Thermal Radiation I
Chapter 6
The Monte Carlo Method and its Application to Radiative Heat Transfer

Example 1: Blackbody Emissive Power (Planck function)

The basic parameters

$$e_{\lambda b} = \frac{2\pi C_1}{\lambda^5 \left(e^{C_2/\lambda T} - 1\right)}$$

Energy emitted per unit area per unit wavelength per unit time

$$\sigma T^4 = \int_0^\infty e_{\lambda b} d\lambda$$

Total Energy emitted over all wavelenth per unit area per unit time

$$\frac{e_{\lambda b} d\lambda}{\sigma T^4}$$

The probability that an energy "bundle" emitted from a black surface at temperature T to be within the wavelength region between λ and $\lambda + d\lambda$

$$\frac{e_{\lambda b}}{\sigma T^4} = P(\lambda)$$

The probability density function (pdf) of energy emission from a black surface at λ

$$\int_0^\lambda \frac{e_{\lambda b} d\lambda}{\sigma T^4} = F_{0-\lambda T}$$

The cumulative distribution function of energy emission from a black surface at λ

Example 1: Blackbody Emissive Power (Planck function)

The Monte Carlo Simulation

$$\int_0^{\lambda} \frac{e_{\lambda b} d\lambda}{\sigma T^4} = F_{0-\lambda T} = R_{\lambda} = \xi$$

Random number from computer

Monte Carlo simulation of radiative emission from a blackbody:

N = number of "bundles" used in the simulation
$e = \sigma T^4/N$ = energy per bundle
For each bundle, pick a random number ξ
Determine the wavelength λ from the relation

$$\xi = R_{\lambda} = F_{0-\lambda}$$

Method of Evaluating Wavelength from Random Number (Empirical Relations)

TABLE 7.1
Inverse Probability Function for Choosing Wavelength of Emission from a Gray or Black Surface (λT in μm·K)

$$\lambda T = D_1 + D_2 R_\lambda^{1/8} + D_3 R_\lambda^{1/4} + D_4 R_\lambda^{3/8} + D_5 R_\lambda^{1/2} \qquad 0.0 < R_\lambda < 0.1$$

$$\lambda T = D_1 + D_2 R_\lambda + D_3 R_\lambda^2 + D_4 R_\lambda^3 + D_5 R_\lambda^4 \qquad 0.1 < R_\lambda < 0.9$$

$$\lambda T = \left[\frac{0.152886 \times 10^{12}}{D_1(1-R_\lambda) + D_2(1-R_\lambda)^2 + D_3(1-R_\lambda)^3 + D_4(1-R_\lambda)^4}\right]^{1/3} \qquad 0.9 < R_\lambda < 1$$

Range of R_λ	D_1	D_2	Coefficients D_3	D_4	D_5
0.0–0.1	503.247	230.243	5,863.85	−10,759.6	8,723.14
0.1–0.4	1,560.84	7,603.61	−15,540.1	31,257.7	−20,844.8
0.4–0.7	2,846.63	−1,430.38	27,936.0	−41,041.9	25,960.9
0.7–0.9	345,197	−1,828,567	3,674,856	−3,284,391	1,108,939
0.9–0.99	1.200	9.476	−44.84	156.9	—
0.99–1.0	1.10064	16.8148	−183.445	890.699	—

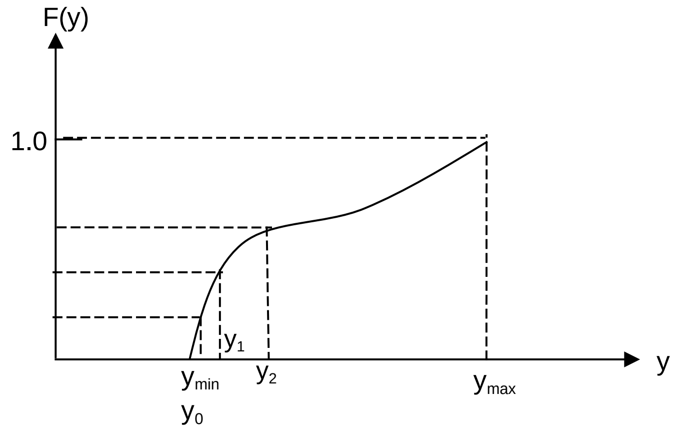
Source: Haji-Sheikh, A.: Monte Carlo Methods, in W. J. Minkowycz, E. M. Sparrow, G. E. Schneider, and R. H. Pletcher (eds.), Chap. 16, *Handbook of Numerical Heat Transfer,* 1st ed., pp. 672–723, Wiley Interscience, New York, 1988. (Slightly modified for $R_\lambda > 0.9$ as a result of personal communication with A. Haji-Sheikh.)

