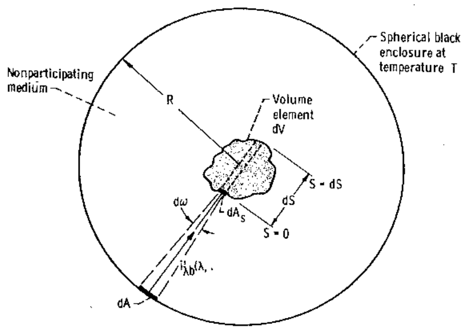
6 Geometrical Problem of Gas-Radiative Interchange

6.1 Emission from a gas volume of temperature T and absorption coefficient a_λ



Emission from the surface

$$i'(0) = i'_{\lambda b}(\lambda, T)$$

Change of intensity in dV as a result of absorption

$$di' = -i'_{\lambda b}(\lambda, T)\, a_\lambda dS$$

Energy absorbed by a differential volume $dSdA_s$

$$d^5 Q'_{\lambda,a} = -i'_{\lambda b}(\lambda, T)\, a_\lambda d\lambda dS dA_s d\omega$$

Where

$$d\omega = \frac{dA}{R^2}$$

With dA being the source area normal to $i'(0) = i'_{\lambda b}(\lambda, T)$

The absorption by all of the volume is

$$d^4Q'_{\lambda,a} = i'_{\lambda b}(\lambda,T)\, a_\lambda d\lambda d\omega \int_{dV} dS dA_s = i'_{\lambda b}(\lambda,T)\, a_\lambda d\lambda d\omega dV$$

The absorption by the volume for energy from all direction is

$$d^3Q'_{\lambda,a} = i'_{\lambda b}(\lambda,T)\, a_\lambda d\lambda dV \int_{4\pi} d\omega = 4\pi i'_{\lambda b}(\lambda,T)\, a_\lambda d\lambda dV$$

For thermodynamic equilibrium, the emission must equal to absorption in an isothermal medium

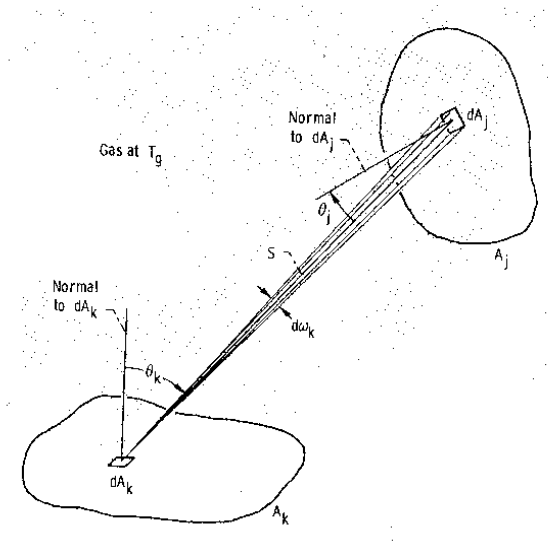
$$d^2Q'_{\lambda,e} = 4\pi i'_{\lambda b}(\lambda,T)\, a_\lambda d\lambda dV$$

Since the emission of a volume should be independent of its environment, the above expression is applicable for emission of a gas volume of temperature T in general.

For a gray medium, a_λ is independent of λ, the total emission from a gas volume is

$$dQ'_e = 4a\sigma T^4 dV$$

6.2. Concept of Direct Exchange Factors (For a Participating Gas and Black Surfaces, factors do not include exchange due to reflection and scattering)



a. Surface-surface Exchange

$$d^3Q_{di-dj,\lambda} = E_{b\lambda,i}\frac{\cos\theta_i\cos\theta_j\tau_\lambda(S)}{\pi S^2}dA_i dA_j d\lambda$$

$$d^3Q_{dj-di,\lambda} = E_{b\lambda,j}\frac{\cos\theta_i\cos\theta_j\tau_\lambda(S)}{\pi S^2}dA_i dA_j d\lambda$$

With

$$\tau_\lambda(S) = e^{-K_\lambda S}$$

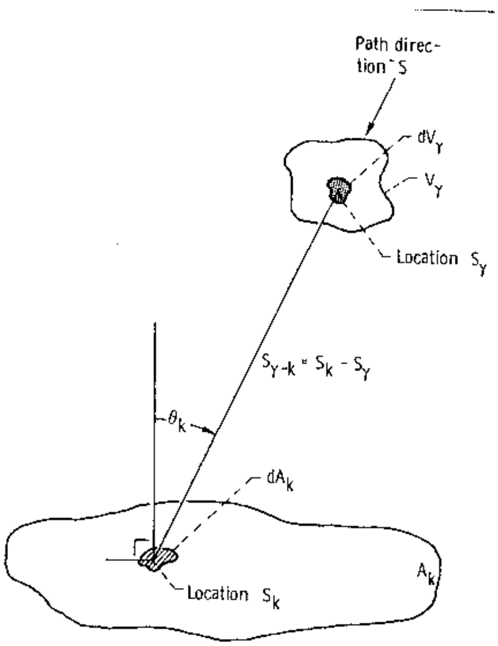
The surface-surface exchange factor is defined by

$$dQ_{\lambda,i\Leftrightarrow j} = \left(E_{b\lambda,i} - E_{b\lambda,j}\right)s_i s_j d\lambda$$

