The Monte Carlo Method and its Application to Heat Transfer Problems

1. Basic Concepts of Probability

An Elementary Random Event is a random event which one cannot (or one does not choose to) break up into simpler event. (e.g. the result (head or tail) of flipping a coin or the result (1 to 6) of throwing a dice is an elementary random event).

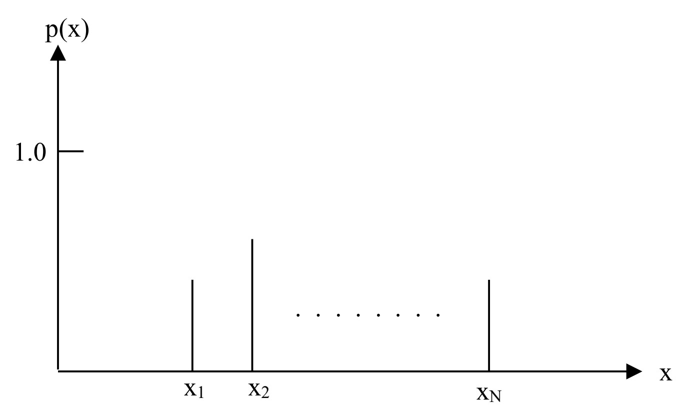
A Random Variable is the numerical value assigned to a particular random event. (e.g. 1 or 0 assigned to head or tail in the flipping of a coin and the number 1 to 6 which are outcomes of the throwing of a dice.)

A set of random variables can be either discrete (1 and 0 for flipping coin, 1 to 6 for throwing of a dice) or continuous (e.g. the wavelength of photon emitting from a black surface).

A probability distribution can be defined for a particular set of random variable and it has the following property

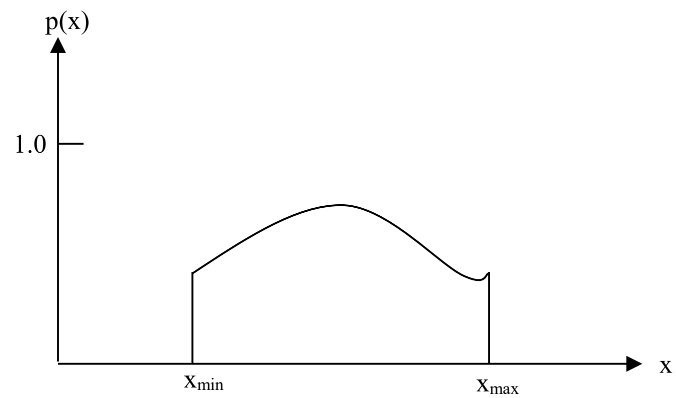
For a set of discrete random variables $\left(x_1 < x_2 < \cdots < x_N\right)$

$$\sum_{i=1}^{N} p\left(x_i\right) = 1.0$$



For a continuous random variable $\left(x_{\min} \leq x \leq x_{\max}\right)$

$$\int_{x_{\min}}^{x_{\max}} p(x) = 1.0$$

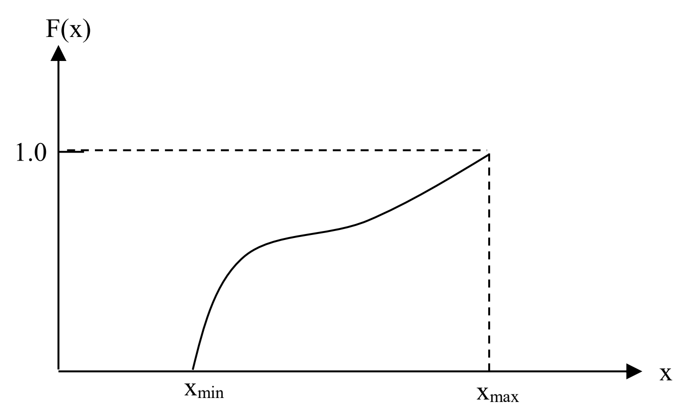


For a continuous random variable X with $x_{\min} < X < x_{\max}$, a <u>cumulative distribution function</u> is defined as

$$F(x) = P\{\text{a random selection of X gives a value less than x}\}$$

Properties of the cumulative distribution function:

1. F(x) is a non-decreasing function for $x_{\min} < x < x_{\max}$, with $F(x_{\min}) = 0$ and $F(x_{\max}) = 1$



2. F(x) may have intervals on which it is differentiable; in these intervals the probability density function (pdf), p(x), is defined as

$$p(x) = \frac{dF(x)}{dx} \geq 0$$

Physically, p(x)dx is the probability of a random selection of X giving a value between x and x +dx and

$$F(x) = \int_{x_{\min}}^{x} p(x')\,dx'$$

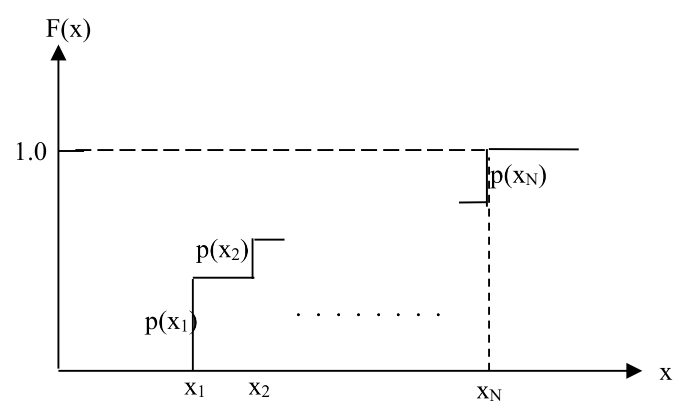
$$F(x_{\min}) = 0 \text{ and } F(x_{\max}) = \int_{x_{\min}}^{x_{\max}} p(x')\,dx' = 1.0$$

3. For a discrete set of random variables $(x_1 < x_2 < \cdots < x_N)$, a cumulative distribution function F(x) is defined as

$$F(x) = \sum_{i=1}^{N} \delta(x - x_i)\, p(x_i)$$

where $\delta(z)$ is the Dirac delta function defined by

$$\delta(z) = \begin{cases} 1 & \text{when } z = 0 \\ 0 & \text{otherwise} \end{cases}$$



2. Sampling Random Variables

Basic problem of sampling: Assuming that we have an infinite supply of random variables with uniform probability, can one use these random variables to generate a set of random variables which follow a specific probability density function p(x) ?

2.1 Transformation of Random Variables

Let x be a random variable with cumulative distribution function $F_x(x)$ and a pdf of

$$f_x(x) = \frac{dF_x(x)}{dx}$$

If y = y(x) is a continuous non-decreasing function of x, i.e.

$$y(X) \le y(x) \quad \text{if and only if} \quad X \le x$$

then y and x should have similar cumulative distribution function

$$P\{y(X) = Y \le y(x)\} = P\{X \le x\}$$

and

$$F_y(y) = F_x(x) \quad \text{when} \quad y = y(x)$$

The relation between the two probability density functions (pdf) is

$$f_y(y)\,dy = f_x(x)\,dx \rightarrow f_y(y)\frac{dy}{dx} = f_x(x)$$

If y = y(x) is a continuous non-increasing function of x, i.e.

$$f_y(y)\frac{dy}{dx} = -f_x(x)$$

and in general, for any random variable x with pdf $f_x(x)$ and $y = y(x)$, then

$$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right| = f_x(x)\left|\frac{dx}{dy}\right|^{-1}$$

Some pdf Examples:

1.

$$f_y(x) = \frac{4}{\pi}\frac{1}{1+x^2} \qquad 0 \le x \le 1$$

$$y = \frac{1}{x} \qquad 1 \le y \le \infty$$

$$f_x(y) = \frac{4}{\pi}\frac{1}{1+y^2} \qquad 1 \le y \le \infty$$

2. Uniform distribution

$$f_x(x) = 1 \qquad x \in [0,1]$$

$$y = a + bx \qquad y \in [a, a+b]$$

$$f_y(y) = \begin{cases} \dfrac{1}{b} & a \le y \le a+b \quad \text{for} \quad b > 0 \\ -\dfrac{1}{b} & a+b \le y \le a \quad \text{for} \quad b < 0 \end{cases}$$

3. Gaussian Distribution

$$f_x(x) = \Phi'(x|0,1) = \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{x^2}{2}\right] \qquad -\infty < x < \infty$$

$$y = \sigma x + \mu \qquad -\infty < y < \infty$$

$$f_y(y) = \Phi'(y|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right] \qquad -\infty < y < \infty$$