Note: An alternative formulation accurate within 1 percent for the range $750 \le \lambda T \le 65 \times 10^3$ ($5.96 \times 10^{-6} \le R_\lambda \le 0.99957$) is given in Haji-Sheikh and Howell (2006):

$$\lambda T = 1 - \exp\left[-1.2\sqrt[3]{R_\lambda/(1-R_\lambda)}\right] - \frac{0.12 + 7.0\times10^{-5}[R_\lambda/(1-R_\lambda)] - 0.005\sqrt{R_\lambda/(1-R_\lambda)}}{\{1+0.30\,[R_\lambda/(1-R_\lambda)]^{-3/4}\}\{1+7.0\times10^{-6}\,[R_\lambda/(1-R_\lambda)]^{3/2}\}} + \frac{0.12 + 6.0\times10^{-4}(1-R_\lambda)^{-2}}{\{1+5.0\,[R_\lambda/(1-R_\lambda)]^{2/3}\}^4}$$

Method of Evaluating Wavelength from Random Number (Discretization)

```
c     lambda T products are in um K
          data prodtable/   555.6,  1666.7,  3055.6,  4166.7,  5277.8,
     1              6388.9,  7500.0,  9722.2, 12777.8, 55555.6/
          data fractable/ 0.17d-7, 0.02537, 0.28576, 0.51029, 0.66685,
     1             0.76838, 0.83435, 0.90819, 0.95307, 1./
```



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     1            0.76838, 0.83435, 0.90819, 0.95307, 1./
```

Example of algorithm

$N[(\lambda T)_i]$ = number of energy bundles with $(\lambda T)_{i-1} < \lambda T < (\lambda T)_i$

Initially, set N[1666.7], N[3055.6], … N[55555.6] = 0

Pick random number ζ, determine λT from lookup table, determine $F_{0-\lambda T}$ update the appropriate $N[(\lambda T)_i]$

Method of Evaluating Wavelength from Random Number (Discretization)