With

$$s_i s_j = \int_{A_i}\int_{A_j}\frac{\cos\theta_i\cos\theta_j\tau_\lambda(S)\,dA_i dA_j}{\pi S^2}$$

b. Surface-volume Exchange



$$d^3Q_{dk-d\gamma,\lambda} = E_{b\lambda,k}\frac{K_\lambda(\gamma)\cos\theta_k\tau_\lambda(S_{\gamma-k})}{\pi S_{\gamma-k}^2}dA_k dV_\gamma d\lambda$$

$$d^3Q_{d\gamma-dk,\lambda} = E_{b\lambda,\gamma}\frac{K_\lambda(\gamma)\cos\theta_k\tau_\lambda(S_{\gamma-k})}{\pi S_{\gamma-k}^2}dA_k dV_\gamma d\lambda$$

and the surface-volume exchange factor is given by

$$dQ_{\lambda,k\Leftrightarrow\gamma} = \left(E_{b\lambda,k} - E_{b\lambda,\gamma}\right)s_k g_\gamma d\lambda$$

With

$$s_k g_\gamma = \int_{A_k}\int_{V_\gamma}\frac{K_\lambda(\gamma)\cos\theta_k\tau_\lambda(S_{\gamma-k})\,dA_k dV_\gamma}{\pi S_{\gamma-k}^2}$$

b. Volume-volume Exchange

$$d^3Q_{dk-d\gamma,\lambda} = E_{b\lambda,k}\frac{K_\lambda(k)K_\lambda(\gamma)\tau_\lambda(S_{\gamma-k})}{\pi S_{\gamma-k}^2}dV_k dV_\gamma d\lambda$$

$$d^3Q_{d\gamma-dk,\lambda} = E_{b\lambda,\gamma}\frac{K_\lambda(k)K_\lambda(\gamma)\tau_\lambda(S_{\gamma-k})}{\pi S_{\gamma-k}^2}dV_k dV_\gamma d\lambda$$

and the volume-volume exchange factor is given by

$$g_i g_j = \frac{dQ_{\lambda,i\Leftrightarrow j}}{(E_{b\lambda,i} - E_{b\lambda,j})d\lambda} = \int_{V_i}\int_{V_j}\frac{K_\lambda^2\tau_\lambda(r)dV_i dV_j}{\pi r^2}$$

$$dQ_{\lambda,k\Leftrightarrow\gamma} = (E_{b\lambda,k} - E_{b\lambda,\gamma})\,g_k g_\gamma d\lambda$$

With

$$g_k g_\gamma = \int_{V_k}\int_{V_\gamma}\frac{K_\lambda(\gamma)K_\lambda(k)\tau_\lambda(S_{\gamma-k})\,dV_k dV_\gamma}{\pi S_{\gamma-k}^2}$$

6.2 Mathematical Properties of Direct Exchange Factors

a. Reciprocity

$s_i s_j = s_j s_i$

$s_i g_j = g_j s_i$

$g_i g_j = g_j g_i$

Note that $s_i s_j = A_i F_{i-j}$ in the limit of K_λ approaches zero

b. Summation Rule for an Enclosure with a Participating Medium

$$\sum_j s_i s_j + \sum_k s_i g_k = A_i$$
$$\sum_j g_i s_j + \sum_k g_i g_k = 4K_i V_i$$

and therefore physical interpretations of the direct exchange factors are

$\frac{s_i s_j}{A_i}$ = fraction of energy radiated from surface A_i and intercepted by surface A_j

$\frac{s_i g_j}{A_i}$ = fraction of energy radiated from surface A_i and intercepted by volume V_j

$\frac{g_i s_j}{4K_i V_i}$ = fraction of energy radiated from volume V_i and intercepted by surface A_j

