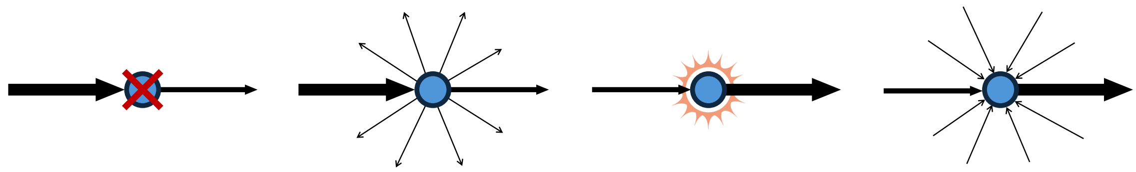
1 Background I: Physically-Based Volume Rendering

Accurately simulating the appearance of participating media such as clouds, smoke, fire, fog or translucent objects, unlike surface-based rendering, which only considers boundary interactions, requires modeling how light is altered as it travels through the volume. Specifically, as light travels along a path through a medium, it is affected by absorption, scattering, and emission phenomena. The visual effects of these light-medium interactions are illustrated in Figure 1.1.



(a) Absorption **(b)** Out-scattering **(c)** Emission **(d)** In-scattering

Figure 1.1: Changes in radiance due to absorption **(a)**, scattering **(b, d)**, and emission **(c)** while traveling through a differential volume element.

1.1 Radiative Transfer Equation

A participating medium is a collection of microscopic particles that interact with photons traveling through it. Rather than modeling each particle explicitly, the medium is described statistically by specifying their total density per unit volume. This density, along with the *absorption coefficient* $\mu_a(\mathbf{x}) \geq 0$ and the *scattering coefficient* $\mu_s(\mathbf{x}) \geq 0$ quantify the probability of a photon undergoing the respective interaction per unit distance traveled.

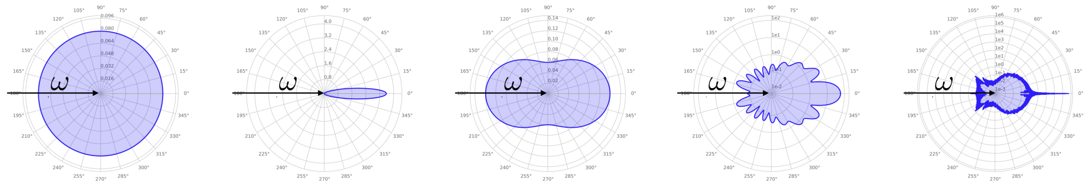
The steady-state *Radiative Transfer Equation* (RTE) [Cha60] mathematically describes how radiance $L(\mathbf{x}, \omega)$ changes due to absorption, scattering and emission, and forms the theoretical foundation for physically-based volume rendering. It states that the radiance at the point $\mathbf{x} \in \mathbb{R}^3$ traveling in the direction $\omega \in S^2$, where S^2 a unit sphere surrounding x, is defined by the differential equation:

$$\begin{aligned}(\omega \cdot \nabla)\, L(\mathbf{x}, \omega) \;=\; &-\; \mu_a(\mathbf{x})\, L(\mathbf{x}, \omega) \;-\; \mu_s(\mathbf{x})\, L(\mathbf{x}, \omega) \\ &+\; \mu_a(\mathbf{x})\, Q(\mathbf{x}, \omega) \;+\; \mu_s(\mathbf{x})\, S(\mathbf{x}, \omega).\end{aligned} \tag{1.1}$$

We can see in the RTE (Equation 1.1) there are four terms, each corresponding to the distinct physical processes that light undergoes in the medium: absorption, out-scattering, emission and in-scattering, as illustrated in Figure 1.1. The first term represents the loss of radiance due to absorption, which is proportional to the incoming radiance and the absorption coefficient at point $\mathbf{x}$. The second term represents loss from out-scattering, where radiance is reduced as light is scattered out of direction ω, similarly a product of the incoming radiance and the scattering coefficient. The third term $Q(\mathbf{x}, \omega)$ models emission, representing radiance added by photons emitted within the medium in direction ω. Finally, the fourth term corresponds to in-scattering, capturing radiance added by light being scattered into the direction ω. The in-scattered radiance $S(\mathbf{x}, \omega)$ is defined as:

