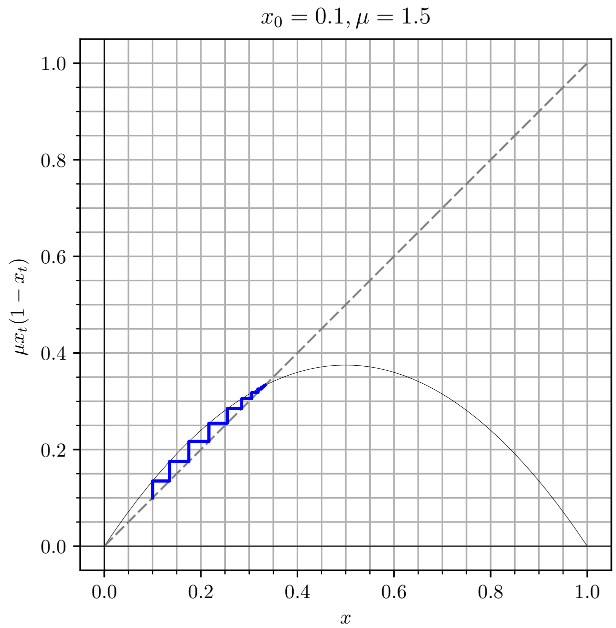

Figure 1.1: Cobweb plot of the logistic map setting $\mu = 1.5$ and $x_0 = 0.1$. The trajectory asymptotically approaches the equilibrium point of $\frac{1}{3}$.

complex behavior beyond a stable equilibrium solution. Once $\mu > 3$, the mapping enters a cyclic region. Much like how fixed points could be solved for by identifying where $x_t = x_{t+1}$, stable oscillatory points can be found by solving for the equilibrium points of higher iterations of the function. A 2-cycle will be such that $x_t = x_{t+2} \neq x_{t+1}$ for example and the stability of a such a cycle can be found using the same methodology as described previously. The logistic map also provides a mechanism to more quantitatively describe what it means for a system to be chaotic. The Lyapunov exponent, named after one of major driving forces in the development of stability analysis, is used to measure the effect of small perturbations in initial conditions on the trajectory of the variable.[12] Conceptually, the logistic map and the systems discussed in this paper are determinisitic. However, chaotic systems have highly divergent trajectories with even small changes in their initial conditions; thus knowing approximately what the initial conditions are does not provide approximate information on the trajectory of the variable.

In order to quantify this, we begin by taking the absolute value of the derivative of the function as this allows us to effectively magnify the effect of an infinitesimal change in the intial conditions. We then take the natural logarithm of this derivative in order to measure the exponential rate of separation of trajectories. Finally, we take the average separation over an arbitrarily high number of iterations n as exponential separation is

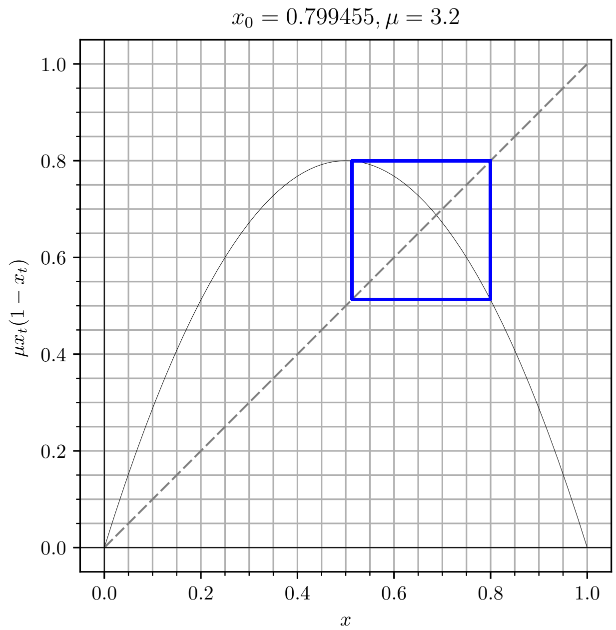

Figure 1.2: 2-period cycle of the logistic map showing only the cyclic behavior.

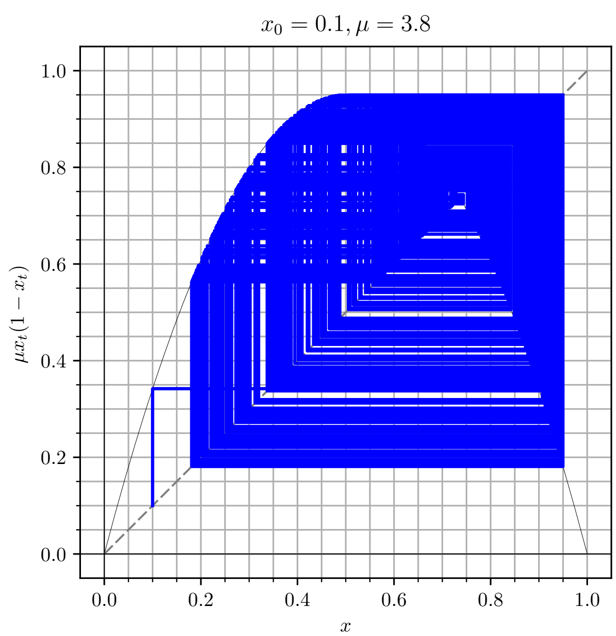

Figure 1.3: Chaotic behavior in the logistic map.