$\frac{g_i g_j}{4K_i V_i}$ = fraction of energy radiated from volume V_i and

intercepted by
volume V_j

6.3 Concepts of Geometric Mean Transmittance and Absorptance

a. Geometric Mean Transmittance, $\tau_{\lambda,i-j}$

$$s_i s_j = A_i F_{i-j} \tau_{\lambda,i-j}$$

$\tau_{\lambda,i-j}$ = Average transmittance of radiation emitted by A_i in the direction of A_j by the medium bounded between the two areas

b. Geometric Mean Absorptance, $\alpha_{\lambda,i-j}$

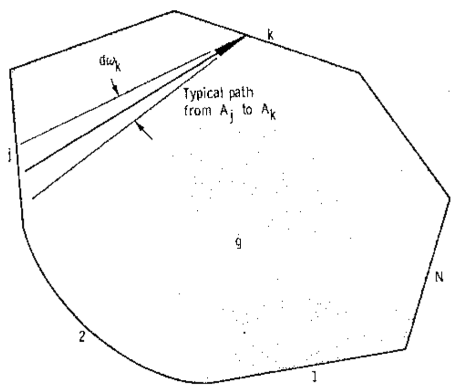
$$\alpha_{\lambda,i-j} = 1 - \tau_{\lambda,i-j}$$

$\alpha_{\lambda,i-j}$ = Average absorptance of radiation emitted by A_i in the direction of A_j by the medium bounded between the two areas

if the medium is isothermal and non-scattering, then

$\alpha_{\lambda,i-j} = \varepsilon_{\lambda,i-j}$ = Average emittance of radiation emitted by the medium bounded between the two areas A_i and A_j in the direction of A_j

6.4 Radiative Exchange in an Enclosure with an Isothermal Absorbing Medium



a. Gray Formulation Based on Direct Exchange Factors

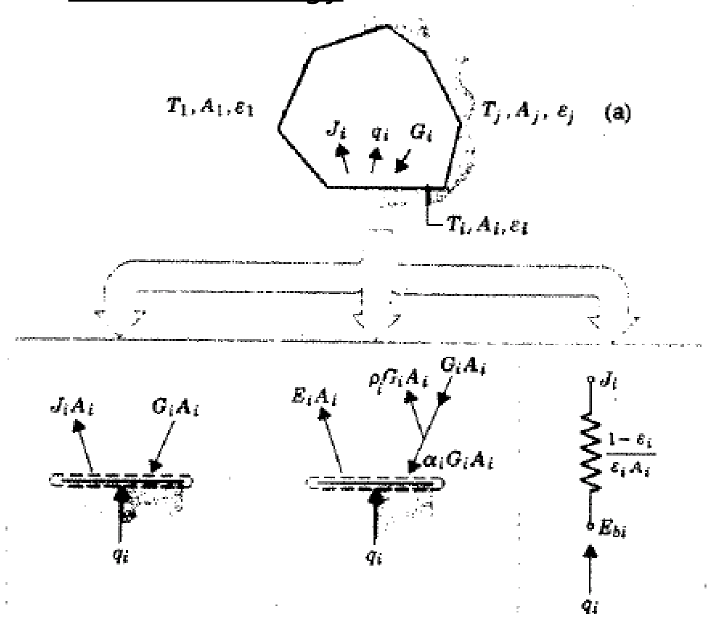
Net Radiation Exchange at a Surface

$$q_i = A_i(J_i - G_i)$$

$$J_i = E_i + \rho_i G_i = \varepsilon_i E_{b,i} + (1 - \varepsilon_i) G_i$$

$$q_i = \frac{E_{b,i} - J_i}{(1-\varepsilon_i)/(A_i \varepsilon_i)}$$

Network Analogy



Radiation Exchange Between Surfaces and Gas Volume

For an area element A_i, the incident radiation is

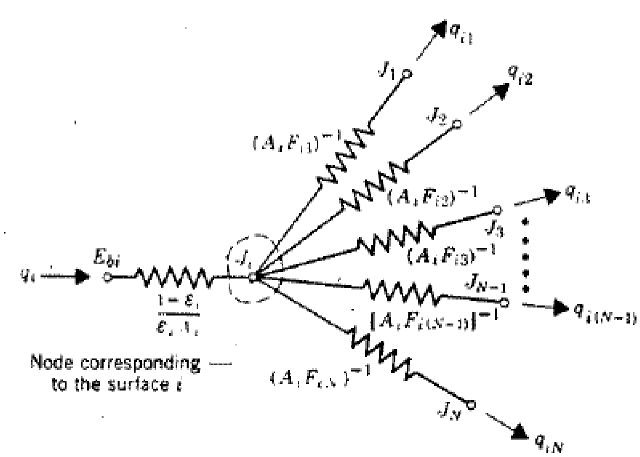
$$\begin{aligned}
A_i G_i &= J_1 (s_1 s_i) + J_2 (s_2 s_i) + \cdots + J_i (s_i s_i) + \cdots + J_N (s_N s_i) + \sigma T_g^4 (g s_i) \\
&= \sum_{k=1}^{N} J_k (s_k s_i) + \sigma T_g^4 (g s_i) \\
&= \sum_{k=1}^{N} J_k (s_i s_k) + \sigma T_g^4 (s_i g) \quad (\text{reciprocity})
\end{aligned}$$