4. Uniform distribution

$$f_x(x) = 1 \qquad x \in [0,1]$$
$$y = x^r$$
$$f_y(y) = \left|\frac{1}{r}\right| y^{1/r-1} \qquad \text{for} \begin{cases} 0 < y \le 1 & \text{for} \quad \text{r} > 0 \\ 1 < y < \infty & \text{for} \quad \text{r} < 0 \end{cases}$$

5. Uniform distribution

$$f_x(x) = 1 \qquad x \in [0,1]$$
$$y = -\log x \qquad \left(x = e^{-y}\right)$$
$$f_y(y) = e^{-y} \qquad 1 < y < \infty$$

2.2 Algorithm for the Sampling of a Random Variable with a Specific Distribution

Let ξ be a random variable between 0 and 1, with a uniform pdf and therefore a cumulative distribution function given by

$$F_\xi(\xi) = \begin{cases} 0, & \xi < 0 \\ \xi, & 0 \le \xi \le 1 \\ 1, & \xi \ge 1 \end{cases}$$

If y is an increasing function of ξ, then the actual relation y(ξ) can be determined by the equation

$$F_y(y) = F_\xi(\xi) = \xi$$

The procedure for sampling of y with a given pdf f(y) is as follow:

1. Find the solution to the equation

$$F_y(y) = F_\xi(\xi) = \xi$$

and make sure that y is an increasing function of ξ

2. Sample the random variable ξ with the random number generator and determine y from the relation developed in (1)

3. The resulting distribution of y will satisfy the cumulative distribution $F_y(y)$ and the pdf f(y)

Examples:

1.

$$f_y(y) = \lambda e^{-\lambda y} \qquad 0 < y < \infty$$

$$F_y(y) = \int_0^y \lambda e^{-\lambda u} du = 1 - e^{-\lambda y} = \xi$$

$$y = -\frac{1}{\lambda}\log(1-\xi)$$

2.

$$f_y(y) = \frac{2}{\pi}\frac{1}{1+y^2} \qquad 0 < y < \infty$$

$$F_y(y) = \int_0^y \frac{2}{\pi}\frac{1}{1+u^2} du = \frac{2}{\pi}\tan^{-1} y = \xi$$

$$y = \tan\frac{\pi}{2}\xi$$

3.

$$f_y(y) = \frac{1}{\pi}\frac{1}{1+y^2} \qquad -\infty < y < \infty$$

$$F_y(y) = \int_{-\infty}^y \frac{1}{\pi}\frac{1}{1+u^2} du = \frac{1}{\pi}\tan^{-1} y + \frac{1}{2} = \xi$$

$$y = \tan\frac{\pi}{2}(2\xi - 1)$$

4.

$$f_r(r) = r\exp\left[-\frac{1}{2}r^2\right] \qquad 0 < r < \infty$$

$$F_r(r) = \int_0^r u\exp\left[-\frac{1}{2}u^2\right]du = 1 - \exp\left[-\frac{r^2}{2}\right] = \xi$$

$$r = \left[-2\log(1-\xi)\right]^{1/2}$$

Example of Sampling of Two Random Variables

Let y_1 and y_2 be two independent random variables for a Gausian distribution pdf

$$f\left(y_1, y_2\right) = \Phi'\left(y_1 \middle| 0,1\right)\Phi'\left(y_2 \middle| 0,1\right) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(y_1^2 + y_2^2\right)\right]$$

the equation can be transformed into polar coordinate by the transformation

$$y_1 = r\cos\phi$$
$$y_2 = y\sin\phi$$

and the pdf becomes

$$f\left(y_1, y_2\right)dy_1dy_2 = \left(\exp\left[-\frac{r^2}{2}\right]rdr\right)\left(\frac{d\phi}{2\pi}\right)$$

Since the angle ϕ is uniformly distributed over $(0, 2\pi)$, it is sampled by

$$\phi = 2\pi\xi_2$$

r is sampled by

$$r = \left[-2\log\left(1-\xi_1'\right)\right]^{1/2} = \left[-2\log\xi_1\right]^{1/2}$$

So the two independent variables are sampled by

$$y_1 = \left[-2\log\xi_1\right]^{1/2}\cos\left(2\pi\xi_2\right)$$
$$y_2 = \left[-2\log\xi_1\right]^{1/2}\sin\left(2\pi\xi_2\right)$$

2.3 Numerical Transformation

For same pdf, the solution to the equation

$$F_y(y) = \xi$$

cannot be done analytically. For example, a random variable with a Gaussian distribution has a pdf

$$f(x) = \Phi(x|0,1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

would yield an equation

$$F_x(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp\left(-\frac{u^2}{2}\right) du = erf(x) = \xi$$

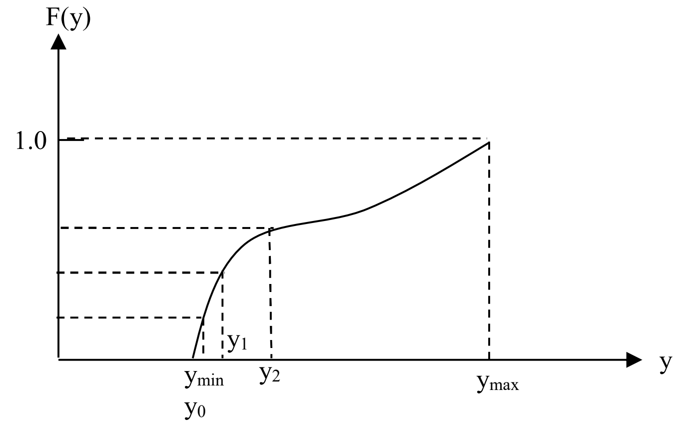
To generate the random variable x from a sampling of ξ would require a numerical inversion.

<u>An alternate numerical approach</u>

For a given $F_y(y)$, find a set of discrete value of y $(y_0, y_1, \cdots y_N)$ such that

$$F_y(y_n) = \int_{y_{\min}}^{y_n} f_y(y)\, dy = \frac{n}{N}, \qquad n = 0, 1, 2, \cdots, N$$

with $y_0 = y_{\min},\ y_N = y_{\max}$



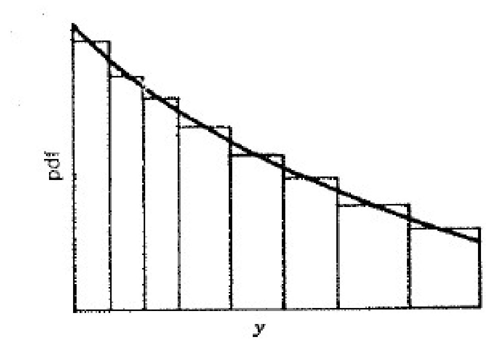
For a random variable ξ, find n such that

$$\frac{n}{N} < \xi < \frac{n+1}{N}$$

Then y(ξ) can be calculated by linear interpolation

$$\begin{aligned} &y(\xi) = y_n + (y_{n+1} - y_n)u \\ &with \\ &u = N\xi - n, \qquad 0 < u < 1 \end{aligned}$$

Note that this approach corresponds to approximating a pdf by a step function as follow:

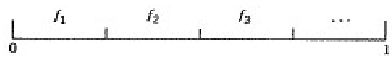


2.4 Sampling of Discrete Distributions

Suppose we have a class of events E_k with discrete probability f_k and we want to sample the events. Since

$$\sum_{k=1}^{N} f_k = 1.0$$

It is possible to take the interval [0,1] and exhaust it by dividing the interval into N segments each of which has a length equal to f_k as follow:



A uniform random variable ξ is generated. The interval to which ξ falls determine the identify of the event E_j by the relation

$$\sum_{k=0}^{j-1} f_k < \xi < \sum_{k=0}^{j} f_k$$

Example: Sampling of K equally likely event

$$f_k = \frac{1}{K}, \quad k = 1, 2, \cdots, K$$

For a given random number ξ within the interval [0,1], the interval at which ξ falls is determined by

$$\frac{j-1}{K} \le \xi \le \frac{j}{K} \to j-1 \le K\xi \le j$$

Computer program (in FORTRAN) simulating the flipping of a coin (with equal probability of two discrete outcomes, head or tail)

```
      INTEGER(4) iseed
      iseed = 425001

      open (10, file = 'coin.out')
      write(10, 10)
10    format(' i ', 2x, '   rflip   ', 2x, ' nh', 2x, ' nt', /)
      n = 100
      nh = 0
      nt = 0
      do i = 1, n
```

```
          rflip = RAN(iseed)
          if (rflip .le. 0.5) then
          nh = nh +1
          else
          nt = nt +1
          endif
          write(10, 100) i, rflip, nh, nt
100       format(i3, 2x, e11.4, 2(2x, i3))
          enddo
          end
```

