The Realistic Graphical Simulation of Skipping a Stone on Water

CS488 (Computer Graphics) Spring 2024

Final Project Proposal

Euan Hughes (20874431, e5hughes)

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Inspiration and Purpose

While at a party on Canada day, I had a long conversation with an engineer regarding the physics of a "perfect" stone skip, as he was designing various silicon moulds to discover the optimal shape of a skipping stone. This interaction was the basis for my inspiration to create my CS488 final graphics project on accurately simulating the fun, interesting, and complex phenomenon of skipping a stone on the surface of water.

Topics: rigid body dynamics; contact mechanics; water surface simulation; advanced fluid-solid interaction

1 Project Overview

Skipping stones on water is a popular pastime that has fascinated people for centuries, involving a captivating blend of physics and skill. Stone skipping involves throwing a flat stone across water in such a way that it bounces off the surface multiple times. For a visual demonstration, see this video. The objective of this project is to develop a reasonably accurate and visually engaging simulation of this phenomenon by significantly extending the existing basic physics-based animation framework.

To achieve this, this project will first involve: modeling the motion of a stone as a rigid body; creating an accurate water surface simulation; and accurately modelling the change to a rigid body's velocity and orientation from interactions with both solid and fluid surfaces. It will then involve implementing complex numerical algorithms which model the motion of a stone skipping on water at various stages (Figure 1).

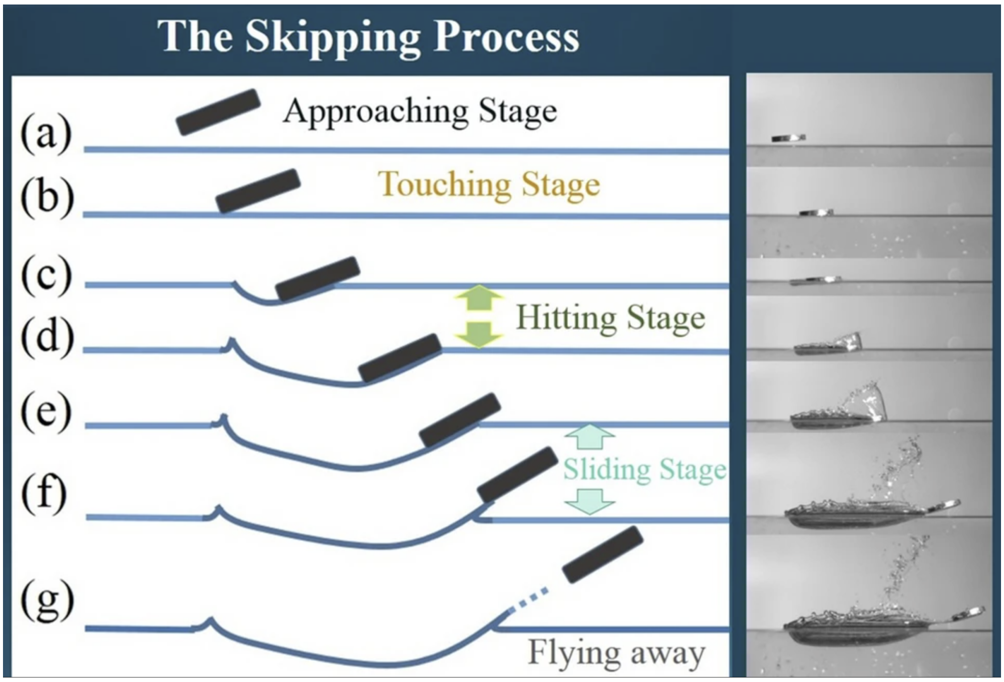


Figure 1: Detailed stages of stone-skipping: (a) approaching, (b) touching, (c & d) hitting stage, (e & f) sliding stage, and (g) flying away (Tsai et. al., 2022).

This project is interesting and challenging because stone skipping is a fun and fascinating physical phenomena that almost everyone knows about, but it will be particularly exciting of a project due to the complex interplay of forces that occur. I will need to learn about many advanced topics in physics-based animation in order to build the foundation to implement realistic stone skipping. Specifically, I will need to accurately model both rigid body and fluid dynamics, and model how a rigid body interacts with both solids and fluids. This includes manipulating the orientations and velocities of the rigid body, while also accounting for water displacement and propagating waves. After building this foundation, I will also have to implement a series of robust and complex numerical algorithms that model each stage of the skipping process (Figure 1) in order to be physically realistic. At the end, I hope to have a fun, exciting, and unique graphics project.