The energy balance

$$\begin{aligned}
q_i &= A_i (J_i - G_i) \\
&= A_i J_i - \sum_{k=1}^{N} J_k (s_i s_k) - \sigma T_g^4 (s_i g) \\
&= \sum_{k=1}^{N} (s_i s_k) (J_i - J_k) + (s_i g) (J_i - \sigma T_g^4) \quad \text{using} \left[\sum_{k=1}^{N} (s_i s_k) + (s_i g) = A_i \right] \\
&= \sum_{k=1}^{N} \frac{(J_i - J_k)}{1/(s_i s_k)} + \frac{(J_i - \sigma T_g^4)}{1/(s_i g)} \\
&= \frac{E_{b,i} - J_i}{(1 - \varepsilon_i)/(A_i \varepsilon_i)}
\end{aligned}$$

$$\frac{1}{s_i s_k} = R_{i-k} = \text{Resistance Between Surface } A_i \text{ and } A_k$$

$$\frac{1}{s_i g} = R_{i-g} = \text{Resistance Between Surface } A_i \text{ and the gas volume g}$$



and for the single gas element V_g

$$\begin{aligned}
4KV_g G_g &= J_1(s_1 g) + J_2(s_2 g) + \cdots + J_i(s_i g) + \cdots + J_N(s_N g) + \sigma T_g^4(gg) \\
&= \sum_{k=1}^{N} J_k(s_k g) + \sigma T_g^4(gg) \\
&= \sum_{k=1}^{N} J_k(g s_k) + \sigma T_g^4(gg) \quad (\text{reciprocity})
\end{aligned}$$

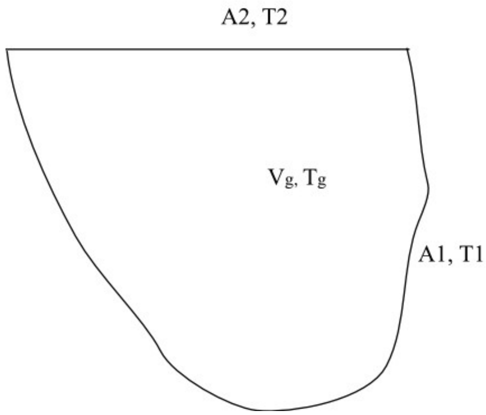
The energy balance

$$\begin{aligned}
q_i &= 4KV_g\left(\sigma T_g^4 - G_g\right) \\
&= 4KV_g \sigma T_g^4 - \sum_{k=1}^{N} J_k(g s_k) - \sigma T_g^4(gg) \\
&= \sum_{k=1}^{N}(g s_k)\left(\sigma T_g^4 - J_k\right) \quad \text{using } \left[\sum_{k=1}^{N}(g s_k) + (gg) = 4KV_g\right] \\
&= \sum_{k=1}^{N} \frac{\left(\sigma T_g^4 - J_k\right)}{1/(g s_k)}
\end{aligned}$$

<u>General Conclusion</u>: In the one-gas-zone approximation, a homogeneous, isothermal absorbing gas volume acts is treated approximately as an effective black surface with an effective area of $4KV_g$

b. Examples of Network Analysis

The Two-Surface Enclosure with an Isothermal Gas Volume with No Internal Heat Generation



$$q_1 = -q_2 = \frac{E_{b,1} - E_{b,2}}{R_{total}}$$

With

$$R_{total} = \frac{1-\varepsilon_1}{A_1\varepsilon_1} + \frac{1-\varepsilon_2}{A_2\varepsilon_2} + R_{net,1-2}$$

$$\frac{1}{R_{net,1-2}} = \frac{1}{1/\left(s_1 s_2\right)} + \frac{1}{1/\left(s_1 g\right) + 1/\left(s_2 g\right)}$$

<u>Remark</u>: Accuracy for a one-gas-zone analysis is generally quite poor in the optically thick limit (i.e., large K). Why?

Case with A1 >> A2 and F_{21} = 1.0, ε_2 = 1.0, T_2 = 0 (a non-evacuated blackbody cavity)

$$R_{total} = \frac{1-\varepsilon_1}{A_1\varepsilon_1} + R_{net,1-2}$$

$$s_1s_2 = A_1F_{12}\bar{\tau}_{12} = A_2F_{21}\bar{\tau}_{12} = A_2F_{21}\bar{\tau}_{12} = A_2\bar{\tau}_{12}$$

$$s_1g = A_1 - s_1s_2 = A_1 - A_2\bar{\tau}_{12}$$

$$s_2g = A_2 - s_2s_1 = A_2 - A_2\bar{\tau}_{12} = A_2(1-\bar{\tau}_{12}) = A_2\bar{\alpha}_{12}$$