Result of the calculation

i	rflip	nh	nt
1	.5531E+00	0	1
2	.8759E+00	0	2
3	.4586E+00	1	2
4	.6844E+00	1	3
5	.1413E+00	2	3
6	.3427E+00	3	3
7	.2402E+00	4	3
8	.9630E+00	4	4
9	.3832E+00	5	4
10	.5354E+00	5	5
11	.8917E+00	5	6
12	.2799E+00	6	6
13	.2022E+00	7	6
14	.6372E+00	7	7
15	.7591E-01	8	7
16	.3466E+00	9	7
17	.2850E+00	10	7
18	.9279E+00	10	8
19	.7132E+00	10	9
20	.7959E+00	10	10
21	.7828E+00	10	11
22	.2777E-01	11	11
23	.4178E+00	12	11
24	.1118E+00	13	11
25	.9732E+00	13	12
26	.5329E+00	13	13
27	.1612E+00	14	13
28	.4771E+00	15	13
29	.8472E+00	15	14
30	.4074E+00	16	14
31	.4647E+00	17	14
32	.3606E+00	18	14
33	.4194E+00	19	14
34	.9996E+00	19	15
35	.4202E-01	20	15
36	.2159E+00	21	15
37	.7820E+00	21	16
38	.4850E+00	22	16
39	.7024E+00	22	17
40	.8063E+00	22	18
41	.5745E+00	22	19
42	.7305E+00	22	20
43	.5792E-01	23	20
44	.6274E-02	24	20
45	.8540E+00	24	21
46	.5200E+00	24	22
47	.3817E+00	25	22
48	.2477E+00	26	22
49	.6113E+00	26	23
50	.6542E+00	26	24

2.5 Sampling of Mixed Distributions

Consider a pdf $f_x(x)$ which has a jump from 0 to 0.5 at x = 0, for example

$$F_x(x) = \begin{cases} 0 & x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & x > 0 \end{cases}$$

We set

$$x = \begin{cases} 0 & \xi \leq \frac{1}{2} \\ \text{-}\log\left[2(1\text{-}\xi)\right]/\lambda & \xi \geq \frac{1}{2} \end{cases}$$

that is, we solve for $F_x(x) = \xi$ only when $\xi \geq 1/2$.

Application to Radiative Heat Transfer

Example 1: Blackbody Emissive Power (Planck function)

$$e_{\lambda,b} = \frac{2\pi C_1}{\lambda^5\left(e^{C_2/\lambda T} - 1\right)}$$ = energy emitted per unit area per unit time per unit wavelength around λ

$$\sigma T^4 = \int_0^\infty e_{\lambda,b}\, d\lambda$$ = total energy emitted per unit area per unit time

$$P(\lambda) = \frac{e_{\lambda,b}}{\sigma T^4}$$ = pdf of a surface emitting radiant energy at wavelength λ

$$R_\lambda = \int_0^\lambda P(\lambda')\, d\lambda' = \frac{\int_0^\lambda e_{\lambda,b}(\lambda')\, d\lambda'}{\sigma T^4} = F_{0-\lambda}$$ = cumulative distribution function

Monte Carlo simulation of radiative emission from a blackbody:

N = number of "bundles" used in the simulation
e = $\sigma T^4/N$ = energy per bundle
For each bundle, pick a random number ξ
Determine the wavelength λ from the relation $\xi = R_\lambda = F_{0-\lambda}$

Example 2: Emission from a non-gray surface

$$e_\lambda = \varepsilon_\lambda e_{\lambda,b} = \varepsilon_\lambda \frac{2\pi C_1}{\lambda^5\left(e^{C_2/\lambda T}-1\right)}$$ = emissive power, energy emitted per unit area per unit time per unit wavelength around λ

$$e = \varepsilon\sigma T^4 = \int_0^\infty \varepsilon_\lambda e_{\lambda,b} d\lambda$$ = total energy emitted per unit area per unit time

$$P(\lambda) = \frac{\varepsilon_\lambda e_{\lambda,b}}{\varepsilon\sigma T^4}$$ = pdf of a surface emitting radiant energy at wavelength λ

$$R_\lambda = \int_0^\lambda P(\lambda')d\lambda' = \frac{\int_0^\lambda \varepsilon_\lambda(\lambda')e_{\lambda,b}(\lambda')d\lambda'}{\varepsilon\sigma T^4}$$ = cumulative distribution function