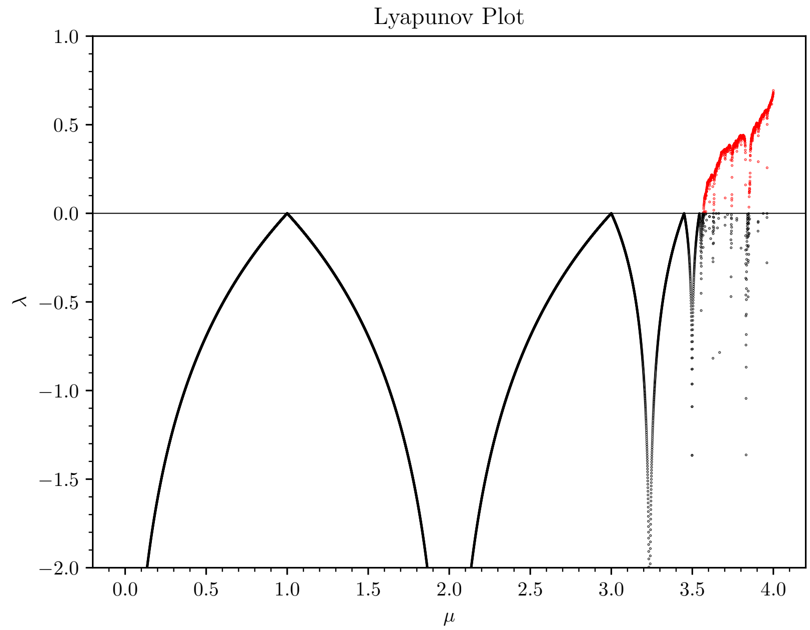

Figure 1.4: Lyapunov exponent plotted against μ for the logistic map. Initial value of 0.1 is used. Red denotes regions where $\lambda \geq 0$, black denotes regions where $\lambda < 0$.

not necessitated over all phase space. This gives us a the equation:

$$\lambda_n(x_0) = \frac{1}{n} \sum_{t=1}^{t=n} \ln|f'(x_{t-1})| \tag{1.7}$$

where $\lambda_n(x_0)$ is the lyapunov exponent for a given intial point when allowed to run for n iterations. The true value of the lyapunov exponent is the limit of the infinite series as $n \to \infty$ divided by n; however, the complexity of these maps often makes it practically impossible to analytically solve for the limit. By choosing arbitrarily high values of n though, it is possible to achieve better approximations at the expense of computational time. It is also important to note that, although the lyapunov exponent is a function of the initial condition, as long as the initial state is not in some stable fixed point or cycle, the trajectories will follow that of the chaotic attractor, thus the value of the lyapunov exponent should be mostly consistent regardless of the choice of initial conditions.

Another way visually see the behavior is with a bifurcation diagram. This diagram shows the longterm behavior of the variable for a given variable. Figure 1.5 qualitatively shows the behavior described previously. For parameter values between 0 and 1, we see the origin fixed point is stable. For parameter values between 1 and 3, their is still a single fixed point that is monotonically increasing; however, we can also clearly see the

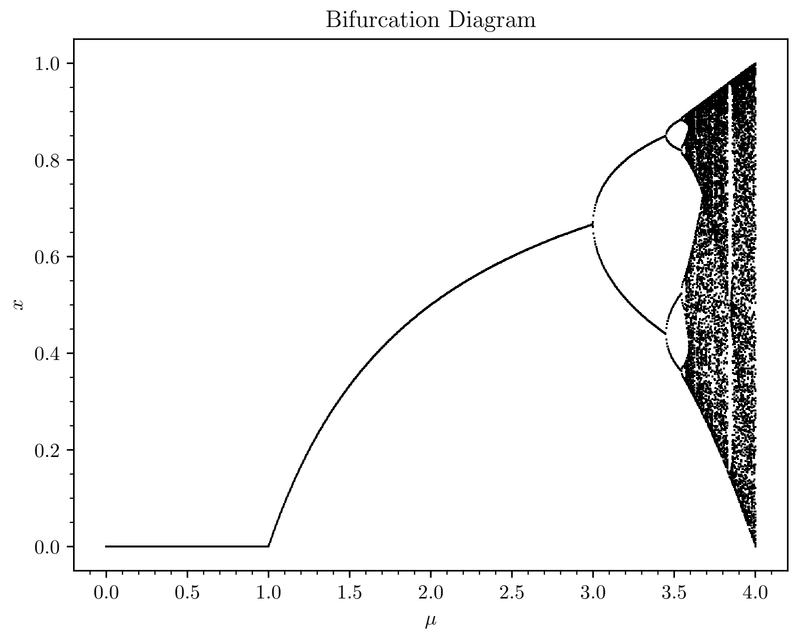

Figure 1.5: Bifurcation diagram plotting x against μ for the logistic map.

beginning of the 2-period cycle once the parameter exceeds 3. It is difficult however, to determine when predictable higher-order cyclic behavior ends and chaotic behavior begins via qualitative observation of the bifurcation diagram. The benefit of the bifurcation diagram is that it allows us to see both where the bifurcation points are and what behavior the bifurcation points signify. Bifurcation points are where where infinitesimally small quantitative changes in the parameter induce significant qualitative or topological change in the behavior of the mapping such as the transition from a stable fixed point to a 2-period cycle.

Research on the logistic map and other iterated maps has shown the existence of what is called Feigenbaum's constant. This constant can be found by observing the behavior of the periodic cycles of the map. The interval of stability decreases and the ratio of subsequence intervals actually approaches a limit $\delta \approx 4.6692$.[12] All other topologically similar maps with a single local maximum share this Feigenbaum constant. Once the mapping exceeds this constant, chaos occurs which allows for another method to determine precisely where chaotic behavior occurs.