$$\frac{1}{R_{net,1-2}} = \frac{1}{1/(A_2\bar{\tau}_{12})} + \frac{1}{1/(A_1 - A_2\bar{\tau}_{12}) + 1/(A_2\bar{\alpha}_{12})} = A_2(\bar{\tau}_{12} + \bar{\alpha}_{12}) = A_2$$

$$q_1 = -q_2 = A_2E_{b,1}$$

<u>A Two-Surface Enclosure with an Isothermal Gas Volume with a Known Gas Temperature</u>

$$\frac{E_{b,1} - J_1}{(1-\varepsilon_1)/(A_1\varepsilon_1)} + \frac{J_2 - J_1}{1/(s_1s_2)} + \frac{\sigma T_g^4 - J_1}{1/(s_1g)} = 0$$

$$\frac{E_{b,2} - J_2}{(1-\varepsilon_2)/(A_2\varepsilon_2)} + \frac{J_1 - J_2}{1/(s_1s_2)} + \frac{\sigma T_g^4 - J_2}{1/(s_2g)} = 0$$

Solution to the above two equations leads to J_1 and J_2

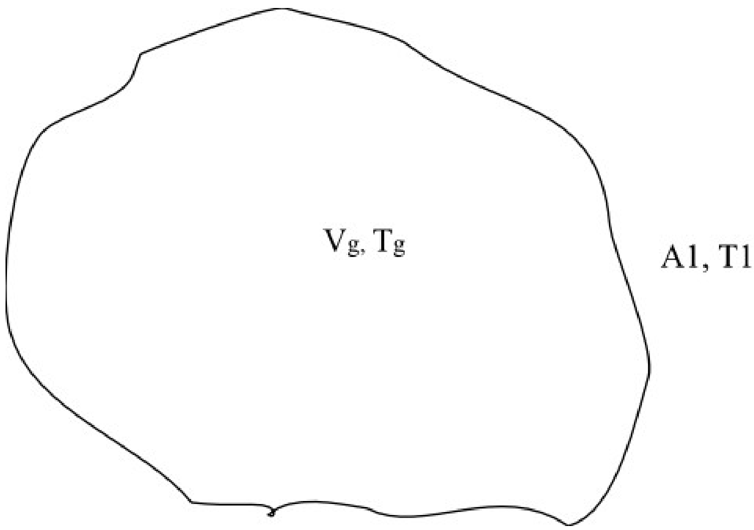
The heat transfer is

$$q_1 = \frac{E_{b,1} - J_1}{(1-\varepsilon_1)/(A_1\varepsilon_1)}$$

$$q_2 = \frac{E_{b,2} - J_2}{(1-\varepsilon_2)/(A_2\varepsilon_2)}$$

$$q_g = \frac{\sigma T_g^4 - J_1}{1/(s_1g)} + \frac{\sigma T_g^4 - J_2}{1/(s_2g)} = -(q_1 + q_2)$$

A One-Surface Enclosure with an Isothermal Gas Volume with a Known Gas Temperature



$$q_1 = -q_g = \left(E_{b,1} - E_{b,g}\right) s_1 g$$

$$s_1 g = A_1 - s_1 s_1 = A_1\left(1 - F_{11}\bar{\tau}_{11}\right) = A_1\left(1 - \bar{\tau}_{11}\right) = A_1\bar{\alpha}_{11}$$

$$q_g = A_1\bar{\alpha}_{11}\left(E_{b,g} - E_{b,1}\right)$$

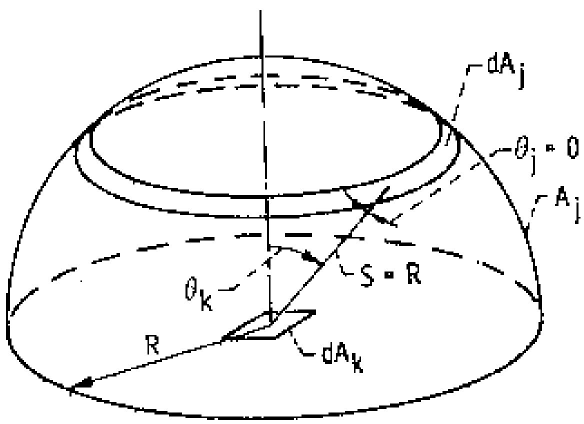
6.5 Total Radiative Exchange Between a Non-Gray Isothermal Non-Scattering Gas and a Black Surface (Concept of Mean Beam Length)

$$q_{g,\lambda} = A_1 \bar{\alpha}_{11,\lambda} \left(E_{b\lambda,g} - E_{b\lambda,1} \right)$$

if $A_i = dA_i$ = a differential area at the center of the base of a hemisphere

and

V_g = a hemispherical gas volume



$$\frac{ds_1 g}{dA_1} = \bar{\alpha}_{d1,g} = \int_{V_g} \frac{a_\lambda \cos\theta \tau_\lambda(r)\, dV_g}{\pi r^2}$$