Monte Carlo simulation of radiative emission from a non-gray surface

N = number of "bundles" used in the simulation
e = $\varepsilon\sigma T^4/N$ = energy per bundle
For each bundle, pick a random number ξ
Determine the wavelength λ from the relation $\xi = R_\lambda$

```
c---------------------------------------------------------------------------c
c                                                                           c
c                                                                           c
c      this program generate the monte carlo simulation of
c      emission from a non-gray surface with step wise emissivity
c                                                                           c
c---------------------------------------------------------------------------c
       program emission

       implicit double precision(a-h,o-z)
       common /data/  xlemit(100), emit(100), xlambda(100)
       common /data1/ nemit
       dimension ncount(100)
      real*8 rand
       data rand/5249347.d0/
       sigma = 0.5672e-04
       xc1 = 0.595e11
       xc2 = 1.439e4
       xpi = 3.14159

c     sigma (erg/(K**4-cm**2-sec)), Stefan-Boltzmann Const.

      open (10, file = 'emit.in')
      open (20, file = 'emit.out')
      open (30, file = 'bundle.out')

       read(10, *) ts, nbundle
       read(10, *) nemit
       do ne = 1, nemit
       read(10, *) xlemit(ne), emit(ne)
       enddo

       if (xlemit(1) .gt. 0.d0) then
       write(20, 10) xlemit(1)
10     format(/, '  first wavelength must be zero',
     1         /, '  xlemit(1) = ', e11.4)
       stop
       endif
c
c      emit(ne) is the emissivity with wavelength greater than
c      xlemit(ne) but less than xlemit(ne +1)
c
c      emit(nemit) is the emissivity with wavelength greater than
c      xlemit(nemit)
c

       read(10, *) nlambda
       do nx = 1, nlambda
       read(10, *) xlambda(nx)
       ncount(nx) = 0.d0
       enddo

c
c      this section calculate the total emissivity
c

       etot = 0.d0
       do nx = 1, nemit
       if (nx .lt. nemit) then
       etot = etot +(ffrac(ts, xlemit(nx +1)) -ffrac(ts, xlemit(nx)))
     1               *emit(nx)
       else
       etot = etot +(1.d0 -ffrac(ts, xlemit(nx)))
     1               *emit(nx)
       endif
       enddo

       ebundle = etot*sigma*ts**4/nbundle

       write(20, 11) ts, etot, ebundle
11     format(/, '  surface temperature = ', e11.4,
```

```
     1        /, '  total emissivity = ', e11.4,
     1        /, '  energy per bundle = ', e11.4, ' erg/cm**2/s')

      do nx = 1, nlambda
      write(20, 22) xlambda(nx), fem(ts, xlambda(nx))
22    format('  xlambda(nx) = ', e11.4, '  fem = ', e11.4)
      enddo

      do i = 1, nbundle

         call random(rx,rand)
c     write(20, 21) rx
21    format('  rx = ', e11.4)

      do nx = 1, nlambda -1
c     write(20, 20) nx, xlambda(nx), xlambda(nx +1),
c    1              rx, fem(ts, xlambda(nx)), fem(ts, xlambda(nx +1))
20    format('  nx = ', i3, '  xlambda(nx) = ', e11.4,
     1        '  xlambda(nx) = ', e11.4,
     1        '  rx = ', e11.4, '  fem(ts, xlambda(nx)) = ', e11.4,
     1        '  fem(ts, xlambda(nx +1)) = ', e11.4)
      if (rx .ge. fem(ts, xlambda(nx))/etot .and.
     1     rx .le. fem(ts, xlambda(nx +1))/etot) then
      ncount(nx) = ncount(nx) +1
      goto 12
      endif
      enddo
      if (rx .gt. fem(ts, xlambda(nlambda))) then
      ncount(nlambda) = ncount(nlambda) +1
      endif
12    continue

      enddo

      write(30, 101)
101   format('   lambda  ', 2x, 'ncount', 2x, '  energy   ', 2x,
     1        '   power   ', 2x, '    ebb    ', 2x, '    em')

      sume = 0.d0
      ntot = 0
      do i = 1, nlambda
      xenergy = ncount(i)*ebundle
      if (i .lt. nlambda) then
      dlam = xlambda(i +1) -xlambda(i)
      else
      dlam = xlambda(nlambda) -xlambda(nlambda -1)
      endif

      do nx = 1, nemit
      if (xlambda(i) .lt. xlemit(nx)) then
      xem = emit(nx -1)
      goto 110
      endif
      enddo
      xem = emit(nemit)
110   continue

      if (xlambda(i) .eq. 0.d0) then
      ebb = 0.d0
      else
      ebb = xem*2.d0*xpi*xc1/xlambda(i)**5
     1          /(dexp(xc2/xlambda(i)/ts) -1.d0)
      endif

      write(30, 100) xlambda(i), ncount(i), xenergy, xenergy/dlam,
     1              ebb, xem
100   format(e11.4, 2x, i6, 4(2x, e11.4))
      sume = sume +ncount(i)*ebundle
      ntot = ntot +ncount(i)
      enddo
```

```
      write(20, 201) ntot, sume, etot*sigma*ts**4
201   format(/, '  ntot = ', i3,
     1        /, '  sume = ', e11.4, '  etot*sigma*ts**4 = ', e11.4)

      end

*----------------------------------*
*  This subroutine generates       *
*  pseudo random number            *
*----------------------------------*
         SUBROUTINE random(RAN,RAND)
C--------------------------------------------------C
C     RANDOM NUMBER GENERATOR                      C
C--------------------------------------------------C
      implicit double precision(a-h,o-z)
         REAL*8 RAND
         RAND=DMOD(RAND*131075.0d0,2147483649.0d0)
c         RAN=SNGL(RAND/2147483649.0D0)
         RAN=DBLE(RAND/2147483649.0D0)
      write(12,'(2x,"rand=",e12.5,"  ran=",f7.4)') rand,ran  !5/8/92
         RETURN
         END


c ********************************************************************
      double precision function ffrac(tfuel,xlm)
       implicit double precision(a-h,o-z)
       real prodtable(10),fractable(10)
c     lambda T products are in um K
       data prodtable/   555.6,  1666.7,  3055.6,  4166.7,  5277.8,
     1               6388.9,  7500.0,  9722.2, 12777.8, 55555.6/
       data fractable/ 0.17d-7, 0.02537, 0.28576, 0.51029, 0.66685,
     1               0.76838, 0.83435, 0.90819, 0.95307, 1./


      prod = tfuel*xlm

       if (prod.ge.0.d0 .and. prod.le.prodtable(1)) then
         ffrac = fractable(1)*prod/prodtable(1)
       return
       else
       if (prod.gt.prodtable(10)) then
         ffrac = fractable(10)
       return
       endif
       endif

       do i = 1, 9
       if (prod .ge. prodtable(i) .and. prod .le. prodtable(i+1)) then
       fraction = (prod - prodtable(i))/(prodtable(i+1) -prodtable(i))
       ffrac = fractable(i) +fraction*(fractable(i+1) -fractable(i))
       goto 100
       endif
       enddo

       write(20, 10) tfuel, xlm
10     format(/, '  error in finding f values with',
     1        /, '  tfuel = ', e11.4,
     2        /, '  xlm   = ', e11.4)
       stop

100    return

       end
c ***************************************************************
c
      double precision function fem(tfuel,xlm)
       implicit double precision(a-h,o-z)
       common /data/ xlemit(100), emit(100), xlambda(100)
       common /data1/ nemit
```

```
      do nx = 1, nemit -1
      if (xlm .ge. xlemit(nx) .and. xlm .le. xlemit(nx +1)) then
      fem = 0.d0
      do ix = 1, nx -1
      fem = fem +emit(ix)*(ffrac(tfuel, xlemit(ix +1))
     1                     -ffrac(tfuel, xlemit(ix)))
      enddo
      fem = fem +emit(nx)*(ffrac(tfuel, xlm)
     1                      -ffrac(tfuel, xlemit(nx)))
      return
      endif
      enddo

      fem = 0.d0
      do ix = 1, nemit -1
      fem = fem +emit(ix)*(ffrac(tfuel, xlemit(ix +1))
     1                     -ffrac(tfuel, xlemit(ix)))
      enddo
      fem = fem +emit(nemit)*(ffrac(tfuel, xlm)
     1                       -ffrac(tfuel, xlemit(nemit)))
      return
      end
```

Input File (emit.in)

```
          2000.   200000
3
0.0       0.35
2.0       0.2
4.0       0.0
21
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
```

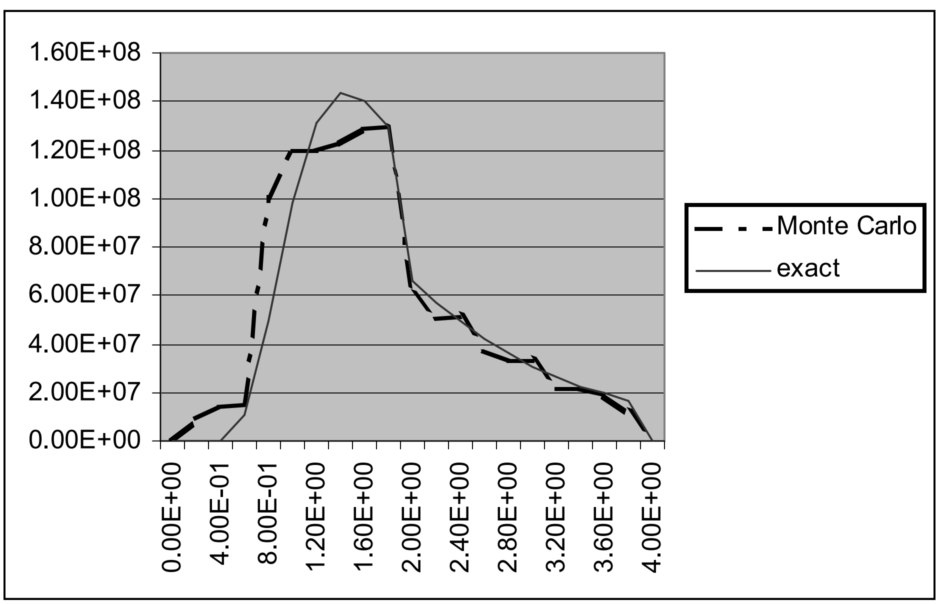
Output file (emit.out)

```
surface temperature =   .2000E+04
total emissivity =   .2417E+00
energy per bundle =   .1097E+04 erg/cm**2/s
xlambda(nx) =   .0000E+00  fem =   .0000E+00
xlambda(nx) =   .2000E+00  fem =   .4284E-08
xlambda(nx) =   .4000E+00  fem =   .1953E-02
xlambda(nx) =   .6000E+00  fem =   .5150E-02
xlambda(nx) =   .8000E+00  fem =   .8346E-02
xlambda(nx) =   .1000E+01  fem =   .3075E-01
xlambda(nx) =   .1200E+01  fem =   .5700E-01
xlambda(nx) =   .1400E+01  fem =   .8324E-01
xlambda(nx) =   .1600E+01  fem =   .1102E+00
xlambda(nx) =   .1800E+01  fem =   .1385E+00
xlambda(nx) =   .2000E+01  fem =   .1668E+00
xlambda(nx) =   .2200E+01  fem =   .1801E+00
xlambda(nx) =   .2400E+01  fem =   .1914E+00
xlambda(nx) =   .2600E+01  fem =   .2027E+00
xlambda(nx) =   .2800E+01  fem =   .2107E+00
xlambda(nx) =   .3000E+01  fem =   .2181E+00
xlambda(nx) =   .3200E+01  fem =   .2253E+00
xlambda(nx) =   .3400E+01  fem =   .2300E+00
xlambda(nx) =   .3600E+01  fem =   .2348E+00
xlambda(nx) =   .3800E+01  fem =   .2390E+00
xlambda(nx) =   .4000E+01  fem =   .2417E+00

ntot = ***
sume =   .2193E+09  etot*sigma*ts**4 =   .2193E+09
```

