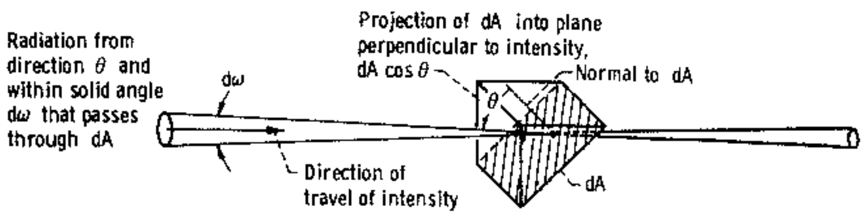
5 Radiation Properties of Absorbing, Emitting and Scattering Media

5.1 Concept of Radiation Intensity in a Medium and Its Properties

$i'_\lambda(x, y, z, \theta, \varphi)$ = energy passing through an area in space per unit time, per unit of the projected area, per unit wavelength interval, and per unit solid angle.



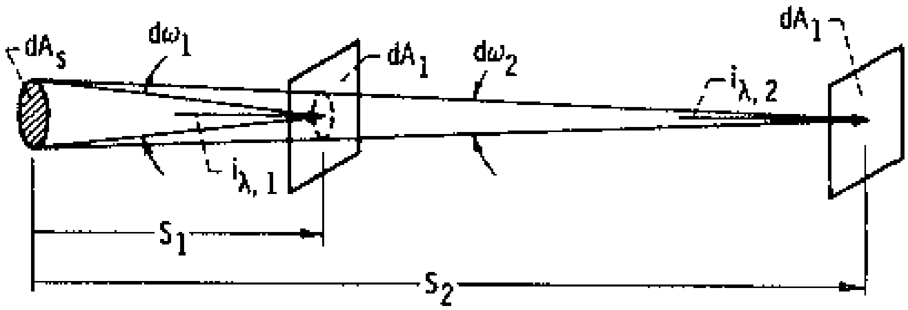
Note: intensity can be defined based on

1. dA at the source and the unit solid angle centered about the direction of travel with its center at dA or
2. dA at the detecting point and the unit solid angle center about the direction of travel looking toward the source

Invariance Property of Intensity in a Vacuum

The intensity, $i'_\lambda(x, y, z, \theta, \varphi)$, in a given direction in a non-attenuating and non-emitting medium is independent of position along that direction

Note: Radiative heat flux is not invariant in a vacuum!



Heat flux through area dA at S_1 is given by

$$d^3Q_{\lambda,1} = i'_\lambda dA_s \frac{dA}{S_1^2} d\lambda$$

Heat flux through area dA at S_2 is given by

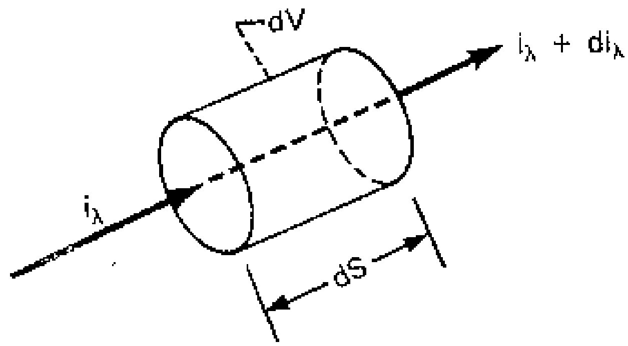
$$d^3Q_{\lambda,2} = i'_\lambda dA_s \frac{dA}{S_2^2} d\lambda$$

Ratio of heat fluxes at the two distances is

$$\frac{d^3Q_{\lambda,1}}{d^3Q_{\lambda,2}} = \frac{S_2^2}{S_1^2}$$

5.2 The Attenuation of Energy by Absorption and Scattering

a. Concepts of Extinction Coefficient, Absorption Coefficient and Scattering Coefficient



Along the direction of propagation of spectral intensity, $i'_\lambda(x, y, z, \theta, \varphi)$, the change of intensity, based on experiments, is given by

$$di'_\lambda = -K_\lambda(S)i'_\lambda dS$$

$K_\lambda(\lambda, T, P, C_i)$ = Extinction Coefficient
$= a_\lambda(\lambda, T, P, C_i) + \sigma_\lambda(\lambda, T, P, C_i)$

$a_\lambda(\lambda, T, P, C_i)$ = Absorption Coefficient
$\sigma_\lambda(\lambda, T, P, C_i)$ = Scattering Coefficient

b. Bouguer's Law (Beer's Law), Radiation Mean Penetration Length, Optical Thickness

Bouguer's Law

$$i'_\lambda(s) = i'_\lambda(0)\exp\left[-\int_0^s K_\lambda(s')\,ds'\right]$$

Fraction of energy absorbed in the layer from S to S +dS

$$\frac{i'_\lambda(S) - i'_\lambda(S+dS)}{i'_\lambda(0)} = -\frac{d}{dS}\left[\frac{i'_\lambda(S)}{i'_\lambda(0)}\right]dS = K_\lambda(S)\exp\left[-\int_0^S K_\lambda(s')\,ds'\right]dS$$

Mean Penetration Length = Average distance "survived" by a photon before it is extincted (either absorbed or scattered)

$$l_m = \int_0^S s' K_\lambda(s')\exp\left[-\int_0^{s'} K_\lambda(s'')\,ds''\right]ds'$$

and when K_λ = constant

$$l_m = \int_0^S s' K_\lambda \exp\left[-\int_0^{s'} K_\lambda ds''\right]ds' = \frac{1}{K_\lambda}$$

Optical Thickness

$$\kappa_\lambda(S) = \int_0^S K_\lambda(s')\,ds'$$

And therefore

$$i'_\lambda(S) = i'_\lambda(0)\exp\left[-\kappa_\lambda(S)\right]$$

when $\kappa_\lambda(S) >> 1$, the medium is optically thick when $\kappa_\lambda(S) << 1$, the medium is optically thin

c. Absorption Coefficients for Some Non-Gaseous Materials

1. For conducting metal, (based on E. M. Theory) with constant index of refraction, (n -iκ)

$$E_x = A_x e^{-\frac{\kappa\omega z}{c_0}} \exp\left\{ i\left[\omega\left(t - \frac{z}{c_0}n \right) + \delta_x \right] \right\} = A_x e^{-\frac{2\pi\kappa z}{\lambda_0}} \exp\left\{ i\left[\omega\left(t - \frac{z}{c_0}n \right) + \delta_x \right] \right\}$$

$$E_y = A_y e^{-\frac{\kappa\omega z}{c_0}} \exp\left\{ i\left[\omega\left(t - \frac{z}{c_0}n \right) + \delta_y \right] \right\} = A_y e^{-\frac{2\pi\kappa z}{\lambda_0}} \exp\left\{ i\left[\omega\left(t - \frac{z}{c_0}n \right) + \delta_y \right] \right\}$$

with

$$i'_\lambda \propto \left(E^2_{x,\lambda} + E^2_{y,\lambda} \right)$$

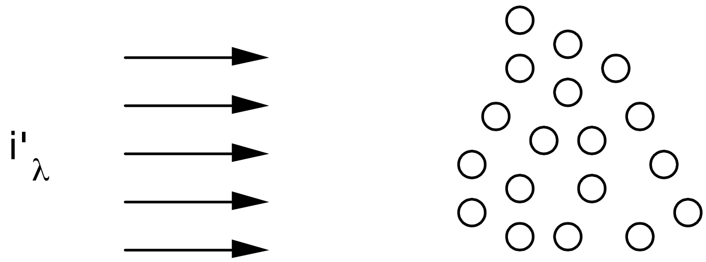
therefore,

$$i'(S) = |E|^2 = i'(0)\, e^{-\frac{4\pi\kappa S}{\lambda_0}}$$

Absorption coefficient for a conducting metal is thus given by

$$a_\lambda = |E|^2 = \frac{4\pi\kappa}{\lambda_0} \quad \text{with} \quad \sigma_\lambda = 0$$