```
c     lambda T products are in um K
        data prodtable/   555.6,  1666.7,  3055.6,  4166.7,  5277.8,
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     1           0.76838, 0.83435, 0.90819, 0.95307, 1./
```

<u>Example of algorithm</u>

e.g. The random number is $\zeta = 0.4287$

The random number is in the range of [0.28576, 0.51079]

From the random number ζ, determine λT from lookup table, determine $F_{0\text{-}\lambda T}$ update the appropriate $N[(\lambda T)_i]$

Increment the energy bundle bin N[4166.7] by 1;

N[4166.7] = N[4166.7] + 1

$\lambda T = 3055.6 + (\zeta - 0.28576)/(0.51029 - 0.28576)*(4166.7 - 3055.6)$

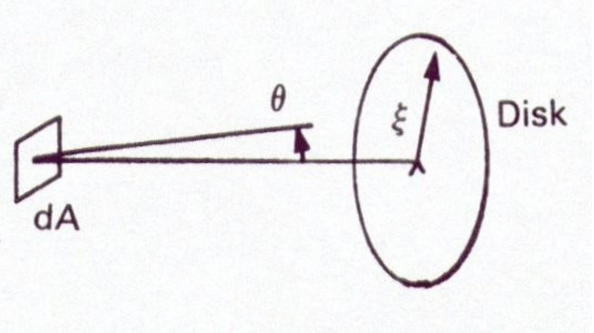
$F_{0\text{-}\lambda T} = \zeta$

Repeat the process until the number of bundle is sufficiently large to achieve convergence

Monte Carlo Simulation of arbitrary probability density distribution with two or more independent variables

Example: Distribution of radiation packets arriving at various disk radii

F(ξ, θ) = number of packets that have arrived at the disk within each small radial increment Δξ and angular increment Δθ about some radius ξ and angle θ



f(ξ, θ) = F(ξ, θ)/(ΔξΔθ) = frequency function, the number of per unit ξ and per unit θ arriving at the disk at (ξ,θ)

P(ξ,θ) = probability density function (in two dimension)

P(ξ,θ)dξdθ = probability that a radiation packet will arrive within an infinitesimal area dξdθ about the position (ξ,θ)

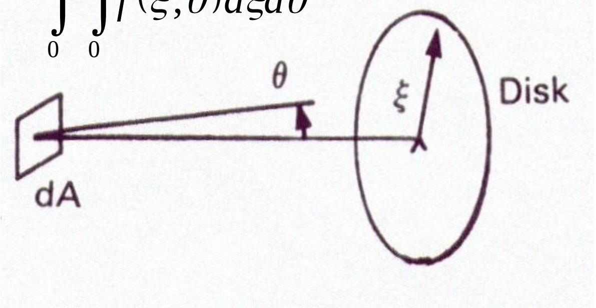
$$P(\xi,\theta) = \frac{f(\xi,\theta)}{\int_0^{\pi}\int_0^{2\pi} f(\xi,\theta)\,d\xi\,d\theta}$$

Question: Can a random number generator be used to simulate the probability density distribution of the radiation packets.

Monte Carlo Simulation of arbitrary probability density distribution with two or more independent variables

Example: Distribution of radiation packets arriving at various disk radii

$$P(\xi,\theta) = \frac{f(\xi,\theta)}{\int_0^{\pi/2}\int_0^{2\pi} f(\xi,\theta)\,d\xi\,d\theta}$$



<u>Question:</u> Can a random number generator be used to simulate the probability density distribution of the radiation packets.

<u>Answer:</u>

1. Pick a random number R_1, determine ξ from

$$R_1 = F(\xi) = \int_{\xi_{\min}}^{\xi}\int_{\theta_{\min}}^{\theta_{\max}} P(\xi,\theta)\,d\theta\,d\xi$$