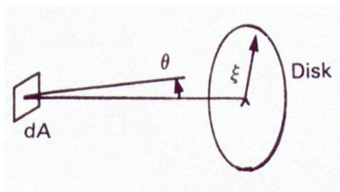
Output file (bundle.out)

lambda	ncount	energy	power	ebb	em
.0000E+00	0	.0000E+00	.0000E+00	.0000E+00	.3500E+00
.2000E+00	1625	.1782E+07	.8910E+07	.9725E-01	.3500E+00
.4000E+00	2616	.2869E+07	.1434E+08	.1971E+06	.3500E+00
.6000E+00	2660	.2917E+07	.1459E+08	.1043E+08	.3500E+00
.8000E+00	18353	.2013E+08	.1006E+09	.4959E+08	.3500E+00
.1000E+01	21799	.2391E+08	.1195E+09	.9825E+08	.3500E+00
.1200E+01	21848	.2396E+08	.1198E+09	.1312E+09	.3500E+00
.1400E+01	22338	.2450E+08	.1225E+09	.1435E+09	.3500E+00
.1600E+01	23407	.2567E+08	.1283E+09	.1406E+09	.3500E+00
.1800E+01	23547	.2582E+08	.1291E+09	.1296E+09	.3500E+00
.2000E+01	10980	.1204E+08	.6021E+08	.6581E+08	.2000E+00
.2200E+01	9206	.1010E+08	.5048E+08	.5729E+08	.2000E+00
.2400E+01	9329	.1023E+08	.5115E+08	.4931E+08	.2000E+00
.2600E+01	6714	.7363E+07	.3681E+08	.4219E+08	.2000E+00
.2800E+01	6038	.6622E+07	.3311E+08	.3602E+08	.2000E+00
.3000E+01	6046	.6630E+07	.3315E+08	.3075E+08	.2000E+00
.3200E+01	3918	.4297E+07	.2148E+08	.2630E+08	.2000E+00
.3400E+01	3904	.4281E+07	.2141E+08	.2254E+08	.2000E+00
.3600E+01	3512	.3851E+07	.1926E+08	.1939E+08	.2000E+00
.3800E+01	2160	.2369E+07	.1184E+08	.1673E+08	.2000E+00
.4000E+01	0	.0000E+00	.0000E+00	.0000E+00	.0000E+00



Monte Carlo Simulation of arbitray probability density distribution with two or more independent variables

Example: Distribution of radiation packets arriving at various disk radii



F(ξ, θ) = number of packets that have arrived at the disk within each small radial increment Δξ and angular increment Δθ about some radius ξ and angle θ

f(ξ) = F(ξ, θ)/(ΔξΔθ) = frequency function, the number of packets per unit ξ and per unit θ arriving at the disk at (ξ,θ)

P(ξ,θ) = probability density function (in two dimension)

P(ξ,θ)dξdθ = probability that a radiation packet will arrive within an infinitesimal area dξdθ about the position (ξ,θ)

$$P(\xi,\theta) = \frac{f(\xi,\theta)}{\int_0^R \int_0^{2\pi} f(\xi,\theta)\, d\theta d\xi}$$

Question: Can a random number generator be used to simulate the probability density distribution of the radiation packets.

Answer:

1. Pick a random number R_1, determine ξ from

$$R_1 = F(\xi) = \int_{\xi_{\min}}^{\xi} \int_{\theta_{\min}}^{\theta_{\max}} P(\xi',\theta')\, d\xi' d\theta'$$

example: for the radiation packets problem:

$$R_1 = F(\xi) = \int_0^{\xi} \int_0^{2\pi} P(\xi',\theta')\, d\xi' d\theta'$$

2. and for a given ξ, pick a second random number R_2 and determine θ from

$$R_2 = G(\xi,\theta) = \int_{\theta_{\min}}^{\theta} P(\xi,\theta')\,d\theta'$$

example: for the radiation packets problem:

$$R_2 = G(\xi,\theta) = \int_{0}^{\theta} P(\xi,\theta')\,d\theta'$$

Sampling procedure for probability density function with three variables: P(λ, ξ, θ)

1. Pick a random number R_1

2. Determine λ from the relation

$$R_1 = F_\lambda(\lambda) = \int_{\lambda_{\min}}^{\lambda}\int_{\xi_{\min}}^{\zeta_{\max}}\int_{\theta_{\min}}^{\theta_{\max}} P(\lambda',\xi',\theta')\,d\lambda' d\xi' d\theta'$$

3. Pick a second random number R_2

4. For the given value of λ and the second random number R_2, determine ξ from

$$R_2 = G_\xi(\lambda,\xi) = \frac{\int_{\xi_{\min}}^{\xi}\int_{\theta_{\min}}^{\theta_{\max}} P(\lambda,\xi',\theta')\,d\xi' d\theta'}{\int_{\xi_{\min}}^{\xi_{\max}}\int_{\theta_{\min}}^{\theta_{\max}} P(\lambda,\xi',\theta')\,d\xi' d\theta'}$$

5. Pick a third random number R_3

6. For the given values of (λ, ξ) and the third random number, determine θ from

$$R_3 = H_\theta(\lambda,\xi,\theta) = \frac{\int_{\theta_{\min}}^{\theta} P(\lambda,\xi,\theta')\,d\theta'}{\int_{\theta_{\min}}^{\theta_{\max}} P(\lambda,\xi,\theta')\,d\theta'}$$

If $\quad P(\lambda,\xi,\theta) = P_1(\lambda)P_2(\xi)P_3(\theta)$

(λ, θ, ξ have mutually independent probablistic density functions)

$$F_\lambda(\lambda) = \int_{\lambda_{\min}}^{\lambda} P_1(\lambda')\,d\lambda' \int_{\xi_{\min}}^{\xi_{\max}} P_2(\xi')\,d\xi' \int_{\theta_{\min}}^{\theta_{\max}} P_3(\theta')\,d\theta'$$
$$= \int_{\lambda_{\min}}^{\lambda} P_1(\lambda')\,d\lambda'$$

$$G_\xi(\lambda,\xi) = \frac{P_1(\lambda)\int_{\xi_{\min}}^{\xi} P_2(\xi')\,d\xi' \int_{\theta_{\min}}^{\theta_{\max}} P_3(\theta')\,d\theta'}{P_1(\lambda)\int_{\xi_{\min}}^{\xi_{\max}} P_2(\xi')\,d\xi' \int_{\theta_{\min}}^{\theta_{\max}} P_3(\theta')\,d\theta'}$$
$$= \int_{\xi_{\min}}^{\xi} P_2(\xi')\,d\xi'$$

$$H_\theta(\lambda,\xi,\theta) = \frac{P_1(\lambda)P_2(\xi)\int_{\theta_{\min}}^{\theta} P_3(\theta')\,d\theta'}{P_1(\lambda)P_2(\xi)\int_{\theta_{\min}}^{\theta_{\max}} P_3(\theta')\,d\theta'} = \int_{\theta_{\min}}^{\theta} P_3(\theta')\,d\theta'$$

General procedure for generating distribution with random numbers for three independent variables

Pick R_1, R_2 and R_3, generate the variables λ, ξ, θ from the relations:

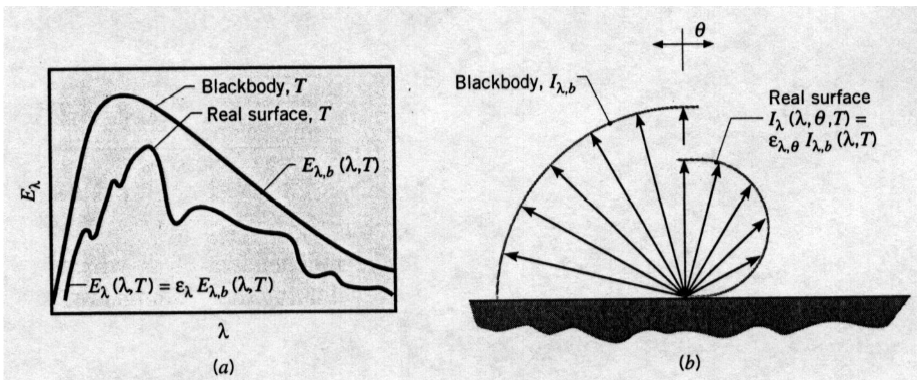
$$R_1 = F(\lambda) = \int_{\lambda_{\min}}^{\lambda} P_1(\lambda')\,d\lambda'$$
$$R_2 = G(\xi) = \int_{\xi_{\min}}^{\xi} P_2(\xi')\,d\xi'$$
$$R_3 = H(\theta) = \int_{\theta_{\min}}^{\theta} P_3(\theta')\,d\theta'$$