It is also beneficial to point out that another mechanism exists for cyclic behavior to exist. The previously described method involved taking a mapping and solving for the stability of its double iteration. This allows for $2n$-period stability cycles to exists. However, there does exist odd-ordered cycles such as the 3 cycle; however, it occurs as a

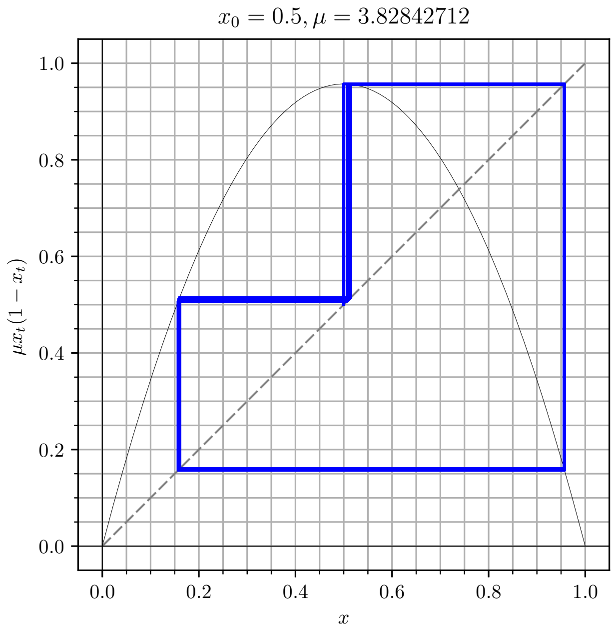

Figure 1.6: 3-period cycle of the logistic map showing only the cyclic behavior.

window of order in the region of chaos. These windows can be seen in Figure 1.4 where the Lyapunov exponent dips into the negative region past the chaotic bifurcation point. These bifurcation points are known as tangent bifurcations. This also allows us to use Sharkovsky's Theorem, which states that any continuous mapping with a 3-period cycle must also have every n-period cycle for every $n \in \mathbb{Z}$.[12] A variety of other mathematical techniques exist to study the dynamics of difference equation mappings but these will be covered more specifically when used for the specific case.

2. Heterogenous Inventory Cycles by Westerhoff et al.

2.1 Background

John Maynard Keynes work revolutionized economic thought; however, he never formalized any of his theories into a mathematical theory. This was performed in a process known as the neoclassical synthesis which was so called for its attempt to bridge classical models with these Keynesian principles. Although nobel prize winning Paul Samuelson was widely credited with providing a mathematical basis to Keynesian economics and popularizing the neoclassical synthesis, he was far from the ony economist involved in the movement[13,14]

Another notable Keynesian was Lloyd Metzler who developed a type of multiplier-accelerator business cycle. This type of model allows for cyclic behavior to occur endogenously, that is to say the model allows for persistent behavior outside of the steady state. This idea runs contrary to the idea that economic booms and recessions are reactions to exogenous shocks which is common in the new classical models. Also known as freshwater economics, these models assume that agents are perfectly rational and are capable of learning from past experiences. This viewpoint came into prominence in the late 1970s with a model by Lucas and Sargent[15] which sought to move past these seemingly outdated Keynesian principles. In the modern day however, Keynesianism has regained popularity with a new neoclassical synthesis that resulted in a new school of thought, New Keynesianism, that seeks to provide stronger microfoundations to macroeconomic models than was previously encountered in neo-Keynesianism.

Many models developed in the neo-Keynesian era of economics still remain in use but have either been expanded or used to study specific relationships. Wegener, Westerhoff, and Zaklan expand upon Metzler's inventory cycle model by introducing heterogenous expectations into firm behavior.[16]

Unlike the well known multiplier-accelerator model developed by Samuelson and Hicks, the dynamics of this model are driven by shifts in the heterogenous mix of behaviors in the firms. This model reduces the decision of firms to one of two behaviors, a regressive type and an extrapolative type. The extrapolative behavior occurs when a firm predicts that future income will deviate from the long-run average. The regressive behavior occurs when firms predict that future income will return to the long-run average. It is this heterogeneity as well as the ability for firms to switch their behavior type that is the defining feature of this model.

2.2 Model Set-Up

In this business cycle model, the economy is closed and income is determined completely by the quantity of goods produced by firms. Goods are completely homogenized but can be produced for 3 purposes: stock S, consumption U, and investment I. This lets us define income as:

$$Y_t = I_t + S_t + U_t \tag{2.1}$$

Investment is held to be exogenously determined and constant, thus:

$$I_t = \bar{I} \tag{2.2}$$

Consumption adapts to income according to the marginal propensity to consume, b. As this model follows closely to the framework developed by Metzler,[17] consumption adapts directly to the current income level as follows:

$$C_t = bY_t \tag{2.3}$$

Producers have a desired level of inventory based on the expected level of consumption goods. Thus setting, $k > 0$:

$$\hat{Q}_t = kU_t \tag{2.4}$$

where Q is the level of inventory. In order to achieve this desired level of inventory, firms produce S amount of output such that:

$$S_t = \hat{Q}_t - Q_{t-1} \tag{2.5}$$

However, the expected level of consumption does not necessarily match the produced level of goods. The realized level of inventory change can thus be determined as:

$$Q_t = \hat{Q}_t - (C_t - U_t) \tag{2.6}$$

The quantity of goods produced for consumption is determined by firms expectations for desired consumption. The extrapolative expectation rule is used by firms that believe consumption levels will continue shifting away from the long-run average quantity. This behavior is interpreted mathematically as:

$$U_t^E = C_{t-1} + c(C_{t-1} - \bar{C}) \tag{2.7}$$

where $c \geq 0$ denotes the speed at which firms expect consumption to deviate from the long-run level.

The other possible strategy firms can take is to expect consumption to return to the equilibrium level. This is described as the regressive expectation rule and is interpreted mathematically as:

$$U_t^R = C_{t-1} + f(\bar{C} - C_{t-1}) \tag{2.8}$$

where $0 \leq f \leq 1$ denotes the expected adjustment speed to the long-run level.