$$= \int_0^{\pi/2} \int_0^R \frac{a_\lambda \cos\theta e^{-a_\lambda r} 2\pi r^2 \sin\theta d\theta dr}{\pi r^2}$$

$$= 1 - e^{-a_\lambda R}$$

Definition of <u>Mean Beam Length, L_e</u>

L_e = Radius of an equivalent hemisphere of the same temperature and optical properties such that the heat flux at the center of its base is equivalent to the actual average heat flux

$$\frac{(s_1 g)_\lambda}{A_1} = 1 - e^{-a_\lambda L_{e,\lambda}}$$

<u>Mathmatical Properties of $L_{e,\lambda}$ for an non-scattering medium</u>

1. Optical thin limit for

   V_g = total gas volume

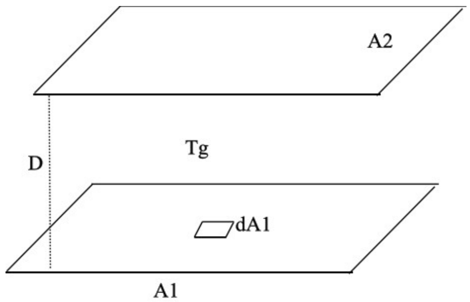
   A = total surface area which encloses the gas

   $$\lim_{a_\lambda \to 0} L_{e,\lambda} = L_{e,0} = \frac{4V_g}{A}$$

2. For a gas volume radiating to its entire surface, $L_{e,\lambda}$ is not a strong function of wavelength, so it is a good approximation to assume that

   $$L_{e,\lambda} = 0.9 L_{e,0} = \frac{3.6 V_g}{A}$$

Example: Mean Beam Length for a Medium between Parallel Plates Radiating to Area on Plate



$$ds_1 g = dA_1 - ds_1 s_2$$

$$ds_1 s_2 = dA_1 \int_{A_2} \frac{e^{-a_\lambda S} \cos^2\theta}{\pi S^2} dA_2$$

Now let

$$S^2 = D^2 + r^2, \quad \cos\theta = \frac{D}{S}, \quad dA_2 = 2\pi r dr = 2\pi S dS$$

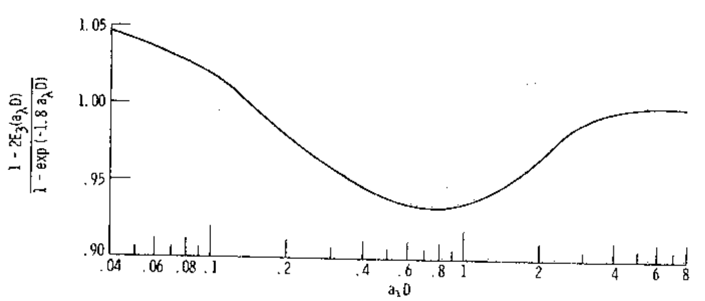
$$ds_1 s_2 = dA_1 2D^2 \int_D^\infty \frac{e^{-a_\lambda S}}{S^3} dS = dA_1 2E_3(a_\lambda D)$$

$$ds_1 g = dA_1 \left[1 - 2E_3(a_\lambda D)\right]$$

The mean beam length is given by

$$\frac{ds_1 g}{dA_1} = \left[1 - 2E_3(a_\lambda D)\right] = 1 - e^{-a_\lambda L_e}$$

$$\frac{L_e}{D} = -\frac{1}{a_\lambda D} \ln\left[2E_3(a_\lambda D)\right]$$



The total average heat flux at the surface of an enclosure with an isothermal non-gray gas is

$$q = \int_0^\infty E_{\lambda b}\left(T_g\right)\frac{\left(s_1 g\right)_\lambda}{A_1}d\lambda$$

$$= \int_0^\infty E_{\lambda b}\left(T_g\right)\left[1 - e^{-a_\lambda L_{e,\lambda}}\right]d\lambda$$

$$= \int_0^\infty E_{\lambda b}\left(T_g\right)\left[1 - e^{-0.9 a_\lambda L_{e,0}}\right]d\lambda$$

$$= \sum_{\text{all band}} E_{\lambda b,i}\left(T_g\right)A_i\left(T_g, P_g, 0.9L_{e,0}\right)$$

Or, using Hottel's chart

$$q = \int_0^\infty E_{\lambda b}\left(T_g\right)\frac{\left(s_1 g\right)_\lambda}{A_1}d\lambda$$