Application to Radiation Heat Transfer:

1. Evaluation of exchange factor

e(λ, θ, φ) dλdθdφ = energy emitted per unit area per unit wavelength in angular interval dθ and dφ

Real Surface Emission (12.4 in Incropera and DeWitt)



$$e(\lambda,\theta,\varphi)\,d\lambda d\theta d\varphi = \frac{1}{\pi}\varepsilon_{\lambda,\theta} e_{\lambda,b}\cos\theta\sin\theta d\lambda d\theta d\varphi$$

with $\varepsilon_{\lambda,\theta}$ = spectral directional emissivity

e(T) = totoal emission

$$E(T) = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\infty}\frac{1}{\pi}\varepsilon_{\lambda,\theta}E_{\lambda,b}\cos\theta\sin\theta d\theta d\varphi$$
$$= \varepsilon(\mathrm{T})\sigma T^4$$

with ε(T) = total emissivity

Probability density function for radiation emission

$$P(\lambda,\theta,\varphi)=\frac{\varepsilon_{\lambda,\vartheta}E_{\lambda,b}\cos\theta\sin\theta}{\pi\varepsilon(T)\sigma T^4}$$

if $$\varepsilon_{\lambda,\theta}=\Phi_1(\lambda)\Phi_2(\theta)$$

then

$$P(\lambda,\theta,\varphi)=P_1(\lambda)P_2(\theta)P_3(\varphi)$$

with

$$P_1(\lambda)=\frac{\Phi_1(\lambda)E_{\lambda,b}}{\varepsilon(T)\sigma T^4}$$
$$P_2(\theta)=2\Phi_2(\theta)\cos\theta\sin\theta$$
$$P_3(\varphi)=\frac{1}{2\pi}$$

For gray diffuse surface ($\varepsilon_{\lambda,\theta}=\varepsilon(T)$)

$$P_1(\lambda)=\frac{E_{\lambda,b}}{\sigma T^4}\qquad R(\lambda)=\int_0^\lambda\frac{E_{\lambda,b}d\lambda}{\sigma T^4}=F_{0-\lambda}$$
$$P_2(\theta)=2\cos\theta\sin\theta\quad R(\theta)=\int_0^\theta 2\cos\theta'\sin\theta'd\theta'=\sin^2\theta$$
$$P_3(\varphi)=\frac{1}{2\pi}\qquad R(\varphi)=\frac{\varphi}{2\pi}$$

Evaluation of Radiative Exchange with the Monte Carlo Method

<u>View Factor (F_{1-2})</u> = Fraction of radiation emitted from a surface A_1 which is absorbed by a black surface A_2 (without accounting for reflection from other surfaces)

1. Emit N_1 energy bundles from area A_1 using the probabilistic distribution

2. Counts all the energy bundles which is intercepted by A_2, N_2

3. $F_{1-2} = N_2/ N_1$

<u>Interchange Factors (F_{1-2})</u> = Fraction of radiation emitted from a surface A_1 which is absorbed by a surface A_2 (accounting for all possible reflection from other surfaces)

(same procedure except the energy bundle is followed through all of its reflection from surfaces)

NEVADA (A computer code to compute view factors and interchange factors for up to 10 diffuse/specular reflecting surfaces)

Download from http://tac1.com/download.html

```
c---------------------------------------------------------------------------
--c
c
c
c
c
c     this program generate the monte carlo simulation of
c     problem 12.8 of Incropera and DeWitt
c
c
c---------------------------------------------------------------------------
--c
      program angular

      implicit double precision(a-h,o-z)
      common /data/  xlemit(100), emit(100), xlambda(100)
      common /data1/ nemit
      real*8 rand
      data rand/5249347.d0/
      sigma = 0.5672e-04
      xc1 = 0.595e11
      xc2 = 1.439e4
      xpi = 3.14159

c     sigma (erg/(K**4-cm**2-sec)), Stefan-Boltzmann Const.

      open (10, file = 'angular.in')
      open (20, file = 'angular.out')
      open (30, file = 'bundle.out')

      read(10, *) theta1, theta2
      read(10, *) nbundle

c
c     this program is to compute the fraction of energy
c     radiated between theta1 and theta2
c
c



      write(20, 11) theta1, theta2, nbundle
11    format(/, '  theta1 = ', e11.4,
     1         /, '  theta2 = ', e11.4,
     1         /, '  number of bundle = ', i7)

c
c     theta1 and theta2 input is in degree
c
      theta1 = theta1/180*xpi
      theta2 = theta2/180*xpi

      write(30, 101)
101   format('  i  ', 2x, '   theta   ', 2x, 'ncount', 2x,
     1         '    f12    ')

      ncount = 0
```

```
      do i = 1, nbundle

           call random(rx,rand)
      theta = dasin(dsqrt(rx))

c     write(20, 21) rx, theta, theta1, theta2
21    format('  rx = ', e11.4, '  theta = ', e11.4,
     1      /, '  theta1 = ', e11.4,
     1         '  theat2 = ', e11.4)

      if (theta .le. theta2 .and. theta .ge. theta1) then
      ncount = ncount +1
      xcount = ncount*1.d0
      f12 = xcount/i
      write(30, 102) i, theta/xpi*180.d0, ncount, f12

      endif

102   format(i5, 2x, e11.4, 2x, i5, 2x, e11.4)

      enddo

      stop
      end

*----------------------------------*
*  This subroutine generates       *
*  pseudo random number            *
*----------------------------------*
          SUBROUTINE random(RAN,RAND)
C---------------------------------------------------C
C     RANDOM NUMBER GENERATOR                       C
C---------------------------------------------------C
      implicit double precision(a-h,o-z)
          REAL*8 RAND
          RAND=DMOD(RAND*131075.0d0,2147483649.0d0)
c          RAN=SNGL(RAND/2147483649.0D0)
          RAN=DBLE(RAND/2147483649.0D0)
      write(12,'(2x,"rand=",e12.5,"  ran=",f7.4)') rand,ran  !5/8/92
          RETURN
          END
```

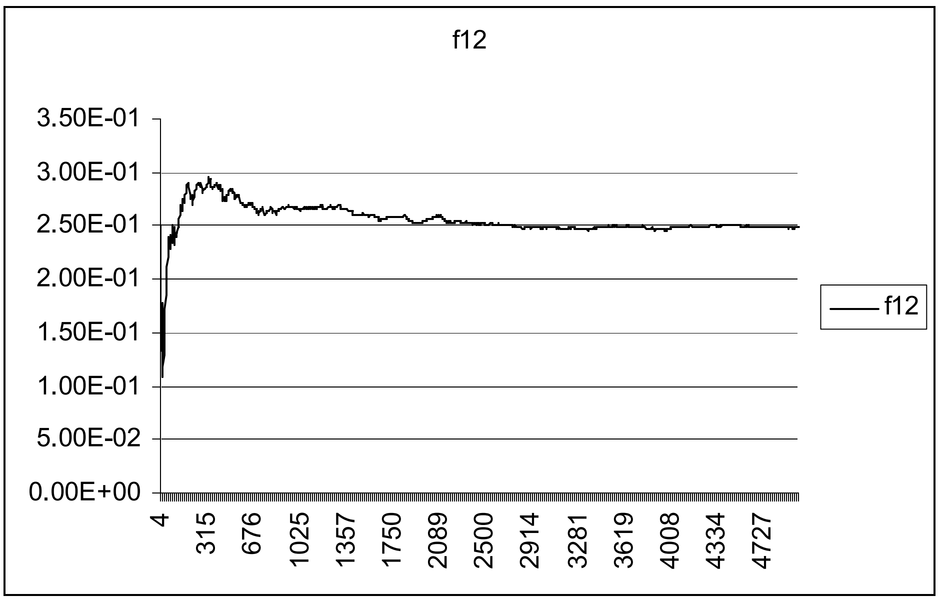
File angular.in

```
0.  30.
5000
```

File angular.out

```
  theta1 =    .0000E+00
  theta2 =    .3000E+02
  number of bundle =    5000
```