2. For a cloud of "large, black" spherical particles



di'_λ / i'_λ = (number of particles per unit volume, N) x
(volume per unit area, dS) x
(projected surface area per particle, $\pi d^2/4 = A/4$)

= NAdS/4

$a_\lambda = NA/4$

Results are valid for:

a. Large particle size relative to wavelength, $\pi d/\lambda >> 1$

b. Spacing of particle such that "shadowing" effect is unimportant

3. For a cloud of spherical particles of arbitrary size (Mie Theory)

$di'_\lambda/i'_\lambda = N(E_\lambda A/4)dS$

E_λ = absorption efficiency factor for a spherical particle (always less than one)

$E_\lambda A/4$ = absorption cross section per particle (has unit of area)

Based on Mie Theory

$$E_\lambda \to 1.0 \qquad \text{when} \quad \frac{\pi D}{\lambda} >> 1$$

$$E_\lambda \to \frac{2\pi D}{\lambda}\left[\frac{n\kappa}{\left(n^2 - \kappa^2 + 2\right)^2 + 4n^2\kappa^2}\right] \qquad \text{when} \quad \frac{\pi D}{\lambda} << 1$$

and for a cloud of "small" spherical particles

$$\begin{aligned} a_\lambda &= N\left(\frac{\pi D^2}{4}\right)\frac{24\pi D}{\lambda}\left[\frac{n\kappa}{\left(n^2 - \kappa^2 + 2\right)^2 + 4n^2\kappa^2}\right] \\ &= N\left(\frac{\pi D^3}{6}\right)\frac{36\pi}{\lambda}\left[\frac{n\kappa}{\left(n^2 - \kappa^2 + 2\right)^2 + 4n^2\kappa^2}\right] \\ &= f_v \frac{36\pi}{\lambda}\left[\frac{n\kappa}{\left(n^2 - \kappa^2 + 2\right)^2 + 4n^2\kappa^2}\right] \end{aligned}$$

With f_v = volume fraction

d. Scattring Coefficient for Spherical Particles (or Voids)

1. Concept of Scattering Cross Section

$di'_\lambda/i'_\lambda = \sigma_{s\lambda}dS = Ns_\lambda dS$

s_λ = scattering cross section per particle (has unit of area)

s_λ, in general, is a functio of particle radius R. For a particle clouds of size distribution, $N_s(R)$, the scattering coefficient, $\sigma_{s\lambda}$, is given by

$$\sigma_{s\lambda} = \int_{R=0}^{\infty} N_s(R)\, s_\lambda(R)\, dR$$

With

$$N = \int_{R=0}^{\infty} N_s(R)\, dR$$

2. Concept of Scattering Phase Function

$di'_{\lambda,s}(\theta, \varphi)$ = energy scattered in the direction (θ, φ) measured relataive to the incident direction per unit solid angle, per unit area normal to the incident beam, and per unit solid angle of the incident radiation

$$di'_{\lambda,s}(\theta,\phi) \equiv \frac{\text{Spectral Energy Scattered in Direction } (\theta,\phi)}{d\omega_s dA d\omega_i d\lambda}$$

$$= \frac{d^5 Q'_{\lambda,s}(\theta,\phi)}{d\omega_s dA d\omega_i d\lambda}$$

And

$$di'_{\lambda,s}(\theta,\phi) = di'_{\lambda,s}\frac{\Phi(\theta,\phi)}{4\pi} = \sigma_{s,\lambda} i'_\lambda dS \frac{\Phi(\theta,\phi)}{4\pi}$$

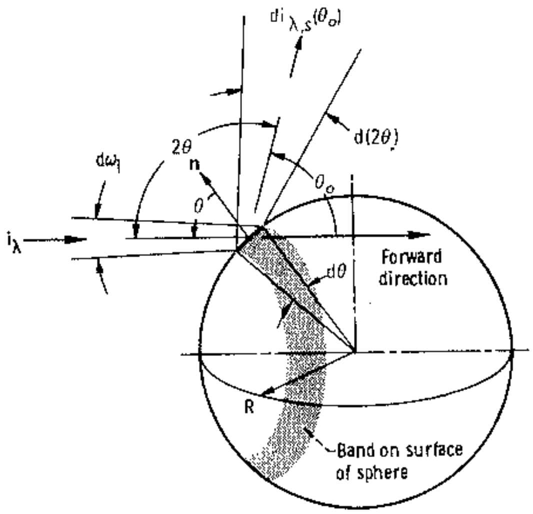
$\Phi(\theta, \varphi)$ is called the scattering phase function and has the properties:

$$\Phi(\theta,\phi) = \frac{di'_{\lambda,s}(\theta,\phi)}{di'_{\lambda,s}/4\pi} = \frac{di'_{\lambda,s}(\theta,\phi)}{1/4\pi \int_{\omega_s=4\pi} di'_{\lambda,s}(\theta,\phi)\, d\omega_s}$$

And

$$\frac{1}{4\pi}\int_{\omega_s=4\pi} \Phi(\theta,\phi)\, d\omega_s = 1$$

3. Scattering from a Cloud of "Large" Specularly-Reflecting Spheres with Reflectivity, ρ_λ



$s_\lambda = \rho_\lambda \pi R^2 = \rho_\lambda A/4$

$\sigma_{s\lambda} = N\rho_\lambda A/4$

and for a cloud of spherical particles with a size distribution, $N_s(R)$,

$$\sigma_{s\lambda} = \int_{R=0}^{\infty} N_s(R)\,\rho_\lambda \pi R^2 dR$$

With

$$N = \int_{R=0}^{\infty} N_s(R)\,dR$$

Scattering phase function becomes

$$\Phi(\beta) = \frac{\rho'_\lambda\left[(\pi - \beta)/2\right]}{\rho_\lambda}$$

With

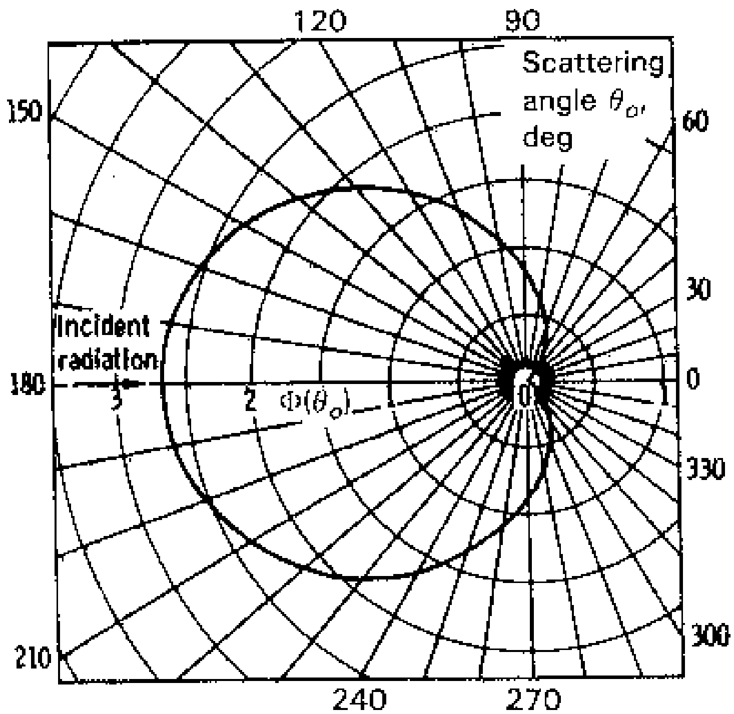
$$\beta = \pi - 2\theta$$

4. Scattering from a Cloud of "Large" Diffusely-Reflecting Spheres with Reflectivity, ρ_λ

Scattering cross section is identical to that of specularly-reflecting spheres

Scattering Phase Function is

$$\Phi(\theta_0) = \frac{8}{3\pi}(\sin\theta_0 - \theta_0\cos\theta_0)$$



5. Scattering Cross Section of Large Dielectric Sphere with Refractive Index Close to Unity

$$\kappa = 0, \qquad n \approx 1$$

$$s_\lambda = \pi R^2 \left[2 - \frac{4}{W} \sin W + \frac{4}{W^2} (1 - \cos W) \right]$$