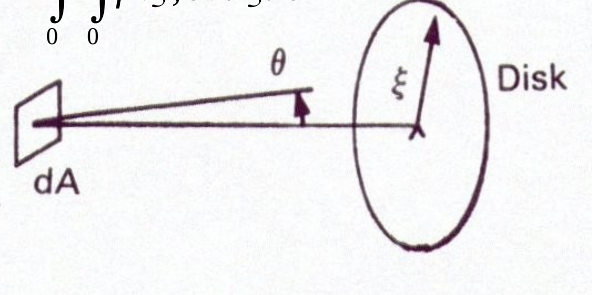
for example, for the radiation packets problem

$$R_1 = F(\xi) = \int_0^{\xi}\int_0^{\pi/2} P(\xi,\theta)\,d\theta\,d\xi$$

Monte Carlo Simulation of arbitrary probability density distribution with two or more independent variables

Example: Distribution of radiation packets arriving at various disk radii

$$P(\xi,\theta) = \frac{f(\xi,\theta)}{\int_0^{\pi/2}\int_0^{2\pi} f(\xi,\theta)\,d\xi\,d\theta}$$



<u>Question:</u> Can a random number generator be used to simulate the probability density distribution of the radiation packets.

<u>Answer:</u>

2. For a given ξ, pick a second random number R_2, determine θ from

$$R_2 = G(\xi,\theta) = \int_{\theta_{\min}}^{\theta} P(\xi,\theta')\,d\theta'$$

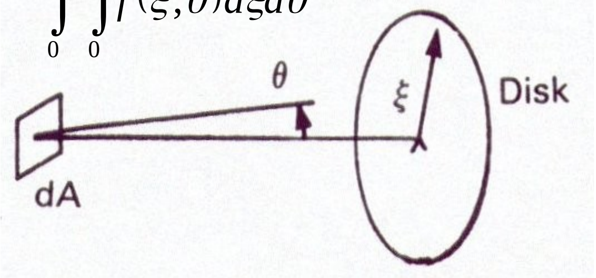
for example, for the radiation packets problem

$$R_2 = G(\xi,\theta) = \int_0^{\theta} P(\xi,\theta')\,d\theta'$$

Monte Carlo Simulation of arbitrary probability density distribution with two or more independent variables

Example: Distribution of radiation packets arriving at various disk radii

$$P(\xi,\theta) = \frac{f(\xi,\theta)}{\int_0^{\pi/2}\int_0^{2\pi} f(\xi,\theta)\,d\xi\,d\theta}$$



Question: Can a random number generator be used to simulate the probability density distribution of the radiation packets.

Answer:

3. If the two-variables probability density function can be expressed as a product of pdf for the two variables

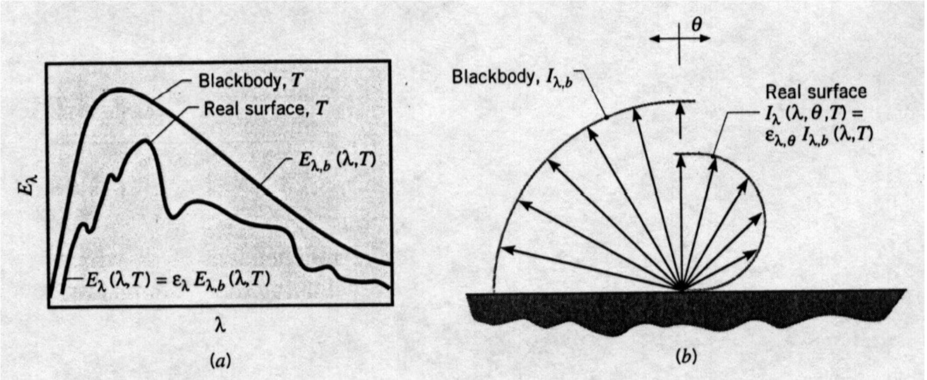
$$P(\xi,\theta) = P_1(\xi)P_2(\theta)$$

then

$$R_1 = F_1(\xi) = \int_0^{\xi} P_1(\xi')\,d\xi'$$

$$R_2 = F_2(\theta) = \int_0^{\theta} P_2(\theta')\,d\theta'$$

Probability density function for general radiation emission



Real Surface Emission

<u>Directional Spectral Emission</u>

$$E(\lambda,\theta,\xi) = \frac{1}{\pi}\varepsilon_\lambda'(\theta,\xi)E_{\lambda b}\cos\theta\sin\theta d\theta d\xi$$

with ε'$_\lambda$ (θ, ξ) = spectral directional emissivity

<u>Total Emission</u>

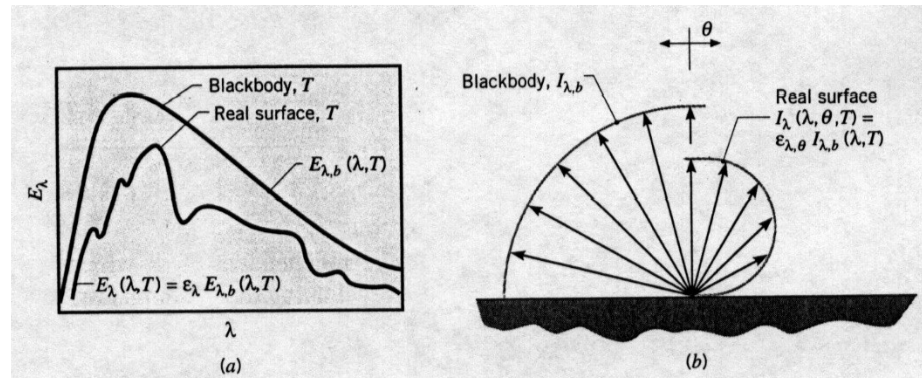
$$E(T) = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\infty}\frac{1}{\pi}\varepsilon_\lambda'(\theta,\xi)E_{\lambda b}\cos\theta\sin\theta d\theta d\xi$$

$$=\varepsilon(T)\sigma T^4$$

$\varepsilon(T) =$ total emissivity