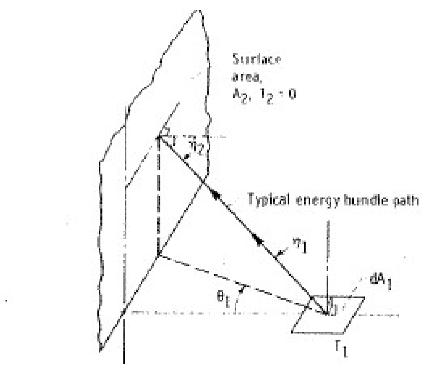
File bundle.out



2. Evaluation of radiative exchange between gray surfaces

Physics of the problem:

Consider the radiative exchange between dA_1, at temperature T_1 and surface A_2, an infinite plane at temperature $T_2 = 0$ as in the following figure:



Let element dA_1 have the emissivity $\varepsilon_1(T_1, \lambda, \eta_1) = \varepsilon_{1,\lambda}(T_1, \lambda)\varepsilon_{1,\eta}(T_1, \eta_1)$

The total emissivity is given by

$$\varepsilon_1(T_1) = \frac{1}{\sigma T_1^4}\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\infty}\frac{1}{\pi}\varepsilon_{1,\lambda}(T_1,\lambda)\varepsilon_{1,\eta}(T_1,\eta_1)E_{\lambda,b}(T_1,\lambda)\cos\eta_1\sin\eta_1 d\eta_1 d\theta_1$$

Then the sampling of emission from dA_1 can be generated by the following pdf and cumulative distribution functions

$$P_1(\lambda) = \frac{\varepsilon_{1,\lambda}(T_1,\lambda)E_{\lambda,b}}{\varepsilon_1(T_1)\sigma T_1^4} \qquad \xi_\lambda = R_1(\lambda) = \frac{\int_0^{\lambda}\varepsilon_{1,\lambda}(T_1,\lambda)E_{\lambda,b}d\lambda}{\varepsilon_1(T_1)\sigma T_1^4}$$

$$P_2(\eta_1) = 2\varepsilon_{1,\eta}(T_1,\eta_1)\cos\eta_1\sin\eta_1 \qquad \xi_\eta = R_2(\eta_1) = 2\int_0^{\eta_1}\varepsilon_{1,\eta}(T_1,\eta_1')\cos\eta_1'\sin\eta_1' d\eta'$$

$$P_3(\theta_1) = \frac{1}{2\pi} \qquad \xi_\theta = R_3(\theta_1) = \frac{\theta_1}{2\pi}$$

For convenience of computation, it is useful to simulate the exact integral by a power series

$$\lambda = A + B\xi_\lambda + C\xi_\lambda^2 + \cdots$$
$$\eta_1 = A' + B'\xi_\eta + C'\xi_\eta^2 + \cdots$$

Monte Carlo simulation of the absorption process

First determine the normal of surface A_2

$$\vec{n}_2 = \hat{x}$$

For a bundle with wavelength λ, emitted in direction (η_1, θ_1), it has a directional vector of

$$\vec{r}_{12} = \left(\sin\eta_1\cos\theta, \sin\eta_2\sin\theta, \cos\eta_1\right)$$

Relative to surface A_2, the bundle has a polar angle given by

$$\cos\eta_2 = \vec{r}_{12}\cdot\vec{n}_2 = \sin\eta_1\cos\theta_1$$

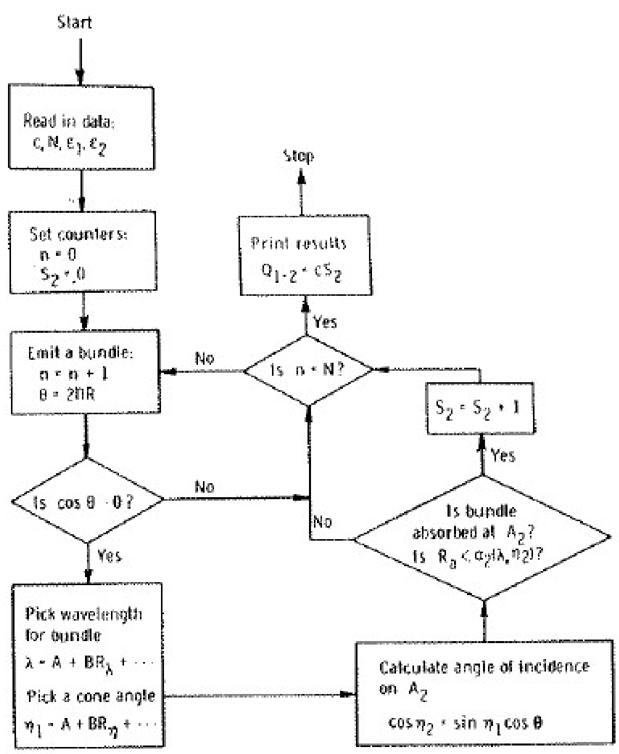
To simulate the absorption of energy bundles by A_2 with an absorptivity (by Kirchoff's law)

$$\alpha_2\left(\lambda,\eta_2\right) = \varepsilon_2\left(\lambda,\eta_2\right)$$

pick a random number ξ_α, the bundle is considered as absorbed when

$$\xi_\alpha \le \varepsilon_2\left(\lambda,\eta_2\right)$$

Flow chart for the computer program



3. Simulation of radiative absorption by a medium

For a beam of energy of initial intensity I_0

Absorption by a medium with absorption coefficient a(S) at an interval between S and S+dS

$$dI = I_0 e^{-\int_0^S a(s')ds'} a(S)\,dS$$

Probability density of a photon to be absorbed by the medium at S is

$$P(S) = \frac{dI}{I_0 dS} = a(S) e^{-\int_0^S a(s')ds'}$$

The Monte Carlo simulation of the absorption process is thus given by:

1. Let N = number of bundle
2. Energy per bundle = I_0 / N
3. pick a random number ξ_L
4. The distance L at which the bundle is absorbed is given by

$$\xi_L = \int_0^L P(S)\,dS = \int_0^L e^{-\int_0^S a(s')ds'} a(S)\,dS = 1 - e^{-\int_0^L a(s')ds'}$$

or determine L by

$$-\int_0^L a(s')\,ds' = \ln(1-\xi_L)$$

For a medium with constant absorption coefficient a, L is determined by

$$\xi_L = 1 - e^{-aL} \quad \text{or} \quad L = -\frac{1}{a}\ln(1-\xi_L)$$

4. Simulation of radiative emission by a medium

Radiative emission from a medium is isotropic

The emission from a volume dV at temperature T with wavelength λ in direction (η, ϕ) is given by

$$d^4Q_e = a(\lambda,T,P)\,i_{\lambda b}(T)\sin\eta\, d\eta\, d\phi\, d\lambda\, dV$$

Total energy emitted by the volume dV is

$$dQ_e = dV\int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} a(\lambda,T,P)\,i_{\lambda b}(T)\sin\eta\, d\eta\, d\phi\, d\lambda$$
$$= 4\pi dV\int_0^{\infty} a(\lambda,T,P)\,i_{\lambda b}(T)\, d\lambda$$

So the pdf for emission from a medium is

$$P(\eta) = \frac{1}{2}\sin\eta$$
$$P(\phi) = \frac{1}{2\pi}$$
$$P(\lambda) = \frac{\pi a(\lambda,T,P)\,i_{\lambda b}(T)}{a_p\sigma T^4}$$

where a_P is called the Planck mean absorption coefficient given by

$$a_p = \frac{\pi\int_0^{\infty} a(\lambda,T,P)\,i_{\lambda b}(T)\, d\lambda}{\sigma T^4}$$

Sampling of emission in (λ,η,ϕ) is given by

$$\xi_\eta = \int_0^{\eta} P(\eta')\,d\eta' = \frac{1}{2}\int_0^{\eta}\sin\eta'\,d\eta' = \frac{1-\cos\eta}{2}$$
$$\xi_\phi = \frac{\phi}{2\pi}$$
$$\xi_\lambda = \frac{\pi\int_0^{\lambda} a(\lambda',T,P)\,i_{\lambda b}(\lambda',T)\, d\lambda'}{a_p\sigma T^4}$$

Condition of radiative equilibrium (energy conservation) in a Monte Carlo calculation