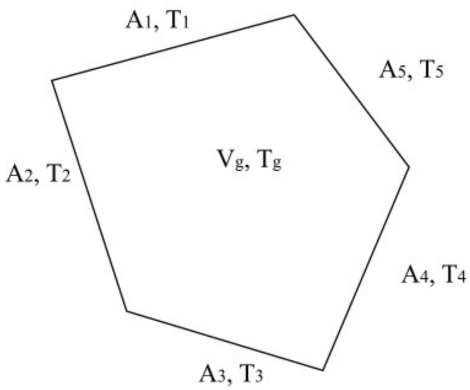
$$= \int_0^\infty E_{\lambda b}\left(T_g\right)\left[1 - e^{-a_\lambda L_{e,\lambda}}\right]d\lambda$$

$$= \int_0^\infty E_{\lambda b}\left(T_g\right)\left[1 - e^{-0.9 a_\lambda L_{e,0}}\right]d\lambda$$

$$= \varepsilon_g E_{\lambda b}\left(T_g\right)$$

Example: As a result of combustion of ethane at 100% excess air, the combustion products are 4 mole of CO2, 6 mole of H2O vapor, 33.3 mole of air, and 26.3 mole of N2. Assume these products are in a cylindrical region 4 m high and 2 m in diameter, are uniformly mixed at a theoretical flame temperature of 1853K, and are at 1 atm pressure. Compute the radiation from the gaseous region.

Concept of Mean Beam Length for Radiative Exchange between two Areas, $L_{e,ij}$



$L_{e,ij}$ = Length of an equivalent one dimensional slab of the same temperature and optical properties such that the transmittance is identical to the geometric mean transmittance

$$\frac{\left(s_i s_j\right)_\lambda}{A_i F_{ij}} = e^{-a_\lambda L_{e,ij,\lambda}}$$

Relationship between the "surface-surface" mean beam length and "surface-volume" mean beam length

$$A_1 = \sum_{i=1}^{5} \left(s_1 s_i\right)_\lambda + \left(s_1 g\right)_\lambda$$
$$= A_1 \sum_{i=1}^{5} F_{ij} e^{-a_\lambda L_{e,1i,\lambda}} + A_1\left(1 - e^{-a_\lambda L_{e,\lambda}}\right)$$

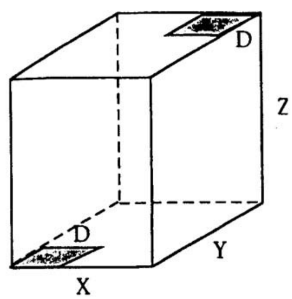
So

$$e^{-a_\lambda L_{e,\lambda}} = \sum_{i=1}^{5} F_{ij} e^{-a_\lambda L_{e,1i,\lambda}}$$

$$L_{e,\lambda} = -\frac{\ln\left(\sum_{i=1}^{5} F_{ij} e^{-a_\lambda L_{e,1i,\lambda}}\right)}{a_\lambda}$$

Concept of Generic Exchange Factor ("The Zonal Method - A Practical Solution Method for Radiative Heat Transfer in Non-Heat Transfer Uniform, Non-Isothermal Absorbing, Emitting and Scattering Media," with E.E. Takara, Annual Review of Heat Transfer, Vol. 8, pp. 153-215, 1995)

<u>Surface-Surface Exchange Factor (Parallel)</u>



$$s_i s_j = \int_{A_i}\int_{A_j} \frac{e^{-\tau}\left|n_i \cdot r\right|\left|n_j \cdot r\right|}{\pi r^4} dA_i dA_j$$

With $r = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{\frac{1}{2}}$, $\tau = \int_{r_i}^{r_j} k(s)\, ds$, $s = |r - r_i|$

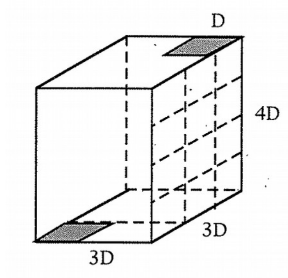
Assuming constant extinction coefficient and introducing

$$\eta_x = \frac{x}{D}, \quad \eta_y = \frac{y}{D}, \quad \eta_z = \frac{z}{D} \quad \eta_r = \frac{r}{D}$$

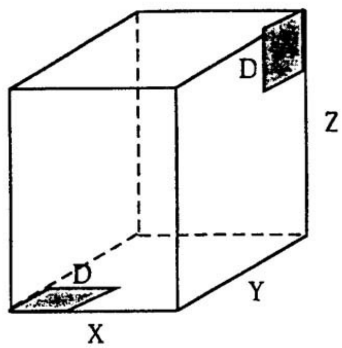
$$\eta_X = \frac{X}{D}, \quad \eta_Y = \frac{Y}{D}, \quad \eta_Z = \frac{Z}{D}$$

$$\frac{s_i s_j}{D^2} = F_{ssp}(kD, \eta_X, \eta_Y, \eta_Z) = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{e^{-kD\eta_r}\left|\eta_{z,i} - \eta_{z,j}\right|^2}{\pi\eta_r^4} d\eta_{x,i} d\eta_{y,i} d\eta_{x,j} d\eta_{y,j}$$