Probability density function for general radiation emission

Real Surface Emission



<u>Probability Density Function for Surface Emission</u>

$$P(\lambda,\theta,\xi) = \frac{\varepsilon_\lambda'(\theta,\xi) E_{\lambda b} \cos\theta \sin\theta}{\pi\varepsilon(T)\sigma T^4}$$

For $\quad \varepsilon_\lambda'(\theta,\xi) = \Phi_1(\lambda)\Phi_2(\theta)$

$$P(\lambda,\theta,\xi) = P_1(\lambda)P_2(\theta)P_3(\xi)$$

$$P_1(\lambda) = \frac{\Phi_1(\lambda)E_{\lambda,b}}{\varepsilon(T)\sigma T^4}$$

$$P_2(\theta) = 2\Phi_2(\theta)\cos\theta\sin\theta$$

$$P_3(\varphi) = \frac{1}{2\pi}$$

For gray diffuse surface ($\varepsilon_{\lambda,\theta} = \varepsilon(T)$)

$$P_1(\lambda) = \frac{E_{\lambda,b}}{\sigma T^4} \qquad R(\lambda) = \int_0^\lambda \frac{E_{\lambda,b}\,d\lambda}{\sigma T^4} = F_{0-\lambda}$$

$$P_2(\theta) = 2\cos\theta\sin\theta \qquad R(\theta) = \int_0^\theta 2\cos\theta'\sin\theta' d\theta' = \sin^2\theta$$

$$P_3(\varphi) = \frac{1}{2\pi} \qquad R(\varphi) = \frac{\varphi}{2\pi}$$

Note that even for a diffuse surface, the pdf is not uniform in θ and ξ

Example of Calculating Configuration Factor

N = total number of energy bundle emitted from surface dA_1

H = total number of energy bundle hitting surface A_2

H = 0, N = 0 initially

Pick random number ξ

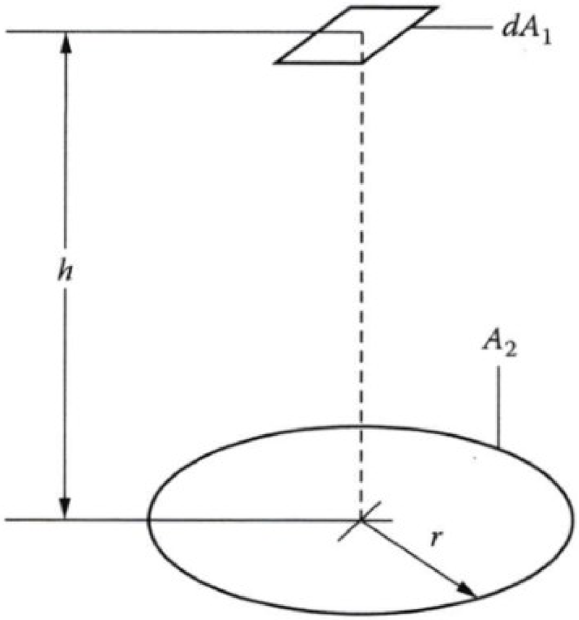
Increment the counter N

$$\xi = \sin^2\theta \Rightarrow \theta = \sin^{-1}\sqrt{\xi}$$

If $\theta < \sin^{-1}\left(\frac{r}{\sqrt{r^2+h^2}}\right)$

or $\sqrt{\xi} = \sin\theta < \frac{r}{\sqrt{r^2+h^2}}$

Increment the counter H



$$F_{d1-2} = \frac{H}{N}$$

ME 240 Radiation I
Midterm Part b

Develop a Monte Carlo simulation of the Planck function.

Compare the simulation results with the exact formulation at 1000 K for three different sampling sizes (1000, 10000, 100000)

Evaluation of Configuration Factor by the Monte Carlo Method

- Configuration Factor (F_{1-2}) = Fraction of radiation emitted from a surface A_1 which is absorbed by a black surface A_2 (without accounting for reflection from other surfaces)

  - Emit N_1 energy bundles from area A_1 using the probabilistic distribution

  - Counts all the energy bundles which is intercepted by A_2, N_2, $F_{1-2} = N_2 / N_1$

Evaluation of Configuration Factor by the Monte Carlo Method