At a given volume dV,

if $w =$ energy per bundle in the simulation
$S_{dV} =$ number of bundles absorbed by dV

then $dQ_{abs} =$ energy absorbed by volume dV
$= wS_{dV}$

Then energy conservation requires

$$dQ_e = 4a_p\sigma T^4 dV = dQ_{abs} = wS_{dV}$$

The temperature can then be determined by

$$T = \left(\frac{wS_{dV}}{4a_p\sigma dV} \right)^{1/4}$$

5. Example Calculation

A gray gas with constant absorption coefficient a is contained between two infinite parallel plates. Plate 1 is at temperature T_1 while plate 2 is at temperature $T_2 = 0$. The two plates are separated by a distance D. Calculate the heat transfer and the gas temperature distribution by the Monte Carlo method.

a. Emission from the lower plate at T_1

N = number of bundle emitted from the lower plate

$w =$ energy per bundle = $\sigma T_1^4 / N$

The direction of each bundle is determined by two random number with

$$\xi_\eta = \sin^2 \eta$$
$$\xi_\phi = \frac{\phi}{2\pi}$$

Since the medium is gray and the surface is black, there is no wavelength dependent and the sampling of wavelength is not required.

b. Absorption of an energy bundle

Since the problem is one-dimensional, the distance D between the two plates can be divided into k equal segments of length $\Delta x = D/k$. For each bundle emitted by the lower plate, the distance it travels prior to absorption, L, is given by

$$\xi_L = 1 - e^{-aL} \quad \text{or} \quad L = -\frac{1}{a}\ln\left(1 - \xi_L\right)$$

The volume element at which this bundle is absorbed is given by

$$j = \left[\frac{L\cos\eta}{\Delta x}\right] + 1$$

where $[x]$ stands for the greatest integer less than x. If $j \le k$, S_j, which is the number of bundles absorbed by the jth volume, will increase by one. If $j > k$, S_{w2}, which is the number of bundles absorbed by the upper wall, will increase by one.

c. Absorption of an energy bundle

Since the medium is at radiative equilibrium, the absorbed energy bundle will be re-emitted from the same element to conserve energy. The direction of emission is sampled by

$$\xi_\eta = \frac{1-\cos\eta}{2}$$

$$\xi_\phi = \frac{\phi}{2\pi}$$

The absorption of this bundle is then determined by the same procedure as in step b. The position of the next absorption point is then determined by

$$x - x_0 = L\cos\eta$$

With x_0 being the position of the previous absorption and L is sampled by

$$\xi_L = 1 - e^{-aL} \quad \text{or} \quad L = -\frac{1}{a}\ln\left(1-\xi_L\right)$$

This process will continue until the bundle reaches a black boundary.

d. Heat transfer and temperature distribution

Let

$S_{w1} =$ number of bundles absorbed by the lower wall
$S_{w2} =$ number of bundles absorbed by the upper wall
$S_j =$ number of bundles absorbed by the jth element

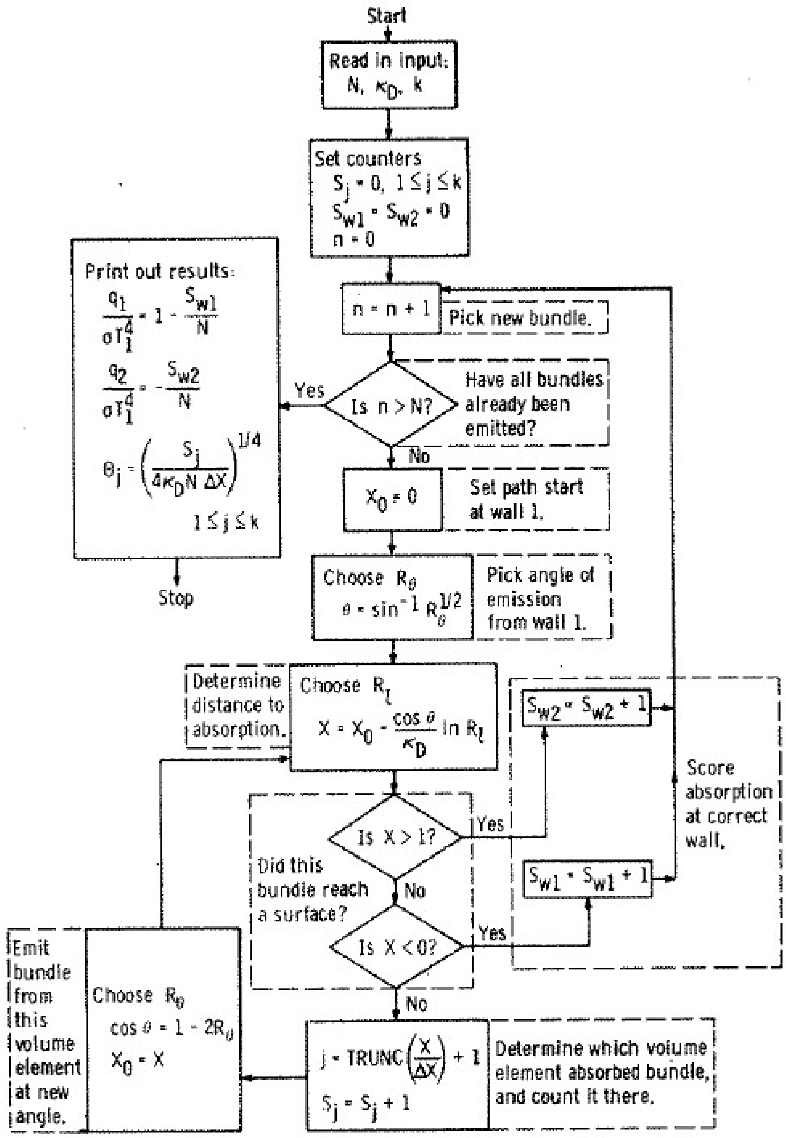
Then

$$\frac{q_1}{\sigma T_1^4} = \frac{w\left(N - S_{w1}\right)}{wN} = 1 - \frac{S_{w1}}{N}$$

$$-\frac{q_2}{\sigma T_1^4} = \frac{S_{w2}}{N} = 1 - \frac{S_{w1}}{N}$$

$$\Theta_j = \frac{T_j}{T_1} = \left(\frac{wS_j}{4a\sigma\Delta x T_1^4}\right)^{1/4} = \left(\frac{S_j}{4aN\Delta x}\right)^{1/4}$$

e. <u>Flow Chart of the program</u>



5. <u>Monte Carlo simulation of an absorbing, emitting and scattering medium</u>

Physics:

The medium is characterized by an extinction coefficient κ_e, an absorption coefficient κ_a and a scattering coefficient κ_s which are related by

$$\kappa_e = \kappa_a + \kappa_s$$

The ratio of the scattering coefficient to the extinction coefficient is called the scattering albedo, defined as

$$\omega = \frac{\kappa_s}{\kappa_e}$$

The scattered energy intensity is related to the incoming intensity by a phase function $S(\eta',\phi')$ with (η',ϕ') being the polar and azimutual angles measured relative to the direction of the incoming intensity.

Mathematically, the probability that a photon (or energy bundle) scattered by the medium, into a direction (η',ϕ') within a solid angle $d\omega'$ relative to the incident direction, is given by

$$\frac{S(\eta',\phi')\,d\omega'}{4\pi} = \left[\frac{S(\eta')\sin\eta' d\eta'}{2}\right]\left(\frac{d\phi'}{2\pi}\right)$$

Note that the scattering phase function generally is only a function of η'

The pdf for the scattering process is thus

$$P(\eta') = \frac{S(\eta')\sin\eta'}{2}$$

$$P(\phi') = \frac{1}{2\pi}$$

The Monte Carlo Simulation of a scattering process

a. Determine the "extinction" length L with a random number ξ_L by

$$L = -\frac{1}{\kappa_e} \ln\left(1 - \xi_L\right)$$

b. Determine whether the bundle is absorbed by a random number ξ_ω by

$\xi_\omega < \omega$ scattered

$\xi_\omega > \omega$ absorbed

c. If the bundle is absorbed and the medium is at radiative equilibrium, re-emit the bundle in the direction given by

$$\xi_\eta = \frac{1 - \cos\eta}{2}$$

$$\xi_\phi = \frac{1}{2\pi}$$

d. If the bundle is scattered, direct the bundle into direction $\left(\eta', \phi'\right)$ by the relation

$$\xi_{\eta'} = \frac{1}{2}\int_0^{\eta'} S\left(x\right)\sin x dx$$

$$\xi_{\phi'} = \frac{\phi'}{2\pi}$$