Where

$$W = 2 \left(\frac{2\pi R}{\lambda} \right) (n - 1)$$

6. Rayleigh Scattering (For Small Particles with $2\pi R/\lambda << 1$)

$$s_\lambda = \frac{24\pi^3 V^2}{\lambda^4} \left| \frac{\overline{n}^2 - 1}{\overline{n}^2 + 2} \right|^2$$

With

$$V = \frac{4}{3} \pi R^3$$

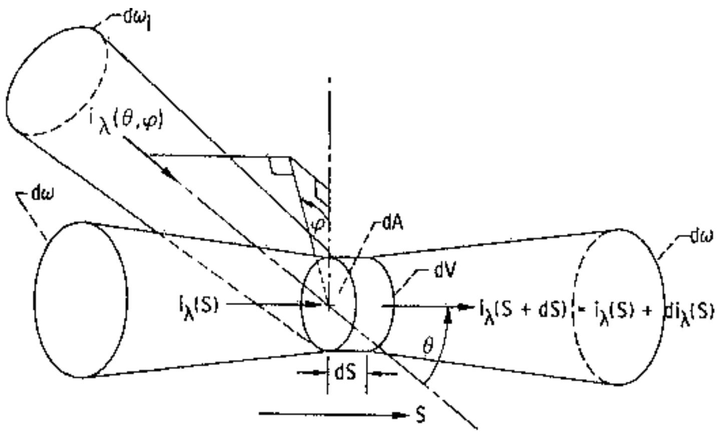
5.3 The Augmentation of Energy by Emission and Scattering

a. The emission of Energy by a Medium

From thermodynamic consideration

$$di'_{\lambda,s} = a_\lambda(\lambda, T, P)i'_{\lambda b}(T)dS$$

b. Augmentation of Intensity by Incoming Scattering



$$di'_\lambda = \frac{\sigma_{s\lambda} dS}{4\pi} \int_{\omega_i = 4\pi} i'_\lambda(\theta,\phi)\,\Phi(\theta,\phi)\,d\omega_i$$

5.4 The Equation of Transfer

$$\frac{di'_\lambda}{dS} = -(a_\lambda + \sigma_{s\lambda})i'_\lambda + a_\lambda i'_{\lambda b}(T) + \frac{\sigma_{s\lambda}}{4\pi} \int_{\omega_i = 4\pi} i'_\lambda(\theta,\phi)\,\Phi(\theta,\phi)\,d\omega_i$$

5.5 Concepts of Absorptance, Emittance and Transmittance for an Isothermal Non Scattering Medium with Uniform Properties

$$di'_{\lambda} = \left(-a_{\lambda} i'_{\lambda} + a_{\lambda} i'_{\lambda b}(T)\right) dS$$

And the solution is

$$i'_{\lambda} = i'_{\lambda}(0)\, e^{-a_{\lambda} S} + \left(1 - e^{-a_{\lambda} S}\right) i'_{\lambda b}(T)$$

Concept of spectral transmittance

$$\tau'_{\lambda} = e^{-a_{\lambda} S}$$

Concept of spectral absorptance

$$\alpha'_{\lambda} = 1 - e^{-a_{\lambda} S}$$

Concept of spectral emittance

$$\varepsilon'_{\lambda} = \alpha'_{\lambda} = 1 - e^{-a_{\lambda} S}$$

Note that $\alpha'_{\lambda} = \varepsilon'_{\lambda} = 1$ in the limit of large $a_{\lambda}S$, i.e., an isothermal absorbing medium acts like a blackbody in the optically thick limit.

The total absorptance (α'), emittance (ε') and transmittance (τ') are given by

$$\varepsilon'(T,P,S) = \frac{\pi \int_0^\infty i'_{\lambda b}(\lambda,T)\left[1 - e^{-a_\lambda S}\right] d\lambda}{\sigma T^4}$$

$$\alpha'(T,P,S,\text{Source}) = \frac{\pi \int_0^\infty i'_\lambda(0)\left[1 - e^{-a_\lambda S}\right] d\lambda}{\int_0^\infty i'_\lambda(0)\, d\lambda}$$

$$\tau'(T,P,S,\text{Source}) = \frac{\pi \int_0^\infty i'_\lambda(0)\, e^{-a_\lambda S} d\lambda}{\int_0^\infty i'_\lambda(0)\, d\lambda}$$

Note that

$$\varepsilon'(T,P,S) \neq \alpha'(T,P,S,\text{Source}) \qquad \text{in general}$$

$$\alpha'(T,P,S,\text{Source}) = 1 - \tau'(T,P,S,\text{Source}) \qquad \text{always}$$

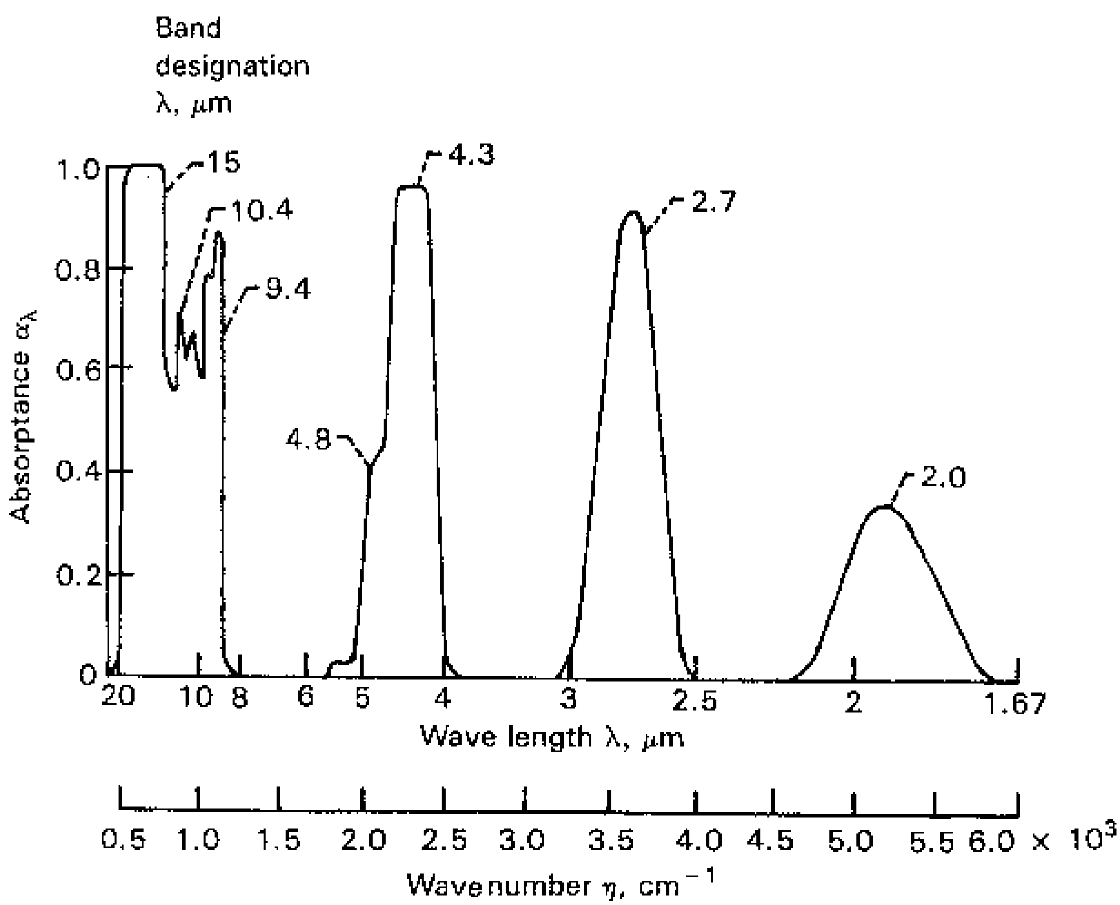
5.6 Absorptance and Emittance of Isothermal Real Gases