As this economy only involves two expectation rules, the aggregate expected level of consumption is a simple weighted average:

$$U_t = w_t U_t^E + (1 - w_t) U_t^R \tag{2.9}$$

The weight is defined as a function of consumption level which allows firms to switch their predictive behavior based on the current consumption level. Intuitively, as consumption deviates further from equilibrium, more firms will believe that the boom or slump will end and adjust in accordance. Likewise, when consumption is close to equilibrium, firms will believe it to be more accurate to extrapolate consumption. Weight is determined by the function:

$$w_t = \frac{1}{1 + d(\bar{C} - C_{t-1})^2} \tag{2.10}$$

where d is determined by the popularity of the regressive rule.

Taking these rules in aggregate, income is a second-order nonlinear iterated mapping. Condensing the model, we arrive at the mapping:

$$Y_t = U_t + kU_t - (1 + k)U_{t-1} + C_{t-1} + \bar{I} \tag{2.11}$$

This gives a single fixed point of:

$$\bar{Y} = \frac{1}{1 - b}\bar{I} \tag{2.12}$$

It is apparent that, as $\bar{Y}$ is constant, the theoretical set-up of the model is only valid

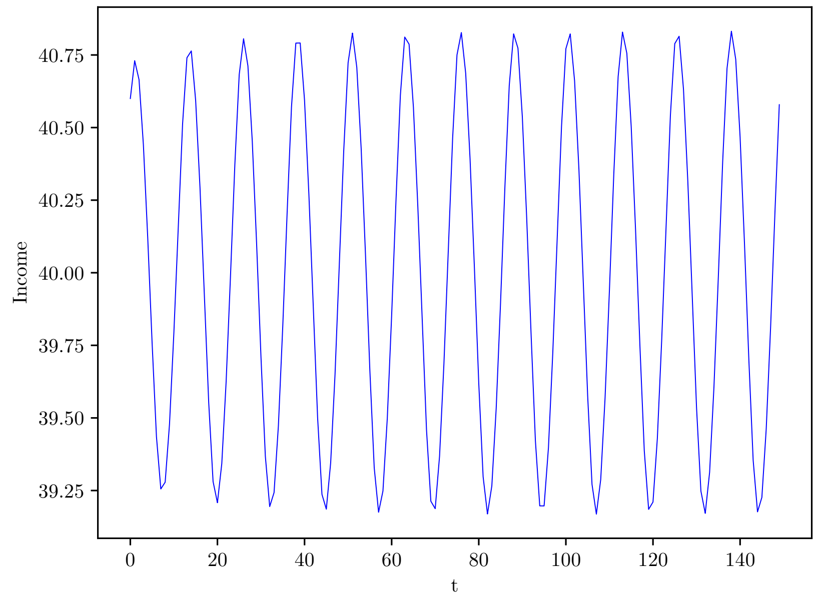

Figure 2.1: Timeseries plot of inventory cycle as described by Westerhoff et al.[16] Simulation is determined given $Y_0 = 40.6$, $U_0 = 30.3$, and $\bar{I} = 10$. The parameters of the model are: $b = 0.75, c = 0.3, d = 1.0, f = 0.1, k = 0.1$. Income does not feature long-run growth but rather oscillates around the steady state level of income.

if their is no long-term growth. This is because the predictive mechanism used by firms is reliant on the difference between the lag in consumption and the steady-state level of consumption. If consumption were to consistently grow, then this difference would also grow over time, eliminating any effective purpose in providing heterogenous expectations as firms would practically unanimously switch to the extrapolative mechanism.

2.3 Analysis of Income Dynamics

In the example trajectory given above, income clearly followed bounded, cyclic behavior. As seen with the analysis of the logistic map however, there are other types of long-term behavior possible. Given the restrictions on parameters, the steady-state level of income is stable if:

$$k < \frac{1 - b - bc}{b(1 + c)} \tag{2.13}$$

The proof for this can be seen in the appendix of Westerhoff et al.[16] The stability of the steady state is only dependent on b, k, and c, thus the behavior of the regressive expectation rule and the ratio of firms behaving with each expectation rule do not affect

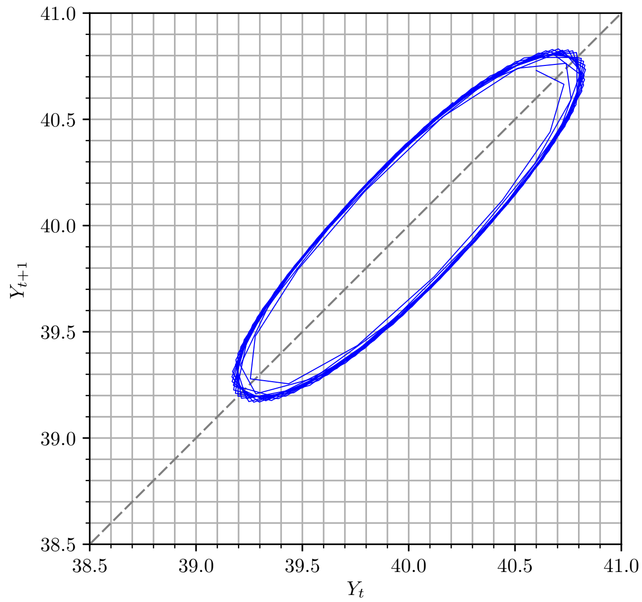

Figure 2.2: Cobweb plot of same trajectory in Figure 2.1. The beginning of the trajectory occurs at the point (40.6, 40.6).

the stability of this fixed point. This can be seen graphically using bifurcation diagrams of the parameters.

There is a Neimark-Sacker bifurcation where $b \approx 0.65$ which marks a transition point from a stable fixed point to a bounded cycle. However, as b approaches 1, the bifurcation diagram does not graphically feature cycles of well defined order. Varying c and k alters the behavior of the cycles as seen in Figure 2.4 and 2.5. Altering d or f affects the magnitude of the inventory cycles but does not have any effect on the presence of the cycles.