Example of $F_{ssp}(kD, 3, 3, 4)$

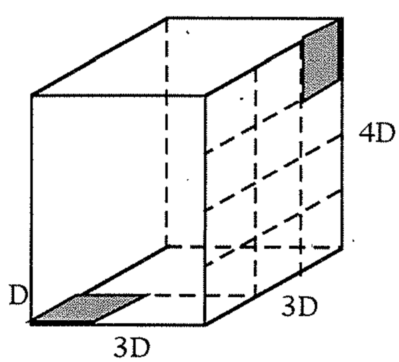


Surface-Surface Exchange Factor (Perpendicular)

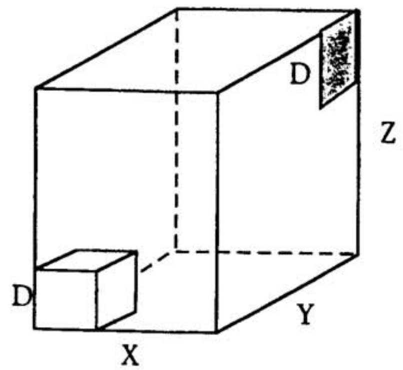


$$\frac{s_i s_j}{D^2} = F_{sst}\left(kD, \eta_X, \eta_Y, \eta_Z\right) = \int_0^1\int_0^1\int_0^1\int_0^1 \frac{e^{-kD\eta_r}\left|\eta_{z,i} - \eta_{z,j}\right|\left|\eta_{x,i} - \eta_{x,j}\right|}{\pi\eta_r^4} d\eta_{x,i} d\eta_{y,i} d\eta_{x,j} d\eta_{y,j}$$

Example of $F_{sst}\left(kD, 3, 3, 4\right)$



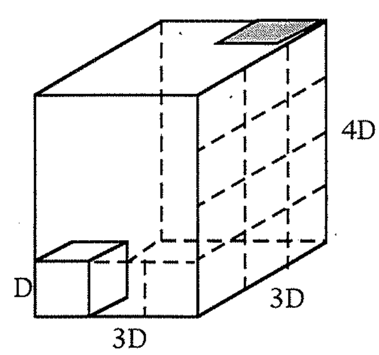
Surface-Volume Exchange Factor



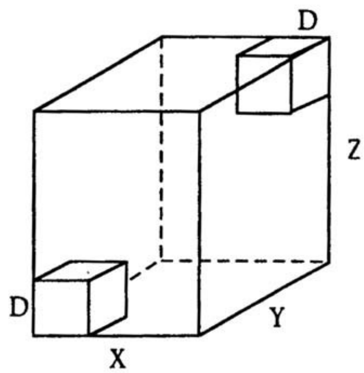
$$g_i s_j = \int_{V_i}\int_{A_j} \frac{k e^{-\tau} \left| n_j \cdot r \right|}{\pi r^3} dV_i dA_j$$

$$\frac{g_i s_j}{kD^3} = F_{gs}\left(kD, \eta_X, \eta_Y, \eta_Z\right) = \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{e^{-kD\eta_r} \left| \eta_{x,i} - \eta_{x,j} \right|}{\pi \eta_r^3} d\eta_{x,i} d\eta_{y,i} d\eta_{z,i} d\eta_{x,j} d\eta_{y,j}$$

Example of $F_{gs}\left(kD, 3, 3, 4\right)$



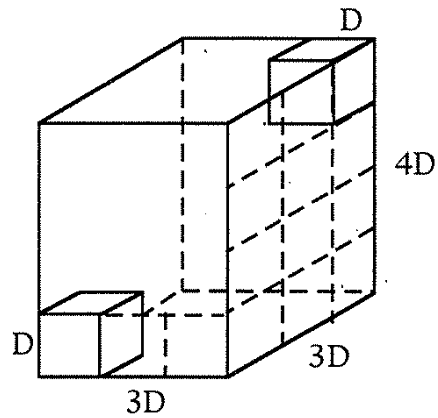
Surface-Volume Exchange Factor



$$g_i g_j = \int_{V_i}\int_{V_j} \frac{k^2 e^{-\tau}}{\pi r^2} dV_i dV_j$$

$$\frac{g_i s_j}{k^2 D^4} = F_{gg}\left(kD, \eta_X, \eta_Y, \eta_Z\right) = \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{e^{-kD\eta_r}}{\pi \eta_r^2} d\eta_{x,i} d\eta_{y,i} d\eta_{z,i} d\eta_{x,j} d\eta_{y,j} d\eta_{z,j}$$

Example of $F_{gg}\left(kD, 3, 3, 4\right)$



Generic Exchange Factors are tabulated for different values of kD, η_X, η_Y and η_Z

Exchange Factors for arbitrary geometry can be generated from generic exchange factor by superposition.