Basic Problem: Evaluation of

$$\varepsilon_g'\left(T_g,P,S\right)=\frac{\pi\int_0^\infty i_{\lambda b}'\left(\lambda,T_g\right)\left[1-e^{-a_\lambda\left(T_g,P\right)S}\right]d\lambda}{\sigma T_g^4}$$

$$\alpha_g'\left(T_0,T_g,P,S\right)=\frac{\pi\int_0^\infty i_{\lambda b}'\left(\lambda,T_0\right)\left[1-e^{-a_\lambda\left(T_g,P\right)S}\right]d\lambda}{\sigma T_0^4}$$

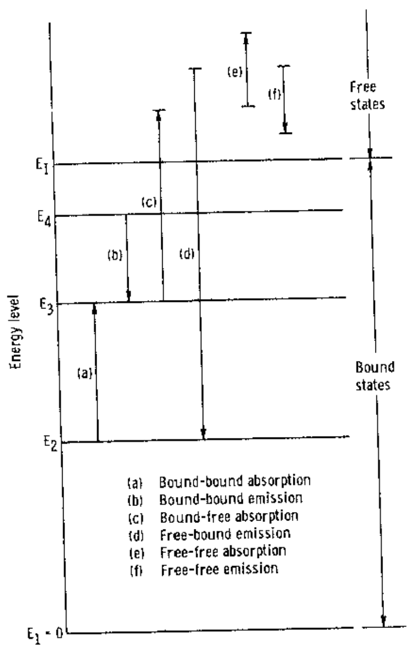
with highly irregular spectral dependence in $\alpha'_\lambda(\varepsilon'_\lambda)$



1. Physics of gas absorption

a. Gas molecules absorb (or emit) in discrete wavelengths corresponded to different transitions between different energy levels. The frequency of the absorbed (or emitted) radiation is

$$\nu = (E_{upper} - E_{lower})/h$$

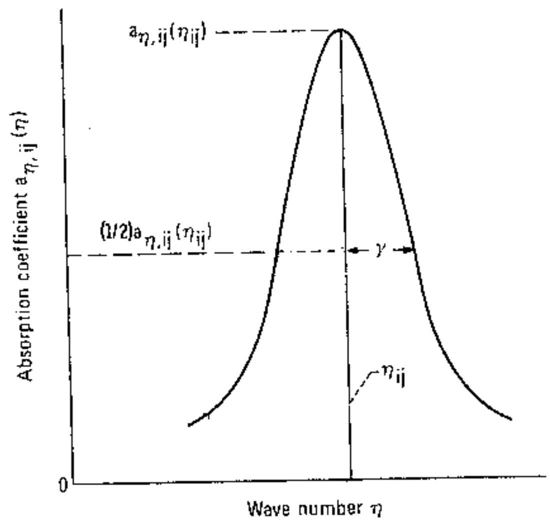


Transition Types		Energy
Different rotational state Same vibrational state Same electronic state	—>	Far Infrared (large λ)
Different rotational state Different vibrational state Same electronic state	—>	Near Infrared (moderate λ)
Different rotational state Different vibrational state Different electronic state	—>	Ultraviolet (small λ)

Remark: Since there is no rotational transition for diatomic symmetric molecules, gases such as N_2 and O_2 are transparent in the far infrared and absorb (and emit) only in the near infrared and ultraviolet range.

b. Line Broadening

Due to the uncertainty principle (the energy level of a given state is not infinitely precise), collision between molecules and other effects, the frequency associated with a given transition is never completely discrete, but in "Lorentz" or "Gaussian" shape profiles.



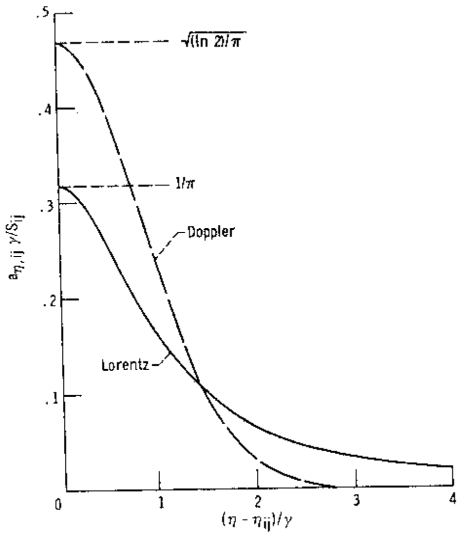
$$\frac{a_{\eta,ij}(\eta)}{S_{ij}} = \frac{\gamma_n / \pi}{\gamma_n^2 + (\eta - \eta_{ij})^2}$$

where S_{ij} is the Line Intensity given by

$$S_{ij} = \int_0^{\infty} a_{\eta,ij}(\eta)\, d\eta = \int_{-\infty}^{\infty} a_{\eta,ij}(\eta)\, d(\eta - \eta_c)$$

and γ_n is called the half-width of the line. Both S_{ij} and γ_n can be sensitive functions of temperature and pressure

2. Absorption and Emission by a Spectral Line



With a gas of uniform temperature and pressure,

The absorption by the gas can be written as

$$\frac{d^2Q_a'}{dA_p d\omega} = \int_0^\infty i_\eta'(0,\eta)\left[1 - e^{-a_\eta S}\right] d\eta$$

And the emission by the gas is given by

$$\frac{d^2Q_e'}{dA_p d\omega} = \int_0^\infty i_{\eta b}'(\eta)\left[1 - e^{-a_\eta S}\right] d\eta$$

For all gases, it can be assumed that the variation of $i'_\eta(0, \eta)$ and $i'_{\eta b}(\eta)$ is small over the line spectrum, The absorption and emission of the gas can be written as

$$\frac{d^2Q_a'}{dA_p d\omega} = i_\eta'(0,\eta_{ij}) \int_0^\infty \left[1 - e^{-a_\eta S}\right] d\eta = i_\eta'(0,\eta_{ij}) \bar{A}_{ij}$$