$$S(\mathbf{x}, \omega) = \int_{S^2} f_p\big(\mathbf{x}, \omega' \to \omega\big)\, L(\mathbf{x}, \omega')\, \mathrm{d}\omega', \tag{1.2}$$

which integrates incoming light from all directions ω' weighted by the phase function f_p. This function defines the probability density of scattering from direction ω' into direction ω at point $\mathbf{x}$. If the medium scatters light uniformly, the phase function is *isotropic*, and constant: $f_p = 1/(4\pi)$. Otherwise, it is *anisotropic*, favoring certain scattering directions, for example, forward scattering in fog or backward scattering in smoke. This directional bias significantly affects the distribution of scattered light and, therefore, the appearance of volumetric effects, leading to more physically accurate and visually realistic scenes. Several commonly used phase functions used in volume rendering are visualized in Figure 1.2.



(a) Isotropic **(b)** Henyey-Greenstein **(c)** Rayleigh **(d)** Lorenz-Mie small particles **(e)** Lorenz-Mie large particles

Figure 1.2: Phase functions illustrating different scattering behavior: **(a)** isotropic (uniform in all directions) vs. **(be)** anisotropic, with directional scattering patterns. Image from Novak *et al.* [NGH+18]

For compactness and better physical intuition, we define two useful quantities: the *extinction coefficient* $\mu_t(\mathbf{x}) = \mu_a(\mathbf{x})+\mu_s(\mathbf{x})$ as the total probability of either absorption or out-scattering events occurring, resulting in the attenuation of incoming radiance; and the *singlescattering albedo* $\alpha(\mathbf{x}) = \frac{\mu_s(\mathbf{x})}{\mu_t(\mathbf{x})}$ as the fraction of extinction events that are scattering rather than absorption. Then, by integrating both sides of the RTE along the ray $\mathbf{x}_t = \mathbf{x}_0 - t\,\omega$, for $t \geq 0$, we obtain the integral form:

$$L(\mathbf{x}_0, \omega) = \int_0^\infty T(\mathbf{x}_0, \mathbf{x}_t)\, \mu_t(\mathbf{x}_t)\, L_o(\mathbf{x}_t, \omega)\, dt \qquad \text{with} \tag{1.3}$$

$$L_o(\mathbf{x}_t, \omega) = (1 - \alpha(\mathbf{x}_t))\, Q(\mathbf{x}_t, \omega) + \alpha(\mathbf{x}_t)\, S(\mathbf{x}_t, \omega). \tag{1.4}$$

This integrates emitted and in-scattered light along the ray in direction $-\omega$ from the point x_0, and extending into the medium. This reflects the fact that when computing the radiance arriving at $\mathbf{x}_0$, we must trace backwards along the ray from where the light came from. The term $T(\mathbf{x}_0, \mathbf{x}_t)$ is the *transmittance*, and accounts for the loss of radiance from extinction events along the ray, specifically absorption or out-scattering. For spatially uncorrelated media, those in which scattering and absorption events are assumed to be independent and randomly distributed along the ray, the transmittance is given by an exponential decay according to the Beer-Lambert law [Lam60]:

$$T(\mathbf{x}_0, \mathbf{x}_t) \;=\; \exp\!\left(-\int_0^t (\mu_t\; \mathbf{x}_{t'})\, \mathrm{d}t'\right), \tag{1.5}$$

where the integral in the exponent is called the *optical thickness* τ, the accumulated extinction along the path.

This exponential model assumes a homogeneous, Poisson distribution of particles. In practice, this assumption may not hold, especially in structured, correlated media such as clouds or smoke. Recently, a linear transmittance model has been proposed [BRM+18] that better fits the behavior in such media. Under this model, transmittance is expressed in a form resembling:

$$T(\mathbf{x}_0, \mathbf{x}_t) = 1 - \frac{1}{d}\left(\int_0^t (\mu_t\; \mathbf{x}_{t'})\, \mathrm{d}t'\right), \tag{1.6}$$

where d is a constant reflecting the expected distance to extinction.