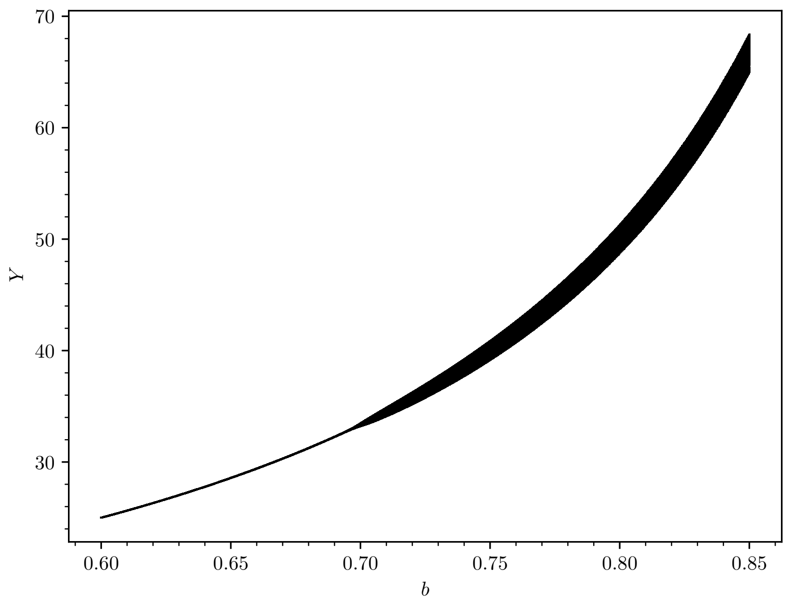

Figure 2.3: Bifurcations diagram varying the parameter b over the range [0.2, 0.85]holding all other parameters and initial conditions as described in Figure 2.1. The simulation was allowed to run for 1000 iterations and the last 50 points are captured in the diagram.

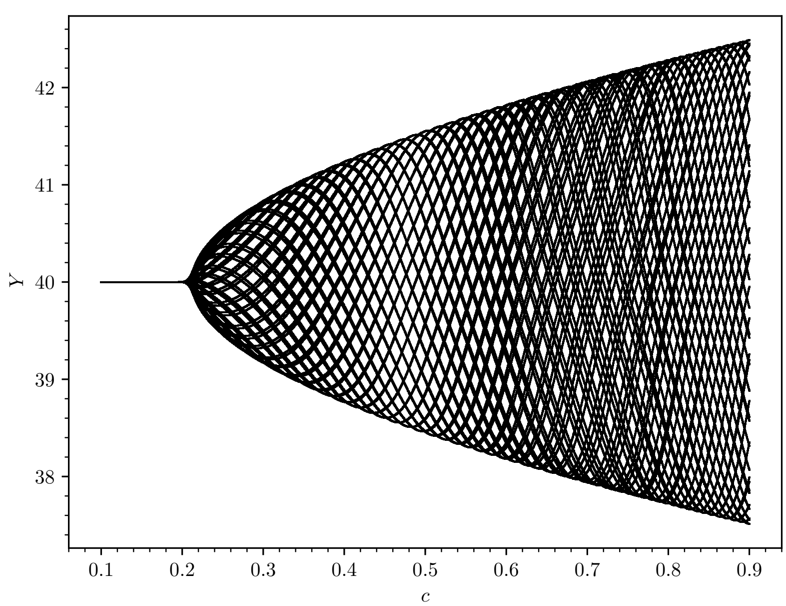

Figure 2.4: Bifurcations diagram varying the parameter c over the range [0.1, 0.9] holding all other parameters and initial conditions as described in Figure 2.1. The simulation was allowed to run for 1000 iterations and the last 50 points are captured in the diagram.

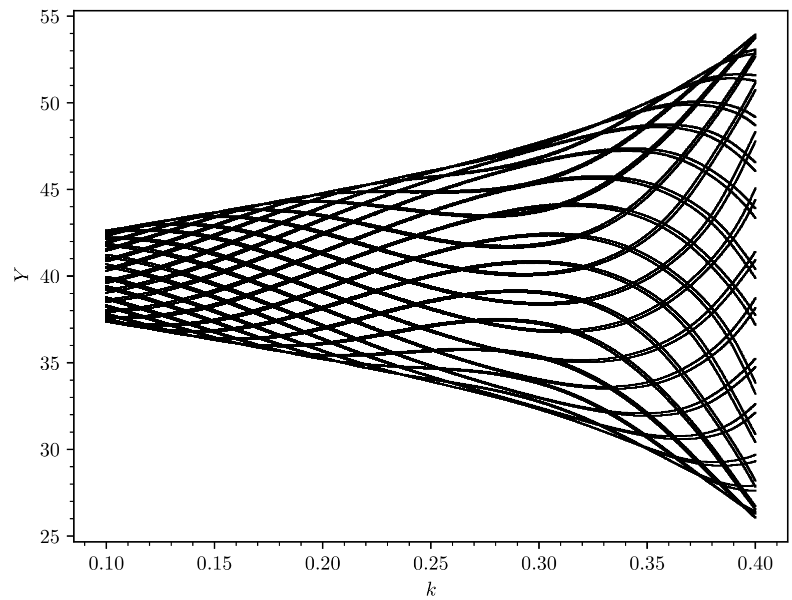

Figure 2.5: Bifurcations diagram varying the parameter k over the range [0 1, 0.4] holding all other parameters and initial conditions as described in Figure 2.1. The simulation was allowed to run for 1000 iterations and the last 50 points are captured in the diagram.