2 Technical Outline

In order to accurately model the stone skipping process, I will aim to complete the objectives outlined in section 4. Below, I explain in more detail how I aim to achieve each of these objectives:

Objective 1: Rigid Body Motion

For simplicity, I will ignore spin around the axis ($\vec{n}$), which would normally be important for maintaining the inclined angle of the stone (α in Figure 2) through the gyroscopic effect. I will also be ignoring air resistance and drag.

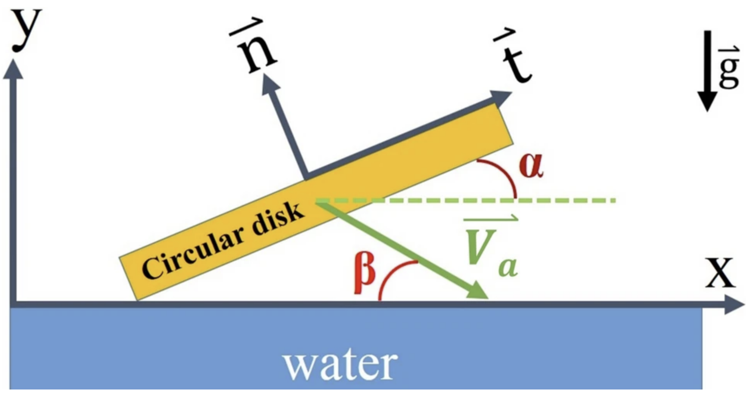


Figure 2: A circular disk approaching the water surface. (Tsai et. al., 2022)

For this project, I will implement a stone as a thin cylindrical disk. I will utilize Newton's laws of motion to define the kinematic equations for a rigid body. The stone data structure will store its centre of mass and maintain an inertia tensor relative to it in order to compute the rotational motion and change to the inclined angle (α). To model the motion of the body over time, I will use Verlet time stepping integration just like in A3 due to its superior stability over other methods.

Objective 2: Solid Surface Collision

I will implement detection for collisions between the stone and a solid surface by seeing if the dynamic bounding volume for the cylindrical disk penetrates the solid surface's fixed bounding volume. Depending on how and what region of the rigid body makes contact with the solid surface, I will apply the collision force to that region of the rigid body, and I will ensure to use a coefficient of restitution to model the elastic properties of the collision. If the stone collides with the solid surface on one of its corners, this will require manipulating the rotational motion via updating the inertia tensor. For this, I will use a simplified solution to the Painlevé's problem which ignores friction.

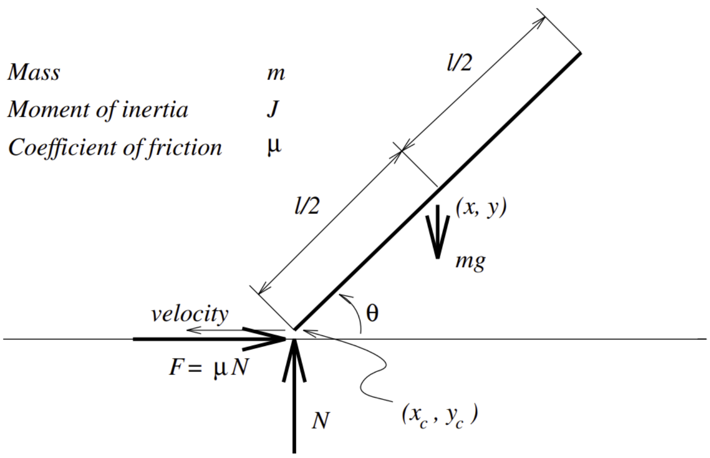


Figure 3: Depiction of Painlevé's problem, solving the collision response for a disk making contact with a flat, solid surface on one of its corners (Stewart DE., 2000).

Objective 3: Sliding Objects

To realistically model the stone sliding across solid flat surfaces with various slopes, velocities, and frictions, I will define an interaction model that includes static and kinetic friction coefficients. The effects of surface inclination will be incorporated by adjusting the normal and frictional forces accordingly. Friction will be modeled as a function of the stone's velocity, possibly incorporating velocity thresholds for static and dynamic friction transitions. Additionally, I will the track energy loss due to friction to ensure the realistic slowing down of the stone.