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  - Emit N_1 energy bundles from area A_1 using the probabilistic distribution

  - Counts all the energy bundles which is intercepted by A_2, N_2, $F_{1-2} = N_2 / N_1$

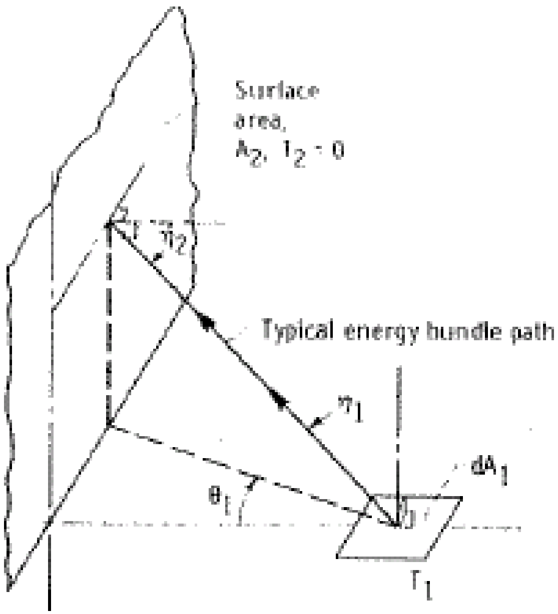
Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

- Physics
  - Consider the radiative transfer into a black surface dA_1, at temperature $T_1 = 0$ from a nongray, nondiffuse surface A_2, an infinite plane at temperature T_2, = T with

$$\varepsilon_2(\lambda,\eta_2,\phi_2,T) = \varepsilon_{2,\lambda}(\lambda,T)\,\varepsilon_{2,\eta}(\eta_2,T)$$

Note that

$$\varepsilon_2(\lambda,\eta_2,\phi,T) = \alpha_2(\lambda,\eta_2,\phi,T)$$



Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

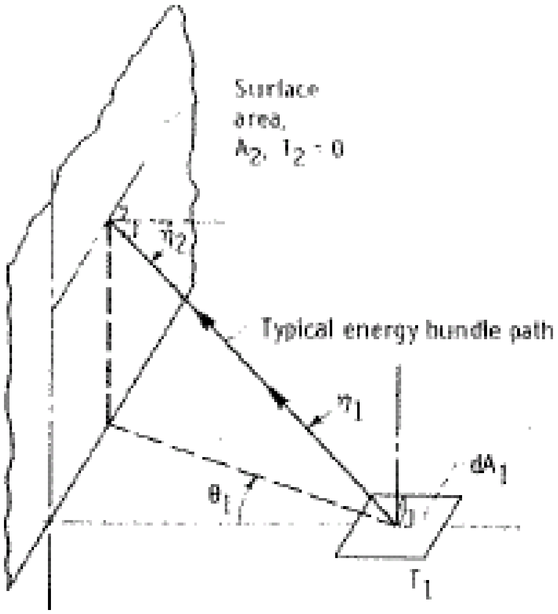
- Physics

- The heat transfer is

$$dQ_{2-d1} =$$

$$dA_1 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\pi} e_{\lambda b}(T)\varepsilon_{\lambda}(\lambda,T)\varepsilon_{\eta}(\eta_2,T)\cos\eta_2 \sin\eta_2 d\phi d\eta_2 d\lambda$$

Note: This model is difficult to be simulated by Monte Carlo



Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

- Physics
  - Consider the radiative transfer from a nondiffuse nongray surface dA_1, at temperature $T_1 = T$ into a black surface A_2, an infinite plane at temperature T_2, = 0

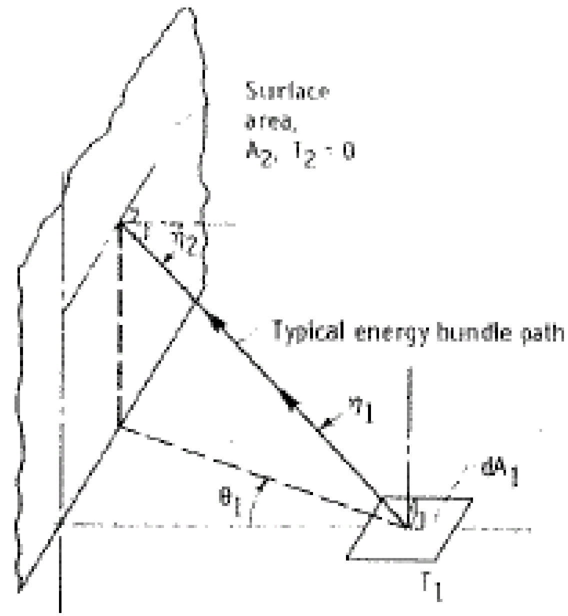
$$dQ_{d1-2} =$$

$$dA_1 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\pi} e_{\lambda b}(T)\varepsilon_{\lambda}(\lambda,T)\varepsilon_{\eta}(\eta_1,T)\cos\eta_1 \sin\eta_1 d\phi d\eta_1 d\lambda$$

$$= dA_1 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\pi} e_{\lambda b}(T)\varepsilon_{\lambda}(\lambda,T)\varepsilon_{\eta}\left(\frac{\pi}{2}-\eta_2,T\right)\cos\eta_2 \sin\eta_2 d\phi d\eta_2 d\lambda$$

$$= dA_1 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\pi} e_{\lambda b}(T)\varepsilon_{\lambda}(\lambda,T)\varepsilon_{\eta}(\eta_2',T)\cos\eta_2' \sin\eta_2' d\phi d\eta_2' d\lambda$$

$$= dQ_{2-d1}$$

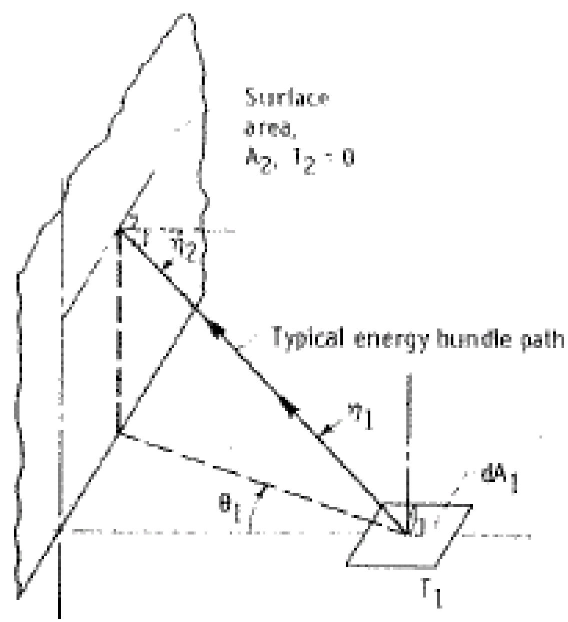


Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

- Physics
  - Consider the radiative transfer from a black surface dA_1, at temperature $T_1 = T$ into a nongray, nondiffuse surface A_2, an infinite plane at temperature $T_2, = 0$