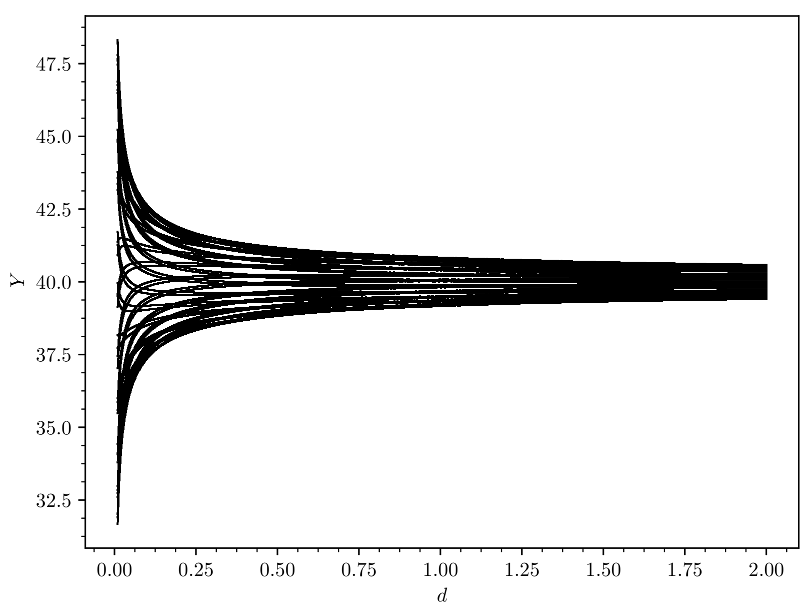

Figure 2.6: Bifurcations diagram varying the parameter d over the range [0.01, 2.0] holding all other parameters and initial conditions as described in Figure 2.1. The simulation was allowed to run for 1000 iterations and the last 50 points are captured in the diagram.

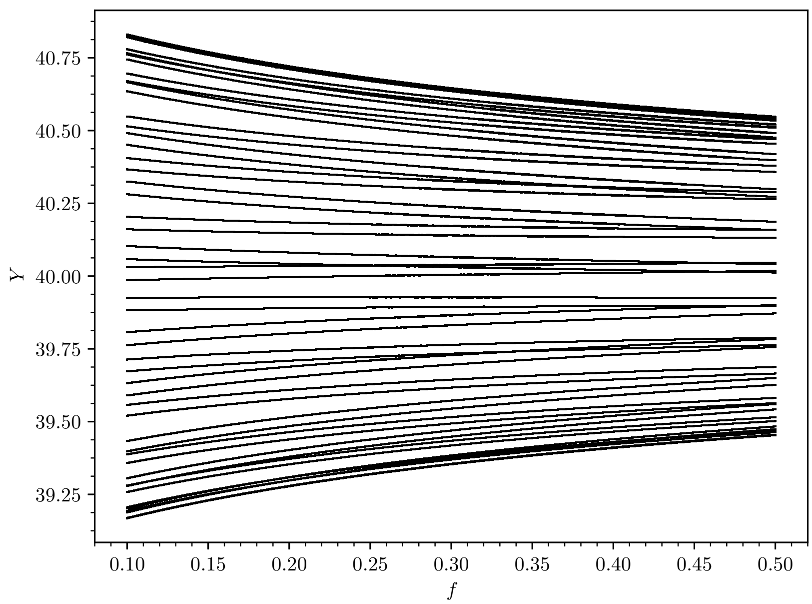

Figure 2.7: Bifurcations diagram varying the parameter f over the range [0.1, 0.5] holding all other parameters and initial conditions as described in Figure 2.1. The simulation was allowed to run for 1000 iterations and the last 50 points are captured in the diagram.

3. Inventory Cycles with Endogenous Investment and Adaptive Expectations

3.1 Background

Metzler's and Westerhoff's model show that it is possible for endogenous business cycles to be induced through inventory cycles and the failure of firms to accurately predict future consumption. These models were designed to focus on the effects of firm expectations with Westerhoff's contribution consisting of introducing a heterogenous expectation rule that allows firms to switch behavior based on the state of the economy. Both of these models however simplify other aspects of the economy that feature more complex behavior in other business cycle models.

The inventory cycle model features 3 main factors of production: predicted consumption, investment, and inventory. Metzler and Westerhoff holds investment as an exogenously determined constant; however, this is both unrealistic and does not allow for long term endogenous growth. To incorporate a mechanism for endogenously adjusting investment, we take inspiration from the discrete time business cycle model described by Tönu Puu.[12] In order for investment to operate under Keynes' accelerator principle, capital stock must be in a proportion to the change in income, thus the investment level would be a function of the rate of change in income.

A linear function for investment captures this premise; however, this leads to unrealistic behavior for higher magnitudes of income change. Suppose income dramatically increased; a linear function implies that a proportionally high level of investment can sustain this higher level of production when in reality other factors of production such as the land, labor, or technology available are the primary limiting factors. Of larger concern with a linear model though is if the economy encounters a sharp decrease in income. This induces a large, negative value for investment which implies that firms would actively destroy their machinery and other forms of capital stock in the event of an economic recession. This is obviously unrealistic and so John Hicks introduced a piecewise linear investment function such that at extreme levels of income change, investment will reach a predetermined maximal or minimal value. This piecewise function was then adapted to be differentiable over all points by Richard Goodwin by approximating the curve with a hyperbolic-tangent function.[12]

Puu approximates the hyperbolic-tangent function with its linear-cubic Taylor series expansion as this introduces a back-bending behavior into the curve. This allows the investment curve to capture not only firm behavior in the private sector but also implicitly include government spending and taxation. This follows from the now common policy for governments to engage in contracyclic behavior, increasing the quantity and size of spending projects and decreasing taxes when income is decreasing. Likewise, when the economy is performing well, the government cuts back on spending projects intended to stimulate the economy while also increasing taxes in order to take advantage of the overheating economy.