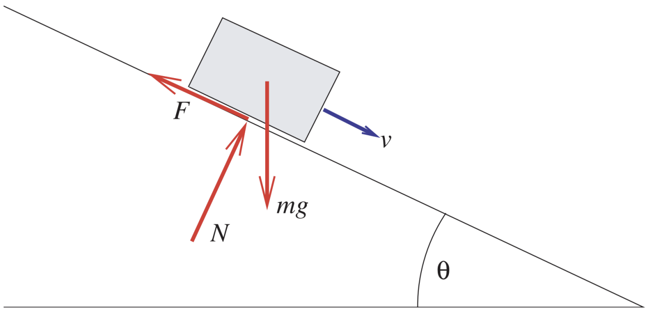


Figure 4: Depiction of a disk sliding down a frictional ramp (Stewart DE., 2000).

Objective 4: Basic Water Surface Simulation

To implement a height-field water-surface simulation with wave propagation, I will represent the water surface as a height field, represented as a 2.5D array of columns, where each column stores a height and velocity value.

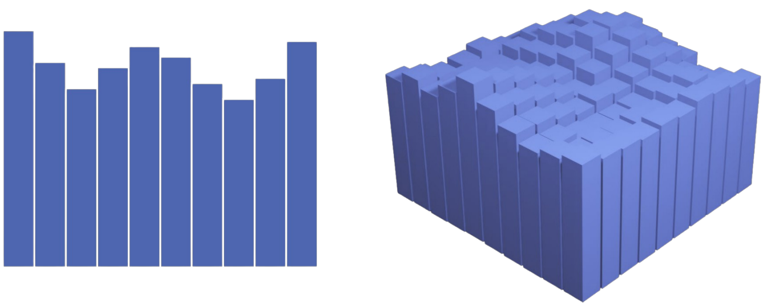
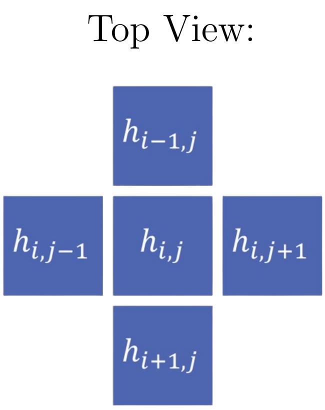


Figure 5: Water as an "array of columns" in 1.5D and 2.5D (Müller, M., 2023)



The simulation of surface wave propagation will be done using a simple semi-implicit Euler integration method which is a discretization of the wave equation second-order PDE. In this method, the velocity of a column $v_{i,j}$ is updated by the acceleration $a_{i,j}$, which is based on the height of the 4 neighbouring columns relative to the height of column itself, multiplied by the constant $\frac{c^2}{s^2}$, where c is the wave propagation speed and s is the column width. Then, the height of a column $h_{i,j}$ is updated based on its new velocity.

The pseudocode for the algorithm is given below:

for all i, j **do**
  $a_{i,j} \leftarrow \frac{c^2}{s^2}(h_{i-1,j} + h_{i+1,j} + h_{i,j-1} + h_{i,j+1} - 4h_{i,j})$
  $v_{i,j} \leftarrow v_{i,j} + \Delta t \ \ a_{i,j}$
end for
for all i, j **do**
  $h_{i,j} \leftarrow h_{i,j} + \Delta t \ \ v_{i,j}$
end for

Where Δt is the time step size, and the stability criterion is $\Delta t \ \ c < s$. To handle wave interactions with the edges of the domain, we will have a reflecting boundary condition where, if $h_{i\pm1,j\pm1}$ is outside of the domain, then we would replace it with $h_{i,j}$.

The wave equation PDE is effective for simulating water surface waves due to its ability to model the dynamics of wave propagation accurately. However, it is not as comprehensive as using the shallow water equations, which can account for more complex fluid dynamics. Given extra time for the assignment, I will attempt to implement a similar discrete form of the shallow water PDEs, using the method outlined in pages 50-52 of (Kass, M., et al., 1990).

Objective 5: Basic Solid-Fluid Interaction

For a simple model of the 2-way interaction of objects with water, in order to prevent volume loss and account for submerged objects, we will maintain an additional field for each column, $b_{i,j}$, which represents the height of column $h_{i,j}$ covered by a vertical cross-section an object.