In essence, Equation 1.3 states that the radiance observed at $\mathbf{x}_0$ from direction ω is the accumulated contribution of in-scattered and emitted light along the path in the direction $-\omega$, each attenuated according to the optical thickness of the medium and the linear or exponential transmittance model.

1.2 Volume Rendering Equation

Currently, the RTE only captures the volumetric component of light transport. However, in most scenes, light interacts with not only participating media but also with hard surfaces. Thus, the RTE needs to be extended to include surface interactions, accounting for both volumetric emission and in-scattering as well as the surface radiance contribution.

Assume a ray travels from $\mathbf{x}_0$ in direction ω and intersects a surface at position $\mathbf{x}_s = \mathbf{x}_0 - s\omega$. The radiance arriving at x_0 from direction ω is the combination of:

1. The accumulated contribution of light emitted and in-scattered within the participating medium along the segment $[0, s]$, attenuated by transmittance.
2. The light radiated by the surface at $\mathbf{x}_s$, attenuated by the medium along $[0, s]$.

The surface radiance at $\mathbf{x}_s$ is governed by the *Rendering Equation* [Kaj86], while the in-medium contributions come from the RTE (1.3). This yields the *Volume Rendering Equation* (VRE):

$$L(\mathbf{x}_0, \omega) = \underbrace{\int_0^s T(\mathbf{x}_0, \mathbf{x}_t)\ \mu_t(\mathbf{x}_t)\ L_o(\mathbf{x}_t, \omega)\,\mathrm{d}t}_{\text{Integral RTE (Equation (1.3))}} + \underbrace{T(\mathbf{x}_0, \mathbf{x}_s)\,L(\mathbf{x}_s, \omega)}_{\substack{\text{medium attenuated}\\ \text{surface radiance}}}\ . \tag{1.7}$$

This decomposition is illustrated in Figure 1.3, which shows how the total radiance along a ray is composed of both volumetric and surface contributions. The contributions are continuously attenuated by the transmittance accumulated as it travels through the medium.

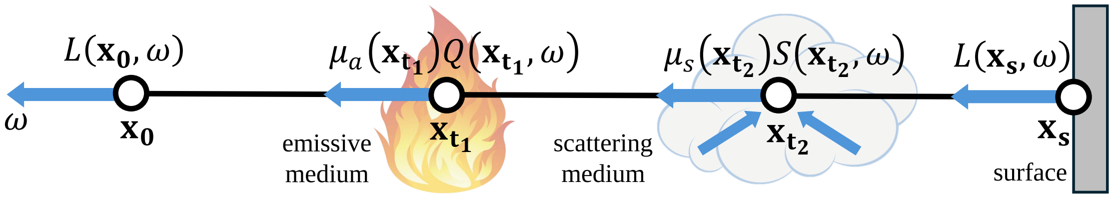


Figure 1.3: Illustration of the VRE (Equation 1.7), showing the contributions of volumetric emission, in-scattering, and surface radiance along a ray.

This equation serves as the mathematical foundation for simulating light transport in scenes containing both volumes and surfaces.

1.3 Solving the Volume Rendering Equation

Solving the VRE (Equation 1.7) is the core computational challenge in physicallybased volume rendering. In particular, the nested integrals over continuous heterogeneous coefficients and inscattering terms are the most difficult to compute efficiently and accurately.

In the simplest case, we have spatially *homogeneous* mediums where the extinction coefficient $\mu_t(\mathbf{x})$ is a constant μ_t. This simplifies the solution of the VRE by enabling a closed-form sampling strategy for the next interaction point along a ray. This process, known as *free-flight sampling*, draws distances directly from the *free-path distribution* that describes the probability of a photon traveling a certain distance before undergoing an interaction (either absorption or scattering).

The key insight is that the culminative transmittance $T(\mathbf{x}_0, \mathbf{x}_t)$, as described by the Beer-Lambert Law (Equation 1.5), reduces to a simple exponential:

$$T(\mathbf{x}_0, \mathbf{x}_t) \ = \ e^{-\mu_t t}. \tag{1.8}$$

The transmittance can be thought of as a survival function, which describes the probability of a photon