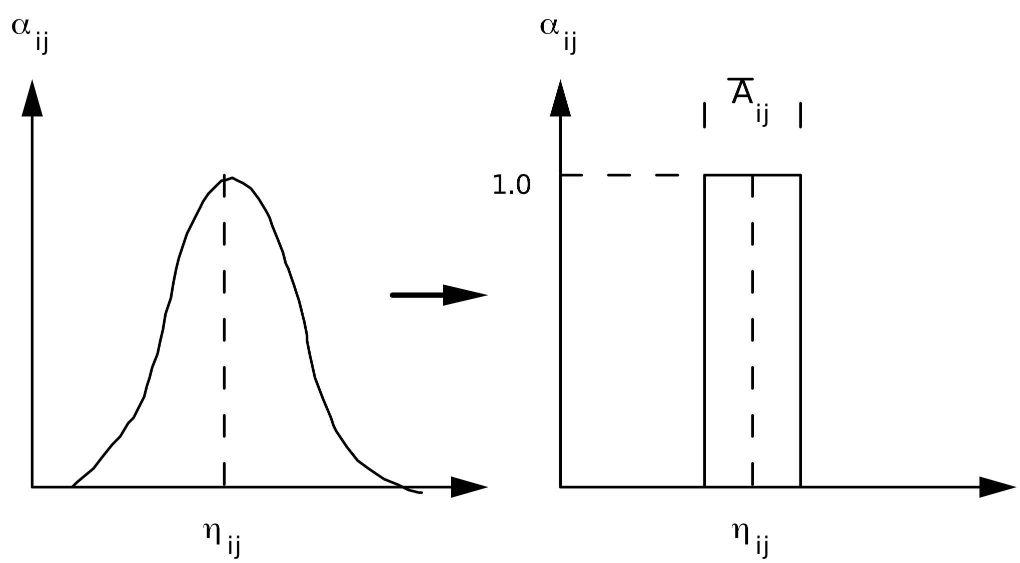
$$\frac{d^2Q_e'}{dA_p d\omega} = i_{\eta b}'(\eta_{ij}) \int_0^\infty \left[1 - e^{-a_\eta S}\right] d\eta = i_{\eta b}'(\eta_{ij}) \bar{A}_{ij}$$

With

$$\bar{A}_{ij} = \int_0^\infty \left[1 - e^{-a_{\eta,ij}(\eta)S}\right] d\eta = \int_{-\infty}^\infty \left[1 - e^{-a_{\eta,ij}(\eta)S}\right] d(\eta - \eta_{ij})$$

= Equivalent Line Width

Physical Interpretation of the Equivalent Line Width



$\bar{A}_{ij} \equiv$ width of an equivalent absorption band with $a_\eta \rightarrow \infty$ everywhere within the band

Result of an actual integration for a band with a Lorentz profile

$$\bar{A}_{ij}(S) = \int_{-\infty}^{\infty}\left\{1 - \exp\left[-\frac{S_{ij}}{\pi}\frac{\gamma_c S}{\gamma_c^2 + (\eta - \eta_{ij})^2}\right]\right\} d(\eta - \eta_{ij})$$

$$= 2\pi\gamma_c \xi e^{-\xi}\left[I_0(\xi) + I_1(\xi)\right]$$

$$\approx 2\pi\gamma_c \xi\left[1 + \left(\frac{\pi\xi}{2}\right)^{5/4}\right]^{-2/5}$$

With

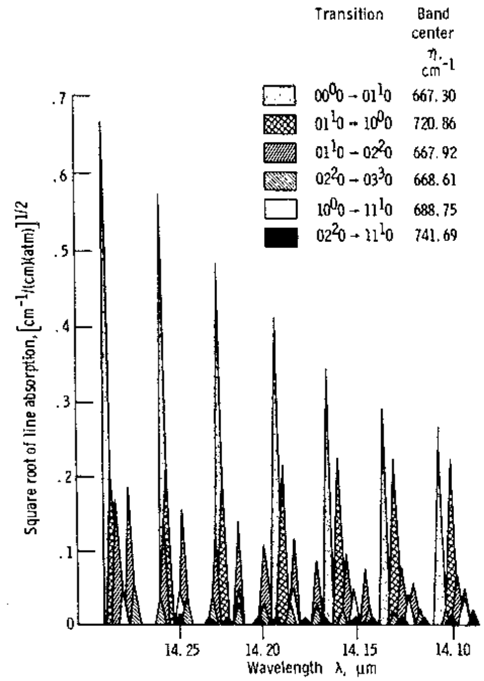
$$\xi = \frac{SS_{ij}}{2\pi\gamma_c}$$

and

$$\xi < 0.1 \qquad \bar{\mathrm{A}}_{ij} \to SS_{ij}\,(\text{Independent of line profile})$$

$$\xi > 3.0 \qquad \bar{\mathrm{A}}_{ij} \to 2\sqrt{S\gamma_c S_{ij}}$$

3. Absorption and Emission by a Group of Overlapping Lines



Total emittance is given by (assuming that the Planck function is constant across each band)

$$\varepsilon(T,P,C_i,S) = \frac{\pi \int_0^\infty i_{\eta b}(\eta,T)\left[1 - e^{-a_\eta S}\right] d\eta}{\sigma T^4}$$

$$= \frac{\pi \sum_l i_{\eta b,l} \int_l \left[1 - e^{-a_\eta S}\right] d\eta}{\sigma T^4}$$

Introducing the concept of effective bandwidth $\overline{A}_l(S)$

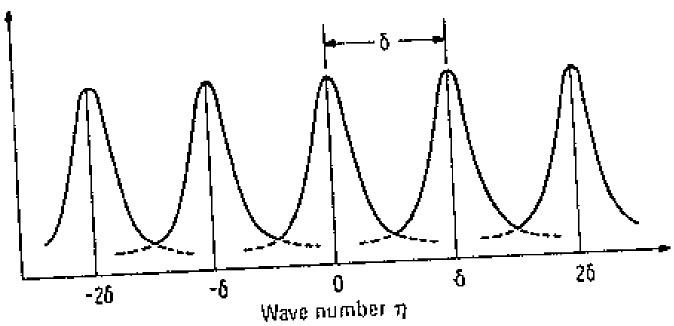
$$\overline{A}_l(S) \equiv \int_l \left[1 - e^{-a_\eta S}\right] d\eta$$

The total emittance is given by

$$\varepsilon(T,P,C_i,S) = \frac{\pi \sum_l i_{\eta b,l} \overline{A}_l(S)}{\sigma T^4}$$

a. Narrow Band Model (Local average of a group of overlapping lines

Elsassar Model: At a small neighborhood around η_{ij} (say, within Δ_c), the effect of overlapping lines can be approximated by an infinite series of lines of identical shape with equal spacing between them.



$$a_\eta(\eta) = \frac{S_c}{\pi}\sum_{n=-\infty}^{\infty}\frac{\gamma_c}{\gamma_c^2 + (\eta - n\delta)^2}$$
$$= \frac{S_c}{\delta}\frac{\sinh(2\beta)}{\cosh(2\beta) - \cos\left[2\pi\left(\eta - \eta_{ij}\right)/\delta\right]}$$

With

$$\beta = \frac{\pi\gamma_c}{\delta}$$

The effective bandwidth for m identical overlapping lines is given by

$$\bar{A}_l(S) = \int_{m\delta}\left[1 - e^{-a_\eta S}\right]d\eta$$
$$= 2m\int_0^{\delta/2}\left[1 - \exp\left\{-\frac{SS_c}{\delta}\frac{\sinh(2\beta)}{\cosh(2\beta) - \cos\left[2\pi\left(\eta - \eta_{ij}\right)/\delta\right]}\right\}\right]d\left(\eta - \eta_{ij}\right)$$
$$= m\delta\left[1 - \frac{1}{2}\int_0^2 \exp\left\{-\frac{\mu\sinh(2\beta)}{\cosh(2\beta) - \cos\left[\pi z/2\right]}\right\}dz\right]$$