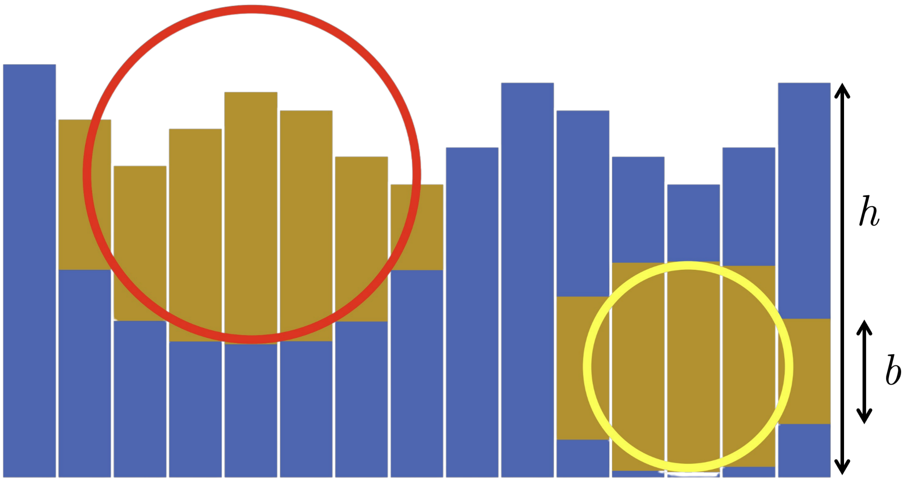


Figure 6: Water column height fields ($h_{i,j}$) and the height occupied by objects on and below the surface ($b_{i,j}$) (Müller, M., 2023)

To model the change in the column heights based on the object's interaction, we will add the change of the occupied space $b_{i,j}$, which will ensure volume conservation.

This will look like the following simple algorithm:

for all i, j **do**
  $h_{i,j} \leftarrow h_{i,j} + (b_{i,j} - b_{i,j}^{\text{prev}})$
end for

To model the change to the object's velocity based on its interaction with the water, I will calculate the upward buoyant force based on Archimedes' principle, where the upward force will be proportional to the displaced water volume and the surface area in contact with the water. An accurate simulation will allow an object with high initial velocity and strong buoyant force to bob up and down on the water surface.

For each column with overlapping volume from an object, the upward buoyant force from the column will be given by:

$$f_{i,j} = mg = \rho_{water} b_{i,j} s^2 g$$

where ρ_{water} is the density of water, $b_{i,j}$ is the overlap of an object in the water column, s is the width of a column, and g is the gravitational constant. The forces from the individual columns will sum and apply an aggregate force onto the object.

The model will be simplified by applying the aggregate force to the centre of mass of the object, keeping the stone's orientation constant. Additionally, the drag and capillary drag forces from the water acting on the stone are also ignored, meaning the horizontal velocity will remain constant as well. This specific objective will focus exclusively on vertical displacement and buoyancy effects. However, I will test to ensure that water displacement and wave propagation work realistically with pre-scripted changes to the orientation and horizontal velocities.

Objective 6: Basic Object Orientation Manipulation from Fluid

This objective will focus on allowing water to influence the orientation of solid objects. Accurate simulation of this means a floating disk at an inclined angle will flatten out as it floats on the water's surface. This is important for accurate stone skipping because, as can be seen in Figure 7, the approach angle α gets reduced throughout the hitting stage as the corner of the disk makes contact with the water's surface.

In objective 5, we computed the upwards buoyant force from a column acting upon an object, but the upward force was applied to the centre of mass. In this objective, I will allow the buoyant force from a column to apply its force to the specific part of the submerged object within that column, which will apply a torque and affect the stone's rotational motion. We then sum the torques from all relevant columns to obtain the net torque acting on the object, and update the angular velocity via the inertia tensor of the stone (as outlined in objective 1) based on the net torque and the moment of inertia.

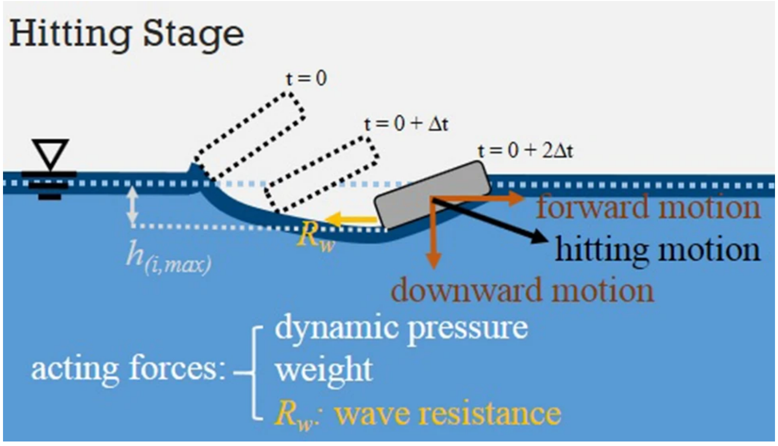


Figure 7: The forces at play during the hitting stage of stone skipping (Tsai et. al., 2022)

In relation to the hitting stage of stone skipping (Figure 7), the force we are computing in from objectives 5 and 6 is the lifting force from the reaction of the water surface, in other words, the dynamic pressure acting upon the stone from the fluid surface.