$$dQ_{d1-2} =$$

$$dA_1 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\pi} e_{\lambda b}(T)\,\varepsilon_{2,\lambda}(\lambda,T)\,\varepsilon_{2,\eta}(\eta_1,T)\cos\eta_1 \sin\eta_1\, d\phi\, d\eta_1\, d\lambda$$



<u>Monte Carlo Simulation</u>

dA_1 is a nongray, nondiffuse surface, emitting energy bundle with the given emissivities

Energy bundle will get absorbed if it reaches A_2

Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

- Physics
  - Consider the radiative transfer from a nongray nondiffuse surface dA_1, at temperature $T_1 = T$ into a black surface A_2, an infinite plane at temperature T_2, = 0

<u>For the emission from surface dA_1</u>

$$P_1(\lambda) = \frac{\varepsilon_\lambda(\lambda) E_{\lambda,b}}{\varepsilon(T)\sigma T^4}$$

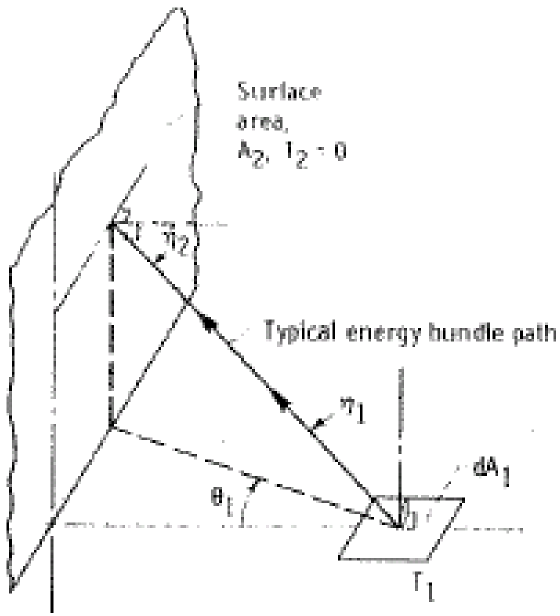
$$P_2(\eta) = 2\varepsilon_\eta(\eta)\cos\eta\sin\eta$$

$$P_3(\phi) = \frac{1}{2\pi}$$

$$R(\lambda) = \frac{1}{\varepsilon(T)\sigma T^4}\int_0^\lambda \varepsilon_\lambda(\lambda) E_{\lambda,b} d\lambda$$

$$R(\eta_1) = \frac{\int_0^{\eta_1} 2\varepsilon_\eta(\eta)\cos\eta\sin\eta d\eta}{\int_0^{\frac{\pi}{2}} 2\varepsilon_\eta(\eta)\cos\eta\sin\eta d\eta}$$

$$R(\phi) = \frac{\phi}{2\pi}$$



Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

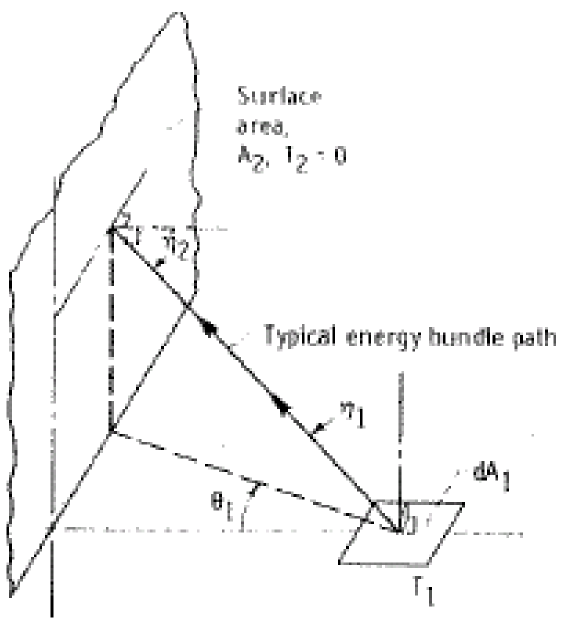
- Physics
  - Consider the radiative transfer from a nongray nondiffuse surface dA_1, at temperature T_1 = T into a black surface A_2, an infinite plane at temperature T_2, = 0

For convenience, it might be useful to generate an empirical correlations for the integral

$$\zeta_\lambda = R(\lambda) = A_1 + A_2\lambda + A_3\lambda^2 + \ldots$$

$$\zeta_\eta = R(\eta_1) = B_1 + B_2\eta_1 + B_3\eta_1^2 + \ldots$$

$$\zeta_\phi = R(\phi) = \frac{\phi}{2\pi}$$



Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

- Physics
  - Consider the radiative transfer from a nongray nondiffuse surface dA_1, at temperature $T_1 = T$ into a black surface A_2, an infinite plane at temperature T_2, = 0

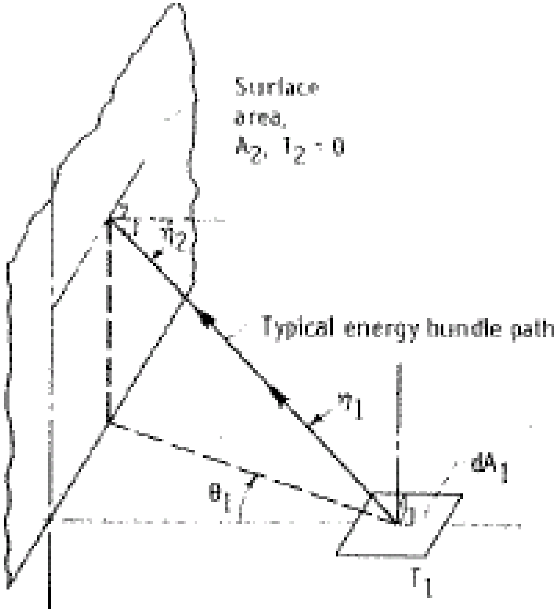
<u>Simulation</u>

Pick random numbers $\zeta_\lambda, \zeta_\eta$ and ζ_ϕ

Determine λ, η_1, ϕ_1

Determine if bundle hits surface $A_2 (0 \leq \phi \leq \pi)$