With $\mu = \frac{SS_c}{\delta}$, $z = \frac{4(\eta - \eta_{ij})}{\delta}$

Limiting Expressions for the Average Emittance

a. Weak non-overlapping lines

$$\frac{\overline{A}_l}{m\delta} = \frac{\overline{A}_l}{\omega} = \frac{\overline{A}_{ij}}{\delta} = \mu \qquad \mu << 1$$

With $\omega = m\delta$ being the overall bandwidth of m line

same as limit of single line absorptance

b. Long pathlength (S is large) with overlapping lines

$$\frac{\overline{A}_l}{\omega} = 1 \qquad \mu >> 1$$

c. Strong (long path length) non-overlapping lines

$$\frac{\overline{A}_l}{\omega} = erf\left(\sqrt{\beta\mu}\right) \qquad \beta << 1, \ \frac{\mu}{\beta} >> 1$$

d. Strong (long path length) non-overlapping lines which are thin and spaced well apart (square root limit)

$$\frac{\overline{A}_l}{\omega} = 2\sqrt{\frac{\beta\mu}{\pi}} \qquad \beta << 1, \ \frac{\mu}{\beta} >> 1, \ \beta\mu << 1$$

e. Broad band limit (lines are very broad compared to the spacing between them

$$\frac{\overline{A}_l}{\omega} = 1 - e^{-\mu} = 1 - e^{-\frac{SS_c}{\delta}} \qquad \beta >> 1, \ \gamma_c >> \delta$$

$$\frac{S_c}{\delta} \equiv \text{average absorption coefficient}$$

4. Band Correlations for Wide-Band Models

Basic idea of Wide-Band Models:

1. Within a single absorption band (wide band), the narrow band model is accurate only in a localized sense (around a particular wave number, η).

2. To get absorption behavior of a wide-band, integrate the narrow-band results allowing the narrow-band parameters S_c, Δ_c, δ and β to be function of wave number within the wide band

3. Physical consideration of quantum mechanical results can lead to semi-empirical spectral dependence of narrow-band parameters. Numerical integration and experiments lead to correlations for wide band models

Wide Band Models and Correlations

For B < 1:

$$\bar{A} = \omega u \qquad 0 \le u \le B$$

$$\bar{A} = \omega\left(2\sqrt{Bu} - B\right) \qquad B \le u \le \frac{1}{B}$$

$$\bar{A} = \omega\left[\ln(Bu) + 2 - B\right] \qquad \frac{1}{B} \le u \le \infty$$

For B $\ge$1:

$$\bar{A} = \omega u \qquad 0 \le u \le 1$$

$$\bar{A} = \omega\left[\ln u + 1\right] \qquad 1 \le u \le \infty$$

Where $B = \beta P_e, \quad P_e = \left[P/P_0 + (p/P_0)(b-1)\right]^n$

P = total pressure (atm)
P_0 = 1 atm
p = partial pressure of the radiating gas

$$u = \frac{X\alpha}{\omega}, \quad X = \rho S, \quad \omega = \omega_0 (T / T_0)^{1/2}, \quad T_0 = 100K$$

$$\alpha(T) = \alpha_0 \frac{1 - \exp\left(- \sum_{k=1}^{m} u_k \delta_k \right)}{1 - \exp\left(- \sum_{k=1}^{m} u_{0,k} \delta_k \right)} \frac{\Psi(T)}{\Psi(T_0)}$$

$$\beta(T) = \beta_0 \left(\frac{T}{T_0} \right)^{1/2} \frac{\Phi(T)}{\Phi(T_0)}$$

$$\Psi(T) = \frac{\prod_{k=1}^{m} \sum_{\upsilon_k = \upsilon_{0,k}}^{\infty} \left[\left(\upsilon_k + g_k + |\delta_k| - 1 \right)! / \left(g_k - 1 \right)! \upsilon_k ! \right] e^{-u_k \upsilon_k}}{\prod_{k=1}^{m} \sum_{\upsilon_k = 0}^{\infty} \left[\left(\upsilon_k + g_k - 1 \right)! / \left(g_k - 1 \right)! \upsilon_k ! \right] e^{-u_k \upsilon_k}}$$

$$\Phi(T) = \frac{\left(\prod_{k=1}^{m} \sum_{\upsilon_k = \upsilon_{0,k}}^{\infty} \left\{ \left[\left(\upsilon_k + g_k + |\delta_k| - 1 \right)! / \left(g_k - 1 \right)! \upsilon_k ! \right] e^{-u_k \upsilon_k} \right\}^{1/2} \right)^2}{\prod_{k=1}^{m} \sum_{\upsilon_k = \upsilon_{0,k}}^{\infty} \left[\left(\upsilon_k + g_k + |\delta_k| - 1 \right)! / \left(g_k - 1 \right)! \upsilon_k ! \right] e^{-u_k \upsilon_k}}$$

With

$$u_k = \frac{hc\eta_k}{kT}, \qquad u_{0,k} = \frac{hc\eta_k}{kT_0}$$

$$\upsilon_{0,k} = 0, \qquad \text{if } \delta_k \geq 0$$
$$\upsilon_{0,k} = |\delta_k|, \qquad \text{if } \delta_k < 0$$

Values of various parameters can be find in the following table

TABLE 11-2 Exponential wide-band parameters [13, 45]

Gas m, η (cm^{-1}), g	Band, μm	Band center η, cm^{-1}	$\delta_1 \ldots \delta_m$	Pressure parameters ($T_0 = 100$ K) b	Pressure parameters ($T_0 = 100$ K) n	α_0, cm^{-1}/g·m^{-2}	β_0	ω_0, cm^{-1}
CO_2	15	667	0, 1, 0	1.3	0.7	19.0	0.06157	12.7
$m=3$, $\eta_1=1351$, $g_1=1$	10.4	960	−1, 0, 1	1.3	0.8	2.47×10^{-9}	0.04017	13.4
$\eta_2=667$, $g_1=2$	9.4	1060	0, −2, 1[b]	1.3	0.8	2.48×10^{-9}[b]	0.11888[b]	10.1
$\eta_3=2396$, $g_3=1$	4.3	2410[a]	0, 0, 1	1.3	0.8	110.0	0.24723	11.2
	2.7	3660	1, 0, 1	1.3	0.65	4.0	0.13341	23.5
	2.0	5200	2, 0, 1	1.3	0.65	0.066	0.39305	34.5
CH_4	7.66	1310	0, 0, 0, 1	1.3	0.8	28.0	0.08698	21.0
$m=4$, $\eta_1=2914$, $g_1=1$	3.31	3020	0, 0, 1, 0	1.3	0.8	46.0	0.06973	56.0
$\eta_2=1526$, $g_2=2$	2.37	4220	1, 0, 0, 1	1.3	0.8	2.9	0.35429	60.0
$\eta_3=3020$, $g_3=3$	1.71	5861	1, 1, 0, 1	1.3	0.8	0.42	0.68598	45.0
$\eta_4=1306$, $g_4=3$								
H_2O	rotational[d]	140	0, 0, 0	$8.6(T_0/T)^{1/2}+0.5$	1	44205	0.14311	69.3
$m=3$, $\eta_1=3652$, $g_1=1$	6.3	1600	0, 1, 0	$8.6(T_0/T)^{1/2}+0.5$	1	41.2	0.09427	56.4
$\eta_2=1595$, $g_2=1$	2.7	3760[c]	0, 2, 0	$8.6(T_0/T)^{1/2}+0.5$	1	0.19	0.13219	60.0
$\eta_3=3756$, $g_3=1$			1, 0, 0			2.30		
			0, 0, 1			22.40		
	1.87	5350	0, 1, 1	$8.6(T_0/T)^{1/2}+0.5$	1	3.0	0.08169	43.1
	1.38	7250	1, 0, 1	$8.6(T_0/T)^{1/2}+0.5$	1	2.5	0.11628	32.0
CO	4.7	2143	1	1.1	0.8	20.9	0.07506	25.5
$m=1$, $\eta_1=2143$, $g_1=1$	2.35	4260	2	1.0	0.8	0.14	0.16758	20.0

[a] Upper band limit.