This orientation manipulation is important for the stone skipping simulation as it accounts for the gradual lowering in the inclined angle, leading to smaller jumps and eventual sinking of the stone.

Objective 7: Object Horizontal Velocity Manipulation from Fluid

In this objective, I will incorporate the final needed force to simulate the hitting stage of the stone skipping process: the effects capillary-gravity drag (wave resistance) on the stone's horizontal motion as it interacts with the water surface.

Capillary-gravity drag, or wave resistance occurs due to the formation of waves as the stone moves across the water surface, which acts opposite to the direction of motion, reducing the horizontal velocity of the stone. This computation will yield the horizontal deceleration of the stone, and then would then apply this deceleration force to the Verlet time stepping integration scheme in the update of the object's velocity.

An accurate simulation will involve launching a semi submerged ball on the water's surface and seeing an accurate slowing down of the sphere while creating a wake.

Objective 8: Sliding on Water Surface

In this objective, we will finally complete the final step needed for the stone skipping process, which is simulating the stone gliding over the water surface after initial impact, or the sliding stage.

When the stone impacts the water, it will reach a maximum depth before being launched at an angle θ. According to the paper (Tsai et al., 2022), the stone reaches its maximum depth and transitions from the hitting stage to the sliding stage when the dynamic forces acting on it are balanced. Specifically, this occurs when the vertical

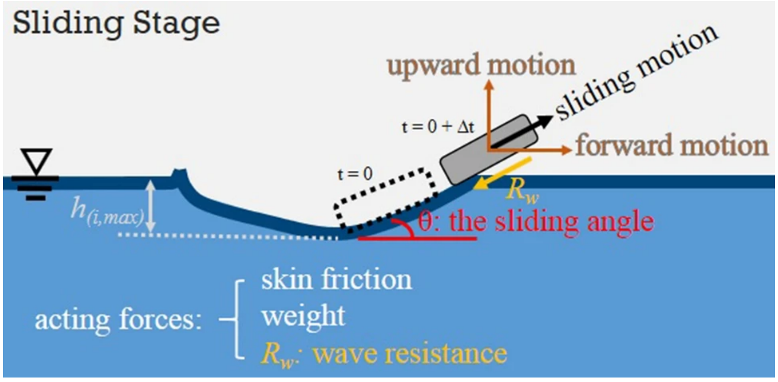


Figure 8: The forces at play during the sliding stage of stone skipping (Tsai et. al., 2022)

component of the stone's velocity becomes zero, marking the end of its downward motion and the beginning of its horizontal gliding on the water surface.

When it is being launched at angle θ, it will maintain the horizontal velocity from before. However, as can be seen from Figure 8, it will experience deceleration due to friction and wave resistance (latter is calculated in objective 7). By combining the friction and wave resistance forces, the total horizontal deceleration of the stone can be modeled, and then would then apply this deceleration force to the Verlet time stepping integration scheme in the update of the object's velocity.

In both objectives 7 and 8, it is crucial to account for the deceleration forces on the water's surface from friction and wave resistance to accurately simulate the gradual slowing down and eventual sinking of the stone as it impacts and skims across the water's surface.

3 References

References

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4 Objectives

UWaterloo UserID: e5hughes **Student ID: 20874431**

___ 1: Accurately model motion of a stone as a rigid body

___ 2: Realistically model stone's collision and resulting bounce off of a solid surface

___ 3: Realistically model stone sliding across solid flat surfaces with various slopes, velocities, and frictions

___ 4: Implement height-field water-surface simulation with wave propagation using discrete wave equation PDE

___ 5: Implement simple model of the 2-way interaction of objects with water surface (dynamic vertical velocity, fixed/scripted orientation, fixed/scripted horizontal velocity)

___ 6: Allow water to manipulate the orientation of solid objects interacting with it (simulated orientation)

___ 7: Slow the horizontal velocity of solid objects interacting with water through wave resistance (capillary-gravity wave drag) forces (simulated horizontal velocity)

___ 8: Accurately model stone sliding up off of water surface

Extras implemented in A1-A3:

(A1) Rasterization:

1. Minor optimization via bounding box for each triangle pre-rasterization
2. Added near-plane clipping

(A2) Ray Tracing:

1. Major optimization via surface area heuristic to BVH building process
2. Minor optimization via mailboxing to BVH transversal process
3. Added image based lighting

(A3) Animation:

1. Major optimization via Barnes-Hut method to gravitational field simulation
2. Major optimization via custom BVH to sphere collision simulation