Update bundle count and energy if it is a "hit"



Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

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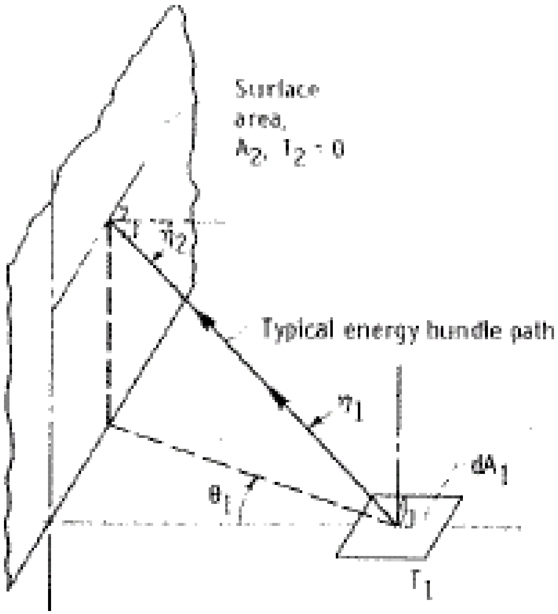
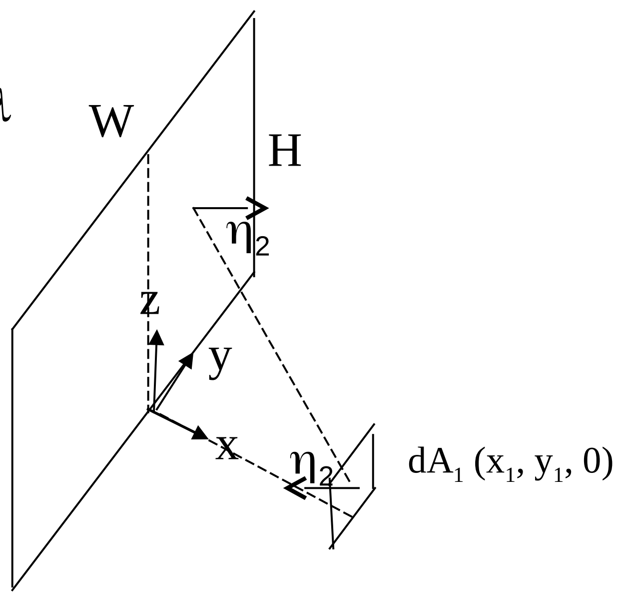


Raditive Exchange between Nongray, Nondiffuse Surfaces by the Monte Carlo Method

- Case with dA_1 parallel to A_2

$$dQ_{2-d1} = dQ_{d1-2}$$

$$dA_1 \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\pi} e_{\lambda b}(T)\,\varepsilon_{\lambda}(\lambda, T)\,\varepsilon_{\eta}(\eta_2, T)\cos\eta_2 \cos\eta_2\, d\phi\, d\eta_2\, d\lambda$$



The development of the “engineering approach” to account for the non-gray effect of combustion mixture

- **The “engineering approach” for particulates**
  - “Small particle” absorption models for soot in luminous flames (Tien and Felske 1973, Sato and Matsumoto 1962)

$$a_{\lambda s}(f_v) = \frac{c}{\lambda} \qquad c = 36\pi f_v \frac{n\kappa}{\left(n^2 - \kappa^2 + 2\right)^2 + 4n^2\kappa^2}$$

$$\varepsilon_s = \frac{1}{\sigma T^4}\int_0^\infty e_{b\lambda}\left(1 - e^{-a_\lambda L}\right)d\lambda = 1 - \frac{15}{\pi^4}\Psi^{(3)}(1 + x)$$

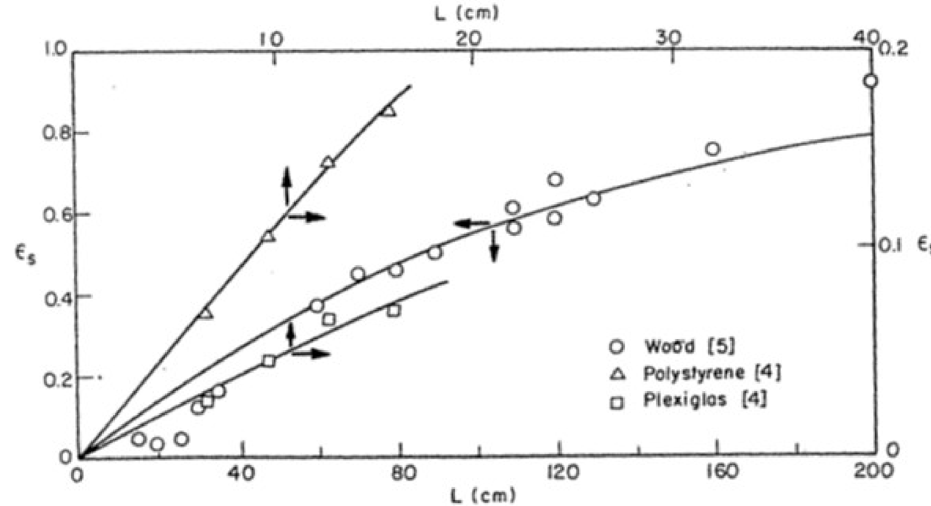
$$x = \frac{cTL}{c_2}$$

The development of the "engineering approach" to account for the non-gray effect of combustion mixture

- **The "engineering approach" for particulates**
  - Gray soot model for luminous flames (Yuen and Tien, 1977)

$$\varepsilon_s = 1 - e^{-kL}$$

$$k = 3.6\frac{cT}{c_2}$$



The development of the "engineering approach" to account for the non-gray effect of combustion mixture

- **The "engineering approach" for particulates**
  - For wood with L = 1 m, ε ≈ 0.5

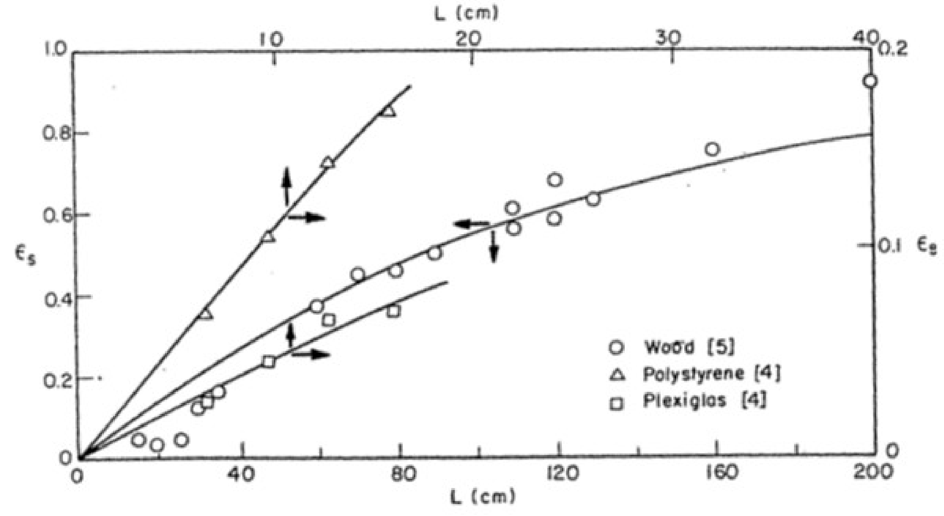
k = 0.69 1/m

c = 0.0028/T(K)

$$a_{\lambda s}(f_v)L = \frac{0.0028}{\lambda T} \quad L = 1m$$

$$\varepsilon_s = 1 - e^{-kL}$$

$$k = 3.6\frac{cT}{c_2}$$



The development of the "engineering approach" to account for the non-gray effect of combustion mixture

- **The "engineering approach" for particulates**
  - For wood with L = 1 m, ε ≈ 0.5

$$\varepsilon_\lambda = 1 - e^{-a_s L} = 1 - e^{-\frac{0.0028}{1300\lambda}} = 1 - e^{-\frac{2.15\times 10^{-6}}{\lambda}}$$

Spectral emissivity at 1300K

λ (μm)	ε_λ
0.1	1.0
0.5	0.987
1.0	0.88
2.0	0.659
5.0	0.35