[b] Use values for the 10.4 μm band instead of those for the 9.4 μm band.

[c] See notes in [45].

[d] For the H_2O rotational band, $\alpha(T) = \alpha_0 \exp[-9.0(T_0/T)^{1/2}]$, $\beta(T) = \beta_0(T_0/T)^{1/2}$, $\omega(T) = \omega_0(T/T_0)^{1/2}$; if the calculated lower band limit is negative, use $\eta = 0$ for the lower limit (the band width then equals the calculated upper limit); further details are in: Modak, A. T., "Exponential Wide Band Parameters for the Pure Rotational Band of Water Vapor," *J. Quant. Spectrosc. Radiat. Transfer*, vol. 21, pp. 131–142, 1979.

Example: Find the effective bandwidth $\bar{A}$ of the 9.4 μm band for pure CO_2 at 1 atm and 500 K for a path length S of 0.364 m

Solution:

First, determine $u = X\alpha / \omega$

To find α, need to evaluate $\Psi(T)$ and $\Psi(T_0)$

From Table:

$$\alpha_0 = 2.48\times 10^{-9}\ \mathrm{m}^2/\mathrm{g}\cdot\mathrm{cm},\ \delta_1 = -1,\ \delta_2 = 0,\ \delta_3 = 1$$

Therefore

$$\upsilon_{0,1} = 1(\delta_1 = -1),\ \upsilon_{0,2} = 0(\delta_2 = 0),\ \upsilon_{0,3} = 0(\delta_3 = 1)$$

From Table:

$$g_1 = 1,\ g_2 = 2,\ g_3 = 1$$
$$\eta_1 = 1351,\ \eta_2 = 667,\ \eta_3 = 2396$$

Using

$$h = 6.626\times 10^{-34}\,\mathrm{J}\cdot\mathrm{s}$$
$$k = 1.381\times 10^{-23}\,\mathrm{J/K}$$
$$c = 2.998\times 10^{-10}\,\mathrm{cm/s}$$

Gives

$$u_1 = 3.887,\ u_2 = 1.919,\ u_3 = 6.893$$
$$u_{0,k} = u_k\left(\frac{T}{T_0}\right) = 5u_k$$

Leading to

$$\left.\begin{aligned}&\Psi(T) = 0.0415,\ \Psi(T_0) = 7.266\times 10^{-9}\\ &\alpha(T) = 0.01341\ \mathrm{m}^2/\mathrm{g}\cdot\mathrm{cm}\end{aligned}\right\}\ (\text{P.S.6, \#1, prove this})$$

To find β and ω

$$\left.\begin{aligned}&\Phi(T)=4.479,\ \Psi(T_0)=1.0237\\&\beta(T)=0.079\\&\omega=22.58\ \text{cm}^{-1}\end{aligned}\right\}\ (\text{P.S.6, \#2, prove this})$$

Use the ideal gas law to find density

$$\rho=\frac{PM}{RT}=\frac{(1\ \text{atm})(44\ \text{kg/kg}\cdot\text{mole})}{\left(0.08206\left(\text{atm}\cdot\text{m}^3/\text{kg}\cdot\text{mol}\cdot\text{K}\right)\right)(500K)}=1.072\ \text{kg/m}^3$$

$$\left.\begin{aligned}&X=390.3\ \text{g/m}^2\\&u=0.232\end{aligned}\right\}\ (\text{P.S.6, \#3, prove this})$$

From Table

$b=1.3,\ n=0.8$

Can show

$$\left.\begin{aligned}&P_e=1.234\\&B=0.097\\&\bar{A}=4.58\ \text{cm}^{-1}\end{aligned}\right\}\ (\text{P.S.6, \#4, prove this})$$

Empirical Wide Band Correlations

Tien-Lowder Correlation

$$\frac{\bar{A}_l}{\omega}=\ln\left[uf(B)\frac{u+2}{u+2f(B)}+1\right]$$

With

$$f(B)=2.94\left[1-\exp(-2.60B)\right]$$

Cess-Tiwari Correlation

$$\frac{\overline{A}_l}{\omega} = 2\ln\left[1 + \frac{u}{2 + u^{1/2}\left(1 + 1/B\right)^{1/2}}\right]$$

Goody-Belton Correlation

$$\frac{\overline{A}_l}{\omega} = 2\ln\left[1 + \frac{\sqrt{B}u}{\left(u + 4B\right)^{1/2}}\right]$$

P.S. 6, #5

Find the effective bandwidth $\overline{A}$ of the 9.4 μm band for pure CO_2 at 1 atm and 500 K for a path length S of 0.364 m using the three empirical correlations.

5. Expression for Gas Emittance and Absorptance Based on Effective Bandwidth

$$\varepsilon(T,P,C_i,S) = \frac{\pi \sum_l i_{\eta b,l} \overline{A}_l(S)}{\sigma T^4}$$

and

$$\alpha(T,P,C_i,S,Souce) = \frac{\pi \sum_l i_{\eta i,l}(0) \overline{A}_l(S)}{\int_0^\infty i_{\eta i}(0)\, d\eta}$$

Overlapping Bands from Different Species

Example: Both CO_2 and H_2O have an absorption band centered at 2.7 μm

Mathematics: Absorption coefficient is additive, but not emittance, absorptance and transmittance

$$
\begin{aligned}
\bar{A}_{a+b} &= \int_l \left[1 - e^{-(a_{a,\eta} + a_{b,\eta})S}\right] d\eta \\
&= \int_l \left[\left(1 - e^{-a_{a,\eta}S}\right) + \left(1 - e^{-a_{b,\eta}S}\right) - \left(1 - e^{-a_{a,\eta}S}\right)\left(1 - e^{-a_{b,\eta}S}\right)\right] d\eta \\
&= \bar{A}_a + \bar{A}_b - \int_l \left(1 - e^{-a_{a,\eta}S}\right)\left(1 - e^{-a_{b,\eta}S}\right) d\eta \\
&< \bar{A}_a + \bar{A}_b
\end{aligned}
$$

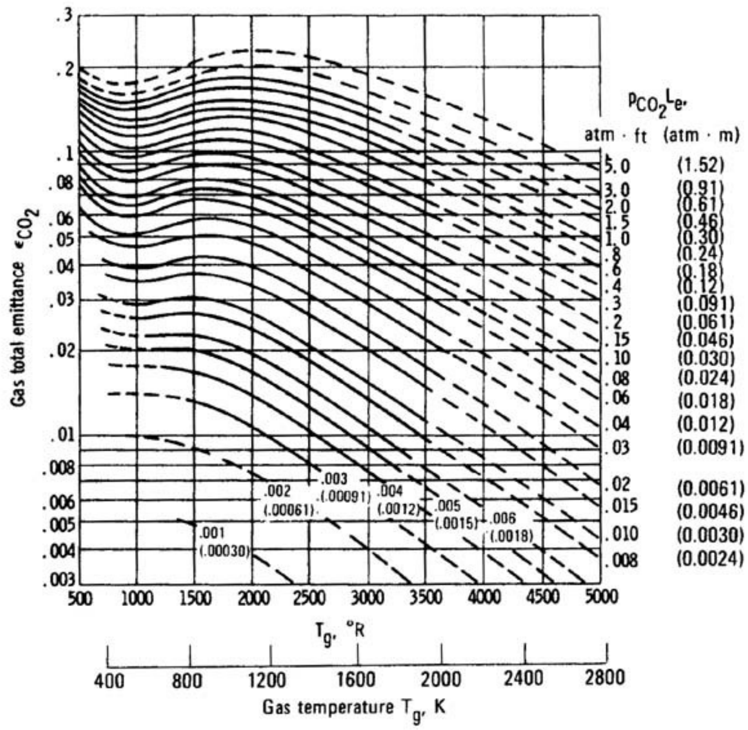
Empirical Approximation:

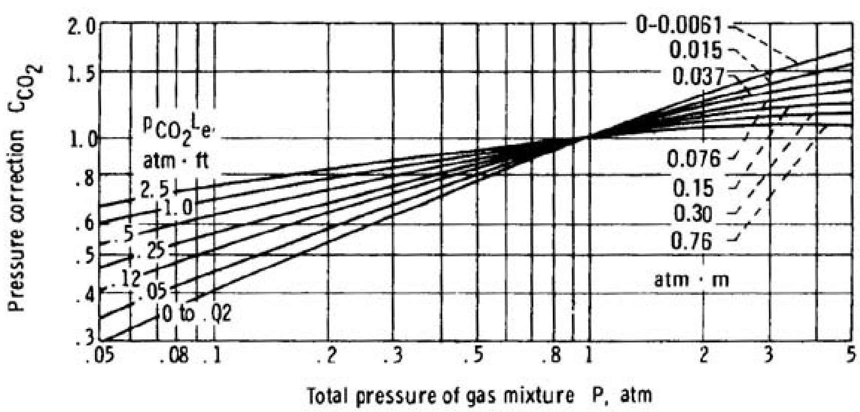
$$
\bar{A}_{a+b} = \bar{A}_a + \bar{A}_b - \frac{\bar{A}_a \bar{A}_b}{\Delta\eta}
$$

6. Evaluation of Gas Emittance and Absorptance Based on Hottel's Chart (Empirical for P = 1 atm)

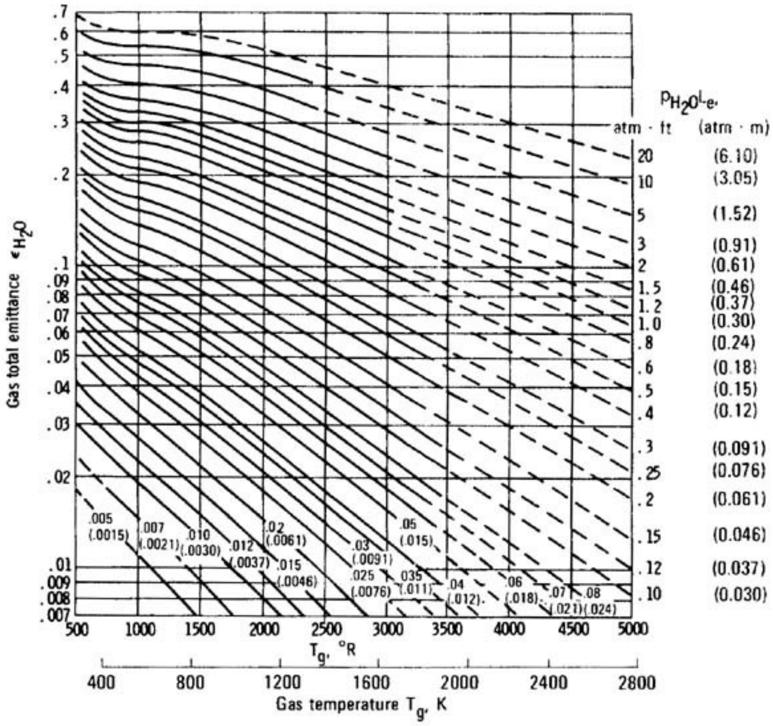
$$\varepsilon_g = C_{CO_2}\varepsilon_{CO_2} + C_{H_2O}\varepsilon_{H_2O} - \Delta\varepsilon$$

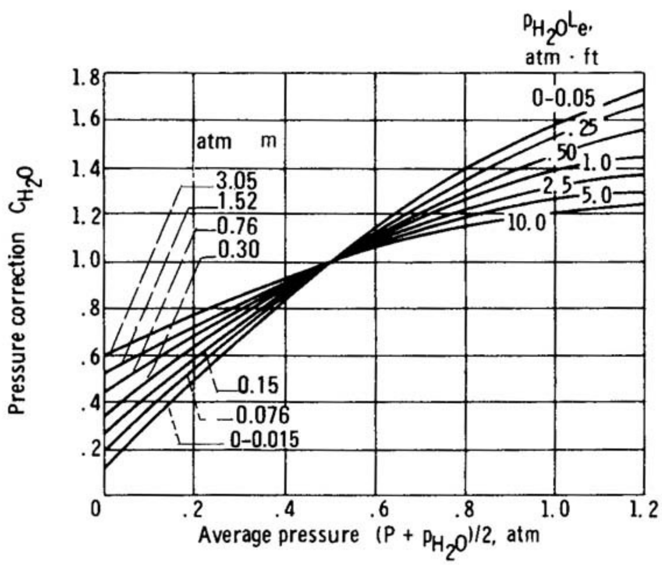
$$\varepsilon_{CO_2} = \varepsilon_{CO_2}\left(T_g, p_{CO_2}L\right),\ \ \mathrm{C}_{CO_2} = C_{CO_2}\left(P_g, p_{CO_2}L\right)$$



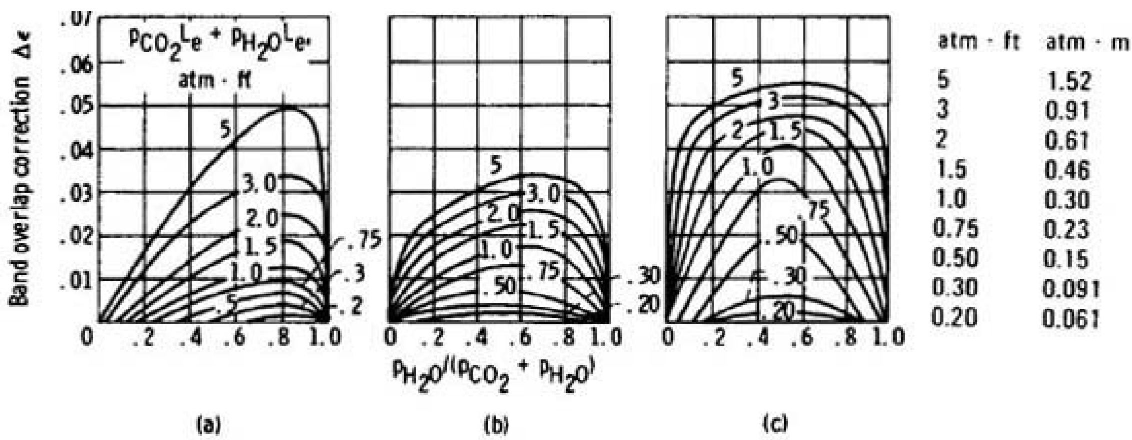


$$\varepsilon_{H_2O} = \varepsilon_{H_2O}\left(T_g, p_{H_2O}L\right),\ \ C_{H_2O} = C_{H_2O}\left(P_g, p_{H_2O}L\right)$$





$$\Delta\varepsilon = \Delta\varepsilon\left(T_g, p_{CO_2}L, p_{H_2O}L\right)$$



P.S. 6, #6

Find the effective bandwidth $\overline{A}$ of the 9.4 μm band for pure CO_2 at 1 atm and 500 K for a path length S of 0.364 m using the Hottel chart.