The problem with the cubic function is that it leads to unbounded behavior in the extremes, much like the linear function. Puu resolves this by ensuring that income growth is bounded by [-1, 1]; this is not the case for our model however. We instead want a function that features similar curvature and behavior as the cubic function but flattens when income change is of significant magnitude. This can be accomplished with the following function:

$$I_t = \frac{\frac{Y_{t-1}-Y_{t-2}}{v}}{(\frac{Y_{t-1}-Y_{t-2}}{v})^4 + q} \tag{3.1}$$

Metzler describes the existence of two important lags in the study of Keynesian models. The Robertson lag is characterized by making current consumption a function of past income, i.e. consumption behavior lags behind current income. The Lundberg lag however concerns a discrepancy between the income level and the production decision of firms[17] . These lags are named after the two economists D. H. Robertson and Erik Lundberg who developed models that contained only their eponymous lag type. Although both lags are likely to exist in reality, most models only incorporate one lag due to the increased complexity associated with it. Metzler and Westerhoff make use of a Lundberg lag by making income a function of the predicted level of consumption as opposed to actual consumption. Tönu Puu's model makes use of a Robertson in order to induce endogenous business cycles. Metzler himself does not claim that the Lundberg sequence is any more realistic than the Robertson sequence. Whichever lag has a longer time-period can be treated as of being greater importance but Metzler actually proposes a variety of scenarios that present contradicting conclusions. Suppose that decided their behavior on a quarterly basis but consumers altered their spending behavior with every paycheck, then it is no longer unrealistic to treat the Robertson lag as being of 0 length, i.e. nonexistent. If consumers revise their spending behavior every 6 months to a year, then it would actually be more realistic to include a non-zero Robertson lag while minimizing the Lundberg lag.

For the purposes of this model, we will include a non-zero Lundberg and Robertson lag. The consumption function is treated exactly as presented in Puu:

$$C_t = (1-s)Y_{t-1} + sY_{t-2} \tag{3.2}$$

This function incorporates a 1-period Lundberg lag where $s \in [0, 1]$ is the marginal propensity to save. This function also contains a 2-period delayed consumption due to the marginal propensity to save, thus all income made in some period t can be though of as being eventually spent in the period $t+1$ and $t+2$. Although intuitive, this explanation is not wholly accurate as the Lundberg lag does not imply saving of income to spend in the next period but rather that spending behavior is influenced only on the information of lagged income level.

As the economy is also making use of a Robertson lag, income is not directly a function of consumption as may be seen in other models. Rather, income is viewed from a production standpoint. This is achieved by explicitly defining a predicted level of consumption as in Metzler or Westerhoff.

$$U_t = C_{t-1} + \eta(C_{t-1} - C_{t-2}) \tag{3.3}$$

This predictive rule differs from that proposed in Chapter 2 because of the inclusion of endogenous investment. This mechanism introduces the possibility of endogenous long-run growth which eliminates the existence of a non-trivial steady-state level of consumption (given a sufficiently long time span where income is equal to 0, the long-run behavior of the system will also equal 0, thus the steady-state level of consumption would equal 0). By this predictive rule then, firms predict that current consumption is a sum of the 1-period lag level of consumption and a proportion of the change in consumption. This proportion $\eta \in [-1, 1]$ is the coefficient of expectation and is bounded as such in Metzler's model although this bound can be exceeded without logical contradiction. The coefficient is held constant in Metzler which implies that, in this economy, firms are unable to learn from the accuracy of their predictions. Although this simplifies the model, it is unrealistic as individual firms benefit from the ability to better predict desired consumption levels in order to prevent underproduction or overproduction of goods. Westerhoff attempted to resolve this by introducing heterogenous expectation rules and allowing firms to switch between an extrapolative and regressive behavior. As this economy often does not have a useful steady-state level of consumption, we must develop a new method to better allow firms to predict consumption changes.

Let firms operate homogenously, that is they all predict consumption via the same mechanism. To allow the coefficient of expectation to adapt to actual consumption, we can change Equation 3.3 to be of the form:

$$U_t = C_{t-1} + \eta_{t-1}(C_{t-1} - C_{t-2}) \tag{3.4}$$

The coefficient of expectation changes because firms learn from past experiences. Suppose then that firms decide their expectation based on what would have provided an accurate prediction in the last period. This can be accomplished solving:

$$C_t = C_{t-1} + \eta_t(C_{t-1} - C_{t-2})$$

This gives the function:

$$\eta_t = \frac{C_t - C_{t-1}}{C_{t-1} - C_{t-2}} \tag{3.5}$$

The adaptive coefficient of expectation is thus a third-order difference equation on actual consumption. This removes the bounds on η set by Metzler as firms can now choose arbitrary coefficients based on the result of the past.

Inventory production proceeds as seen in Chapter 2 with firms producing S_t explicitly to maintain Inventory at optimal levels:

$$S_t = kU_t - Q_{t-1} \tag{3.6}$$

where Q_t is the level of inventory maintained at the end of time t. This can be solved for as the sum of the previous inventory level, production intended for inventory, and the difference between production intended for consumption and the actual consumption level:

$$Q_t = Q_{t-1} + S_t + (U_t - C_t) \tag{3.7}$$

Income level, or output, can thus be written as a sum of production:

$$Y_t = I_t + S_t + U_t \tag{3.8}$$

However, as income has the capability of sustained growth under this model, it is difficult to analyze the behavior of the model purely by the value of income. It is preferable then to derive a function for the rate of change of income:

$$\begin{aligned}
Z_t = {} & \frac{\frac{Z_{t-1}}{v}}{\left(\frac{Z_{t-1}}{v}\right)^4 + q} - \frac{\frac{Z_{t-2}}{v}}{\left(\frac{Z_{t-2}}{v}\right)^4 + q} + \\
& [(1-s)Z_{t-2} + sZ_{t-3}] + \left[\frac{[(1-s)Z_{t-2} + sZ_{t-3}]^2}{(1-s)Z_{t-3} + sZ_{t-4}}\right] - \left[\frac{[(1-s)Z_{t-3} + sZ_{t-4}]^2}{(1-s)Z_{t-4} + sZ_{t-5}}\right] + \\
& k\left[[(1-s)Z_{t-2} + sZ_{t-3}] + \left[\frac{[(1-s)Z_{t-2} + sZ_{t-3}]^2}{(1-s)Z_{t-3} + sZ_{t-4}}\right] - \left[\frac{[(1-s)Z_{t-3} + sZ_{t-4}]^2}{(1-s)Z_{t-4} + sZ_{t-5}}\right]\right] - \\
& [(k+1)\left[[(1-s)Z_{t-3} + sZ_{t-4}] + \left[\frac{[(1-s)Z_{t-3} + sZ_{t-4}]^2}{(1-s)Z_{t-4} + sZ_{t-5}}\right] - \left[\frac{[(1-s)Z_{t-4} + sZ_{t-5}]^2}{(1-s)Z_{t-5} + sZ_{t-6}}\right]\right] \\
& - (1-s)Z_{t-2} - sZ_{t-3}]
\end{aligned} \tag{3.9}$$

4. Conclusion

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

5. References

(1) Aubin, D.; Dalmedico, A. D. Writing the history of dynamical systems and chaos: Longue durée and revolution, disciplines and cultures. *Historia Mathematica* **2002**, *29*, 273–339.

(2) Winfree, A. T. The prehistory of the Belousov-Zhabotinsky oscillator. *Journal of Chemical Education* **1984**, *61*, 661.

(3) Field, R. J.; Foersterling, H. D. On the oxybromine chemistry rate constants with cerium ions in the Field-Koeroes-Noyes mechanism of the Belousov-Zhabotinskii reaction: the equilibrium $HBrO_2 + BrO_3^- + H^+ \rightleftharpoons 2\,BrO_2 + H_2O$. *The Journal of Physical Chemistry* **1986**, *90*, 5400–5407.

(4) Beutel, K. M.; Peacock-López, E. Chemical Oscillations: Two-Variable Chemical Models. **2007**, *12*, 224–235.

(5) Hudson, J. L.; Tsotsis, T. T. Electrochemical reaction dynamics: a review., 1994.

(6) Matalon, M. Flame dynamics. *Proceedings of the Combustion Institute* **2009**, *32 I*, 57–82.

(7) Poincaré, H.; Goroff, D. L., *New methods of celestial mechanics*; American Institute of Physics: 1993, p 1077.

(8) Györgyi, L.; Field, R. J. A three-variable model of deterministic chaos in the Belousov-Zhabotinsky reaction. *Nature* **1992**, *355*, 808–810.

(9) Kavanagh, A. J.; Richards, O. W. The Autocatalytic Growth-Curve. *The American Naturalist* **1934**, *68*, 480.

(10) Zwanzig, R. Generalized Verhulst laws for population growth. *Proceedings of the National Academy of Sciences of the United States of America* **1973**, *70*, 3048–51.

(11) May, R. M. Simple mathematical models with very complicated dynamics. *Nature* **1976**, *261*, 459–467.

(12) Puu, T., *Attractors, Bifurcations, & Chaos*, 2nd Editio; Springer-Verlag: Berlin & Heidelberg, 2003, p 549.

(13) Skousen, M. The Perseverance of Paul Samuelson's Economics. *Journal of Economic Perspectives* **1997**, *11*, 137–152.

(14) Samuelson, P. A. Interactions between the Multiplier Analysis and the Principle of Acceleration. *The Review of Economics and Statistics* **1939**, *21*, 75–78.

(15) Lucas, R.; Sargent, T. After Keynesian macroeconomics [conference paper]. *Federal Reserve Bank of Minneapolis Quarterly Review* **1979**, *3*, 1–16.

(16) Wegener, M.; Westerhoff, F.; Zaklan, G. A Metzlerian business cycle model with nonlinear heterogeneous expectations. *Economic Modelling* **2009**, *26*, 715–720.

(17) Metzler, L. A. The Nature and Stability of Inventory Cycles. *The Review of Economics and Statistics* **1941**, *23*, 113.

A. Solving for Change in Income

We can define the growth rate Z_t as the difference in income between contiguous time periods:

$$Z_{t-1} = Y_t - Y_{t-1} \tag{A.1}$$

This equation can be expanded to be of the form:

$$Z_{t-1} = (U_t - U_{t-1}) + (I_t - I_{t-1}) + (S_t - S_{t-1})$$

The change in investment can be trivially solved for as investment is already a function of the change in income.

$$I_t = \frac{\frac{Z_{t-2}}{v}}{\left(\frac{Z_{t-2}}{v}\right)^4 + q}$$

Thus the change in investment is:

$$I_t - I_{t-1} = \frac{\frac{Z_{t-2}}{v}}{\left(\frac{Z_{t-2}}{v}\right)^4 + q} - \frac{\frac{Z_{t-3}}{v}}{\left(\frac{Z_{t-3}}{v}\right)^4 + q} \tag{A.2}$$

As predicted consumption is a function of the change in actual consumption, we must derive a function for the change in consumption in terms of the growth rate.

$$C_t - C_{t-1} = (1-s)Y_{t-1} + sY_{t-2} - [(1-s)Y_{t-2} + sY_{t-3}]$$

This can be simplified to:

$$C_t - C_{t-1} = (1-s)(Y_{t-1} - Y_{t-2}) + s(Y_{t-2} - Y_{t-3}) = (1-s)Z_{t-2} + sZ_{t-3} \tag{A.3}$$